

A note on capacity Muckenhoupt weights with respect to the Hausdorff content

Yangzhi Zhang and Ciqiang Zhuo

ABSTRACT. Let $p \in [1, \infty)$ and $\delta \in (0, n]$. Recently, to characterize the weighted boundedness of the maximal operator on Choquet integrals based on Hausdorff content $\mathcal{H}_\infty^\delta$, a class of capacity Muckenhoupt weights is introduced, denoted by $\mathcal{A}_{p,\delta}$, and, for any fixed $\delta \in (0, n]$, this new class of weights is proved to satisfy the self-improving property with respect to the index $p \in (1, \infty)$. In this note, we show that, for every fixed $p \in [1, \infty)$, the class of capacity Muckenhoupt weights $\mathcal{A}_{p,\delta}$ fails to satisfy the self-improving property with respect to the index $\delta \in (0, n]$.

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1. Introduction

Let $\delta \in (0, n]$ and $\mathcal{H}_\infty^\delta$ be the *Hausdorff content* (or δ -dimensional Hausdorff capacity) of a subset $E \subset \mathbb{R}^n$ (see Section 2 for the definition). For any subset $E \subset \mathbb{R}^n$ and any non-negative function f defined on E , the *Choquet integral* of f on E is defined by setting

$$\int_E f(x) d\mathcal{H}_\infty^\delta := \int_0^\infty \mathcal{H}_\infty^\delta(\{x \in E : f(x) > t\}) dt.$$

For any $p \in (0, \infty)$, the p -Choquet integral with respect to the capacity of a function f on E is defined by setting

$$\|f\|_{L^p(E, \mathcal{H}_\infty^\delta)} := \left[\int_E |f(x)|^p d\mathcal{H}_\infty^\delta \right]^{\frac{1}{p}}.$$

For any $p \in (0, \infty)$ and $E \subset \mathbb{R}^n$, we always use $L^p(E, \mathcal{H}_\infty^\delta)$ to denote the space of all functions f such that the quasi-norm $\|f\|_{L^p(E, \mathcal{H}_\infty^\delta)}$ is finite, and denote by

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Zhuo is the corresponding author.

$L^1_{\text{loc}}(\mathbb{R}^n, \mathcal{H}^\delta_\infty)$ the set of all functions f satisfying that $f \in L^1(K, \mathcal{H}^\delta_\infty)$ for any compact set $K \subset \mathbb{R}^n$. For the further theory of Choquet integrals with respect to Hausdorff content, we refer the reader to [1, 2, 11, 12].

Recently, inspired by the recent extensive research on capacity (see, for example, [3, 6, 9]), Huang et al. [7] introduced a class of capacity Muckenhoupt weight with respect to the Hausdorff content. Precisely, for $p \in [1, \infty)$, the *capacity Muckenhoupt weight* $\mathcal{A}_{p,\delta}$ is defined to be the set of all $\mathcal{H}^\delta_\infty$ -capacity weight on $\mathbb{R}^n$ such that $[w]_{\mathcal{A}_{p,\delta}} < \infty$, where for $p \in (1, \infty)$,

$$[w]_{\mathcal{A}_{p,\delta}} := \sup_{Q \subset \mathbb{R}^n} \left\{ \frac{1}{\mathcal{H}^\delta_\infty(Q)} \int_Q w(x) d\mathcal{H}^\delta_\infty \right\} \times \left\{ \frac{1}{\mathcal{H}^\delta_\infty(Q)} \int_Q w(x)^{-\frac{1}{p-1}} d\mathcal{H}^\delta_\infty \right\}^{p-1}$$

with the supremum being taken over all cubes Q of $\mathbb{R}^n$, and for $p = 1$,

$$[w]_{\mathcal{A}_{1,\delta}} := \inf \left\{ K : \mathcal{M}_{\mathcal{H}^\delta_\infty} w \leq Kw \text{ for } \mathcal{H}^\delta_\infty\text{-almost everywhere} \right\}.$$

Here, a function w is called a $\mathcal{H}^\delta_\infty$ -capacity weight on $\mathbb{R}^n$, we mean that $w \in L^1_{\text{loc}}(\mathbb{R}^n, \mathcal{H}^\delta_\infty)$ and $w \in (0, \infty)$ almost everywhere on $\mathbb{R}^n$ with respect to the Hausdorff content $\mathcal{H}^\delta_\infty$, and $\mathcal{M}_{\mathcal{H}^\delta_\infty}$ stands for the Hardy-Littlewood maximal operator associated with the Hausdorff content defined by setting

$$\mathcal{M}_{\mathcal{H}^\delta_\infty} f(x) := \sup_{Q \ni x} \frac{1}{\mathcal{H}^\delta_\infty(Q)} \int_Q |f(y)| d\mathcal{H}^\delta_\infty,$$

with the supremum ranging over all cubes Q containing x .

When $\delta = n$, the capacity Muckenhoupt weight $\mathcal{A}_{p,n}$, restricted on all Lebesgue measurable subsets of $\mathbb{R}^n$, is equivalent to the classical Muckenhoupt weight A_p introduced in [10] and have several interesting applications. For example, such capacity Muckenhoupt weight gives a sufficiency and necessity for the boundedness of the maximal operator $\mathcal{M}_{\mathcal{H}^\delta_\infty}$ on $L^p(\mathcal{H}^\delta_\infty)$ (see [7, Theorems 1.1 and 1.3]). As an analogue of the classical BMO space, functions with δ -dimensional bounded mean oscillation also enjoy an equivalent characterization in terms of the capacity Muckenhoupt weight (see [8]).

Besides the above interesting application, Huang et al. in [7] further established several properties of the class of capacity Muckenhoupt weights $\mathcal{A}_{p,\delta}$, such as the reverse Hölder inequality, the Jones factorization theorem, and monotonicity with respect to p . Particularly, for a given $\delta \in (0, n]$, $\mathcal{A}_{p,\delta}$ has the self-improving property with respect to the index p , namely, if $w \in \mathcal{A}_{p,\delta}$ with $p \in (1, \infty)$, then there exists $q \in (1, p)$ such that $w \in \mathcal{A}_{q,\delta}$. On the other hand, by [7, Proposition 2.18], we know that, for any fixed $p \in [1, \infty)$, if $0 < \delta < \beta \leq n$, then $\mathcal{A}_{p,\delta} \subsetneq \mathcal{A}_{p,\beta}$. Therefore, a natural interesting question arises:

Question. Does the class of capacity Muckenhoupt weights have the self-improving property with respect to the dimension δ ? That is, for any given $p \in [1, \infty)$ and $w \in \mathcal{A}_{p,\delta}$ with $\delta \in (0, n]$, does there exist $\delta_1 \in (0, \delta)$ such that $w \in \mathcal{A}_{p,\delta_1}$?

The main result of this note gives a negative answer to the above question.

Theorem 1.1. *Let $\delta \in (0, n]$ and $p \in [1, \infty)$. Then there exists a capacity weight w such that $w \in \mathcal{A}_{p,\delta}$ but $w \notin \mathcal{A}_{p,\beta}$ for any $\beta \in (0, \delta)$, namely, the capacity weights $\mathcal{A}_{p,\delta}$ do not have the self-improving property with respect to the index δ .*

The proof of Theorem 1.1 is constructive, which relies on the critical properties of Hausdorff dimension. For any set $E \subset \mathbb{R}^n$, the Hausdorff dimension $\dim_{\mathcal{H}}(E)$ and the s -dimensional Hausdorff measure $\mathcal{H}^s(E)$ satisfy

$$\mathcal{H}^s(E) = \begin{cases} \infty, & 0 \leq s < \dim_{\mathcal{H}}(E), \\ 0, & s > \dim_{\mathcal{H}}(E), \end{cases}$$

see [5, (3.11)]. Here, for any subset E , $s = \dim_{\mathcal{H}}(E)$ is the critical value for the Hausdorff measure $\mathcal{H}^s(E)$ from ∞ to 0. Moreover, it was pointed out in [5] that the Hausdorff measure at the critical value would be zero, a finite positive number or infinity. However, it seems that there is no specific example satisfying

$$\dim_{\mathcal{H}}(E) = s \quad \text{and} \quad \mathcal{H}^s(E) = 0. \quad (1.1)$$

In this note, we construct a specific subset $E \subset \mathbb{R}^n$ satisfying (1.1), through a method from the iterative construction of generalized Cantor sets (see [5]), by adjusting the sparsity of the Cantor set and the decay rate of the interval lengths. Using this example, we obtain a weight w satisfying the requirement of Theorem 1.1.

2. Proof of Theorem 1.1

We first recall the definitions of Hausdorff measure and Hausdorff dimension (see [4, 5]).

Definition 2.1. Let $\delta \in (0, n]$. For any $\epsilon \in (0, \infty]$ and any subset $E \subset \mathbb{R}^n$, the δ -dimensional ϵ -Hausdorff outer measure of E is defined by setting

$$\mathcal{H}_\epsilon^\delta(E) := \inf \left\{ \sum_i [l(Q_i)]^\delta : E \subset \bigcup_i Q_i, l(Q_i) < \epsilon \right\},$$

where the infimum is taken over all finite or countable cubes from $\mathbb{R}^n$ coverings $\{Q_i\}_i$ of E and $l(Q)$ denotes the edge length of the cube Q .

Particularly, $\mathcal{H}_\infty^\delta(E)$ is called the *Hausdorff content* (or δ -dimensional Hausdorff capacity) of E . Furthermore, the δ -dimensional Hausdorff measure of E is defined as

$$\mathcal{H}^\delta(E) := \lim_{\epsilon \rightarrow 0^+} \mathcal{H}_\epsilon^\delta(E). \quad (2.1)$$

We remark that the limit in (2.1) exists for any subset $E \subset \mathbb{R}^n$, although the limiting value can be (and usually is) zero or infinity. Moreover,

$$\mathcal{H}^\delta(E) = \sup_{\epsilon > 0} \mathcal{H}_\epsilon^\delta(E).$$

Definition 2.2. Let $E \subset \mathbb{R}^n$ with $E \neq \emptyset$. Then the *Hausdorff dimension* of E is defined by

$$\dim_{\mathcal{H}}(E) := \inf \{s \geq 0 : \mathcal{H}^s(E) = 0\}$$

In particular, the Hausdorff dimension of the empty set is defined to be 0.

We point out that, for any set E , $\dim_{\mathcal{H}}(E)$ is also the critical value of s at which $\mathcal{H}^s(E)$ “jumps” from ∞ to 0 (see [5, p.48]). Alternatively, when $s < \dim_{\mathcal{H}}(E)$, $\mathcal{H}^s(E) = \infty$ and, when $s > \dim_{\mathcal{H}}(E)$, $\mathcal{H}^s(E) = 0$. Thus,

$$\dim_{\mathcal{H}}(E) = \sup \{\delta \geq 0 : \mathcal{H}^\delta(E) = +\infty\}.$$

When $\dim_{\mathcal{H}}(E) = s$, $\mathcal{H}^s(E)$ may be zero, a finite positive number or infinity. However, it seems that there is no specific example for the case “zero” or “infinity” to our knowledge. The following conclusion is the main step for proof of Theorem 1.1, which confirms the possibility that, for any $s \in (0, n]$, when $\dim_{\mathcal{H}}(E) = s$, $\mathcal{H}^s(E) = 0$.

Proposition 2.3. Let $\delta \in (0, n]$. Then there exists a subset $E \subset \mathbb{R}^n$ such that

$$\dim_{\mathcal{H}}(E) = \delta \quad \text{and} \quad \mathcal{H}^\delta(E) = 0.$$

To prove Proposition 2.3, we need the following mass distribution principle (see [5, p.67]). Let μ be a measure on $\mathbb{R}^n$. The smallest closed set $F \subset \mathbb{R}^n$ satisfying $\mu(\mathbb{R}^n \setminus F) = 0$ is called the support of μ , denoted by $\text{spt } \mu$. Moreover, for any F a subset of $\mathbb{R}^n$, if $\text{spt } \mu \subset F$ and $0 < \mu(F) < \infty$, then μ is called a mass distribution on F .

Lemma 2.4 (Mass Distribution Principle). Let $s \in (0, \infty)$, F be a subset of $\mathbb{R}^n$ and μ a mass distribution on F . If there exist positive constants K and ϵ such that, for every set $U \subset \mathbb{R}^n$ with $\text{diam } U \leq \epsilon$,

$$\mu(U) \leq K(\text{diam } U)^s,$$

then

$$\mathcal{H}^s(F) \geq \frac{\mu(F)}{K} \quad \text{and} \quad \dim_{\mathcal{H}}(F) \geq s.$$

We are now ready to prove Proposition 2.3.

Proof of Proposition 2.3. We only give the proof of the case $n = 1$ and $\delta \in (0, 1]$, since it is not difficulty to extend to the general n .

Step 1: The Construction of E . The subset E is constructed by induction. For the initial step ($k = 0$), let $I_{0,1} = [0, 1]$. For $k \in \mathbb{N}$, we take operation on closed intervals $I_{k-1,i}$ ($i = 1, 2, \dots, N_{k-1}$) from the $(k-1)$ -th iteration as follows, where N_k denotes the number of all intervals in the k -step ($N_0 = 1$).

- (i) Select two closed subintervals at the endpoints of the closed interval $I_{k-1,i}$ with length

$$L_k := 2^{-\frac{k}{\delta} - \frac{k}{\ln(k+1)}},$$

and remove the middle unselected part;

- (ii) Denote all selected closed intervals in the k -th iteration by $\{I_{k,i}\}_{i=1}^{N_k}$, where the number of intervals satisfies

$$N_k = 2 \cdot N_{k-1} = 2^k.$$

Let $E_k := \bigcup_{i=1}^{N_k} I_{k,i}$. Then the sparse generalized Cantor set is defined as the intersection of the retained intervals from all iterations:

$$E := \bigcap_{k=1}^{\infty} E_k.$$

By the nested closed intervals theorem, $E \neq \emptyset$ and $E \subset [0, 1] \subset \mathbb{R}$.

Step 2: Verification for $\mathcal{H}^\delta(E) = 0$. For any $k \geq 1$, let $\epsilon_k := L_k$ (the diameter of the intervals at the k -th iteration). Then $\{I_{k,i}\}_{i=1}^{N_k}$ forms an ϵ_k -cover of E and hence

$$\mathcal{H}_{\epsilon_k}^\delta(E) \leq \sum_{i=1}^{N_k} (\text{diam } I_{k,i})^\delta = 2^k \cdot \left(2^{-\frac{k}{\delta} - \frac{k}{\ln(k+1)}} \right)^\delta = 2^{-\frac{k\delta}{\ln(k+1)}}.$$

As $k \rightarrow \infty$, $\epsilon_k = L_k \rightarrow 0^+$ and $\frac{k\delta}{\ln(k+1)} \rightarrow +\infty$. Therefore,

$$\mathcal{H}^\delta(E) = \lim_{k \rightarrow \infty} \mathcal{H}_{\epsilon_k}^\delta(E) \leq \lim_{k \rightarrow \infty} 2^{-\frac{k\delta}{\ln(k+1)}} = 0,$$

which implies $\mathcal{H}^\delta(E) = 0$.

Step 3: Proof for $\dim_{\mathcal{H}}(E) = \delta$. By the definition of Hausdorff dimension, it is enough to show the following two statements:

- (1) For any $s > \delta$, $\mathcal{H}^s(E) = 0$;
- (2) For any $s < \delta$, $\dim_{\mathcal{H}}(E) \geq s$.

For (1), we first have

$$\sum_{i=1}^{N_k} (\text{diam } I_{k,i})^s = 2^k \cdot 2^{-\frac{ks}{\delta} - \frac{ks}{\ln(k+1)}} = 2^{k(1 - \frac{s}{\delta}) - \frac{ks}{\ln(k+1)}}. \quad (2.2)$$

Since $1 - \frac{s}{\delta} < 0$ by $s > \delta$, it follows that the right-hand side of (2.2) tends to 0 as $k \rightarrow \infty$. Thus, $\mathcal{H}^s(E) = 0$ when $s > \delta$.

To show (2), we first define a mass distribution μ on E as follows. For each interval $I_{k,i}$ at the k -th iteration, set $\mu(I_{k,i}) := \frac{1}{N_k} = \frac{1}{2^k}$; for any set $A \subset \mathbb{R}$, if $A \cap E_k = \emptyset$ for some $k \in \mathbb{N}$, then we set $\mu(A) = 0$. By [5, Proposition 1.7], μ can be extended to a measure on $\mathbb{R}$ with $\text{spt } \mu \subset E$ and

$$\mu(E) = \lim_{k \rightarrow \infty} \mu(E_k) = 1.$$

Thus μ is a mass distribution on E . Moreover, for any $k \in \mathbb{N}$ and any $i = 1, 2, \dots, N_k$,

$$\mu(I_{k,i}) = \frac{1}{2^k} \leq K(\text{diam } I_{k,i})^s, \quad (2.3)$$

where

$$K := \max_{k \in \mathbb{N}} 2^{-k(1-\frac{s}{\delta}) + \frac{ks}{\ln(k+1)}},$$

which is finite due to $s < \delta$.

Let $U \subset \mathbb{R}$ be any set with $\rho := \text{diam } U \in (0, 2^{-\frac{1}{\delta}-2}]$. Since L_k is strictly decreasing to 0 and $L_1 = 2^{-\frac{1}{\delta}-\frac{1}{\ln 2}} > 2^{-\frac{1}{\delta}-2}$, it follows that

$$(0, 2^{-\frac{1}{\delta}-2}] \subset \bigcup_{k \in \mathbb{N}} [L_{k+1}, L_k). \quad (2.4)$$

Furthermore, for any $k \in \mathbb{N}$,

$$\frac{L_k}{L_{k+1}} = 2^{\frac{1}{\delta} + \frac{k+1}{\ln(k+2)} - \frac{k}{\ln(k+1)}} \leq 2^{\frac{1}{\delta} + \frac{2}{\ln 3} - \frac{1}{\ln 2}} =: c_0.$$

Hence, $[L_{k+1}, L_k) \subset [\frac{1}{c_0} L_k, L_k)$. Combining this with (2.4), we obtain

$$(0, 2^{-\frac{1}{\delta}-2}] \subset \bigcup_{k \in \mathbb{N}} [L_{k+1}, L_k) \subset \bigcup_{k \in \mathbb{N}} \left[\frac{1}{c_0} L_k, L_k \right).$$

Consequently, there exists $k_0 \in \mathbb{N}$ such that $\frac{1}{c_0} L_{k_0} \leq \rho < L_{k_0}$. If U intersects with three or more intervals of the k_0 -th iteration with $k_0 \geq 2$, then $\rho > L_{k_0}$, which is a contradiction. Hence $U \cap E$ is covered by at most two intervals from the k_0 -th iteration, denoted by $I_{k_0, i_1}, I_{k_0, i_2}$. By subadditivity and (2.3),

$$\mu(U) = \mu(U \cap E) \leq \mu(I_{k_0, i_1}) + \mu(I_{k_0, i_2}) = \frac{1}{2^{k_0-1}} \leq 2K(L_{k_0})^s \leq 2Kc_0^s \rho^s.$$

Therefore, from the mass distribution principle (see Lemma 2.4), we deduce that $\dim_{\mathcal{H}}(E) \geq s$. Thus, (2) holds true. This completes the proof of Proposition 2.3. $\square$

Proof of Theorem 1.1. According to the proof of Proposition 2.3, for the given cube $Q_0 := [0, 2]^n$, there exists a subset $E \subset Q_0$ such that $\dim_{\mathcal{H}}(E) = \delta$ and $\mathcal{H}^\delta(E) = 0$. Define

$$w(x) := \begin{cases} 1, & x \in \mathbb{R}^n \setminus E, \\ 0, & x \in E. \end{cases}$$

Since the δ -Hausdorff capacity and δ -Hausdorff measure share the same null sets (see [4, Lemma 1, p.64]), it follows that $\mathcal{H}_\infty^\delta(E) = 0$ and hence $w(x) = 1$ for $\mathcal{H}_\infty^\delta$ -a.e. $x \in \mathbb{R}^n$ which further implies $w \in \mathcal{A}_{p, \delta}$ for any $p \in [1, \infty)$.

However, for any $\beta \in (0, \delta)$, we have $\mathcal{H}^\beta(E) = \infty$ by the definition of Hausdorff measure. Using again that the δ -Hausdorff capacity and δ -Hausdorff measure have the same null sets, we obtain $\mathcal{H}_\infty^\beta(E) > 0$. This means that, for the above cube Q_0 ,

$$\int_{Q_0} w(x)^{-\frac{1}{p-1}} d\mathcal{H}_\infty^\beta \geq \int_E w(x)^{-\frac{1}{p-1}} d\mathcal{H}_\infty^\beta = \infty,$$

and hence $w \notin \mathcal{A}_{p,\beta}$ for any $p \in (1, \infty)$. Moreover, it is easy to know that $\mathcal{M}_{\mathcal{H}_\infty^\delta} w(x) > 0$ on E and, therefore, $w \notin \mathcal{A}_{1,\beta}$. This completes the proof of Theorem 1.1. $\square$

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(Yangzhi Zhang) MOE-LCSM, SCHOOL OF MATHEMATICS AND STATISTICS, HUNAN NORMAL UNIVERSITY, CHANGSHA, HUNAN 410081, P. R. CHINA
yzzhang@hunnu.edu.cn

(Ciqiang Zhuo) MOE-LCSM, SCHOOL OF MATHEMATICS AND STATISTICS, HUNAN NORMAL UNIVERSITY, CHANGSHA, HUNAN 410081, P. R. CHINA
cqzhuo87@hunnu.edu.cn

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