

Weaving phase retrieval and weaving norm retrieval in tensor product space

Suman Dowerah and Saikat Mukherjee

ABSTRACT. Phase retrieval and norm retrieval sequences for Hilbert spaces were introduced by Balan et al. in 2006 and by Bahmanpour et al. in 2015, respectively. In signal processing, phase retrieval plays a crucial role in reconstructing a signal from its intensity measurements. Similarly, norm retrieval restores the norm of a signal from its intensity measurements. Bemrose et al., in 2016, instigated weaving frames for Hilbert spaces. Weaving phase and norm retrieval sequences for a Hilbert space were recently studied by Dowerah et al.

In this paper, we study weaving phase and weaving norm retrieval sequences in the tensor product of Hilbert spaces defined by Folland, in the year 1995, as a space of bounded antilinear maps between Hilbert spaces. We investigate the preservation and transfer of the weaving property of spanning sequences, the complement property, phase retrieval, and norm retrieval when passing from Hilbert spaces to their tensor product spaces. Several characterizations and methods of construction of weaving phase and norm retrieval sequences in tensor product spaces are discussed.

CONTENTS

1. Introduction	1404
2. Preliminaries	1406
3. Main Results	1410
4. Conclusion	1418
References	1419

1. Introduction

Reconstruction of signals is one of the important and significant problems in engineering. Frames are mathematical tools for reconstructing signals. Duffin and Schaeffer [10] introduced frames for Hilbert spaces in 1952. Frames, just like orthonormal bases, help to recover lost signals in a stable manner. Frames have a more flexible structure than orthonormal bases. It gained popularity through Daubechies' work [9] in 1986. Many researchers have made significant

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contributions to frame theory, such as fusion frames [7], K-frames [14], wavelet frames [8], weaving frames [5], and so on.

The concept of weaving frames was introduced by Bemrose et al. [5] in 2016. Weaving of two sequences means entry-wise substitution of some elements of one sequence by corresponding elements of another sequence so that the resulting sequence preserves the characteristics of the original sequences. The idea of weaving frames has proven to be beneficial in wireless sensor networks and signal processing since it allows the replacement of a damaged sensor.

The process of reconstruction of signals using frames becomes difficult whenever signal information is lost partially. For example, if the signal's phase is lost, phase retrieval sequences are used to regain the lost signal. The concept of phase retrieval for Hilbert spaces was introduced by Balan et al. [2] in 2006. Mathematicians have worked on phase retrieval by frames, subspaces, and operators. Phase retrieval has many applications, such as speech recognition technology [21], X-ray crystallography [15], etc.

Similar to the concept of phase retrieval, Bahmanpour et al. [1] introduced norm retrieval for Hilbert spaces in 2015. Norm retrieval means regaining the norm of the lost signal from the phaseless measurements. Researchers have studied norm retrieval by frames and subspaces.

Phase retrieval and norm retrieval have been extensively studied in Hilbert spaces due to their significant theoretical importance and practical applications. As many real-world problems involve multidimensional or composite systems, it is natural to investigate whether these properties can be extended to tensor product Hilbert spaces. Lower dimensional techniques can be expanded to higher dimensions using the tensor product of spaces. The geometry of tensor product Hilbert spaces is richer than that of the individual Hilbert spaces. Studying geometric properties such as orthogonality, inner products, and operator representations in tensor product spaces provides a deeper understanding of composite systems. This geometric viewpoint is particularly useful in investigating phase retrieval, norm retrieval, and weaving, and it broadens their applications to multidimensional signal processing, quantum information theory, and other areas involving composite Hilbert spaces.

Over the last few years, researchers worked on various generalizations of weaving frames, namely, weaving K-frames, weaving G-frames, weaving fusion frames, weaving Schauder frames, etc. Dowerah and Mukherjee [11] introduced weaving phase retrieval and weaving norm retrieval sequences in 2022. Khosravi and Asgari [19], in 2003, briefly discussed frames in the tensor product of Hilbert spaces. Many researchers have worked on various types of frames in tensor product spaces, which include fusion frames and G-frames [18], dual frames [24], phase retrieval by dual frames [22], 2-frames [23], frames in tensor product space [4, 20], G-frames and operator valued-frames [16], etc. The above-mentioned research work motivates us to explore and analyze weaving phase retrieval sequences and weaving norm retrieval sequences in tensor product spaces.

Following notations are used throughout the paper. $\mathcal{H}$, $\mathcal{H}_1$, $\mathcal{H}_2$ denote separable Hilbert spaces, $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ denotes the space of bounded linear operators from $\mathcal{H}_1$ into $\mathcal{H}_2$, and we write $\mathcal{L}(\mathcal{H})$ for $\mathcal{L}(\mathcal{H}, \mathcal{H})$. I, J are countable index sets. $I_{\mathcal{H}}$ stands for the identity operator on $\mathcal{H}$. We write $\{u_i\}_{i \in I}$ spans $\mathcal{H}$ if $\overline{\text{span}\{u_i : i \in I\}} = \mathcal{H}$. We use $\|\cdot\|$ to denote the norm of a vector in $\mathcal{H}$ or the norm of a bounded operator; whereas $|||\cdot|||$ denotes the norm of an element in the tensor product of Hilbert spaces.

In this paper, we study weaving phase retrieval and weaving norm retrieval sequences in tensor product of Hilbert spaces and establish several characterizations and construction methods. We also introduce the notions of weaving spanning sequences and sequences with the weaving complement property in both Hilbert spaces and tensor product of Hilbert spaces, and investigate their fundamental properties. Furthermore, we present a counterexample to a previously published result, demonstrating that it does not hold in full generality and thereby correcting an existing result in the literature. Through these results, we aim to contribute to a more accurate understanding of weaving properties in Hilbert spaces and tensor product Hilbert spaces. The paper is organized as follows: Section 2 contains the basic definitions and preliminary results necessary for a better understanding of the paper. The main results of the paper are presented in Section 3. The conclusion of the paper is presented in Section 4.

2. Preliminaries

In this section we recall basic definitions and results needed for this paper.

2.1. Frame. Frames are mathematical tools for signal reconstruction in Hilbert spaces. We refer to the book by Ole Christensen [8] for a detailed study about frames.

Definition 2.1. [8] A sequence $\{u_i\}_{i \in I}$ in $\mathcal{H}$ is said to be a frame for $\mathcal{H}$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_{i \in I} |\langle x, u_i \rangle|^2 \leq B\|x\|^2, \quad \forall x \in \mathcal{H}.$$

The constants A and B are called lower and upper frame bounds respectively. In case $B = A$ then the frame is called A -tight frame.

For example, let $\{e_i\}$ be an orthonormal basis for $\mathcal{H}$ then $\{e_1, e_1, e_2, e_3, e_4, \dots\}$ is a frame for $\mathcal{H}$ with $A = 1$ and $B = 2$.

2.2. Phase retrieval and norm retrieval. Reconstruction of a signal becomes challenging if there is a partial loss of information. Reconstructing signals from phaseless measurements is possible using phase retrieval sequences.

Definition 2.2. [2] A sequence of vectors $\{u_i\}_{i \in I}$ in $\mathcal{H}$ does phase retrieval for $\mathcal{H}$ if for $x, y \in \mathcal{H}$ satisfying

$$|\langle x, u_i \rangle| = |\langle y, u_i \rangle| \quad \forall i \in I$$

imply $x = cy$ for some $|c| = 1$.

Unlike finite dimensional spaces, a phase retrieval sequence may not be a frame in infinite dimensional spaces.

Definition 2.3. [6] A sequence of closed subspaces $\{W_i\}_{i \in I}$ with corresponding orthogonal projections $\{P_i\}_{i \in I}$ in $\mathcal{H}$ does phase retrieval for $\mathcal{H}$ if for $x, y \in \mathcal{H}$ satisfying

$$\|P_i x\| = \|P_i y\| \quad \forall i \in I$$

imply $x = cy$ for some $|c| = 1$.

Norm retrieval was introduced by Bahmanpour et al. [1] in 2015. It was shown that norm retrieval is the necessary condition a phase retrieval subspace must have so that its orthogonal complement also does phase retrieval.

Definition 2.4. [1] A sequence of vectors $\{u_i\}_{i \in I}$ in $\mathcal{H}$ does norm retrieval for $\mathcal{H}$ if for $x, y \in \mathcal{H}$ satisfying

$$|\langle x, u_i \rangle| = |\langle y, u_i \rangle| \quad \forall i \in I$$

imply $\|x\| = \|y\|$.

Thus, a norm retrieval sequence helps to restore the norm of the lost signal. Every tight frame does norm retrieval. It is obvious that a phase retrieval sequence is also a norm retrieval sequence.

One of the important properties of a phase retrieval sequence is the complement property.

Definition 2.5. [2] A sequence of vectors $\{u_i\}_{i \in I}$ in $\mathcal{H}$ is said to have complement property if for every $\sigma \subset I$, either $\overline{\text{span}}\{u_i\}_{i \in \sigma} = \mathcal{H}$ or $\overline{\text{span}}\{u_i\}_{i \in \sigma^c} = \mathcal{H}$

A phase retrieval sequence has complement property. In fact, in [2], it was proved that phase retrieval is equivalent to complement property for real Hilbert spaces.

2.3. Weaving phase retrieval and weaving norm retrieval. Consider two frames $\{u_i\}_{i \in I}$ and $\{u'_i\}_{i \in I}$ having similar characteristics. $\{u_i\}_{i \in I}$ may cease to be a frame if some of the frame elements are removed. However, we may still be able to reconstruct the signals by replacing the corresponding elements from $\{u'_i\}_{i \in I}$ in the first frame. This process of reconstructing signals by mixing two frames is the basic idea of weaving frames. Weaving frames or woven frames was introduced by Bemrose et al. [5] in the year 2016.

Definition 2.6. [5] Two frames $\{u_i\}_{i \in I}$ and $\{u'_i\}_{i \in I}$ in $\mathcal{H}$ are called weaving frames for $\mathcal{H}$ if for every $\sigma \subset I$, $\{u_i\}_{i \in \sigma} \cup \{u'_i\}_{i \in \sigma^c}$ is a frame for $\mathcal{H}$.

Weaving phase retrieval and weaving norm retrieval sequences are defined as follows.

Definition 2.7. [11] Given two sequences $u = \{u_i\}_{i \in I}$ and $u' = \{u'_i\}_{i \in I}$ in $\mathcal{H}$, u and u' are said to do weaving phase(norm) retrieval for $\mathcal{H}$ if for every non-trivial $\sigma \subset I$, the collection $\{u_i\}_{i \in \sigma} \cup \{u'_i\}_{i \in \sigma^c}$ does phase(norm) retrieval for $\mathcal{H}$.

In $\mathbb{R}^2$, $\{2e_1, -e_2, e_1 + e_2\}$ and $\{e_1, 6e_2, e_1 - 4e_2\}$ do weaving phase retrieval and hence weaving norm retrieval.

Phase retrieval, norm retrieval and their wovenness have been widely studied in Hilbert spaces. To extend these concepts to composite Hilbert spaces, we first discuss the notion of tensor product of Hilbert spaces.

2.4. Tensor product. Tensor product spaces are used to combine two or more spaces into a larger space, making it possible to study more complex mathematical systems. Tensor product Hilbert spaces help us study the geometry of a larger Hilbert space formed by combining two Hilbert spaces. They allow us to understand how properties such as inner products, orthogonality, and operators behave in the combined space. This makes tensor product space a natural setting for extending the ideas of frames, phase retrieval, norm retrieval, and their weaving, and for studying problems arising in quantum mechanics, signal processing, and other applications. Tensor product of Hilbert spaces are used to model multi-particle systems in quantum mechanics, multi-variable function spaces, quantum information networks, etc.

There are multiple ways to define tensor product spaces. Folland [13], in 1995, discussed the tensor product of Hilbert spaces. He referred it as the space of bounded antilinear operators. This operator-based approach is simple and useful because it naturally involves concepts such as boundedness, adjoints, and inner products. Since phase retrieval and norm retrieval rely heavily on these operator concepts, Folland's definition provides a suitable and effective framework for studying these problems in tensor product Hilbert spaces. In this paper we consider the tensor product spaces introduced by Folland.

Definition 2.8. [13] Given two vector spaces V_1, V_2 over the same field $\mathbb{F}$, a map $T : V_1 \rightarrow V_2$ is said to be antilinear if $T(\alpha x + y) = \bar{\alpha}T(x) + T(y)$, for all $x, y \in V_1$ and $\alpha \in \mathbb{F}$.

The operator norm for the antilinear map, T , between two normed spaces is defined as

$$\|T\| = \sup_{\|x\|=1} \|Tx\|.$$

Continuity and boundedness are also equivalent for antilinear maps as in the case of linear maps.

This paper deals with bounded antilinear maps between Hilbert spaces. Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces and $T : \mathcal{H}_2 \rightarrow \mathcal{H}_1$ be a bounded, antilinear map. The adjoint, T^* , of T is defined as

$$\langle T^*x, y \rangle = \langle Ty, x \rangle, \forall x \in \mathcal{H}_1, y \in \mathcal{H}_2.$$

T^* is a bounded antilinear map from $\mathcal{H}_1$ into $\mathcal{H}_2$.

Let $\{u_i\}, \{v_j\}$ be orthonormal bases for $\mathcal{H}_1, \mathcal{H}_2$, respectively. Then using Parseval identity we have

$$\sum_j \|Tv_j\|^2 = \sum_{i,j} |\langle Tv_j, u_i \rangle|^2 = \sum_i \|T^*u_i\|^2. \quad (1)$$

Hence, $\sum_j \|Tv_j\|^2$ does not depend on the choice of the orthonormal basis.

Definition 2.9. [13] Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. Then the tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$, denoted by $\mathcal{H}_1 \otimes \mathcal{H}_2$, is the collection of all bounded antilinear maps $T : \mathcal{H}_2 \rightarrow \mathcal{H}_1$ satisfying $\sum_{j \in J} \|Tv_j\|^2 < \infty$, where $\{v_j\}_{j \in J}$ is an

orthonormal basis for $\mathcal{H}_2$. The tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_2$ is a Hilbert space with respect to the inner product and the induced norm given by

$$\langle T_1, T_2 \rangle = \sum_{j \in J} \langle T_1 v_j, T_2 v_j \rangle \text{ and } \|T\|^2 = \sum_j \|Tv_j\|^2.$$

We note from equation (1), $\|T^*\| = \|T\|$. Further, if $T \in \mathcal{H}_1 \otimes \mathcal{H}_2$, $S_1 \in \mathcal{L}(\mathcal{H}_1)$ and $S_2 \in \mathcal{L}(\mathcal{H}_2)$, then $S_1 T \in \mathcal{H}_1 \otimes \mathcal{H}_2$, $S_2 T^* \in \mathcal{H}_2 \otimes \mathcal{H}_1$ and $\|S_1 T\| \leq \|S_1\| \|T\|$, $\|S_2 T^*\| \leq \|S_2\| \|T\|$.

The tensor product of vectors is defined as follows. Let $u \in \mathcal{H}_1$ and $v \in \mathcal{H}_2$, define the map $A_{uv} : \mathcal{H}_2 \rightarrow \mathcal{H}_1$ such that $A_{uv}(w) = \langle v, w \rangle u$, where $w \in \mathcal{H}_2$. Clearly, A_{uv} is an antilinear map; further, it can be shown that $\|A_{uv}\| = \|u\| \|v\|$, and hence $A_{uv} \in \mathcal{H}_1 \otimes \mathcal{H}_2$. We denote this map by $u \otimes v$. We also observe that for $u_1, u_2 \in \mathcal{H}_1$ and $v_1, v_2 \in \mathcal{H}_2$, we have

$$\langle u_1 \otimes v_1, u_2 \otimes v_2 \rangle = \langle u_1, u_2 \rangle \langle v_1, v_2 \rangle.$$

Remark 2.10. We note that $u \otimes v = 0 \iff u = 0$ or $v = 0$. Further, $(u \otimes v)^* = v \otimes u$.

Proposition 2.11. [13] If $\{u_i\}_{i \in I}$ and $\{v_j\}_{j \in J}$ are orthonormal bases for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, then $\{u_i \otimes v_j\}$ is an orthonormal basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$.

The tensor product for linear operators is defined as follows. Let $S_1 \in \mathcal{L}(\mathcal{H}_1)$ and $S_2 \in \mathcal{L}(\mathcal{H}_2)$. The tensor product of S_1 and S_2 , denoted $S_1 \otimes S_2$, is a map from $\mathcal{H}_1 \otimes \mathcal{H}_2$ into $\mathcal{H}_1 \otimes \mathcal{H}_2$ such that

$$(S_1 \otimes S_2)T = S_1 T S_2^*, \forall T \in \mathcal{H}_1 \otimes \mathcal{H}_2.$$

We observe that the composition of a linear and an antilinear map is antilinear. Since T is antilinear, $S_1 \otimes S_2$ is linear. That is $S_1 \otimes S_2 \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

Theorem 2.12. [13] Let $U_1, U_2 \in \mathcal{L}(\mathcal{H}_1)$ and $T_1, T_2 \in \mathcal{L}(\mathcal{H}_2)$. Then each of the following holds:

- (i) $U_1 \otimes T_1 \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ and $\|U_1 \otimes T_1\| = \|U_1\| \|T_1\|$.
- (ii) $(U_1 \otimes T_1)(x \otimes y) = U_1 x \otimes T_1 y$ for all $x \in \mathcal{H}_1$, $y \in \mathcal{H}_2$.
- (iii) $(U_1 \otimes T_1)(U_2 \otimes T_2) = (U_1 U_2) \otimes (T_1 T_2)$.
- (iv) $(U_1 \otimes T_1)^* = U_1^* \otimes T_1^*$.
- (v) If U_1 and T_1 are unitary then so is $U_1 \otimes T_1$.

2.5. Phase and norm retrieval in tensor product space. In [12], the authors studied phase and norm retrieval sequences in tensor product of Hilbert spaces. Some of the main results established in [12] are stated below and will be referred to in subsequent section of this paper.

Theorem 2.13. [12] Let $\{u_i\}_{i \in I}$ and $\{v_j\}_{j \in J}$ be two sequences in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then the statements are equivalent.

- i $\{u_i\}_{i \in I}$ does phase retrieval for $\mathcal{H}_1$ and $\{v_j\}_{j \in J}$ does phase retrieval for $\mathcal{H}_2$,
- ii $\{u_i \otimes v_j\}_{i \in I, j \in J}$ does phase retrieval for $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Theorem 2.14. [12] Suppose $\{u_i\}_{i \in I}$ does norm retrieval for $\mathcal{H}_1$ and $\{v_j\}_{j \in J}$ does norm retrieval for $\mathcal{H}_2$; and one of these two sequences contains a tight frame. Then $\{u_i \otimes v_j\}_{i \in I, j \in J}$ does norm retrieval for $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Theorem 2.15. [12] Consider sequences of vectors $\{u_i\}_{i \in I}$ in $\mathcal{H}_1$ and $\{v_j\}_{j \in J}$ in $\mathcal{H}_2$. Suppose $\{u_i \otimes v_j\}_{i \in I, j \in J}$ does norm retrieval for $\mathcal{H}_1 \otimes \mathcal{H}_2$. Then $\{u_i\}_{i \in I}$ does norm retrieval for $\mathcal{H}_1$ and $\{v_j\}_{j \in J}$ does norm retrieval for $\mathcal{H}_2$.

2.6. Moore-Penrose pseudo inverse. Suppose $\mathcal{H}_1$ and $\mathcal{H}_2$ are two Hilbert spaces. Consider the closed range operator $U \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. Then the Moore-Penrose pseudo inverse [3] of U is uniquely given by $U^\dagger : \mathcal{H}_2 \rightarrow \mathcal{H}_1$ such that

- (i) $UU^\dagger = P_{U(\mathcal{H}_1)}$, $U^\dagger U = P_{U^*(\mathcal{H}_2)}$.
- (ii) $UU^\dagger U = U$, $U^\dagger UU^\dagger = U^\dagger$, $(UU^\dagger)^* = UU^\dagger$, $(U^\dagger U)^* = U^\dagger U$.

3. Main Results

The main goal of the paper is to study phase retrieval and norm retrieval sequences and their weaving properties in tensor product of Hilbert spaces. In this section we present the main results of the paper.

Since a phase retrieval sequence has complement property and a frame is a spanning sequence, it is worthwhile to understand the behavior of spanning sequences and their weaving property in tensor product spaces. It provides a basic framework for more general results on weaving frames and weaving phase retrieval.

We begin this section by introducing weaving spanning sequences. Further, we discuss weaving spanning sequences in the tensor product of Hilbert spaces.

3.1. Weaving spanning sequences. The definition of weaving spanning sequences is introduced in this subsection. Some results and examples are given for weaving spanning sequences.

Definition 3.1. Two sequences $\{u_i\}_{i \in I}$ and $\{u'_i\}_{i \in I}$ in $\mathcal{H}$ are said to be weaving spanning sequences if for all $\sigma \subset I$, $\overline{\text{span}}\{\{u_i\}_{i \in \sigma} \cup \{u'_i\}_{i \in \sigma^c}\} = \mathcal{H}$.

Let $\{e_i\}_{i \in I}$ be an orthonormal basis for $\mathcal{H}$ then $\{e_1, e_2, e_1 + 2e_2, e_3, e_4, e_5, \dots\}$ and $\{e_1 + e_2, e_1 + 3e_2, 5e_1 + 6e_2, e_3, e_4, e_5, \dots\}$ are weaving spanning sequences for ℓ^2 .

In [12], it was proved that $\{u_i\}_{i \in I}, \{v_j\}_{j \in J}$ spans $\mathcal{H}_1, \mathcal{H}_2$, respectively, is equivalent to $\{u_i \otimes v_j\}_{i \in I, j \in J}$ spans $\mathcal{H}_1 \otimes \mathcal{H}_2$. However, the equivalence does not hold for weaving spanning sequences. The following theorem provides methods to construct weaving spanning sequences in $\mathcal{H}_1$ and $\mathcal{H}_2$ from weaving spanning sequences in $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Theorem 3.2. Suppose $\{u_i \otimes v_j\}_{(i,j) \in I \times J}$, $\{u'_i \otimes v'_j\}_{(i,j) \in I \times J}$ are weaving spanning sequences for $\mathcal{H}_1 \otimes \mathcal{H}_2$. Then $\{u_i\}_{i \in I}$, $\{u'_i\}_{i \in I}$ are weaving spanning sequences for $\mathcal{H}_1$ and $\{v_j\}_{j \in J}$, $\{v'_j\}_{j \in J}$ are weaving spanning sequences for $\mathcal{H}_2$.

Proof. If possible, suppose $\{u_i\}_{i \in I}$, $\{u'_i\}_{i \in I}$ are not weaving spanning sequences for $\mathcal{H}_1$. Then, there exists $\sigma \subset I$ such that some non-zero $x \in \mathcal{H}_1$ satisfies $\langle x, u_i \rangle = 0$ for all $i \in \sigma$ and $\langle x, u'_i \rangle = 0$ for all $i \in \sigma^c$. By choosing $y (\neq 0) \in \mathcal{H}_2$, we get

$$\begin{aligned} \langle x \otimes y, u_i \otimes v_j \rangle &= \langle x, u_i \rangle \langle y, v_j \rangle = 0 \text{ for all } i \in \sigma, j \in J, \\ \langle x \otimes y, u'_i \otimes v'_j \rangle &= \langle x, u'_i \rangle \langle y, v'_j \rangle = 0 \text{ for all } i \in \sigma^c, j \in J. \end{aligned}$$

Since $x \otimes y \neq 0$, this contradicts the assumption. Hence, no such σ can exist.

In the same way, we can show that $\{v_j\}_{j \in J}$, $\{v'_j\}_{j \in J}$ are weaving spanning sequences for $\mathcal{H}_2$. $\square$

However, the converse of the above theorem does not hold, which is evident from the following example.

Example 3.3. Consider the spanning sequences $\{u_i\} = \{(1, 0), (0, 1)\}$ and $\{u'_i\} = \{(1, 1), (1, -1)\}$ in $\mathbb{R}^2$, and spanning sequences $\{v_j\} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and $\{v'_j\} = \{(1, 0, -1), (0, 1, -1), (0, 1, 1)\}$ in $\mathbb{R}^3$. Here $I = \{1, 2\}$, $J = \{1, 2, 3\}$.

It is quite easy to check $\{u_i\}_{i \in I}$ and $\{u'_i\}_{i \in I}$ are weaving spanning sequences for $\mathbb{R}^2$. Further, for $\sigma = \{1\}$ and $\sigma^c = J \setminus \{1\}$, the set $\{v_1, v'_2, v'_3\}$ is linear independent and hence spans $\mathbb{R}^3$. Similarly, for every $\sigma \subset J$, we can show that the set $\{\{v_j\}_{j \in \sigma} \cup \{v'_j\}_{j \in \sigma^c}\}$ spans $\mathbb{R}^3$. Hence, $\{v_j\}_{j \in J}$ and $\{v'_j\}_{j \in J}$ are weaving spanning sequences for $\mathbb{R}^3$.

Set $S = \{(1, 3), (2, 1), (2, 2)\} \subset I \times J$. Then for any $w = (x, y, z) \in \mathbb{R}^3$, on the index set S , $(u_1 \otimes v_3)(w) = (z, 0)$, $(u_2 \otimes v_1)(w) = (0, x)$, $(u_2 \otimes v_2)(w) = (0, y)$, and on S^c , $(u'_1 \otimes v'_1)(w) = (x - z, x - z)$, $(u'_1 \otimes v'_2)(w) = (y - z, y - z)$, $(u'_2 \otimes v'_3)(w) = (y + z, -y - z)$.

Consider the antilinear map $A : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined as $A(x, y, z) = (x + y, z)$. By choosing the standard orthonormal basis $\{e_1, e_2, e_3\}$ for $\mathbb{R}^3$, where $e_i = (\delta_{ik})_{k=1}^3$, $i = 1, 2, 3$, we have

$$\begin{aligned} \langle A, u_1 \otimes v_3 \rangle &= \sum_{i=1}^3 \langle Ae_i, (u_1 \otimes v_3)e_i \rangle \\ &= \langle (1, 0), (0, 0) \rangle + \langle (1, 0), (0, 0) \rangle + \langle (0, 1), (1, 0) \rangle \\ &= 0. \end{aligned}$$

Similarly, we also get $\langle A, u_2 \otimes v_1 \rangle = 0$, $\langle A, u_2 \otimes v_2 \rangle = 0$, $\langle A, u'_1 \otimes v'_1 \rangle = 0$, $\langle A, u'_1 \otimes v'_2 \rangle = 0$, $\langle A, u'_2 \otimes v'_3 \rangle = 0$. Therefore, $A \perp \{\{u_i \otimes v_j\}_{(i,j) \in S} \cup \{u'_i \otimes v'_j\}_{(i,j) \in S^c}\}$.

Hence, $\{u_i \otimes v_j\}$, $\{u'_i \otimes v'_j\}$ fail to be weaving spanning sequences for $\mathbb{R}^2 \otimes \mathbb{R}^3$.

We recall that for a finite dimensional space, frames are equivalent to spanning sets.

Remark 3.4. Example 3.3 shows weaving spanning property may not be preserved when passing from Hilbert spaces to their tensor product spaces. For finite dimensional Hilbert spaces, frames are equivalent to spanning sequences. Therefore, this is evident that even if $\{u_i\}$, $\{u'_i\}$ are woven frames in $\mathcal{H}_1$ and $\{v_j\}$, $\{v'_j\}$ are woven frames in $\mathcal{H}_2$, the sequences $\{u_i \otimes v_j\}$, $\{u'_i \otimes v'_j\}$ need not be woven frames in $\mathcal{H}_1 \otimes \mathcal{H}_2$. Consequently, Example 3.3 serves as a counterexample to the conclusion of Theorem 3.1 in [17].

However, it is interesting to note that the following equivalence holds for weaving spanning sequences.

Theorem 3.5. Suppose $\{u_i\}_{i \in I}$, $\{u'_i\}_{i \in I}$ are two sequences of vectors in $\mathcal{H}_1$ and $\{v_j\}_{j \in J}$ is a sequence of vectors in $\mathcal{H}_2$.

Then the following statements are equivalent.

- (i) $\{u_i\}_{i \in I}$, $\{u'_i\}_{i \in I}$ are weaving spanning sequences for $\mathcal{H}_1$ and $\{v_j\}_{j \in J}$ spans $\mathcal{H}_2$.
- (ii) $\{u_i \otimes v_j\}_{(i,j) \in I \times J}$, $\{u'_i \otimes v_j\}_{(i,j) \in I \times J}$ are weaving spanning sequences for $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Proof. (i) $\implies$ (ii) Suppose, if possible, there exist $S \subset I \times J$ such that $\overline{\text{span}}\{\{u_i \otimes v_j\}_{(i,j) \in S} \cup \{u'_i \otimes v_j\}_{(i,j) \in S^c}\} \neq \mathcal{H}_1 \otimes \mathcal{H}_2$. Then there is a $T (\neq 0) \in \mathcal{H}_1 \otimes \mathcal{H}_2$ such that $T \perp \overline{\text{span}}\{\{u_i \otimes v_j\}_{(i,j) \in S} \cup \{u'_i \otimes v_j\}_{(i,j) \in S^c}\}$. This implies $\langle Tv_j, u_i \rangle = 0$ on S and $\langle Tv_j, u'_i \rangle = 0$ on S^c . For each $j \in J$ set $\sigma_j = \{i \in I : (i, j) \in S\}$, then

$$\langle Tv_j, u_i \rangle = 0 \text{ for all } i \in \sigma_j, \quad \langle Tv_j, u'_i \rangle = 0 \text{ for all } i \in \sigma_j^c.$$

Using the fact that $\{u_i\}_{i \in I}$, $\{u'_i\}_{i \in I}$ are weaving spanning sequences for $\mathcal{H}_1$, we get $Tv_j = 0$ for each $j \in J$. Since $\{v_j\}_{j \in J}$ spans $\mathcal{H}_2$, this is a contradiction to $T \neq 0$. Hence proved.

(ii) $\implies$ (i) Replacing v'_j by v_j in Theorem 3.2, it follows immediately. $\square$

Interchanging the roles of $\{u_i\}_{i \in I}$ and $\{v_j\}_{j \in J}$ the following remark holds.

Remark 3.6. Suppose $\{u_i\}_{i \in I}$ is a sequence of vectors in $\mathcal{H}_1$ and $\{v_j\}_{j \in J}$, $\{v'_j\}_{j \in J}$ are two sequences of vectors in $\mathcal{H}_2$. Then the following are interchangeable.

- (i) $\{u_i\}_{i \in I}$ spans $\mathcal{H}_1$ and $\{v_j\}_{j \in J}$, $\{v'_j\}_{j \in J}$ are weaving spanning sequences for $\mathcal{H}_2$,
- (ii) $\{u_i \otimes v_j\}_{(i,j) \in I \times J}$, $\{u_i \otimes v'_j\}_{(i,j) \in I \times J}$ are weaving spanning sequences for $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Proof. (i) $\implies$ (ii) Suppose, if possible, there exist $S \subset I \times J$ such that $\overline{\text{span}}\{\{u_i \otimes v_j\}_{(i,j) \in S} \cup \{u_i \otimes v'_j\}_{(i,j) \in S^c}\} \neq \mathcal{H}_1 \otimes \mathcal{H}_2$. Then there is a $T (\neq 0) \in \mathcal{H}_1 \otimes \mathcal{H}_2$ such that $T \perp \overline{\text{span}}\{\{u_i \otimes v_j\}_{(i,j) \in S} \cup \{u_i \otimes v'_j\}_{(i,j) \in S^c}\}$. This implies $\langle T^*u_i, v_j \rangle = 0$ on

S and $\langle T^*u_i, v'_j \rangle = 0$ on S^c . For each $i \in I$ set $\sigma_i = \{j \in J : (i, j) \in S\}$, then

$$\langle T^*u_i, v_j \rangle = 0 \text{ for all } j \in \sigma_i, \quad \langle T^*u_i, v'_j \rangle = 0 \text{ for all } j \in \sigma_i^c.$$

Using the fact that $\{v_j\}_{j \in J}, \{v'_j\}_{j \in J}$ are weaving spanning sequences for $\mathcal{H}_2$, we get $T^*u_i = 0$ for each $i \in I$. Since $\{u_i\}_{i \in I}$ spans $\mathcal{H}_1$, we have $T^* = 0$. This is a contradiction to $T \neq 0$. Hence proved.

(ii) $\implies$ (i) Immediately follows from Theorem 3.2. $\square$

3.2. Weaving complement property. The complement property is an important concept in phase retrieval. In finite-dimensional Hilbert spaces, it is useful in determining the minimum number of vectors required for phase retrieval. In fact, it was proved in [2] that, for real Hilbert spaces, a family of vectors does phase retrieval if and only if it satisfies the complement property. This motivates us to introduce and study the weaving complement property before discussing weaving phase retrieval sequences.

Definition 3.7. Two sequences $\{u_i\}_{i \in I}$ and $\{u'_i\}_{i \in I}$ are said to have weaving complement property if for every $\sigma \subset I$, $\{u_i\}_{i \in \sigma} \cup \{u'_i\}_{i \in \sigma^c}$ has complement property.

For instance, $\{(1, 0), (0, 1), (1, 1)\}$ and $\{(4, 0), (0, 5), (1, -2)\}$ have weaving complement property in $\mathbb{R}^2$.

Theorem 3.8. Let $\{u_i\}_{i \in I}, \{v_j\}_{j \in J}$ be two sequences of vectors in $\mathcal{H}_1, \mathcal{H}_2$ respectively. Then the following two statements are interchangeable.

- (i) $\{u_i\}_{i \in I}$ and $\{v_j\}_{j \in J}$ have complement property.
- (ii) $\{u_i \otimes v_j\}_{(i,j) \in I \times J}$ has complement property.

Proof. (i) $\implies$ (ii) Suppose $\{u_i \otimes v_j\}_{(i,j) \in I \times J}$ does not have complement property. Then there exists $S \subset I \times J$ such that for some non-zero $T_1, T_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2$, $T_1 \perp (u_i \otimes v_j) \forall (i, j) \in S$ and $T_2 \perp (u_i \otimes v_j) \forall (i, j) \in S^c$. Since T_1, T_2 are non-zero and $\{v_j\}_{j \in J}$ have complement property there exist $j_0, j_1 \in J$ such that $T_1 v_{j_0} \neq 0$ and $T_2 v_{j_1} \neq 0$.

Fix $i \in I$ and set $\sigma_i = \{j \in J : (i, j) \in S\}$. Then $\langle T_1, u_i \otimes v_j \rangle = 0 \forall j \in \sigma_i$ and $\langle T_2, u_i \otimes v_j \rangle = 0 \forall j \in \sigma_i^c$. This implies

$$\langle T_1^*u_i, v_j \rangle = 0 \forall j \in \sigma_i, \quad \langle T_2^*u_i, v_j \rangle = 0 \forall j \in \sigma_i^c.$$

Therefore, at least one of $T_1^*u_i$ and $T_2^*u_i$ is zero.

Define $\gamma = \{i \in I : T_1^*u_i = 0\}$. This implies $T_2^*u_i = 0 \forall i \in \gamma^c$. We observe that since T_1, T_2 are non-zero, γ is a non-trivial subset of I . Now, $\langle T_1 v_j, u_i \rangle = 0 \forall j \in J, i \in \gamma$ and $\langle T_2 v_j, u_i \rangle = 0 \forall j \in J, i \in \gamma^c$. In particular, $\langle T_1 v_{j_0}, u_i \rangle = 0 \forall i \in \gamma$ and $\langle T_2 v_{j_1}, u_i \rangle = 0 \forall i \in \gamma^c$. Since $T_1 v_{j_0}$ and $T_2 v_{j_1}$ are non-zero, this is a contradiction to the complement property of $\{u_i\}_{i \in I}$.

(ii) $\implies$ (i) Suppose, if possible, $\{u_i\}_{i \in I}$ fails to have complement property. Then $\exists \sigma \subset I$ such that for some non-zero $x_1, x_2 \in \mathcal{H}_1$ we have,

$$\langle x_1, u_i \rangle = 0 \text{ for all } i \in \sigma, \quad \langle x_2, u_i \rangle = 0 \text{ for all } i \in \sigma^c.$$

Then for non-zero $y \in \mathcal{H}_2$, we get

$$\begin{aligned}\langle x_1 \otimes y, u_i \otimes v_j \rangle &= 0 \text{ for all } i \in \sigma, j \in J, \\ \langle x_2 \otimes y, u_i \otimes v_j \rangle &= 0 \text{ for all } i \in \sigma^c, j \in J.\end{aligned}$$

Thus, we obtain a contradiction to the fact that $\{u_i \otimes v_j\}_{(i,j) \in I \times J}$ has complement property in $\mathcal{H}_1 \otimes \mathcal{H}_2$. Hence, $\{u_i\}_{i \in I}$ has complement property in $\mathcal{H}_1$.

Similarly, we can show that $\{v_j\}_{j \in J}$ has complement property in $\mathcal{H}_2$. $\square$

If we have sequences that have weaving complement property in $\mathcal{H}_1$ and $\mathcal{H}_2$, then we can construct sequences having weaving complement property in $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Theorem 3.9. *Let $\{u_i\}_{i \in I}$, $\{u'_i\}_{i \in I}$ be two sequences of vectors in $\mathcal{H}_1$ and $\{v_j\}_{j \in J}$, $\{v'_j\}_{j \in J}$ be sequences of vectors in $\mathcal{H}_2$. Then the following are equivalent.*

- (i) $\{u_i\}_{i \in I}$, $\{u'_i\}_{i \in I}$ have weaving complement property and $\{v_j\}_{j \in J}$, $\{v'_j\}_{j \in J}$ have weaving complement property.
- (ii) $\{u_i \otimes v_j\}_{(i,j) \in I \times J}$ and $\{u'_i \otimes v'_j\}_{(i,j) \in I \times J}$ have weaving complement property.

Proof. (i) $\implies$ (ii) Suppose $\{u_i \otimes v_j\}_{(i,j) \in I \times J}$, $\{u'_i \otimes v'_j\}_{(i,j) \in I \times J}$ do not have weaving complement property. Then $\exists S \subset I \times J$ such that for some non-zero $T_1, T_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2$, we have $T_1 \perp (u_i \otimes v_j) \forall (i, j) \in S$ and $T_2 \perp (u'_i \otimes v'_j) \forall (i, j) \in S^c$. Since T_1 is non-zero and $\{v_j\}_{j \in J}$ has complement property $\exists j_0 \in J$ such that $T_1 v_{j_0} \neq 0$. Similarly, $\exists j_1 \in J$ such that $T_2 v'_{j_1} \neq 0$.

Fix $i \in I$ and set $\sigma_i = \{j \in J : (i, j) \in S\}$. Then

$$\langle T_1^* u_i, v_j \rangle = 0 \forall j \in \sigma_i, \quad \langle T_2^* u'_i, v'_j \rangle = 0 \forall j \in \sigma_i^c.$$

Since $\{v_j\}_{j \in J}$, $\{v'_j\}_{j \in J}$ have weaving complement property, at least one of $T_1^* u_i$ and $T_2^* u'_i$ is zero.

Define $\gamma = \{i \in I : T_1^* u_i = 0\}$. Therefore, $T_2^* u'_i = 0 \forall i \in \gamma^c$. We observe that since T_1, T_2 are non-zero, γ is a non-trivial subset of I . Now, $\langle T_1 v_j, u_i \rangle = 0 \forall j \in J, i \in \gamma$ and $\langle T_2 v'_j, u'_i \rangle = 0 \forall j \in J, i \in \gamma^c$. In particular, $\langle T_1 v_{j_0}, u_i \rangle = 0 \forall i \in \gamma$ and $\langle T_2 v'_{j_1}, u'_i \rangle = 0 \forall i \in \gamma^c$. Since $T_1 v_{j_0}$ and $T_2 v'_{j_1}$ are non-zero, this is a contradiction to weaving complement property of $\{u_i\}_{i \in I}$ and $\{u'_i\}_{i \in I}$.

(ii) $\implies$ (i) Suppose $\{u_i\}_{i \in I}$, $\{u'_i\}_{i \in I}$ fails to have weaving complement property. Then $\exists \sigma \subset I$ such that some non-zero $x_1, x_2 \in \mathcal{H}_1$ satisfy $\langle x_1, u_i \rangle = 0$ for all $i \in \sigma$ and $\langle x_2, u'_i \rangle = 0$ for all $i \in \sigma^c$. Then for $y(\neq 0) \in \mathcal{H}_2$ we have

$$\begin{aligned}\langle x_1 \otimes y, u_i \otimes v_j \rangle &= 0 \text{ for all } i \in \sigma, j \in J, \\ \langle x_2 \otimes y, u'_i \otimes v'_j \rangle &= 0 \text{ for all } i \in \sigma^c, j \in J.\end{aligned}$$

Thus, we obtain a contradiction to the fact that $\{u_i \otimes v_j\}_{(i,j) \in I \times J}$, $\{u'_i \otimes v'_j\}_{(i,j) \in I \times J}$ have complement property. Hence, $\{u_i\}_{i \in I}$, $\{u'_i\}_{i \in I}$ have weaving complement property.

Similarly, we can show that $\{v_j\}_{j \in J}, \{v'_j\}_{j \in J}$ have weaving complement property. $\square$

3.3. Weaving phase retrieval and weaving norm retrieval. In [11], weaving phase retrieval and weaving norm retrieval were introduced for Hilbert spaces. It has been proved in [12] that $\{u_i\}_{i \in I}$ does phase retrieval for $\mathcal{H}_1$ and $\{v_j\}_{j \in J}$ does phase retrieval for $\mathcal{H}_2$ is equivalent to $\{u_i \otimes v_j\}_{(i,j) \in I \times J}$ does phase retrieval for $\mathcal{H}_1 \otimes \mathcal{H}_2$. Here we discuss similar results for weaving phase retrieval sequences.

Theorem 3.10. *If $\{u_i \otimes v_j\}_{(i,j) \in I \times J}, \{u'_i \otimes v'_j\}_{(i,j) \in I \times J}$ do weaving phase retrieval for $\mathcal{H}_1 \otimes \mathcal{H}_2$ then $\{u_i\}_{i \in I}, \{u'_i\}_{i \in I}$ do weaving phase retrieval for $\mathcal{H}_1$ and $\{v_j\}_{j \in J}, \{v'_j\}_{j \in J}$ do weaving phase retrieval for $\mathcal{H}_2$.*

Proof. For $x_1, x_2 \in \mathcal{H}_1$ and for all non trivial $\sigma \subset I$, let $|\langle x_1, u_i \rangle| = |\langle x_2, u_i \rangle|$ for all $i \in \sigma$ and $|\langle x_1, u'_i \rangle| = |\langle x_2, u'_i \rangle|$ for all $i \in \sigma^c$. Consider $y (\neq 0) \in \mathcal{H}_2$, we get

$$\begin{aligned} |\langle x_1 \otimes y, u_i \otimes v_j \rangle| &= |\langle x_2 \otimes y, u_i \otimes v_j \rangle| \quad \forall (i, j) \in \sigma \times J, \\ |\langle x_1 \otimes y, u'_i \otimes v'_j \rangle| &= |\langle x_2 \otimes y, u'_i \otimes v'_j \rangle| \quad \forall (i, j) \in \sigma^c \times J. \end{aligned}$$

Using the given condition we get $x_1 \otimes y = c(x_2 \otimes y)$ for some $|c| = 1$. Thus, we get $x_1 = cx_2$ with $|c| = 1$. Hence, $\{u_i\}_{i \in I}, \{u'_i\}_{i \in I}$ do weaving phase retrieval for $\mathcal{H}_1$.

In a similar way, it can be shown that $\{v_j\}_{j \in J}$ and $\{v'_j\}_{j \in J}$ do weaving phase retrieval for $\mathcal{H}_2$. $\square$

If we further assume that $\{u_i \otimes v_j\}_{i \in I, j \in J}, \{u'_i \otimes v'_j\}_{i \in I, j \in J}$ do phase retrieval for $\mathcal{H}_1 \otimes \mathcal{H}_2$ then $\{u_i\}_{i \in I}, \{u'_i\}_{i \in I}$ do phase retrieval for $\mathcal{H}_1$ and $\{v_j\}_{j \in J}, \{v'_j\}_{j \in J}$ do phase retrieval for $\mathcal{H}_2$.

We have the following result for the converse of Theorem 3.10.

Theorem 3.11. *If $\{u_i\}_{i \in I}, \{u'_i\}_{i \in I}$ do phase retrieval for $\mathcal{H}_1$ and do weaving phase retrieval, and $\{v_j\}_{j \in J}$ does phase retrieval for $\mathcal{H}_2$, then $\{u_i \otimes v_j\}_{(i,j) \in I \times J}, \{u'_i \otimes v'_j\}_{(i,j) \in I \times J}$ do phase retrieval for $\mathcal{H}_1 \otimes \mathcal{H}_2$ and do weaving phase retrieval.*

Proof. Since $\{u_i\}_{i \in I}, \{u'_i\}_{i \in I}$ do phase retrieval for $\mathcal{H}_1$ and $\{v_j\}_{j \in J}$ does phase retrieval for $\mathcal{H}_2$, by Theorem 2.13 it is immediate that $\{u_i \otimes v_j\}_{(i,j) \in I \times J}, \{u'_i \otimes v'_j\}_{(i,j) \in I \times J}$ do phase retrieval for $\mathcal{H}_1 \otimes \mathcal{H}_2$.

For the weaving part, let $S \subset I \times J$, and consider two non-zero operators $T_1, T_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2$, such that $|\langle T_1, u_i \otimes v_j \rangle| = |\langle T_2, u_i \otimes v_j \rangle|$ on S and $|\langle T_1, u'_i \otimes v_j \rangle| = |\langle T_2, u'_i \otimes v_j \rangle|$ on S^c , that is, $|\langle T_1 v_j, u_i \rangle| = |\langle T_2 v_j, u_i \rangle|$ on S and $|\langle T_1 v_j, u'_i \rangle| = |\langle T_2 v_j, u'_i \rangle|$ on S^c . For each $j \in J$, we set $\sigma_j = \{i \in I : (i, j) \in S\}$. Then

$$|\langle T_1 v_j, u_i \rangle| = |\langle T_2 v_j, u_i \rangle| \quad \forall i \in \sigma_j \text{ and } |\langle T_1 v_j, u'_i \rangle| = |\langle T_2 v_j, u'_i \rangle| \quad \forall i \in \sigma_j^c.$$

Using the fact that $\{u_i\}_{i \in I}$, $\{u'_i\}_{i \in I}$ do phase retrieval for $\mathcal{H}_1$ and do weaving phase retrieval, we get for all $j \in J$, $T_1 v_j = c_j T_2 v_j$ for some $|c_j| = 1$. It suffices to show that all such c_j are equal. Now for all $j \in J$, we have

$$|\langle T_1^* u_i, v_j \rangle| = |\langle T_1 v_j, u_i \rangle| = |\langle c_j T_2 v_j, u_i \rangle| = |\langle T_2^* u_i, v_j \rangle|.$$

Thus, we get $T_1^* u_i = d_i T_2^* u_i$ for all $i \in I$. Again, we get

$$\langle T_1 v_j, u_i \rangle = \langle T_1^* u_i, v_j \rangle = \langle d_i T_2^* u_i, v_j \rangle = d_i \langle T_2 v_j, u_i \rangle = \frac{d_i}{c_j} \langle T_1 v_j, u_i \rangle.$$

That is

$$(c_j - d_i) \langle T_1 v_j, u_i \rangle = 0 \quad \forall i \in I, j \in J. \quad (2)$$

We choose $j_0, j_1 \in J$ such that $T_1 v_{j_0}, T_1 v_{j_1}$ are non-zero. Set

$$\sigma_{j_0} = \{i \in I : \langle T_1 v_{j_0}, u_i \rangle \neq 0\}, \quad \sigma_{j_1} = \{i \in I : \langle T_1 v_{j_1}, u_i \rangle \neq 0\}.$$

Since $\{u_i\}$ does phase retrieval for $\mathcal{H}_1$, σ_{j_0} and σ_{j_1} are non-empty. We note that if $\sigma_{j_0} \cap \sigma_{j_1} = \emptyset$, $\exists \delta \subset I$ such that $\delta \subset \sigma_{j_0}^c$ and $\delta^c \subset \sigma_{j_1}^c$ satisfies $T_1 v_{j_0} \perp u_i \quad \forall i \in \delta$ & $T_1 v_{j_1} \perp u_i \quad \forall i \in \delta^c$, which contradicts the fact that $\{u_i\}$ has complement property and hence $\sigma_{j_0} \cap \sigma_{j_1} \neq \emptyset$. Hence, from equation (2) we get $c_{j_0} = d_i, \forall i \in \sigma_{j_0}$ and $c_{j_1} = d_i, \forall i \in \sigma_{j_1}$. Since $\sigma_{j_0} \cap \sigma_{j_1} \neq \emptyset, c_{j_0} = c_{j_1} = c$.

If $T_1 v_j = 0$ for some $j \in J$, then from above we have,

$$|\langle T_2 v_j, u_i \rangle| = |\langle 0, u_i \rangle| \quad \forall i \in \sigma_j \text{ and } |\langle T_2 v_j, u'_i \rangle| = |\langle 0, u'_i \rangle| \quad \forall i \in \sigma_j^c.$$

Again, since $\{u_i\}_{i \in I}, \{u'_i\}_{i \in I}$ do phase retrieval for $\mathcal{H}_1$ and do weaving phase retrieval, we get $T_2 v_j = c \cdot 0 = 0$ for any unimodular constant c .

Hence, we can conclude that $T_1 = c T_2$. $\square$

Consequently, by interchanging the roles of $\{u_i\}$ and $\{v_j\}$ the following remark holds.

Remark 3.12. Suppose $\{u_i\}_{i \in I}$ does phase retrieval for $\mathcal{H}_1$ and $\{v_j\}_{j \in J}, \{v'_j\}_{j \in J}$ do phase retrieval and weaving phase retrieval for $\mathcal{H}_2$. Then $\{u_i \otimes v_j\}_{(i,j) \in I \times J}, \{u_i \otimes v'_j\}_{(i,j) \in I \times J}$ do weaving phase retrieval for $\mathcal{H}_1 \otimes \mathcal{H}_2$.

We now discuss weaving norm retrieval sequences in $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Theorem 3.13. If $\{u_i \otimes v_j\}_{i \in I, j \in J}, \{u'_i \otimes v'_j\}_{i \in I, j \in J}$ do weaving norm retrieval for $\mathcal{H}_1 \otimes \mathcal{H}_2$, then $\{u_i\}_{i \in I}, \{u'_i\}_{i \in I}$ do weaving norm retrieval for $\mathcal{H}_1$ and $\{v_j\}_{j \in J}, \{v'_j\}_{j \in J}$ do weaving norm retrieval for $\mathcal{H}_2$.

Proof. This can be proved using a similar approach as in the proof of Theorem 3.10. $\square$

Further if $\{u_i \otimes v_j\}_{i \in I, j \in J}, \{u'_i \otimes v'_j\}_{i \in I, j \in J}$ do norm retrieval then $\{u_i\}_{i \in I}, \{u'_i\}_{i \in I}$ do norm retrieval for $\mathcal{H}_1$ and $\{v_j\}_{j \in J}, \{v'_j\}_{j \in J}$ do norm retrieval for $\mathcal{H}_2$.

For the converse, we have the following.

Theorem 3.14. *If $\{u_i\}_{i \in I}$, $\{u'_i\}_{i \in I}$ do norm retrieval for $\mathcal{H}_1$ and do weaving norm retrieval, and $\{v_j\}_{j \in J}$ is a sequence in $\mathcal{H}_2$ that contains a tight frame, then $\{u_i \otimes v_j\}_{(i,j) \in I \times J}$, $\{u'_i \otimes v_j\}_{(i,j) \in I \times J}$ do norm retrieval for $\mathcal{H}_1 \otimes \mathcal{H}_2$ and do weaving norm retrieval.*

Proof. Since any sequence containing a tight frame does norm retrieval, using Theorem 2.14 it is immediate that $\{u_i \otimes v_j\}_{(i,j) \in I \times J}$, $\{u'_i \otimes v_j\}_{(i,j) \in I \times J}$ do norm retrieval for $\mathcal{H}_1 \otimes \mathcal{H}_2$.

For weaving, let $S \subset I \times J$, and consider non-zero operators $T_1, T_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2$, such that $|\langle T_1, u_i \otimes v_j \rangle| = |\langle T_2, u_i \otimes v_j \rangle|$ on S and $|\langle T_1, u'_i \otimes v_j \rangle| = |\langle T_2, u'_i \otimes v_j \rangle|$ on S^c . Similar to the proof of Theorem 3.11 we get, $\|T_1 v_j\| = \|T_2 v_j\|$ for all $j \in J$. Let $\{e_n\}$ be an orthonormal basis for $\mathcal{H}_1$, then we note that $\|T_1 v_j\|^2 = \sum_n |\langle T_1 v_j, e_n \rangle|^2 = \sum_n |\langle T_1^* e_n, v_j \rangle|^2$. From the given condition there exists a $J_1 \subset J$ and $c > 0$ such that $\{v_j\}_{j \in J_1}$ is a c -tight frame. Now,

$$\begin{aligned} \sum_{j \in J_1} \|T_1 v_j\|^2 &= \sum_{j \in J_1} \sum_n |\langle T_1 v_j, e_n \rangle|^2 = \sum_n \sum_{j \in J_1} |\langle T_1^* e_n, v_j \rangle|^2 \\ &= c \sum_n \|T_1^* e_n\|^2 = c \|T_1\|^2. \end{aligned}$$

Thus, we obtain $\|T_1\| = \|T_2\|$.

If one of T_1, T_2 is zero, say $T_1 = 0$, it follows that for every fix $j \in J$, with $\sigma_j = \{i \in I : (i, j) \in S\}$ we have

$$|\langle T_2 v_j, u_i \rangle| = 0, \forall i \in \sigma_j \text{ and } |\langle T_2 v_j, u'_i \rangle| = 0, \forall i \in \sigma_j^c.$$

Using the fact that $\{u_i\}_{i \in I}$, $\{u'_i\}_{i \in I}$ do norm retrieval for $\mathcal{H}_1$ and do weaving norm retrieval, we get for all $j \in J$, $T_2 v_j = 0$. Therefore, as above

$$\|T_2\|^2 = \sum_n \|T_2^* e_n\|^2 = \frac{1}{c} \sum_{j \in J_1} \|T_2 v_j\|^2 = 0 = \|T_1\|^2.$$

Hence, the conclusion follows. $\square$

By interchanging the roles of $\{u_i\}_{i \in I}$ and $\{v_j\}_{j \in J}$, we have the following remark.

Remark 3.15. Suppose $\{u_i\}_{i \in I}$ is a sequence in $\mathcal{H}_1$ that contains a tight frame, and $\{v_j\}_{j \in J}, \{v'_j\}_{j \in J}$ do norm retrieval for $\mathcal{H}_2$ and do weaving norm retrieval. Then $\{u_i \otimes v_j\}_{(i,j) \in I \times J}, \{u_i \otimes v'_j\}_{(i,j) \in I \times J}$ do norm retrieval for $\mathcal{H}_1 \otimes \mathcal{H}_2$ and do weaving norm retrieval.

The next theorem shows that bounded linear operators under certain conditions preserve the weaving phase retrieval property.

Theorem 3.16. Suppose $U \in \mathcal{L}(\mathcal{H}_1)$ and $T \in \mathcal{L}(\mathcal{H}_2)$ are closed range operators. If $\{A_i\}_{i \in I}, \{A'_i\}_{i \in I}$ do weaving phase retrieval for $\mathcal{H}_1 \otimes \mathcal{H}_2$, then $\{U A_i T\}_{i \in I}, \{U A'_i T\}_{i \in I}$ do weaving phase retrieval for $U(\mathcal{H}_1) \otimes T^*(\mathcal{H}_2)$.

Proof. For $\sigma \subset I$ and $B_1, B_2 \in U(\mathcal{H}_1) \otimes T^*(\mathcal{H}_2)$, suppose $|\langle B_1, UA_iT \rangle| = |\langle B_2, UA_iT \rangle|$ for all $i \in \sigma$ and $|\langle B_1, UA'_iT \rangle| = |\langle B_2, UA'_iT \rangle|$ for all $i \in \sigma^c$. We note that

$$\begin{aligned}\langle B_1, UA_iT \rangle &= \langle B_1^*U, T^*A_i^* \rangle = \langle U^*B_1T^*, A_i \rangle \forall i \in \sigma, \\ \langle B_1, UA'_iT \rangle &= \langle B_1^*U, T^*A_i'^* \rangle = \langle U^*B_1T^*, A'_i \rangle \forall i \in \sigma^c.\end{aligned}$$

Since $\{A_i\}_{i \in I}, \{A'_i\}_{i \in I}$ do weaving phase retrieval for $\mathcal{H}_1 \otimes \mathcal{H}_2$, we get $U^*B_1T^* = cU^*B_2T^*$, for some c with $|c| = 1$. Pre-multiplication and post-multiplication by $(U^*)^\dagger$ and $(T^*)^\dagger$ to the above equality, lead to

$$(U^*)^\dagger U^*(B_1 - cB_2)T^*(T^*)^\dagger = 0 \implies P|_{U(\mathcal{H}_1)}(B_1 - cB_2)P|_{T^*(\mathcal{H}_2)} = 0.$$

Thus, we get $B_1 = cB_2$ on $T^*(\mathcal{H}_2)$ for some $|c| = 1$. Hence, we can conclude that $\{UA_iT\}_{i \in I}, \{UA'_iT\}_{i \in I}$ do weaving phase retrieval for $U(\mathcal{H}_1) \otimes T^*(\mathcal{H}_2)$. $\square$

Remark 3.17. In Theorem 3.16, if $T = I_{\mathcal{H}_2}$, then $\{UA_i\}_{i \in I}, \{UA'_i\}_{i \in I}$ do weaving phase retrieval for $U(\mathcal{H}_1) \otimes \mathcal{H}_2$. Further, if U is invertible, then $\{UA_i\}_{i \in I}, \{UA'_i\}_{i \in I}$ do weaving phase retrieval for $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Similarly, if $U = I_{\mathcal{H}_1}$ in Theorem 3.16, then $\{A_iT\}_{i \in I}, \{A'_iT\}_{i \in I}$ do weaving phase retrieval for $\mathcal{H}_1 \otimes T^*(\mathcal{H}_2)$. Therefore, if T is invertible, then $\{A_iT\}_{i \in I}, \{A'_iT\}_{i \in I}$ do weaving phase retrieval for $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Hence, the weaving phase retrieval property is preserved under pre- and post-multiplication by invertible operators.

4. Conclusion

Phase retrieval and norm retrieval are significant and growing research areas in mathematics. Weaving properties of frames, phase retrieval sequences, and norm retrieval sequences facilitate the reconstruction of signals and functions and have found applications in wireless sensor networks, where distributed processing is often required. In this paper, we have studied weaving properties for phase retrieval, norm retrieval, spanning sequences, and complement property in tensor product space.

In Section 3.2, we have shown that the construction of sequences having complement property/weaving complement property in the product space is equivalent to taking the tensor product of such sequences in the respective Hilbert spaces.

Spanning sequences in the tensor product space can be easily constructed by taking the tensor product of the spanning sequences in the respective Hilbert spaces. However, with the help of Example 3.3, we have shown that weaving spanning sequences for the tensor product space cannot be constructed by taking the tensor product of the weaving spanning sequences in the respective Hilbert spaces. For finite dimensional Hilbert spaces, frames are equivalent to spanning sequences. Therefore, Example 3.3 clearly demonstrates that the claim made in Theorem 3.1 of [17] is, in general, not true.

We have seen that phase retrieval sequences for the tensor product space can be constructed by taking the tensor product of phase retrieval sequences

in the respective Hilbert spaces. A natural question that arises is whether the construction of weaving phase retrieval sequences for the product space is possible by taking the tensor product of weaving phase retrieval sequences in the respective Hilbert spaces. We have given methods of construction of weaving phase retrieval sequence in $\mathcal{H}_1 \otimes \mathcal{H}_2$ from a phase retrieval/weaving phase retrieval sequence in $\mathcal{H}_1$ and a weaving phase retrieval/phase retrieval sequence in $\mathcal{H}_2$. We have also given methods of construction of weaving norm retrieval sequences in the tensor product space.

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(Suman Dowerah) DEPARTMENT OF MATHEMATICS, THONG NOKBE COLLEGE, KARBI ANGLONG -782441, ASSAM, INDIA
sumandowerah@gmail.com

(Saikat Mukherjee) DEPARTMENT OF MATHEMATICS, NIT MEGHALAYA, SOHRA 793108, MEGHALAYA, INDIA
saikat.mukherjee@nitm.ac.in

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