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Bochner-Riesz commutators on Métivier groups: boundedness and compactness

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ABSTRACT. In this paper, we prove the boundedness and compactness properties of the Bochner-Riesz commutator associated with the sub-Laplacians on Métivier groups. We show that the smoothness parameter can be expressed in terms of the topological dimension, rather than the homogeneous dimension of the Métivier groups. One of the main novel aspects of our proof is the use of *weighted Plancherel estimates* and *truncated restriction-type estimates* to obtain the threshold in terms of the topological dimension.

CONTENTS

1. Introduction and statement of main results

Let G be a two-step stratified Lie group with $\mathfrak{g}$ as its Lie algebra, that is to say that G is connected, simply connected nilpotent Lie group and the Lie algebra $\mathfrak{g}$ endowed with a vector space decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ into two non-trivial subspaces $\mathfrak{g}_1, \mathfrak{g}_2 \subseteq \mathfrak{g}$ such that $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2$ and $\mathfrak{g}_2$ is the center of $\mathfrak{g}$. We call $\mathfrak{g}_1$, $\mathfrak{g}_2$ as first and second layer respectively and let dim $\mathfrak{g}_1 = d_1$, dim $\mathfrak{g}_2 = d_2$. Let us assume that $X_1, \dots, X_{d_1}$ be a basis of the first layer $\mathfrak{g}_1$ and $T_1, \dots, T_{d_2}$

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be a basis of the second layer $\mathfrak{g}_2$. We further assume that we have an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ such that $X_1,\ldots,X_{d_1},T_1,\ldots,T_{d_2}$ is an orthonormal basis of $\mathfrak{g}$. The inner product $\langle\cdot,\cdot\rangle$ induces a norm on the dual space $\mathfrak{g}_2^*$ of $\mathfrak{g}_2$, which we denote by $|\cdot|$. Then any element $\mu \in \mathfrak{g}_2^* \setminus \{0\}$ gives rise to a skew-symmetric bilinear form $\omega_\mu(x,x') = \mu([x,x'])$ for $x,x' \in \mathfrak{g}_1$. Let J_μ be the skew-symmetric endomorphism such that

$$\omega_\mu(x,x') = \langle J_\mu x, x'\rangle, \quad x,x' \in \mathfrak{g}_1.$$

We then say G is a Métivier group if J_μ is invertible for all $\mu \in \mathfrak{g}_2^* \setminus \{0\}$, and a Heisenberg type group if $J_\mu^2 = -|\mu|^2 \operatorname{id}_{\mathfrak{g}_1}$ for all $\mu \in \mathfrak{g}_2^* \setminus \{0\}$. In the latter case, the group will be denoted by $\mathbb{H}$. In what follows, G will denote the Métivier groups unless or otherwise specified. Note that the class of Métivier group contains Heisenberg type group, in fact the containment is strict, see [34] for more details.

Corresponding to the basis $X_1,\ldots,X_{d_1}$ of the first layer $\mathfrak{g}_1$, we consider the associated sub-Laplacian $\mathcal{L}$ on G which is defined by

$$\mathcal{L} := -(X_1^2 + \cdots + X_{d_1}^2).$$

The sub-Laplacian $\mathcal{L}$ is positive and (essentially) self-adjoint on $L^2(G)$. Hence, via functional calculus, given a Borel measurable function $F : \mathbb{R} \to \mathbb{C}$ one can define spectral multiplier operator

$$F(\mathcal{L}) = \int_0^\infty F(\lambda)\, dE(\lambda),$$

where E denotes the spectral measure of $\mathcal{L}$. Then from spectral theory, $F(\mathcal{L})$ is a bounded operator on $L^2(G)$ if and only if F is E-essentially bounded. However, for $p \neq 2$, it is well known that the boundedness condition on F alone is not enough to get the L^p-boundedness of $F(\mathcal{L})$, one requires more regularity on the multiplier F. In the context of Euclidean space $\mathbb{R}^n$, one such sufficient condition is given by celebrated Mihlin-Hörmander multiplier theorem, which implies the L^p-boundedness of the spectral multiplier $F(-\Delta)$, which is basically radial Fourier multiplier. It states that $F(-\Delta)$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, whenever

$$\|F\|_{L^2_{s,\mathrm{sloc}}} := \sup_{t>0} \|F(t\cdot)\eta\|_{L^2_s(\mathbb{R})} < \infty \tag{1.1}$$

for some $s > n/2$, where η is some smooth function supported in $(0,\infty)$. In fact the condition $s > n/2$ is sharp (see [41]). After that, Mihlin-Hörmander multiplier theorem has been generalized in many different settings. For instance, in the setting of stratified Lie group $\mathcal{G}$, such results were independently proved by Mauceri and Meda [31] and by Christ [12]. In particular, they showed that for the sub-Laplacian L on $\mathcal{G}$, the operator $F(L)$ extends to a bounded operator on $L^p(\mathcal{G})$ for $p \in (1,\infty)$ whenever F satisfies (1.1) for some $s > Q/2$, where Q is the homogeneous dimension of $\mathcal{G}$. For all homogeneous sub-Laplacian on the Heisenberg group $\mathbb{H}$, Müller and Stein [35] proved that the smoothness condition $s > Q/2$ can be replaced by $s > d/2$, where $d = d_1 + d_2$ being the topological dimension of $\mathbb{H}$. In case of Heisenberg type group, similar result is independently obtained by Hebisch [19]. Moreover, the threshold $s > d/2$ for $\mathbb{H}$ turned out to be sharp in the sense that it can not be improved further. This discovery resulted in a surge of attention among many researchers and sparked numerous current research in the theory of spectral multiplier, see [30], [27], [26], [25], [1], [6], [13] and references therein. Let us remark that, however in general for any 2-step Stratified Lie group, it still remains an open question whether the homogeneous dimension Q in smoothness condition can be replaced by topological dimension d. Nevertheless, in case of Métivier group G, Martini [25] proved that the threshold can be pushed down again from $Q/2$ to $d/2$. In particular, this result also implies that Bochner-Riesz means associated to $\mathcal{L}$,

$$S^\alpha(\mathcal{L}) := (1 - t\mathcal{L})_+^\alpha, \quad t > 0$$

is bounded on $L^p(G)$ for all $1 \le p \le \infty$ whenever $\alpha > (d-1)/2$, and the result is sharp. Therefore if one wants to study the boundedness of $S^\alpha(\mathcal{L})$ for $0 < \alpha \le (d-1)/2$, one should not expect boundedness to hold for all $p \in [1, \infty]$. Towards this direction very recently, Chen and Ouhabaz [9] and Niedorf [36] in the case of Grushin operators, Niedorf in Heisenberg type groups [38] and Métivier groups [39] obtained the p-specific type Bochner-Riesz multiplier results. In fact for Métivier groups, Niedorf proved the following theorem.

For any $n \in \mathbb{N} \setminus \{0\}$, we define the Stein-Tomas exponent by $p_n = \frac{2(n+1)}{n+3}$. Let $p_{d_1,d_2} = p_{d_2}$ for $(d_1, d_2) \notin \{(8,6), (8,7)\}$, $p_{8,6} = 17/12$ and $p_{8,7} = 14/11$.

Theorem 1.1. [39, Theorem 1.2] *Let $1 \le p \le p_{d_1,d_2}$. Suppose that $\alpha > d(1/p - 1/2) - 1/2$ with $d = d_1 + d_2$. Then the Bochner-Riesz means $S^\alpha(\mathcal{L})$ is bounded on $L^p(G)$ uniformly in $t \ge 0$.*

In this paper, we study boundedness and compactness properties of the Bochner-Riesz commutator generated by $S^\alpha(\mathcal{L})$ and $b \in BMO^g(G)$, the space of functions of bounded mean oscillation on G (for definition see section 2). For $f \in \mathcal{S}(G)$, the Bochner-Riesz commutator associated to the sub-Laplacian $\mathcal{L}$ is denoted by $[b, S^\alpha(\mathcal{L})]$ and is defined by

$$[b, S^\alpha(\mathcal{L})]f = b\, S^\alpha(\mathcal{L})f - S^\alpha(\mathcal{L})(bf).$$

Before presenting our results, it would be convenient to revisit some relevant results from the literature. In the classical setting, the Bochner-Riesz commutator was studied by many mathematicians, see for example [20], [24] and [23]. In a very recent article [11], the authors obtained L^q-boundedness of commutators $[b, S_R^\alpha(L)]$ of a BMO function b and the Bochner-Riesz means $S_R^\alpha(L)$ for a class of nonnegative self-adjoint elliptic operators L on doubling metric measure space (X, μ) with ball volume being polynomial type. To be precise, assuming that L satisfies the finite speed of propagation property and the crucial spectral measure estimate:

$$\|dE_{\sqrt{L}}(\lambda)\|_{L^p\to L^{p'}} \leq C\lambda^{n(1/p-1/p')-1}, \quad \lambda > 0, \tag{1.2}$$

for some $1 \leq p < 2$, they obtained L^q-boundedness of $[b, S_R^\alpha(L)]$ for all $p < q < p'$ with the smooth condition $\alpha > n(1/p-1/2)-1/2$, n being the homogeneous dimension of X. It is known that for metric measure spaces if the ball volume is polynomial type, then (1.2) is equivalent to the Stein-Tomas restriction condition: for any $R > 0$ and all Borel functions F such that $\operatorname{supp} F \subseteq [0, R]$,

$$\|F(\sqrt{L})\chi_{B(x,r)}\|_{L^p\to L^2} \leq CR^{n(\frac{1}{p}-\frac{1}{2})}\|\delta_R F\|_{L^2}, \tag{1.3}$$

for all $x \in X$ and $r \geq 1/R$. For further details, see [10, Proposition I.4].

In the present context of Métivier groups G, although the ball volume is polynomial type and the sub-Laplacian $\mathcal{L}$ satisfies the finite speed of propagation property, to the best of our knowledge, it is still remains open question whether one can establish (1.3) type estimates for $\mathcal{L}$. A lot of investigation has been made in this direction, for instance in case of Heisenberg group, the range of p for which the restriction estimate hold is $p = 1$, this is because of the one-dimensional center of the Heisenberg group [33]. For Heisenberg type groups, dimension of the center is bigger than 1, there restriction type estimate was known to be true, see [21], [43]. But outside the class of Heisenberg type groups, unfortunately restriction estimate is still unknown, see [22, Reviewer's remark], also see remark of [38] on [5]. Therefore, it is a natural question whether L^p-boundedness of $[b, S^\alpha(\mathcal{L})]$ still holds with smoothness parameter $\alpha > Q(1/p-1/2)-1/2$, where Q being the homogeneous dimension of G. Because of the sub-Riemannian geometry of the underlying group, one can also ask whether smoothness parameter can be replaced by a weaker condition $\alpha > d(1/p-1/2)-1/2$, with d as the topological dimension of G. We answer this question positively, using a weaker version of restriction type estimate proved in [37].

For all $d_1, d_2 \geq 1$, let p_{d_1,d_2} be the same as just before the Theorem (1.1) and $L^2_\beta(\mathbb{R})$ denote the L^2 Sobolev space of order $\beta \geq 0$. Then following is our first main result of this paper.

Theorem 1.2. *Let $1 \leq p \leq p_{d_1,d_2}$ and $\beta > d(1/p-1/2)$. Then for any even Borel function F with $\operatorname{supp} F \subseteq [-1,1]$ and $F \in L^2_\beta(\mathbb{R})$, we have*

$$\|[b, F(t\sqrt{\mathcal{L}})]f\|_{L^q(G)} \leq C\|b\|_{BMO^\varrho(G)}\|F\|_{L^2_\beta(\mathbb{R})}\|f\|_{L^q(G)},$$

for all $p < q < p'$ and uniformly in $t \in (0,\infty)$.

The following result is a consequence of the above theorem, which gives the boundedness of Bochner-Riesz commutator $[b, S^\alpha(\mathcal{L})]$.

Theorem 1.3. *If $1 \leq p \leq p_{d_1,d_2}$ and $\alpha > d(1/p-1/2)-1/2$, then the Bochner-Riesz commutator $[b, S^\alpha(\mathcal{L})]$ is bounded on $L^q(G)$ whenever $p < q < p'$.*

Since Heisenberg type groups falls into the class of Métivier groups, in particular, the above result is also applicable there. Notice that in the above theorems the range of p is in between 1 and p_{d_2}, for all $d_1, d_2 \geq 1$ except the two points $(8,6)$ and $(8,7)$. For $(d_1, d_2) \in \{(8,6),(8,7)\}$, $p_6 = 14/9 > 17/12$ and $p_7 = 8/5 > 14/11$. However, if one restricts attention to the sub-Laplacian on the Heisenberg type groups, then we can improve the above result for all $d_1, d_2 \geq 1$, with the range of p is $1 \leq p \leq p_{d_2}$. In fact, we have the following result.

Theorem 1.4. *Let $\mathbb{H}$ be a Heisenberg type group and $\mathcal{L}$ be its associated sub-Laplacian. If $1 \leq p \leq 2(d_2+1)/(d_2+3)$ and $\alpha > d(1/p - 1/2) - 1/2$, then the Bochner-Riesz commutator $[b, S^\alpha(\mathcal{L})]$ is bounded on $L^q(\mathbb{H})$ whenever $p < q < p'$.*

Our next discussion will be regarding the compactness property of the commutators. In [45], Uchiyama proved that the commutator of Riesz transform on $\mathbb{R}^n$ is compact on $L^p(\mathbb{R}^n)$ with $1 < p < \infty$ if and only if the symbol belongs to $CMO(\mathbb{R}^n)$, the closure of $C_c^\infty(\mathbb{R}^n)$ functions in the $BMO(\mathbb{R}^n)$ norm. In recent times, the compactness property of commutators for many different class of operators in various settings has been studied widely, such as the Riesz transform associated with the sub-Laplacian on stratified Lie groups, the Riesz transform associated with Bessel operator, Bochner-Riesz commutator on $\mathbb{R}^n$, the Cauchy's integrals on $\mathbb{R}$, and the Calderón-Zygmund operator with homogeneous kernel to name a few. Interested readers are referred to [8], [15], [3], [7], [42] and reference therein for further details. Another aim of this paper is to study the compactness property of $[b, S^\alpha(\mathcal{L})]$ in the setting of Métivier groups G. Let $CMO^\varrho(G)$ be the closure of $C_c^\infty(G)$ under the norm of $BMO^\varrho(G)$. Then the following is the second main result of this paper.

Theorem 1.5. *Let $1 \leq p \leq p_{d_1,d_2}$ and $\alpha > d(1/p - 1/2) - 1/2$. If $b \in CMO^\varrho(G)$, then the Bochner-Riesz commutator $[b, S^\alpha(\mathcal{L})]$ is a compact operator on $L^q(G)$ for all $p < q < p'$.*

Remark 1.6. *It is worth mentioning that all the above results, Theorem* 1.2*, Theorem* 1.3*, Theorem* 1.4 *and Theorem* 1.5 *are stated with the smoothness parameter α in terms of the topological dimension, d of the underlying group G. The smoothness parameter what we get here is $\alpha > d(1/p - 1/2) - 1/2$, which is same as the Bochner-Riesz means appeared in* [39, Theorem 1.1] *and* [38, Theorem 1.1]*, and which are sharp in view of* [29]*. One can ask the similar questions for other sub-Laplacians. We are also working on Grushin operator, but this result will appear in somewhere else* [32].

An outline of our strategy for proving Theorem 1.2*.* As mentioned before, one of the major difficulties arising in our setting is the absence of spectral measure estimate (1.2) for $\mathcal{L}$. We would like to mention that the arguments given in [11] have a limited scope as even if one has (1.2) and follows [11], yet one can merely obtain L^q-boundedness of $[b, S^\alpha(\mathcal{L})]$ with the smoothness condition being expressed in terms of the homogeneous dimension of G. One of the main tools that has been used there to reduce the commutator boundedness to the boundedness of the corresponding operator via [11, Lemma 3.2]. We can not use this lemma as a black box, as this will lead to the smoothness parameter in terms of Q. At this point, one may desire to use weighted restriction estimate for $\mathcal{L}$ but unfortunately, we still lack of such tools because such estimates does not hold here, see [38, Section 8]. Thus a refinement of arguments in [11] is necessary. Next, to circumvent the absence of (1.2), we make use of a weaker version of restriction type estimates, Proposition 3.1, which is due to [39], where the operator $F(\mathcal{L})$ is further truncated dyadically along the spectrum of $T := (-(T_1^2 + \cdots + T_{d_2}^2))^{1/2}$. In this approach, we also need to take care of a discrete norm of F appearing in Proposition (3.1). Nevertheless, such an approach is limited to dealing with the error part of this additional truncation (see (4.14)). In order to address the main part, the sub-Riemannian geometry of the underlying manifold is taken into account, in particular we use (2.4) crucially. This helps us to further split the main part into two summands (see (4.16)). The negligible part of the summand is tackled through the weighted Plancherel estimates (Proposition 3.3), whereas the significant part of the summand is taken care of by Proposition 3.1 and Proposition 2.1, which is about the numerology of the Métivier groups. Because of Proposition 2.1, we have to remove three points for the Stein-Tomas range. Nevertheless one can recover the full range of p, by carefully analyzing the point $(d_1, d_2) = (4, 3)$ as done in [39, Section 8].

An outline of our strategy for proving Theorem 1.5. A careful reading of the proof of the corresponding result in euclidean spaces [3] reveals that the proof relies on the three main ingredients namely, Fourier transform estimates, approximation to the identity, and some refined estimates obtained by C. Fefferman in [18]. We would like to point out that such arguments are not very effective in the setting of Métivier group G. A significant aspect of our proof is the use of the fact that the norm limit of compact operators is again a compact operator. First we decompose the operator in Fourier transform side and on each pieces we employ the Kolmogorov-Riesz compactness type theorem Proposition 5.1. In order to do so we need good pointwise control over the kernel and the gradient of the kernel. These estimates we have proved in Proposition 3.4, which helps us to get our required result in this settings as well which could be an independent interest.

The plan of the paper is as follows. In the next Section (2) we gather some well known results concerning the sub-Riemannian geometry of G, the space of bounded mean oscillation, spectral decomposition of $-J_\mu^2$, an explicit formula for convolution kernel, integration of a norm function and an useful result. Section 3 is devoted to developing some key devices related to truncated restriction estimates, weighted Plancherel estimates and pointwise weighted kernel estimates that we need for further study. Finally, in Section 4 and Section 5 we present the proof of main results of this paper.

Let $\mathbb{N} = \{0, 1, 2, ...\}$. For a Lebesgue measurable subset E of $\mathbb{R}^d$, we denote by χ_E the characteristic function of the set E. We use letter C to indicate a positive constant independent of the main parameters, but may vary from line to line. We shall use the notation $f \lesssim g$ to indicate $f \leq Cg$ for some $C > 0$, and whenever $f \lesssim g \lesssim f$, we shall write $f \sim g$. Also, we write $f \lesssim_\epsilon g$ when the implicit constant C may depend on a parameter like ϵ. Whenever $A > 0$ is very small we frequently replace the quantity 2^A by 2^{ϵ_1} for some $\epsilon_1 > 0$ very small. Also, for any ball $B := B(x, r)$ with centered at x and radius r, the notation $\kappa B = B(x, \kappa r)$ stands for the concentric dilation of B by $\kappa > 0$. Moreover, for a measurable function f, we denote the average of f over B as $f_B = \frac{1}{|B|}\int_B f(x)\,dx$. For any function G on $\mathbb{R}$, we define $\delta_R G(\eta) = G(R\eta)$ for any $R > 0$. For $f, g \in \mathcal{S}(G)$, let $f * g$ denotes their group convolution given by

$$(f * g)(x, u) = \int_G f(x', u')g((x', u')^{-1}(x, u))\,d(x', u'), \qquad (x, u) \in G.$$

2. Preliminaries for the Métivier groups

Let G be the Métivier group and $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be the stratification of its Lie algebra $\mathfrak{g}$. Let $X_1, \dots, X_{d_1}$ and $T_1, \dots, T_{d_2}$ be a basis of first layer $\mathfrak{g}_1$ and of the second layer $\mathfrak{g}_2$ respectively. By means of the global diffeomorphism $\exp : \mathfrak{g} \to G$, the group G is identified with its Lie algebra $\mathfrak{g}$, which can be again identified with $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Thus, we write an element g of G as (x, u), where $x \in \mathbb{R}^{d_1}$ and $u \in \mathbb{R}^{d_2}$. Moreover, the group product formula on G is determined by Baker-Campbell-Hausdorff formula,

$$(x, u)(x', u') = (x + x', u + u' + \tfrac{1}{2}[x, x']). \tag{2.1}$$

And the identity element of G is $(0, 0)$, while the inverse $(x, u)^{-1}$ of an element (x, u) is given by $(x, u)^{-1} = (-x, -u)$. The Haar measure $d(x, u)$ on G corresponds to the Lebesgue measure on $\mathfrak{g}$, via the exponential map. There is a natural family of dilations $\{\delta_R : R > 0\}$ on G which is given by

$$\delta_R(x, u) = (Rx, R^2 u), \quad (x, u) \in G. \tag{2.2}$$

With respect to this dilation the following

$$\|(x, u)\| := (|x|^4 + |u|^2)^{1/4}, \quad (x, u) \in G, \tag{2.3}$$

defines a homogeneous norm on G.

Let ϱ denote the Carnot-Carathéodory metric associated to the vector fields $X_1, \dots, X_{d_1}$. In view of Chow–Rashevskii theorem, ϱ induces the Euclidean topology of G. By the property that any two homogeneous norms on a homogeneous Lie group are equivalent, we have

$$\varrho(g, h) \sim \|g^{-1}h\|, \quad \text{for all } g, h \in G.$$

Moreover, since these vector fields are left-invariant, ϱ is left-invariant in the sense that

$$\varrho(ag, ah) = \varrho(g, h) \quad \text{for all } a, g, h \in G.$$

Let $B^{\varrho}((x,u),R)$ and $\bar{B}^{\varrho}((x,u),R)$ denote respectively the open and closed ball of radius $R > 0$ centered at $(x,u) \in G$ with respect to ϱ. In the sequel, we omit the superscript ϱ and simply write B or $B((x,u),R)$ to indicate an open ball with respect to the Carnot-Carathéodory distance. The volume of the ball is given by

$$|B((x,u),R)| \sim R^{Q}|B(0,1)|,$$

where $|\cdot|$ denote the Lebesgue measure and $Q = d_1 + 2d_2$. Thus, the metric measure space $(G, \varrho, |\cdot|)$ is indeed a space of homogeneous type, with homogeneous dimension Q. We also call $d = d_1 + d_2$ to be the topological dimension of G.

From (2.3), one can easily see that, there exists a constant $C > 0$ such that

$$B(0,R) \subseteq B^{|\cdot|}(0,CR) \times B^{|\cdot|}(0,CR^2) \subseteq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}, \tag{2.4}$$

where $B^{|\cdot|}(a,R)$ denotes the ball of radius R and centered at a with respect to Euclidean distance.

Note that on Métivier groups we always have $d_1 > d_2$. This is true because the map $\mu \to \omega_\mu(\cdot, x')$ from $\mathfrak{g}_2^* \to (\mathfrak{g}_1/\mathbb{R}x')^*$ is injective for $x' \neq 0$. The next proposition tells us that for Métivier groups d_1, the dimension of the first layer is much larger than d_2, the dimension of the second layer, except for few cases.

Proposition 2.1. [39, Proposition 7.1] *Let G be a Métivier group of topological dimension $d = d_1 + d_2$ with center of dimension d_2. Then for $(d_1, d_2) \notin \{(4,3), (8,6), (8,7)\}$, we have $d_1 > 3d_2/2$.*

The theory of BMO spaces on space of homogeneous type is well known. The function space of bounded mean oscillation on G is denoted by $BMO^{\varrho}(G)$ and is defined by

$$BMO^{\varrho}(G) = \{f \in L^1_{\text{loc}}(G) : \|f\|_{BMO^{\varrho}(G)} < \infty\}, \tag{2.5}$$

where

$$\|f\|_{BMO^{\varrho}(G)} := \sup_{B} \left(\frac{1}{|B|} \int_B |f(y,t) - f_B| \, d(y,t) \right). \tag{2.6}$$

The celebrated John–Nirenberg inequality states that any function in $BMO^{\varrho}(G)$ has exponential decay of its distribution function. An important consequence of this distribution inequality is the L^q characterization of $BMO^{\varrho}(G)$ norms:

$$\|f\|_{BMO^{\varrho}(G)} \sim \sup_{B} \left(\frac{1}{|B|} \int_B |f(y,t) - f_B|^q \, d(y,t) \right)^{\frac{1}{q}}, \tag{2.7}$$

for any $1 < q < \infty$. More details can be found in [14]. A simple but useful property of $BMO^{\varrho}(G)$ function is the following. Let f be in $BMO^{\varrho}(G)$. Given a ball B and a positive integer k we have

$$|b_B - b_{2^k B}| \leq 2^Q\, k\, \|f\|_{BMO^{\varrho}(G)}. \tag{2.8}$$

Now we will discuss about the kernel representation of $m(\mathcal{L})$, for some appropriate function m. Recall that corresponding to the basis $\{X_1, \ldots, X_{d_1}\}$ and $\{T_1, \ldots, T_{d_2}\}$ of the first layer $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively, the sub-Laplacian $\mathcal{L}$ on G is defined by $\mathcal{L} = -(X_1^2 + \cdots + X_{d_1}^2)$. Then the operators $\mathcal{L}, -iT_1, -iT_2, \cdots, -iT_{d_2}$ are essentially self-adjoint on $L^2(G)$ and commutes strongly, therefore they admit a joint functional calculus. In particular, the operator $\mathcal{L}$ and $T := [-(T_1^2 + \cdots + T_{d_2}^2)]^{1/2}$ admit a joint functional calculus. Therefore, for any bounded Borel function $m : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, the operator $m(\mathcal{L}, T)$ is well defined and bounded on $L^2(G)$. In addition, by imposing an appropriate condition on m, it can be shown that $m(\mathcal{L}, T)$ possesses a convolution kernel $\mathcal{K}_{m(\mathcal{L},T)}$ and moreover, an explicit formula for $\mathcal{K}_{m(\mathcal{L},T)}$ can be obtained in terms of rescaled Laguerre functions. Since G is a Métivier group, the endomorphism J_μ is invertible for all $\mu \in \mathfrak{g}_2^*$. The next proposition gives us the spectral decomposition of $-J_\mu^2$ for μ lies in some Zariski open subset of $\mathfrak{g}_2^*$.

Proposition 2.2. [39, Proposition 3.1] *There exists a non-empty, homogeneous Zariski-open subset $\mathfrak{g}_{2,r}^*$ of $\mathfrak{g}_2^*$, numbers $N \in \mathbb{N} \setminus \{0\}$, vectors $\mathbf{r} = (r_1, \ldots, r_N) \in (\mathbb{N} \setminus \{0\})^N$, a function $\mu \to \mathbf{b}^\mu = (b_1^\mu, \ldots, b_N^\mu) \in [0, \infty)^N$ on $\mathfrak{g}_2^*$, functions $\mu \mapsto P_{n,\mu}$ on $\mathfrak{g}_{2,r}^*$ with $P_{n,\mu} : \mathfrak{g}_1 \to \mathfrak{g}_1$, $n \in \{1, \ldots, N\}$ and a function $\mu \mapsto R_\mu \in O(d_1)$ on $\mathfrak{g}_{2,r}^*$ such that*

$$-J_\mu^2 = \sum_{n=1}^{N} (b_n^\mu)^2 P_{n,\mu} \quad \textit{for all } \mu \in \mathfrak{g}_{2,r}^*,$$

with $P_{n,\mu} R_\mu = R_\mu P_n$, $J_\mu(ran\, P_{n,\mu}) \subseteq ran\, P_{n,\mu}$ for the range of $P_{n,\mu}$ for all $\mu \in \mathfrak{g}_{2,r}^*$, and all $n \in \{1, \ldots, N\}$, where P_n denotes the projection from $\mathbb{R}^{d_1} = \mathbb{R}^{2r_1} \oplus \cdots \oplus \mathbb{R}^{2r_N}$ onto the n-th layer, where

1. *the function $\mu \to b_n^\mu$ are homogeneous of degree 1 and continuous on $\mathfrak{g}_2^*$, real analytic on $\mathfrak{g}_{2,r}^*$, and satisfy $b_n^\mu > 0$ for all $\mu \in \mathfrak{g}_{2,r}^*$ and $n \in \{1, \ldots, N\}$, and $b_n^\mu \neq b_{n'}^\mu$ if $n \neq n'$ for all $\mu \in \mathfrak{g}_{2,r}^*$ and $n, n' \in \{1, \ldots, N\}$,*
2. *the functions $\mu \to P_{n,\mu}$ are (component wise) real analytic on $\mathfrak{g}_{2,r}^*$, homogeneous of degree 0, and the maps $P_{n,\mu}$ are orthogonal projections on $\mathfrak{g}_1$ of rank $2r_n$ for all $\mu \in \mathfrak{g}_{2,r}^*$, with pairwise orthogonal ranges,*
3. *$\mu \to R_\mu$ is a Borel measurable function on $\mathfrak{g}_{2,r}^*$ which is homogeneous of degree 0 and there is a family $(U_\ell)_{\ell \in \mathbb{N}}$ of disjoint Euclidean open subsets $U_\ell \subseteq \mathfrak{g}_{2,r}^*$ whose union is $\mathfrak{g}_{2,r}^*$, up to a set of measure zero such that $\mu \to R_\mu$ is (component wise) real analytic on each U_ℓ.*

For $k, r \in \mathbb{N}$ and $\lambda > 0$, let $\varphi_k^{(\lambda,r)}$ denote the λ-rescaled Laguerre function defined by

$$\varphi_k^{(\lambda,r)}(z) = \lambda^r L_k^{r-1}(\tfrac{1}{2}\lambda|z|^2)e^{-\frac{1}{2}\lambda|z|^2}, \quad z \in \mathbb{R}^{2r},$$

where L_k^{r-1} is the k-th Laguerre polynomial of type $r-1$ (see [44]).

With the same notation as in the above proposition, the following result contains an explicit formula for the kernel of $H(\mathcal{L}, T)$.

Proposition 2.3. [39, Proposition 3.10] *If $m : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is a Schwartz function, then $m(\mathcal{L}, T)$ possesses a convolution kernel $\mathcal{K}_{m(\mathcal{L},T)} \in \mathcal{S}(G)$, that is,*

$$m(\mathcal{L}, T)f = f * \mathcal{K}_{m(\mathcal{L},T)} \quad \textit{for all } f \in \mathcal{S}(G).$$

Moreover, for $x \in \mathfrak{g}_1$ and $u \in \mathfrak{g}_2$, we have

$$\mathcal{K}_{m(\mathcal{L},T)}(x,u) = \frac{1}{(2\pi)^{d_2}} \int_{\mathfrak{g}_{2,r}^*} \sum_{\mathbf{k}\in\mathbb{N}^N} m(\lambda_{\mathbf{k}}^{\mu}, |\mu|) \left[\prod_{n=1}^{N} \varphi_{k_n}^{(b_n^{\mu}, r_n)}(R_\mu^{-1} P_{n,\mu} x)\right] e^{i\langle \mu, u\rangle}\, d\mu,$$

with $\lambda_{\mathbf{k}}^{\mu} = \sum_{n=1}^{N}(2k_n + r_n)b_n^{\mu}$.

We sometimes need the integration of the homogeneous norm on outside of some ball. The proof of the following lemma is easy and follows from just breaking the integral in disjoint union of annular regions, therefore we omit it here.

Lemma 2.4. *Let $R, r > 0$. Then for any $s > Q$ we have*

$$\int_{\|(x,u)\|>r} \frac{d(x,u)}{\big(1 + R\|(x,u)\|\big)^s} \leq CR^{-s}r^{-s+Q}.$$

We conclude this section by stating the following useful lemma, that we need in the proof of our results which can be proved using similar argument from the proof of [11, Lemma 3.2].

Lemma 2.5. *Let U be a linear operator with kernel $\mathcal{K}_U$ such that*

$$\operatorname{supp} \mathcal{K}_U \subseteq \mathcal{D}_r := \{(x,y) \in G \times G : \varrho(x,y) \leq r\}$$

for some $r > 0$. Assume $b \in BMO^{\varrho}(G)$. Then there exists a sequence $\{(x_k, u_k)\}_{k\in\mathbb{N}}$ in G, disjoint sets $\widetilde{B}_k \subseteq B_k := B\big((x_k, u_k), r\big)$ such that

$$B((x_k, u_k), \tfrac{r}{20}) \cap B((x_j, u_j), \frac{r}{20}) = \emptyset$$

for $j \neq k$ and for $1 \leq q < \infty$ we have

$$\|[b, U]f\|_{L^q(G)}^q \leq C \sum_k \Big(\|\chi_{4B_k}(b - b_{B_0})\, U(\chi_{\widetilde{B}_k} f)\|_{L^q(G)}^q + \|\chi_{4B_k}\, U(\chi_{\widetilde{B}_k}(b - b_{B_0})f)\|_{L^q(G)}^q \Big),$$

where $B_0 = B(0, r)$.

3. Restriction and kernel estimates

In this section we discuss about the restriction type estimates (Proposition 3.1, Corollary 3.2), weighted Plancherel estimates with respect to the first layer (Proposition 3.3) and pointwise weighted kernel estimates (Proposition 3.4) for the sub-Laplacians on Métivier groups. One of the main ingredients in proving the boundedness of Bochner-Riesz means in the Stein-Tomas range is the use of restriction type estimates, which goes back to the idea of Stein [17]. Recently in [37], the author proved truncation restriction type estimate on any two step stratified Lie groups, but with the presence of discrete norm instead of L^2-norm, which we discuss below.

For any bounded Borel function $\sigma : \mathbb{R} \to \mathbb{C}$ such that $\operatorname{supp} \sigma \subseteq [0, 8]$, we define the following discrete norm

$$\|\sigma\|_{N,2} = \left(\frac{1}{N} \sum_{K \in \mathbb{Z}} \sup_{\lambda \in [\frac{K-1}{N}, \frac{K}{N}]} |\sigma(\lambda)|^2 \right)^{\frac{1}{2}}, \quad N > 0.$$

Regarding the above discrete norm, we have the following estimates:

$$\|\sigma\|_{N,2} \leq C \, \|\sigma\|_{L^\infty}, \tag{3.1}$$

and for $s > 1/2$,

$$\|\sigma\|_{L^2} \leq \|\sigma\|_{N,2} \leq C \left(\|\sigma\|_{L^2} + N^{-s} \|\sigma\|_{L^2_s} \right), \tag{3.2}$$

where the first estimate follows from the definition and for the second estimate, see [37, eq. (1.7)].

Let $\Theta : \mathbb{R} \to [0, 1]$ be an even C_c^∞ function supported in $[-2, -1/2] \cup [1/2, 2]$, such that

$$\sum_{M \in \mathbb{Z}} \Theta_M(\tau) = 1, \tag{3.3}$$

where $\Theta_M(\tau) = \Theta(2^M \tau)$. Also, let $F : \mathbb{R} \to \mathbb{C}$ is a bounded Borel function supported in $[1/8, 8]$. Then for $M \in \mathbb{Z}$, we define $F_M : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ by

$$F_M(\kappa, \tau) := F(\sqrt{\kappa}) \Theta(2^M \tau) \quad \text{for} \quad \kappa \geq 0, \tag{3.4}$$

and $F_M(\kappa, \tau) = 0$ else.

Then we have the following *truncated* restriction type estimate.

Proposition 3.1. [39, Theorem 1.1] *Suppose* $1 \leq p \leq \min\{p_{d_1}, p_{d_2}\}$. *Let* F *and* F_M *be as defined in* (3.4). *Then*

$$\|F_M(\mathcal{L}, T) f\|_{L^2(G)} \leq C 2^{-M d_2 (1/p - 1/2)} \, \|F\|_{L^2}^{1-\theta_p} \, \|F\|_{2^M, 2}^{\theta_p} \, \|f\|_{L^p(G)},$$

where $\theta_p \in [0, 1]$ *satisfies* $1/p = (1 - \theta_p) + \theta_p / \min\{p_{d_1}, p_{d_2}\}$.

Using (3.2), the following result follows immediately from Proposition 3.1.

Corollary 3.2. *Suppose* $1 \leq p \leq \min\{p_{d_1}, p_{d_2}\}$*. Let* F *and* F_M *be as in* (3.4)*. Then*

$$\|F_M(\mathcal{L}, T)f\|_{L^2(G)} \leq C 2^{-Md_2(1/p-1/2)} \|F\|_{2^M,2} \|f\|_{L^p(G)}.$$

The following proposition, which we sometimes call weighted Plancherel estimate with respect to the first layer, plays an important role later in our proof.

Proposition 3.3. [39, Proposition 8.1] *Let* $\mathcal{K}_{F_M(\mathcal{L},T)}$ *denote the convolution kernel of* $F_M(\mathcal{L}, T)$*. Then for any* $\alpha \geq 0$*, we have*

$$\int_G \left||x|^\alpha \mathcal{K}_{F_M(\mathcal{L},T)}(x, u)\right|^2 d(x, u) \leq C\, 2^{M(2\alpha - d_2)} \|F\|_{L^2}^2.$$

In the sequel, we also require the pointwise weighted kernel estimate. Let us set $X = (X_1, \dots, X_{d_1}, T_1, \dots, T_{d_2})$.

Proposition 3.4. *Let* $\Gamma \in \mathbb{N}^{d_1+d_2}$*. Then for all bounded Borel functions* $F : \mathbb{R} \to \mathbb{C}$ *supported in* $[0, R]$ *and for all* $\beta \geq 0$*, whenever* $s > \beta$ *we have*

$$|(1 + R\|(x, u)\|)^\beta \mathcal{K}_{F(\sqrt{\mathcal{L}})}(x, u)| \leq C\, R^Q\, \|\delta_R F\|_{L^\infty_s} \tag{3.5}$$

$$\text{and} \quad |(1 + R\|(x, u)\|)^\beta X^\Gamma \mathcal{K}_{F(\sqrt{\mathcal{L}})}(x, u)| \leq C\, R^{|\Gamma|}\, R^Q\, \|\delta_R F\|_{L^\infty_s}. \tag{3.6}$$

Moreover, we also have

$$|\mathcal{K}_{F(\sqrt{\mathcal{L}})}(x, u)| \leq C\, R^Q\, \|\delta_R F\|_{L^\infty}. \tag{3.7}$$

Proof. Note that enough to prove the estimate (3.6), as the estimate (3.5) follows from (3.6) by just taking $\Gamma = 0$. The main ingredients to proof this result is the following Gaussian bound for the heat kernel and its gradient estimate.

It is known that for all $t > 0$, the heat kernel $\mathcal{K}_{\exp(-t\mathcal{L})}$ satisfies the following estimates (see [46], [2]).

$$|X^\Gamma \mathcal{K}_{\exp(-t\mathcal{L})}(x, u)| \leq C t^{-|\Gamma|/2} t^{-Q/2} \exp\left\{-c \frac{\|(x,u)\|^2}{t}\right\} \quad \text{for} \quad t > 0.$$

Then we have

$$\begin{aligned}
&\int_{G \setminus B(0,r)} \left|X^\Gamma \mathcal{K}_{\exp(-t\mathcal{L})}(x, u)\right|^2 d(x, u) \\
&\leq C t^{-|\Gamma|} t^{-Q} e^{-c\frac{r^2}{t}} \int_{G \setminus B(0,r)} \exp\left\{-c \frac{\|(x,u)\|^2}{t}\right\} d(x, u) \\
&\leq C t^{-|\Gamma|} t^{-Q/2} e^{-c\frac{r^2}{t}},
\end{aligned}$$

where we have used the fact $\int_G \exp\{-c\frac{\|(x,u)\|^2}{t}\}\, d(x, u) \leq C\, t^{Q/2}$. In particular, we get

$$\int_G \left|X^\Gamma \mathcal{K}_{\exp(-t\mathcal{L})}(x, u)\right|^2 d(x, u) \leq C t^{-|\Gamma|} t^{-Q/2}. \tag{3.8}$$

We will use some ideas from [4, Lemma 4.2, 4.3]. Let us set $F_1(\lambda) = F(\sqrt{\lambda})e^{\lambda/R^2}$ and $F_2(\lambda) = e^{-\lambda/R^2}$. Then $F(\sqrt{\lambda}) = F_1(\lambda)F_2(\lambda)$ and therefore we can write

$$\begin{aligned}\mathcal{K}_{F(\sqrt{\mathcal{L}})}(x,u) &= \mathcal{K}_{F_1(\mathcal{L})} * \mathcal{K}_{F_2(\mathcal{L})}(x,u) \\ &= \int_G \mathcal{K}_{F_1(\mathcal{L})}((z,s)^{-1}(x,u))\mathcal{K}_{F_2(\mathcal{L})}(z,s)\, d(z,s).\end{aligned} \tag{3.9}$$

So that from (3.8) and (3.9), we get

$$\begin{aligned}\int_G |X^\Gamma \mathcal{K}_{F(\sqrt{\mathcal{L}})}(x,u)|^2\, d(x,u) &\leq C\|F_1(\mathcal{L})\|^2_{L^2\to L^2}\|X^\Gamma \mathcal{K}_{e^{-R^{-2}\mathcal{L}}}\|^2_{L^2} \\ &\leq C\|F_1\|^2_{L^\infty} R^{2|\Gamma|} R^Q \\ &\leq CR^{2|\Gamma|} R^Q \|F\|^2_{L^\infty}.\end{aligned} \tag{3.10}$$

Therefore, using Hölder's inequality, (3.8) and (3.10) from (3.9) we can see

$$\begin{aligned}&|X^\Gamma \mathcal{K}_{F(\sqrt{\mathcal{L}})}(x,u)| \\ &\leq \left(\int_G |\mathcal{K}_{F_1(\mathcal{L})}((z,s)^{-1}(x,u))|^2\, d(z,s)\right)^{\frac{1}{2}} \left(\int_G |X^\Gamma \mathcal{K}_{e^{-R^{-2}\mathcal{L}}}(z,s)|^2\, d(z,s)\right)^{\frac{1}{2}} \\ &\leq CR^{Q/2}\|F\|_{L^\infty} R^{|\Gamma|} R^{Q/2} \\ &\leq CR^{|\Gamma|} R^Q \|\delta_R F\|_{L^\infty}.\end{aligned} \tag{3.11}$$

Now, let us set $G(\lambda) = F(\sqrt{R^2\lambda})e^{\lambda}$. Then $F(\sqrt{\lambda}) = G(R^{-2}\lambda)e^{-R^{-2}\lambda}$. Therefore, by Fourier inversion formula we can write

$$F(\sqrt{\lambda}) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{G}(\tau)\exp\left(-(1-i\tau)R^{-2}\lambda\right) d\tau,$$

From this we can easily see that

$$\mathcal{K}_{F(\sqrt{\mathcal{L}})}(x,u) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{G}(\tau)\mathcal{K}_{\exp(-(1-i\tau)R^{-2}\mathcal{L})}(x,u)\, d\tau. \tag{3.12}$$

Then similarly as in [40, Theorem 7.3], one can show that

$$|X^\Gamma \mathcal{K}_{\exp(-(1-i\tau)R^{-2}\mathcal{L})}(x,u)| \leq CR^{|\Gamma|} R^Q \exp\left\{-c\frac{R^2\|(x,u)\|^2}{(1+\tau^2)}\right\}.$$

From this for any $\beta \geq 0$ we get

$$|X^\Gamma \mathcal{K}_{\exp(-(1-i\tau)R^{-2}\mathcal{L})}(x,u)|(1+R\|(x,u)\|)^\beta \leq CR^{|\Gamma|} R^Q (1+|\tau|)^\beta. \tag{3.13}$$

Therefore, from (3.12), (3.13) and using Cauchy-Schwartz inequality for some $\varepsilon > 0$ we have

$$\begin{aligned}&|X^\Gamma \mathcal{K}_{F(\sqrt{\mathcal{L}})}(x,u)|(1+R\|(x,u)\|)^\beta \\ &\leq C\int_{\mathbb{R}} |\widehat{G}(\tau)||X^\Gamma \mathcal{K}_{\exp(-(1-i\tau)R^{-2}\mathcal{L})}(x,u)|(1+R\|(x,u)\|)^\beta\, d\tau\end{aligned} \tag{3.14}$$

$$\leq CR^{|\Gamma|} R^Q \left(\int_{\mathbb{R}} |\widehat{G}(\tau)|^2 (1+|\tau|^2)^{\beta+1/2+\varepsilon} \, d\tau \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} (1+|\tau|^2)^{-1/2-\varepsilon} \, d\tau \right)^{\frac{1}{2}}$$
$$\leq CR^{|\Gamma|} R^Q \, \|G\|_{L^2_{\beta+1/2+\varepsilon}}$$
$$\leq CR^{|\Gamma|} R^Q \, \|\delta_R F\|_{L^\infty_{\beta+1/2+\varepsilon}}.$$

Then interpolating (3.11) and (3.14) as in [16, Lemma 4.3] for any $\beta \geq 0$, we get

$$|X^\Gamma \mathcal{K}_{F(\sqrt{\mathcal{L}})}(x,u)|(1+R\|(x,u)\|)^\beta \leq CR^{|\Gamma|} R^Q \, \|\delta_R F\|_{L^\infty_{\beta+\varepsilon}}. \tag{3.15}$$

Indeed, consider the linear operator $K_R : L^\infty([0,1]) \to L^\infty(G)$ given by

$$K_R(\mathfrak{F}) = X^\Gamma \mathcal{K}_{\delta_{1/R}\mathfrak{F}(\sqrt{\mathcal{L}})}.$$

Note that from (3.11) we have

$$\|K_R(\mathfrak{F})\|_{L^\infty([0,1]) \to L^\infty(G)} \leq CR^{|\Gamma|+Q}.$$

Let us set

$$L^\infty_{\beta,R} := L^\infty(G, \mu_{\beta,R}) \qquad \text{where} \qquad d\mu_{\beta,R}(x,u) = (1+R\|(x,u)\|)^\beta d(x,u).$$

Then from (3.14) we have

$$\|K_R(\mathfrak{F})\|_{L^\infty_{\beta+1/2+\varepsilon}([0,1]) \to L^\infty_{\beta,R}} \leq CR^{|\Gamma|+Q}.$$

Consequently, by interpolation [31, proof of Lemma 1.2] for every $\theta \in (0,1)$, there exists a positive constant C such that for all $\beta > 0$, $\varepsilon' > 0$,

$$\|X^\Gamma \mathcal{K}_{\delta_{1/R}\mathfrak{F}(\sqrt{\mathcal{L}})}\|_{L^\infty_{\beta\theta,R}} \leq CR^{|\Gamma|+Q} \|\mathfrak{F}\|_{L^\infty_{\beta\theta+\theta/2+\varepsilon'}([0,1])}.$$

Taking $\beta' = \beta\theta$ and choosing θ sufficiently small we obtain

$$\|X^\Gamma \mathcal{K}_{\delta_{1/R}\mathfrak{F}(\sqrt{\mathcal{L}})}\|_{L^\infty_{\beta',R}} \leq CR^{|\Gamma|+Q} \|\mathfrak{F}\|_{L^\infty_{\beta'+\varepsilon''}([0,1])}, \tag{3.16}$$

for all $\beta' > 0$ and $\varepsilon'' > 0$.

Then the required estimate (3.15) follows immediately from the above estimate (3.16).

This completes the proof of the estimate (3.6). Now the proof of (3.7) follows from (3.11) by taking $\Gamma = 0$. □

4. Boundedness of Bochner-Riesz commutator

In this section, we prove Theorem 1.2, Theorem 1.3 and Theorem 1.4. Note that Theorem 1.3 is an easy consequence of Theorem 1.2 by setting $F(\lambda) = (1-\lambda^2)^\alpha_+$, $t = 1$ and using the fact that $F \in L^2_\beta$ if and only if $\alpha > \beta - 1/2$. Therefore let us start with the proof of Theorem 1.2.

Proof of Theorem 1.2. By duality and interpolation argument, in order to prove Theorem 1.2, it is enough to consider the case $q \in (p, 2)$. Let δ_R be as in (2.2).

Then we have $\delta_{t^{-1}}\circ\sqrt{\mathcal{L}}\circ\delta_t = \sqrt{\mathcal{L}}$, and consequently $\delta_{t^{-1}}\circ F(\sqrt{\mathcal{L}})\circ\delta_t = F(t\sqrt{\mathcal{L}})$ for any bounded Borel function $F : \mathbb{R} \to \mathbb{C}$. Therefore we obtain

$$\|[b, F(t\sqrt{\mathcal{L}})]\|_{L^p(G)\to L^p(G)} = \|[b_t, F(\sqrt{\mathcal{L}})]\|_{L^p(G)\to L^p(G)}, \tag{4.1}$$

where $b_t = \delta_t b$ and $\|b_t\|_{BMO^\varrho(G)} = \|b\|_{BMO^\varrho(G)}$. In view of the above relation (4.1) it is enough to consider $t = 1$.

We first break F in the Fourier transform side. To this end, we choose an even bump function η supported in $[-1, -1/4] \cup [1/4, 1]$ such that it satisfies $\sum_{i\in\mathbb{Z}} \eta(2^{-i}\lambda) = 1$ for all $\lambda \neq 0$. Set $\eta_0(\lambda) = 1 - \sum_{i\geq 1} \eta(2^{-i}\lambda)$ and $\eta_i(\lambda) = \eta(2^{-i}\lambda)$, $i \in \mathbb{N}$. Using this let us define

$$F^{(i)}(\lambda) = \frac{1}{2\pi}\int_{\mathbb{R}} \eta_i(\tau)\hat{F}(\tau)\cos\lambda\tau \, d\tau, \quad \text{for } i \geq 0.$$

Then via functional calculus, we have

$$F(\sqrt{\mathcal{L}})f = \sum_{i\geq 0} F^{(i)}(\sqrt{\mathcal{L}})f. \tag{4.2}$$

Note that from [39, Lemma 4.4], the sub-Laplacian $\mathcal{L}$ satisfies the finite speed propagation property, therefore by [11, Lemma 2.1] we have $\operatorname{supp} K_{F^{(i)}(\sqrt{L})} \subseteq \mathcal{D}_{2^i}$. So that from Lemma (2.5), choosing $r = 2^i$ we get,

$$\begin{aligned}
&\|[b, F^{(i)}(\sqrt{\mathcal{L}})]f\|^q_{L^q(G)} \\
&\leq C\sum_k \Big(\|\chi_{4B_k}(b - b_{B_0}) F^{(i)}(\sqrt{\mathcal{L}})\chi_{\widetilde{B}_k} f\|^q_{L^q(G)} \\
&\qquad\qquad + \|\chi_{4B_k} F^{(i)}(\sqrt{\mathcal{L}})(\chi_{\widetilde{B}_k}(b - b_{B_0})f)\|^q_{L^q(G)} \Big).
\end{aligned}$$

Thus, in view of (4.2), to prove Theorem (1.2), it is enough to show that for some $\delta > 0$ and for all $i \geq 0$,

$$\|[b, F^{(i)}(\sqrt{\mathcal{L}})]f\|_{L^q(G)} \leq C 2^{-\delta i}\|b\|_{BMO^\varrho(G)}\|F\|_{L^2_\beta(\mathbb{R})}\|f\|_{L^q(G)}. \tag{4.3}$$

Since $\mathcal{L}$ is left-invariant, in view of the previous two inequalities, we are reduced to showing that

$$\begin{aligned}
&\|\chi_{4B_0}(b - b_{B_0}) F^{(i)}(\sqrt{\mathcal{L}})\chi_{\widetilde{B}_0} f\|_{L^q(G)} \\
&\qquad\qquad \leq C 2^{-i\delta}\|b\|_{BMO^\varrho(G)}\|F\|_{L^2_\beta(\mathbb{R})}\|\chi_{\widetilde{B}_0} f\|_{L^q(G)},
\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}
&\|\chi_{4B_0} F^{(i)}(\sqrt{\mathcal{L}})(\chi_{\widetilde{B}_0}(b - b_{B_0})f)\|_{L^q(G)} \\
&\qquad\qquad \leq C 2^{-i\delta}\|b\|_{BMO^\varrho(G)}\|F\|_{L^2_\beta(\mathbb{R})}\|\chi_{\widetilde{B}_0} f\|_{L^q(G)},
\end{aligned} \tag{4.5}$$

for some $\delta > 0$, where $\tilde{B}_{0,k}$ is the translation of $\tilde{B}_k$ via $(-x_k, -u_k)$ to the point $(0, 0)$ and by abuse of notation we write $\tilde{B}_{0,k}$ as $\tilde{B}_0$ (see Lemma 2.5).

Proof of (4.4)**.** We first choose a bump function $\psi \in C_c^\infty(\mathbb{R})$ with support in $[1/16, 4]$ and $\psi = 1$ in $[1/8, 2]$. Then

$$\begin{aligned}&\|\chi_{4B_0}(b-b_{B_0})F^{(i)}(\sqrt{\mathcal{L}})\chi_{\widetilde{B}_0}f\|_{L^q(G)}^q \\ &\leq C\|\chi_{4B_0}(b-b_{B_0})(\psi F^{(i)})(\sqrt{\mathcal{L}})\chi_{\widetilde{B}_0}f\|_{L^q(G)}^q \\ &\qquad + C\|\chi_{4B_0}(b-b_{B_0})((1-\psi)F^{(i)})(\sqrt{\mathcal{L}})\chi_{\widetilde{B}_0}f\|_{L^q(G)}^q.\end{aligned} \tag{4.6}$$

Let us first focus our attention to the term

$$\|\chi_{4B_0}(b-b_{B_0})(\psi F^{(i)})(\sqrt{\mathcal{L}})(\chi_{\widetilde{B}_0}f)\|_{L^q(G)}.$$

We decompose the multiplier $\psi F^{(i)}$ by a dyadic decomposition in the $|\mu|$ variable. From (3.3) and (3.4), for $M \in \mathbb{Z}$, we define $F_M^{(i)} : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ given by

$$F_M^{(i)}(\kappa,\tau) = (\psi F^{(i)})(\sqrt{\kappa})\,\Theta(2^M\tau) \quad \text{for } \kappa \geq 0 \tag{4.7}$$

and $F_M^{(i)}(\kappa,\tau) = 0$ else. Then we can write

$$(\psi F^{(i)})(\sqrt{\mathcal{L}}) = \sum_{M\in\mathbb{Z}} F_M^{(i)}(\mathcal{L}, T).$$

Recall that the function $\mu \mapsto b^\mu$ is homogeneous of degree 1, therefore $b^\mu = |\mu| b^{\bar{\mu}}$, where $\bar{\mu} = |\mu|^{-1}\mu$. Now as we have $\sqrt{\lambda_{\mathbf{k}}^\mu} \in \operatorname{supp}\psi F^{(i)}$ and $2^M|\mu| \in \operatorname{supp}\Theta$, therefore

$$1 \sim \lambda_{\mathbf{k}}^\mu \geq \sum_{n=1}^N r_n b_n^\mu = |\mu| \sum_{n=1}^N r_n b_n^{\bar{\mu}} \sim 2^{-M}\sum_{n=1}^N r_n b_n^{\bar{\mu}}. \tag{4.8}$$

Note that from [28, equation 2.7, Lemma 5] we have

$$\sum_{n=1}^N r_n b_n^\mu \sim \left(\sum_{n=1}^N 2r_n (b_n^\mu)^2\right)^{1/2} = (tr(J_\mu^* J_\mu))^{1/2}.$$

As $(tr(J_\mu^* J_\mu))^{1/2}$ is the pullback of the Hilbert-Schmidt norm via the injective map $\mu \mapsto J_\mu$, therefore $\sum_{n=1}^N r_n b_n^\mu$ does not vanish for all $\mu \neq 0$. Also from Proposition 2.2 all maps $\mu \mapsto b_n^\mu$ are continuous on $\mathfrak{g}_2^*$. As $\bar{\mu} \in \{\mu \in \mathfrak{g}_2^* : |\mu| = 1\}$, from (4.8) we get $2^{-M} \lesssim 1$. Therefore there exists $l_0 \in \mathbb{N}$ such that

$$(\psi F^{(i)})(\sqrt{\mathcal{L}}) = \sum_{M\geq -l_0} F_M^{(i)}(\mathcal{L}, T).$$

With the aid of this decomposition, we write

$$\begin{aligned}&\chi_{4B_0}(b-b_{B_0})(\psi F^{(i)})(\sqrt{\mathcal{L}})(\chi_{\widetilde{B}_0}f) \\ &= \chi_{4B_0}(b-b_{B_0})\Big(\sum_{M=-l_0}^{i} + \sum_{M=i+1}^{\infty}\Big)F_M^{(i)}(\mathcal{L},T)(\chi_{\widetilde{B}_0}f).\end{aligned} \tag{4.9}$$

We distinguish the following cases. The case $M \geq i+1$ is easier to handle. Applying Hölder's inequality, we get

$$\begin{aligned}
&\|\chi_{4B_0}(b-b_{B_0}) \sum_{M=i+1}^{\infty} F_M^{(i)}(\mathcal{L},T)(\chi_{\widetilde{B}_0} f)\|_{L^q(G)} \\
&\leq \sum_{M=i+1}^{\infty} \|\chi_{4B_0}(b-b_{B_0})\|_{L^{\frac{2q}{2-q}}(G)} \|F_M^{(i)}(\mathcal{L},T)(\chi_{\widetilde{B}_0} f)\|_{L^2(G)}.
\end{aligned}$$

Then by (2.7) and (2.8), we get

$$\|\chi_{4B_0}(b-b_{B_0})\|_{L^{\frac{2q}{2-q}}(G)} \leq C\, \|b\|_{BMO^{\varrho}(G)}\, |B_0|^{\frac{2-q}{2q}}. \tag{4.10}$$

Therefore, using the above estimate we can see

$$\begin{aligned}
&\|\chi_{4B_0}(b-b_{B_0}) \sum_{M=i+1}^{\infty} F_M^{(i)}(\mathcal{L},T)(\chi_{\widetilde{B}_0} f)\|_{L^q(G)} \qquad (4.11)\\
&\leq C \sum_{M=i+1}^{\infty} \|b\|_{BMO^{\varrho}(G)} 2^{iQ(1/q-1/2)}\, \|F_M^{(i)}(\mathcal{L},T)(\chi_{\widetilde{B}_0} f)\|_{L^2(G)}.
\end{aligned}$$

Now, applying Corollary 3.2 and (3.2), for $\beta > 1/2$ we get

$$\begin{aligned}
&\|F_M^{(i)}(\mathcal{L},T)(\chi_{\widetilde{B}_0} f)\|_{L^2(G)} \qquad (4.12)\\
&\lesssim 2^{-Md_2(1/p-1/2)}\, \|\psi F^{(i)}\|_{2^M,2}\, \|\chi_{\widetilde{B}_0} f\|_{L^p(G)} \\
&\lesssim 2^{-Md_2(1/p-1/2)}\, 2^{iQ(1/p-1/q)}\Big(\|F^{(i)}\|_{L^2} + 2^{-M\beta}\|F^{(i)}\|_{L^2_\beta}\Big)\|\chi_{\widetilde{B}_0} f\|_{L^q(G)}.
\end{aligned}$$

Substituting the estimate (4.12) into (4.11), leads us

$$\begin{aligned}
&\|\chi_{4B_0}(b-b_{B_0}) \sum_{M=i+1}^{\infty} F_M^{(i)}(\mathcal{L},T)(\chi_{\widetilde{B}_0} f)\|_{L^q(G)} \qquad (4.13)\\
&\lesssim \|b\|_{BMO^{\varrho}(G)} 2^{id(1/p-1/2)}\big(\|F^{(i)}\|_{L^2} + 2^{-i\beta}\|F^{(i)}\|_{L^2_\beta}\big)\, \|\chi_{\widetilde{B}_0} f\|_{L^q(G)} \\
&\lesssim \|b\|_{BMO^{\varrho}(G)} 2^{i(d(1/p-1/2)-\beta)}\, \|F\|_{L^2_\beta}\, \|\chi_{\widetilde{B}_0} f\|_{L^q(G)} \\
&\lesssim \|b\|_{BMO^{\varrho}(G)} 2^{-i\varepsilon}\, \|F\|_{L^2_\beta}\, \|\chi_{\widetilde{B}_0} f\|_{L^q(G)},
\end{aligned}$$

provided $\beta > d(1/p-1/2)$.

Note that for $1 \leq p \leq p_{d_2}$, we have $d(1/p-1/2) > 1/2$. Therefore, for $\beta > d(1/p-1/2)$, we obtain

$$\begin{aligned}
&\left\|\chi_{4B_0}(b-b_{B_0}) \sum_{M=i+1}^{\infty} F_M^{(i)}(\mathcal{L},T)(\chi_{\widetilde{B}_0} f)\right\|_{L^q(G)} \qquad (4.14)\\
&\lesssim \|b\|_{BMO^{\varrho}(G)} 2^{-i\delta}\, \|F\|_{L^2_\beta}\, \|\chi_{\widetilde{B}_0} f\|_{L^q(G)},
\end{aligned}$$

for some $\delta > 0$.

Now we deal with the case $-l_0 \leq M \leq i$. Using the fact (2.4), for each M, first we decompose $\operatorname{supp}(\chi_{\widetilde{B}_0} f) \subseteq B\big((0,0), \frac{2^i}{10}\big)$ in the first layer as

$$\operatorname{supp}(\chi_{\widetilde{B}_0} f) = \bigcup_{m=1}^{N_M} B_m^M, \tag{4.15}$$

where $B_m^M \subseteq B^{|\cdot|}(x_m^M, C\frac{2^M}{10}) \times B^{|\cdot|}(0, C\frac{2^{2i}}{100})$ are disjoint subsets and $|x_m^M - x_{m'}^M| > 2^M/20$ for $m \neq m'$. Note that N_M is bounded by some constant times $2^{(i-M)d_1}$. For $\gamma > 0$, we also define

$$\tilde{B}_m^M := B^{|\cdot|}(x_m^M, 2C\, \frac{2^M}{10}\, 2^{i\gamma}) \times B^{|\cdot|}(0, 4C\frac{2^{2i}}{100}).$$

Then the number of overlapping balls N_γ can be bounded by $2^{C\gamma i}$.

With the aid of this decomposition, we write

$$\begin{aligned} &\chi_{4B_0}(b - b_{B_0}) \sum_{M=-l_0}^{i} F_M^{(i)}(\mathcal{L}, T)\chi_{\widetilde{B}_0} f \\ &= \chi_{4B_0}(b - b_{B_0}) \sum_{M=-l_0}^{i} \sum_{m=1}^{N_M} \chi_{\tilde{B}_m^M} F_M^{(i)}(\mathcal{L}, T)\chi_{B_m^M} f \\ &\qquad + \chi_{4B_0}(b - b_{B_0}) \sum_{M=-l_0}^{i} \sum_{m=1}^{N_M} (1 - \chi_{\tilde{B}_m^M}) F_M^{(i)}(\mathcal{L}, T)\chi_{B_m^M} f. \end{aligned} \tag{4.16}$$

For the second term in the right hand side, to estimate

$$\left\| \chi_{4B_0} \sum_{M=-l_0}^{i} \sum_{m=1}^{N_M} (1 - \chi_{\tilde{B}_m^M}) F_M^{(i)}(\mathcal{L}, T)\chi_{B_m^M} f \right\|_{L^p(G)\to L^2(G)},$$

via Riesz-Thorin interpolation theorem, we interpolate between the $L^1(G) \to L^2(G)$ estimate and $L^2(G) \to L^2(G)$ estimate. Let us first start with $L^1(G) \to L^2(G)$ estimate.

Let $\mathcal{K}_M^{(i)}$ be the convolution kernel of $F_M^{(i)}(\mathcal{L}, T)$. Then, by Minkowski's integral inequality,

$$\begin{aligned} &\left\| \chi_{4B_0} \sum_{M=-l_0}^{i} \sum_{m=1}^{N_M} (1 - \chi_{\tilde{B}_m^M}) F_M^{(i)}(\mathcal{L}, T)\chi_{B_m^M} f \right\|_{L^2(G)} \\ &\leq \sum_{M=-l_0}^{i} \sum_{m=1}^{N_M} \int_{B_m^M} |f(x', u')| \\ &\qquad \times \left(\int_G \chi_{4B_0}^2 (1 - \chi_{\tilde{B}_m^M})^2 |\mathcal{K}_M^{(i)}\big((x', u')^{-1}(x, u)\big)|^2 \, d(x, u) \right)^{\frac{1}{2}} d(x', u'). \end{aligned} \tag{4.17}$$

Suppose $(x,u) \in \chi_{4B_0}(1-\chi_{\tilde{B}_m^M})$ and $(x',u') \in B_m^M$, then we have $|x'-x| \gtrsim 2^M\, 2^{i\gamma}$. Now for $\bar{N} \in \mathbb{N}$, by Proposition 3.3, we have

$$\begin{aligned}
&\Big(\int_G \chi_{4B_0}^2(1-\chi_{\tilde{B}_m^M})^2|\mathcal{K}_M^{(i)}\big((x',u')^{-1}(x,u)\big)|^2\, d(x,u)\Big)^{\frac{1}{2}}\\
&\leq C(2^M 2^{i\gamma})^{-\bar{N}}\Big(\int_G \Big||x-x'|^{\bar{N}}\mathcal{K}_M^{(i)}\big(x-x',u-u'-\tfrac{1}{2}[x',x]\big)\Big|^2\, d(x,u)\Big)^{\frac{1}{2}}\\
&\leq C(2^M 2^{i\gamma})^{-\bar{N}}\Big(\int_G \Big||x|^{\bar{N}}\mathcal{K}_M^{(i)}(x,u)\Big|^2\, d(x,u)\Big)^{\frac{1}{2}}\\
&\lesssim (2^M 2^{i\gamma})^{-\bar{N}}\, 2^{M(\bar{N}-d_2/2)}\, \|\psi F^{(i)}\|_{L^2}.
\end{aligned}$$

Using this estimate in (4.17), we get

$$\begin{aligned}
&\Big\|\chi_{4B_0}\sum_{M=-l_0}^{i}\sum_{m=1}^{N_M}(1-\chi_{\tilde{B}_m^M})F_M^{(i)}(\mathcal{L},T)\chi_{B_m^M}f\Big\|_{L^2(G)} \qquad (4.18)\\
&\lesssim \|F^{(i)}\|_{L^2}2^{-i\gamma\bar{N}}\, 2^{i\varepsilon}\, \|\chi_{\widetilde{B}_0}f\|_{L^1(G)}.
\end{aligned}$$

On the other hand, for $L^2(G) \to L^2(G)$ estimate, by Plancherel theorem we have

$$\begin{aligned}
&\Big\|\chi_{4B_0}\sum_{M=-l_0}^{i}\sum_{m=1}^{N_M}(1-\chi_{\tilde{B}_m^M})F_M^{(i)}(\mathcal{L},T)\chi_{B_m^M}f\Big\|_{L^2(G)}\\
&\leq \|\psi F^{(i)}\|_{L^\infty}\sum_{M=-l_0}^{i}\sum_{m=1}^{N_M}\|\chi_{B_m^M}f\|_{L^2(G)}.
\end{aligned}$$

Note that for $s > 1/2$, Sobolev embedding theorem gives $\|F^{(i)}\|_{L^\infty} \leq \|F^{(i)}\|_{L_s^2} \sim 2^{is}\|F^{(i)}\|_{L^2}$. Then using Cauchy-Schwartz inequality together with the upper bound of N_M, we get

$$\begin{aligned}
&\Big\|\chi_{4B_0}\sum_{M=-l_0}^{i}\sum_{m=1}^{N_M}(1-\chi_{\tilde{B}_m^M})F_M^{(i)}(\mathcal{L},T)\chi_{B_m^M}f\Big\|_{L^2(G)} \qquad (4.19)\\
&\leq 2^{is}\|F^{(i)}\|_{L^2}2^{i\varepsilon}2^{id_1/2}\, \|\chi_{\widetilde{B}_0}f\|_{L^2(G)} \lesssim 2^{is'}\|F^{(i)}\|_{L^2}\, \|\chi_{\widetilde{B}_0}f\|_{L^2(G)},
\end{aligned}$$

for $s' > s + d_1/2$.

Therefore, interpolating between the estimates (4.18) and (4.19), we have for $1 \leq p < 2$,

$$\begin{aligned}
&\Big\|\chi_{4B_0}\sum_{M=-l_0}^{i}\sum_{m=1}^{N_M}(1-\chi_{\tilde{B}_m^M})F_M^{(i)}(\mathcal{L},T)\chi_{B_m^M}f\Big\|_{L^2(G)} \qquad (4.20)\\
&\lesssim 2^{i\varepsilon}2^{-2i\gamma\bar{N}(1/p-1/2)}\, 2^{2is'(1-1/p)}\, \|F^{(i)}\|_{L^2}\, \|\chi_{\widetilde{B}_0}f\|_{L^p(G)}.
\end{aligned}$$

This estimate together with Hölder's inequality and (4.10) gives

$$\begin{aligned}
&\left\|\chi_{4B_0}(b-b_{B_0})\sum_{M=-l_0}^{i}\sum_{m=1}^{N_M}(1-\chi_{\tilde{B}_m^M})F_M^{(i)}(\mathcal{L},T)\chi_{B_m^M}f\right\|_{L^q(G)} \qquad (4.21)\\
&\lesssim 2^{i\varepsilon}2^{iQ(1/q-1/2)}\,\|b\|_{BMO^\varrho(G)}\\
&\qquad\qquad\times\left\|\chi_{4B_0}\sum_{M=-l_0}^{i}\sum_{m=1}^{N_M}(1-\chi_{\tilde{B}_m^M})F_M^{(i)}(\mathcal{L},T)\chi_{B_m^M}f\right\|_{L^2(G)}\\
&\lesssim 2^{i\varepsilon}2^{iQ(1/q-1/2)}\,\|b\|_{BMO^\varrho(G)}\,2^{-2i\gamma\bar{N}(1/p-1/2)}\,2^{2is'(1-1/p)}\,\|F\|_{L^2}\,\|\chi_{\widetilde{B}_0}f\|_{L^p(G)}\\
&\lesssim 2^{i\varepsilon}2^{iQ(1/q-1/2)}\,\|b\|_{BMO^\varrho(G)}\,2^{-2i\gamma\bar{N}(1/p-1/2)}\,2^{2is'(1-1/p)}\\
&\qquad\qquad\|F\|_{L^2}\,2^{iQ(1/p-1/q)}\|\chi_{\widetilde{B}_0}f\|_{L^q(G)}\\
&\lesssim 2^{-\delta i}\,\|b\|_{BMO^\varrho(G)}\,\|F\|_{L^2}\|\chi_{\widetilde{B}_0}f\|_{L^q(G)},
\end{aligned}$$

for some $\delta > 0$ by choosing $\bar{N}$ large enough.

It remains to estimate the first summand of (4.16). Choose s such that $q < s < 2$. Then by (4.10) for $L^{\frac{sq}{s-q}}(G)$ and application of Hölder's inequality yields

$$\begin{aligned}
&\left\|\sum_{M=-l_0}^{i}\sum_{m=1}^{N_M}\chi_{4B_0}(b-b_{B_0})\chi_{\widetilde{B}_m^M}F_M^{(i)}(\mathcal{L},T)\chi_{B_m^M}f\right\|_{L^q(G)}^q \qquad (4.22)\\
&\le C(i+1+l_0)^{q-1}N_\gamma^{q-1}\\
&\qquad\times\sum_{M=-l_0}^{i}\sum_{m=1}^{N_M}\left\|\chi_{4B_0}(b-b_{B_0})\chi_{\widetilde{B}_m^M}F_M^{(i)}(\mathcal{L},T)\chi_{B_m^M}f\right\|_{L^q(G)}^q\\
&\le C2^{iQ(1/q-1/s)q}\,\|b\|_{BMO^\varrho(G)}^q(i+1+l_0)^{q-1}2^{C\gamma j}\\
&\qquad\qquad\times\sum_{M=-l_0}^{i}\sum_{m=1}^{N_M}\|\chi_{\widetilde{B}_m^M}F_M^{(i)}(\mathcal{L},T)\chi_{B_m^M}f\|_{L^s(G)}^q\\
&\le C2^{iQ(1/q-1/s)q}\,\|b\|_{BMO^\varrho(G)}^q2^{i\varepsilon}2^{C\gamma i}\\
&\quad\times\sum_{M=-l_0}^{i}\sum_{m=1}^{N_M}\left(2^M\,2^{\gamma i}\right)^{d_1(1/s-1/2)q}\,2^{i2d_2(1/s-1/2)q}\,\|F_M^{(i)}(\mathcal{L},T)\chi_{B_m^M}f\|_{L^2(G)}^q.
\end{aligned}$$

Now, using Proposition (3.1) in (4.22), for $1 \le p \le p_{d_2}$ (as $\min\{p_{d_1}, p_{d_2}\} = p_{d_2}$ for $d_1 > d_2$), by choosing q and s very close, that is $1/q - 1/s < \varepsilon$, we get

$$\left\|\sum_{M=-l_0}^{i}\sum_{m=1}^{N_M}\chi_{4B_0}(b-b_{B_0})\chi_{\widetilde{B}_m^M}F_M^{(i)}(\mathcal{L},T)\chi_{B_m^M}f\right\|_{L^q(G)}^q \qquad (4.23)$$

$$\lesssim 2^{iQ(1/q-1/s)q}\,\|b\|^q_{BMO^\varrho(G)}2^{i\varepsilon}2^{C\gamma i}\sum_{M=-l_0}^{i}\sum_{m=1}^{N_M}\left(2^M\,2^{\gamma i}\right)^{d_1(1/s-1/2)q}$$
$$2^{i2d_2(1/s-1/2)q}2^{-Md_2(1/p-1/2)q}\,\|\psi F^{(i)}\|^{q(1-\theta_p)}_{L^2}\,\|\psi F^{(i)}\|^{q\theta_p}_{2^M,2}\,\|\chi_{B^M_m}f\|^q_{L^p(G)}$$
$$\lesssim 2^{iQ(1/q-1/s)q}\|b\|^q_{BMO^\varrho(G)}2^{i\varepsilon}2^{C\gamma i}\sum_{M=-l_0}^{i}\sum_{m=1}^{M_l}\left(2^M\,2^{\gamma i}\right)^{d_1(1/s-1/2)q}$$
$$2^{i2d_2(1/s-1/2)q}2^{-Md_2(1/p-1/2)q}\,\|\psi F^{(i)}\|^{q(1-\theta_p)}_{L^2}\,\|\psi F^{(i)}\|^{q\theta_p}_{2^M,2}$$
$$2^{Md_1(1/p-1/q)q}\,2^{i2d_2(1/p-1/q)q}\,\|\chi_{B^M_m}f\|^q_{L^q(G)}$$
$$\lesssim 2^{i\varepsilon}\,2^{i2d_2(1/p-1/2)q}\,\|b\|^q_{BMO^\varrho(G)}\,2^{C\gamma i}\,2^{i\gamma d_1(1/s-1/2)q}\sum_{M=-l_0}^{i}\|\psi F^{(i)}\|^{q(1-\theta_p)}_{L^2}$$
$$\|\psi F^{(i)}\|^{q\theta_p}_{2^M,2}2^{Md_1(1/p-1/2)q}\,2^{-Md_2(1/p-1/2)q}\|\chi_{\widetilde{B}_0}f\|^q_{L^q(G)},$$

where $\theta_p = p'_{d_2}(1-1/p)$ and p'_{d_2} denotes the conjugate exponent of p_{d_2}.

Set $1/r = 1/p - 1/2$ and $\nu = \frac{1}{3}(\beta - d/r)$. Since we have

$$\|F^{(i)}\|_{L^2} \sim 2^{-\beta i}\|F^{(i)}\|_{L^2_\beta} = 2^{-3i\nu}2^{-id/r}\|F^{(i)}\|_{L^2_\beta},$$

therefore using (3.2), for $\tilde{s} > 1/2$, we have

$$\begin{aligned}\|\psi F^{(i)}\|^{(1-\theta_p)}_{L^2}\,\|\psi F^{(i)}\|^{\theta_p}_{2^M,2} &\lesssim \|F^{(i)}\|^{(1-\theta_p)}_{L^2}2^{(i-M)\tilde{s}\theta_p}\|F^{(i)}\|^{\theta_p}_{L^2}\\ &\lesssim 2^{(i-M)\tilde{s}\theta_p}\,2^{-3i\nu}2^{-id/r}\|F\|_{L^2_\beta}.\end{aligned} \tag{4.24}$$

Using this observation, and since ε and γ can be chosen arbitrary small, from (4.23) we get

$$\left\|\sum_{M=-l_0}^{i}\sum_{m=1}^{N_M}\chi_{4B_0}(b-b_{B_0})\chi_{\widetilde{B}^M_m}F^{(i)}_M(\mathcal{L},T)\chi_{B^M_m}f\right\|^q_{L^q(G)} \tag{4.25}$$
$$\lesssim 2^{i\varepsilon}2^{i2d_2q/r}\,\|b\|^q_{BMO^\varrho(G)}\sum_{M=-l_0}^{i}2^{(i-M)\tilde{s}q\theta_p}\,2^{-3iq\nu}2^{-iqd/r}\|F\|^q_{L^2_\beta}$$
$$\times\,2^{Md_1q/r}\,2^{-Md_2q/r}\|\chi_{\widetilde{B}_0}f\|^q_{L^q(G)}$$
$$\lesssim\,2^{i\varepsilon}2^{i2d_2q/r}\|b\|^q_{BMO^\varrho(G)}2^{-2iq\nu}\|F\|^q_{L^2_\beta}$$
$$\sum_{M=-l_0}^{i}\left[2^{(M-i)(d_1-d_2)}\,2^{-(M-i)\tilde{s}\theta_p r}\,2^{(M-i)\nu r}\right]^{q/r}$$
$$\times\,2^{-M\nu q}2^{i(d_1-d_2)q/r}\,2^{-idq/r}\,\|\chi_{\widetilde{B}_0}f\|^q_{L^q(G)}$$

$$\lesssim 2^{-2iq\nu}2^{i\varepsilon}\,\|b\|^q_{BMO^\varrho(G)}\|F\|^q_{L^2_\beta}\sum_{M=-l_0}^{i}\left[2^{(M-i)(d_1-d_2-\tilde{s}\theta_p r+\nu r)}\right]^{\frac{q}{r}}\|\chi_{\widetilde{B}_0}f\|^q_{L^q(G)}.$$

Set $M-i=-\ell$. Choosing $\tilde{s}$ sufficiently close to $1/2$, we see that the series

$$\sum_{\ell=0}^{\infty}2^{-\ell(d_1-d_2-\tilde{s}\theta_p r+\nu r)q/r}$$

is convergent provided $d_1-d_2-\frac{r\theta_p}{2}+\nu r>0$. Then for $1\le p\le 2$, the condition $d_1-d_2-\frac{r\theta_p}{2}+\nu r>0$ is equivalent to

$$1\le p\le \frac{p'_{d_2}+2(d_1-d_2)}{p'_{d_2}+d_1-d_2}:=P_{d_1,d_2}.$$

Now in order to find the required range of p we make use of the Proposition 2.1 depending on the values of (d_1,d_2). Therefore we make following two cases.

Case (a) $(\boldsymbol{d_1},\boldsymbol{d_2})\notin\{(\mathbf{4},\mathbf{3}),(\mathbf{8},\mathbf{6}),(\mathbf{8},\mathbf{7})\}$**:** By Proposition 2.1, we have that $d_1>3d_2/2$. Which in turn implies that $p_{d_2}\le P_{d_1,d_2}$. Therefore, choosing $\varepsilon>0$ very small we obtain that

$$\left\|\sum_{M=-l_0}^{i}\sum_{m=1}^{M_l}\chi_{4B_0}(b-b_{B_0})\chi_{\widetilde{B}^M_m}F^{(i)}_M(\mathcal{L},T)\chi_{B^M_m}f\right\|^q_{L^q(G)} \tag{4.26}$$
$$\lesssim C2^{-iq\nu}\|F\|^q_{L^2_\beta}\|b\|^q_{BMO^\varrho(G)}\,\|\chi_{\widetilde{B}_0}f\|^q_{L^q(G)}.$$

Hence, aggregating the estimates obtained in (4.13), (4.21) and (4.26), we conclude that

$$\left\|\sum_{M=-l_0}^{i}\sum_{m=1}^{N_M}\chi_{4B_0}(b-b_{B_0})\chi_{\widetilde{B}^M_m}F^{(i)}_M(\mathcal{L},T)\chi_{B^M_m}f\right\|_{L^q(G)}$$
$$\lesssim 2^{-i\delta}\|b\|_{BMO^\varrho(G)}\,\|F\|_{L^2_\beta}\,\|\chi_{\widetilde{B}_0}f\|_{L^q(G)},$$

for some $\delta>0$.

Case (b) $(\boldsymbol{d_1},\boldsymbol{d_2})\in\{(\mathbf{4},\mathbf{3}),(\mathbf{8},\mathbf{6}),(\mathbf{8},\mathbf{7})\}$**:** Note that $P_{4,3}=6/5$, $P_{8,6}=17/12$, $P_{8,7}=14/11$ and $p_3=4/3$, $p_6=14/9$, $p_7=8/5$. Therefore from (4.25) we conclude the following result.

For $(d_1,d_2)=(4,3)$, we have

$$\left\|\sum_{M=-l_0}^{i}\sum_{m=1}^{N_M}\chi_{4B_0}(b-b_{B_0})\chi_{\widetilde{B}^M_m}F^{(i)}_M(\mathcal{L},T)\chi_{B^M_m}f\right\|_{L^q(G)} \tag{4.27}$$
$$\lesssim 2^{-i\delta}\|b\|_{BMO^\varrho(G)}\,\|F\|_{L^2_\beta}\,\|\chi_{\widetilde{B}_0}f\|_{L^q(G)},$$

for some $\delta>0$, whenever $1\le p\le 6/5$.

For $(d_1, d_2) = (8, 6)$, we have

$$\left\| \sum_{M=-l_0}^{i} \sum_{m=1}^{N_M} \chi_{4B_0}(b - b_{B_0}) \chi_{\widetilde{B}_m^M} F_M^{(i)}(\mathcal{L}, T) \chi_{B_m^M} f \right\|_{L^q(G)}$$
$$\lesssim 2^{-i\delta} \|b\|_{BMO^\varrho(G)} \|F\|_{L^2_\beta} \|\chi_{\widetilde{B}_0} f\|_{L^q(G)},$$

for some $\delta > 0$, whenever $1 \leq p \leq 17/12$.

And for $(d_1, d_2) = (8, 7)$, we have

$$\left\| \sum_{M=-l_0}^{i} \sum_{m=1}^{N_M} \chi_{4B_0}(b - b_{B_0}) \chi_{\widetilde{B}_m^M} F_M^{(i)}(\mathcal{L}, T) \chi_{B_m^M} f \right\|_{L^q(G)}$$
$$\lesssim 2^{-i\delta} \|b\|_{BMO^\varrho(G)} \|F\|_{L^2_\beta} \|\chi_{\widetilde{B}_0} f\|_{L^q(G)},$$

for some $\delta > 0$, whenever $1 \leq p \leq 14/11$.

Now let us move our attention to the second summand of the right hand side of (4.6). To this end, we first estimate the term

$$\|\chi_{4B_0}(b - b_{B_0})((1-\psi)F^{(i)})(\sqrt{\mathcal{L}})\chi_{\widetilde{B}_0} f\|_{L^q(G)}.$$

We further decompose the function $1 - \psi$ as follows. Since $1 - \psi$ is supported outside the interval $(1/8, 2)$, we can choose a function $\phi \in C_c^\infty(2, 8)$ such that

$$1 - \psi(\xi) = \sum_{\iota \geq 0} \phi(2^{-\iota}\xi) + \sum_{\iota \leq -6} \phi(2^{-\iota}\xi) =: \sum_{\iota \geq 0} \phi_\iota(\xi) + \sum_{\iota \leq -6} \phi_\iota(\xi),$$

for all $\xi > 0$.

Therefore, by spectral theorem we can write

$$((1-\psi)F^{(i)})(\sqrt{\mathcal{L}}) = \sum_{\iota \geq 0} (\phi_\iota F^{(i)})(\sqrt{\mathcal{L}}) + \sum_{\iota \leq -6} (\phi_\iota F^{(i)})(\sqrt{\mathcal{L}}).$$

Again, as in (4.7), define the function $F_{M,t}^{(i,\iota)}$ as

$$F_M^{(i,\iota)}(\kappa, \tau) = (\phi_\iota F^{(i)})(\sqrt{\kappa})\, \Theta(2^M \tau) \quad \text{for} \quad \kappa \geq 0,$$

and $F_M^{(i,\iota)}(\kappa, \tau) = 0$ else. Then similar to (4.9) allows us to write

$$\begin{aligned} &\chi_{4B_0}(b - b_{B_0})((1-\psi)F^{(i)})(\sqrt{\mathcal{L}})\chi_{\widetilde{B}_0} f \\ &= \chi_{4B_0}(b - b_{B_0}) \Big(\sum_{\iota \geq 0} + \sum_{\iota \leq -6}\Big) \left(\sum_{M=-l_0}^{i} + \sum_{M=i+1}^{\infty} \right) F_M^{(i,\iota)}(\mathcal{L}, T) \chi_{\widetilde{B}_0} f \\ &=: \Big(\sum_{\iota \geq 0} + \sum_{\iota \leq -6}\Big) g_{j,\leq i}^{\iota} + \Big(\sum_{\iota \geq 0} + \sum_{\iota \leq -6}\Big) g_{j,>i}^{\iota}. \end{aligned} \tag{4.28}$$

Notice that as $\operatorname{supp} F \subseteq [-1, 1]$, $\operatorname{supp} \phi \subseteq [2, 8]$ and $\check{\eta}$ is Schwartz class function, for any $\bar{N} > 0$ we have

$$\|\phi_\iota F^{(i)}\|_{L^\infty} = 2^i \|\phi_\iota (F * \delta_{2^i} \check{\eta})\|_{L^\infty} \leq C 2^{-\bar{N}(i + \max\{\iota, 0\})} \|F\|_{L^2}. \tag{4.29}$$

For the second term of (4.28), applying Corollary 3.2 and using (3.1) and (4.29), we get

$$
\begin{aligned}
&\left\|\chi_{4B_0}\Big(\sum_{\iota\geq 0}+\sum_{\iota\leq -6}\Big)\sum_{M=i+1}^{\infty}F_M^{(i,\iota)}(\mathcal{L},T)\chi_{\widetilde{B}_0}f\right\|_{L^2}\\
&\leq C\Big(\sum_{\iota\geq 0}+\sum_{\iota\leq -6}\Big)2^{\iota Q(1/p-1/2)}\,2^{-id_2(1/p-1/2)}\,\|(\phi_\iota F^{(i)})(2^{\iota+3}\cdot)\|_{2^M,2}\,\|\chi_{\widetilde{B}_0}f\|_{L^p}\\
&\leq C\Big(\sum_{\iota\geq 0}+\sum_{\iota\leq -6}\Big)2^{\iota Q(1/p-1/2)}\,2^{-id_2(1/p-1/2)}\,\|\phi_\iota F^{(i)}\|_{L^\infty}\,\|\chi_{\widetilde{B}_0}f\|_{L^p}\\
&\leq C\Big(\sum_{\iota\geq 0}+\sum_{\iota\leq -6}\Big)2^{\iota Q(1/p-1/2)}\,2^{-id_2(1/p-1/2)}\,2^{-\bar{N}(i+\max\{\iota,0\})}\|F\|_2\,\|\chi_{\widetilde{B}_0}f\|_{L^p}\\
&\leq C2^{-id_2(1/p-1/2)}\,2^{-\bar{N}i}\|F\|_{L^2}\,\|\chi_{\widetilde{B}_0}f\|_{L^p}.
\end{aligned}
$$

This estimate together with Hölder's inequality and (4.10), we get

$$
\begin{aligned}
&\left\|\Big(\sum_{\iota\geq 0}+\sum_{\iota\leq -6}\Big)g^{\iota}_{j,>i}\right\|_{L^q} \qquad (4.30)\\
&\leq \|\chi_{4B_0}(b-b_{B_0})\|_{L^{\frac{2q}{2-q}}}\left\|\chi_{4B_0}\,\Big(\sum_{\iota\geq 0}+\sum_{\iota\leq -6}\Big)\sum_{M=i+1}^{\infty}F_M^{(i,\iota)}(\mathcal{L},T)\chi_{\widetilde{B}_0}f\right\|_{L^2}\\
&\leq C\|b\|_{BMO^{\varrho}(G)}2^{iQ(1/q-1/2)}\,2^{-id_2(1/p-1/2)}\,2^{-\bar{N}i}\|F\|_{L^2}\,\|\chi_{\widetilde{B}_0}f\|_{L^p}\\
&\leq C\|b\|_{BMO^{\varrho}(G)}2^{iQ(1/q-1/2)}\,2^{-id_2(1/p-1/2)}\,2^{-\bar{N}i}\|F\|_{L^2}2^{iQ(1/p-1/q)}\|\chi_{\widetilde{B}_0}f\|_{L^q}\\
&\leq C2^{id(1/p-1/2)}\,2^{-\bar{N}i}\|b\|_{BMO^{\varrho}(G)}\|F\|_{L^2}\,\|\chi_{\widetilde{B}_0}f\|_{L^q}\\
&\leq C2^{-i\delta}\|b\|_{BMO^{\varrho}(G)}\|F\|_{L^2}\,\|\chi_{\widetilde{B}_0}f\|_{L^q},
\end{aligned}
$$

for some $\delta>0$, by choosing $\bar{N}$ large enough.

Now, for the first term $\Big(\sum_{\iota\geq 0}+\sum_{\iota\leq -6}\Big)g^{\iota}_{j,\leq i}$ of (4.28) we use the decomposition carried out for the case $l\leq i$ in (4.15). Therefore we can write

$$
\begin{aligned}
&\chi_{4B_0}(b-b_{B_0})\sum_{M=-l_0}^{i}\Big(\sum_{\iota\geq 0}+\sum_{\iota\leq -6}\Big)F_M^{(i,\iota)}(\mathcal{L},T)\chi_{\tilde{B}_0}f \qquad (4.31)\\
&=\chi_{4B_0}(b-b_{B_0})\Big(\sum_{\iota\geq 0}+\sum_{\iota\leq -6}\Big)\sum_{M=-l_0}^{i}\sum_{m=1}^{N_M}\\
&\qquad\qquad\big(\chi_{\tilde{B}_m^M}+(1-\chi_{\tilde{B}_m^M})\big)F_M^{(i,\iota)}(\mathcal{L},T)\chi_{B_m^M}f.
\end{aligned}
$$

Let $\mathcal{K}_M^{(i,\iota)}$ be the convolution kernel of $F_M^{(i,\iota)}(\mathcal{L},T)$. Then as estimated in (4.17), an application of Minkowski's inequality together with Proposition 3.3 yields that

$$
\begin{aligned}
&\Big\| \sum_{M=-l_0}^{i} \sum_{m=1}^{N_M} (1-\chi_{\tilde{B}_m^M}) F_M^{(i,\iota)}(\mathcal{L},T)\chi_{B_m^M} f \Big\|_{L^2(G)} \\
&\leq \sum_{M=-l_0}^{i} \sum_{m=1}^{N_M} \int_{B_m^M} |f(x',u')| \Big(\int_G |\mathcal{K}_M^{(i,\iota)}\big((x',u')^{-1}(x,u)\big)|^2 \, d(x,u) \Big)^{\frac{1}{2}} d(x',u') \\
&\leq C \sum_{M=-l_0}^{i} \sum_{m=1}^{N_M} \int_{B_m^M} |f(x',u')| \, (2^{\iota Q/2}\, 2^{-Md_2/2} \, \|(\phi_\iota F^{(i)})(2^{\iota+3}\cdot)\|_{L^2} \, d(x',u') \\
&\leq C \sum_{M=-l_0}^{i} \sum_{m=1}^{N_M} \|\phi_\iota F^{(i)}\|_{L^\infty} \, 2^{\iota Q/2}\, 2^{-Md_2/2} \int_{B_m^M} |f(x',u')| \, d(x',u') \\
&\leq C 2^{i\varepsilon} \|\phi_\iota F^{(i)}\|_{L^\infty} \, 2^{\iota Q/2} \, \|\chi_{\tilde{B}_0} f\|_{L^1(G)}.
\end{aligned}
$$

Moreover, similar to (4.19) using Plancherel estimates and upper bound of N_M,

$$
\begin{aligned}
&\Big\| \sum_{M=-l_0}^{i} \sum_{m=1}^{N_M} (1-\chi_{\tilde{B}_m^M}) F_M^{(i,\iota)}(\mathcal{L},T)\chi_{B_m^M} f \Big\|_{L^2(G)} \\
&\leq C \|\phi_\iota F^{(i)}\|_{L^\infty} 2^{i\varepsilon} 2^{id_1/2} \, \|\chi_{\tilde{B}_0} f\|_{L^2(G)}.
\end{aligned}
$$

Then, by Riesz-Thorin interpolation theorem, for $1 \leq p \leq 2$ we have

$$
\begin{aligned}
&\Big\| \sum_{M=-l_0}^{i} \sum_{m=1}^{N_M} (1-\chi_{\tilde{B}_m^M}) F_M^{(i,\iota)}(\mathcal{L},T)\chi_{B_m^M} f \Big\|_{L^2(G)} \\
&\leq C 2^{\iota Q(1/p-1/2)} \, 2^{id_1(1-1/p)} \, 2^{i\varepsilon} \|\phi_\iota F^{(i)}\|_{L^\infty} \, \|\chi_{\tilde{B}_0} f\|_{L^p(G)}.
\end{aligned}
$$

Using this estimate together with (4.10), Hölder's inequality, (4.29) and again Hölder's inequality, we get

$$
\begin{aligned}
&\Big\| \chi_{4B_0}(b-b_{B_0}) \Big(\sum_{\iota \geq 0} + \sum_{\iota \leq -6} \Big) \sum_{M=-l_0}^{i} \sum_{m=1}^{N_M} (1-\chi_{\tilde{B}_m^M}) F_M^{(i,\iota)}(\mathcal{L},T)\chi_{B_m^M} f \Big\|_{L^q(G)} \qquad (4.32) \\
&\leq C 2^{iQ(1/q-1/2)} \, \|b\|_{BMO^\varrho(G)} \Big(\sum_{\iota \geq 0} + \sum_{\iota \leq -6} \Big) \\
&\qquad\qquad \times \Big\| \sum_{M=-l_0}^{i} \sum_{m=1}^{N_M} (1-\chi_{\tilde{B}_m^M}) F_M^{(i,\iota)}(\mathcal{L},T)\chi_{B_m^M} f \Big\|_{L^2(G)} \\
&\leq C 2^{iQ(1/q-1/2)} \, \|b\|_{BMO^\varrho(G)} \Big(\sum_{\iota \geq 0} + \sum_{\iota \leq -6} \Big) 2^{\iota Q(1/p-1/2)} \, 2^{id_1(1-1/p)} \, 2^{i\varepsilon} \\
&\qquad\qquad \times 2^{-\bar{N}(i+\max\{\iota,0\})} \|F\|_{L^2} \, 2^{iQ(1/p-1/q)} \|\chi_{\tilde{B}_0} f\|_{L^q(G)} \\
&\leq C 2^{iQ(1/p-1/2)} \, \|b\|_{BMO^\varrho(G)} \, 2^{id_1(1-1/p)} \, 2^{-\bar{N}i} \|F\|_{L^2} \, \|\chi_{\tilde{B}_0} f\|_{L^q(G)}
\end{aligned}
$$

$$\leq C2^{-\delta i}\,\|b\|_{BMO^{\varrho}(G)}\,\|F\|_{L^2}\|\chi_{\tilde{B}_0}f\|_{L^q(G)},$$

for some $\delta > 0$, by choosing $\bar{N}$ large enough.

We are left with the estimation of

$$\chi_{4B_0}(b-b_{B_0})\Big(\sum_{\iota\geq 0}+\sum_{\iota\leq -6}\Big)\sum_{M=-l_0}^{i}\sum_{m=1}^{N_M}\chi_{\tilde{B}_m^M}F_M^{(i,\iota)}(\mathcal{L},T)\chi_{B_m^M}f. \tag{4.33}$$

Choose s such that $q < s < 2$. Then using Hölder's inequality and (4.10), we see that

$$\begin{aligned}&\Bigg\|\chi_{4B_0}(b-b_{B_0})\sum_{M=-l_0}^{i}\sum_{m=1}^{N_M}\Big(\sum_{\iota\geq 0}+\sum_{\iota\leq -6}\Big)\chi_{\tilde{B}_m^M}F_M^{(i,\iota)}(\mathcal{L},T)\chi_{B_m^M}f\Bigg\|_{L^q(G)}^q\\ &\leq C2^{iQ(1/q-1/s)q}\,\|b\|_{BMO^{\varrho}(G)}^q(i+1-l_0)^{q-1}N_\gamma^{q-1}\\ &\qquad\times\sum_{M=-l_0}^{i}\sum_{m=1}^{N_M}\Bigg\|\Big(\sum_{\iota\geq 0}+\sum_{\iota\leq -6}\Big)\chi_{\tilde{B}_m^M}F_M^{(i,\iota)}(\mathcal{L},T)\chi_{B_m^M}f\Bigg\|_{L^s(G)}^q.\end{aligned}$$

Now applying Hölder's inequality, Corollary 3.2, the fact (3.1) and (4.29) we get

$$\begin{aligned}&\Bigg\|\Big(\sum_{\iota\geq 0}+\sum_{\iota\leq -6}\Big)\chi_{\tilde{B}_m^M}F_M^{(i,\iota)}(\mathcal{L},T)\chi_{B_m^M}f\Bigg\|_{L^s(G)}\\ &\leq C\Big(\sum_{\iota\geq 0}+\sum_{\iota\leq -6}\Big)\big(2^l2^{i\gamma}\big)^{d_1(1/s-1/2)}2^{i2d_2(1/s-1/2)}\,\|F_M^{(i,\iota)}(\mathcal{L},T)\chi_{B_m^M}f\|_{L^2(G)}\\ &\leq C\Big(\sum_{\iota\geq 0}+\sum_{\iota\leq -6}\Big)\big(2^l2^{i\gamma}\big)^{d_1(1/s-1/2)}(2^i)^{2d_2(1/s-1/2)}2^{\iota Q(1/p-1/2)}\,2^{-Md_2(1/p-1/2)}\\ &\qquad\qquad\times\|\phi_\iota F^{(i)}\|_{2^M,2}\,\|\chi_{B_m^M}f\|_{L^p(G)}\\ &\leq C\Big(\sum_{\iota\geq 0}+\sum_{\iota\leq -6}\Big)\big(2^l2^{i\gamma}\big)^{d_1(1/s-1/2)}2^{i2d_2(1/s-1/2)}2^{\iota Q(1/p-1/2)}\,2^{-ld_2(1/p-1/2)}\\ &\qquad\times 2^{-\bar{N}(i+\max\{\iota,0\})}\|F\|_{L^2}2^{Md_1(1/p-1/q)}\,2^{i2d_2(1/p-1/q)}\,\|\chi_{B_m^M}f\|_{L^q(G)}\\ &\leq C2^{i\varepsilon}2^{i\gamma d_1(1/s-1/2)}2^{Md_1(1/p-1/2+1/s-1/q)}2^{-Md_2(1/p-1/2)}\\ &\qquad\qquad\times 2^{-\bar{N}i}2^{i2d_2(1/p-1/2+1/s-1/q)}\|F\|_{L^2}\|\chi_{\widetilde{B}_0}f\|_{L^q(G)}.\end{aligned}$$

Let us set $l-i=-\ell$ and recall that we have $d_1 > d_2$ on Métivier groups. Therefore, from the previous two estimates by choosing $\gamma > 0$ sufficiently small and s arbitrary close to q,

$$\begin{aligned}&\Bigg\|\chi_{4B_0}(b-b_{B_0})\sum_{M=-l_0}^{i}\sum_{m=1}^{M_l}\Big(\sum_{\iota\geq 0}+\sum_{\iota\leq -6}\Big)\chi_{\tilde{B}_m^M}F_M^{(i,\iota)}(\mathcal{L},T)\chi_{B_m^M}f\Bigg\|_{L^q(G)}^q \qquad (4.34)\\ &\leq C\|b\|_{BMO^{\varrho}(G)}^q2^{i\varepsilon}\|F\|_{L^2}^q\sum_{M=-l_0}^{i}2^{(M-i)(d_1-d_2)(1/p-1/2)q}\,2^{i(d_1-d_2)(1/p-1/2)q}\end{aligned}$$

$$2^{-\bar{N}qi}2^{i2d_2(1/p-1/2)q}\,\|\chi_{\tilde{B}_0}f\|^q_{L^q(G)}$$
$$\leq C2^{id(1/p-1/2)q}\|b\|^q_{BMO^\varrho(G)}(i+1+l_0)^{q-1}\|F\|^q_{L^2}$$
$$\sum_{\ell=0}^{\infty}2^{-\ell(d_1-d_2)(1/p-1/2)q}\,2^{-\bar{N}qi}\,\|\chi_{\widetilde{B}_0}f\|^q_{L^q(G)}$$
$$\leq C2^{-\delta iq}\,\|b\|^q_{BMO^\varrho(G)}\,\|F\|^q_{L^2}\|\chi_{\widetilde{B}_0}f\|^q_{L^q(G)},$$

for some $\delta>0$, by choosing $\bar{N}$ sufficiently large.

Finally, aggregating the estimates obtained in (4.30), (4.32) and (4.34), we get

$$\begin{aligned}&\|\chi_{4B_0}(b-b_{B_0})((1-\psi)F^{(i)}(\sqrt{\mathcal{L}}))\chi_{\widetilde{B}_0}f\|_{L^q(G)}\\ &\lesssim 2^{-\delta i}\|b\|_{BMO^\varrho(G)}\,\|F\|_{L^2}\,\|\chi_{\widetilde{B}_0}f\|_{L^q(G)},\end{aligned}$$

This completes the proof of (4.4). □

Using similar idea as of proof of (4.4) with obvious modification, one can also prove (4.5) and therefore we omit the details.

This completes the proof of Theorem 1.2 except for the point $(d_1,d_2)=(4,3)$. Note that at $(4,3)$, from (4.27) we only get the range $1\leq p\leq 6/5$. Therefore in order to get the full range of p at $(4,3)$, that is for $1\leq p\leq 4/3$, one needs further analysis at that particular point, which we are going to explain now.

Improvement at $(d_1,d_2)=(4,3)$. Here we will use ideas from [39, Section 8]. As the proof at this point is nearly same as the Theorem (1.2) for other points, so we will be brief. Following the argument of Theorem (1.2), one can see that restriction on the range of p comes from the estimate (4.25), which is deduced from the estimate (4.22). Therefore from (4.22) we have

$$\begin{aligned}&\left\|\sum_{M=-l_0}^{i}\sum_{m=1}^{N_M}\chi_{4B_0}(b-b_{B_0})\chi_{\widetilde{B}^M_m}F^{(i)}_M(\mathcal{L},T)\chi_{B^M_m}f\right\|^q_{L^q(G)} \qquad (4.35)\\ &\leq C2^{iQ(\frac{1}{q}-\frac{1}{s})q}\,\|b\|^q_{BMO^\varrho(G)}2^{i\varepsilon}2^{C\gamma j}\sum_{M=-l_0}^{i}\sum_{m=1}^{N_M}\|\chi_{\widetilde{B}^M_m}F^{(i)}_M(\mathcal{L},T)\chi_{B^M_m}f\|^q_{L^s(G)}.\end{aligned}$$

Now similarly as in the [39, Page 32], first using the left invariance of $F^{(i)}_M(\mathcal{L},T)$ we get $\|\chi_{\widetilde{B}^M_m}F^{(i)}_M(\mathcal{L},T)\chi_{B^M_m}f\|_{L^s(G)} = \|\chi_{\widetilde{A}^M}F^{(i)}_M(\mathcal{L},T)\chi_{A^M}f\|_{L^s(G)}$ and then decomposing χ_{A^M} in the second layer, the above quantity inside the L^s norm we can write

$$\begin{aligned}&\chi_{\widetilde{A}^M}F^{(i)}_M(\mathcal{L},T)\chi_{A^M}f \qquad (4.36)\\ &=\sum_{k=1}^{K_M}\chi_{\widetilde{A}^M_k}\chi_{\widetilde{A}^M}F^{(i)}_{M,t}(\mathcal{L},T)\chi_{A^M_k}f+\sum_{k=1}^{K_M}(1-\chi_{\widetilde{A}^M_k})\chi_{\widetilde{A}^M}F^{(i)}_M(\mathcal{L},T)\chi_{A^M_k}f,\end{aligned}$$

where

$$A^M\subseteq B^{|\cdot|}(0,C\frac{2^M}{10})\times B^{|\cdot|}(0,C\frac{2^{2i}}{100}),$$

$$\widetilde{A}^M := B^{|\cdot|}(0, 2C\,\frac{2^M}{10}\,2^{i\gamma}) \times B^{|\cdot|}(0, 4C\,\frac{2^{2i}}{100}2^{i\gamma})$$

and

$$\widetilde{A}_k^M := \{(x,u) \in \widetilde{A}^M : \langle u - u_k^M, v\rangle < 4\bar{C}\frac{2^M 2^i}{100}2^{i\gamma}\}$$

with $v \in \mathfrak{g}_2$ and $|v| = 1$ as in the [39, Proposition 8.2].

Then from (4.35) we have

$$\begin{aligned}
&\left\| \sum_{M=-l_0}^{i} \sum_{m=1}^{N_M} \chi_{4B_0}(b - b_{B_0}) \chi_{\widetilde{B}_m^M} F_M^{(i)}(\mathcal{L}, T) \chi_{B_m^M} f \right\|_{L^q(G)}^q \qquad (4.37)\\
&\leq C 2^{iQ(1/q-1/s)q} \, \|b\|_{BMO^\varrho(G)}^q 2^{i\varepsilon} 2^{C\gamma j} \\
&\qquad \sum_{M=-l_0}^{i} \sum_{m=1}^{N_M} \Bigg[\left\| \sum_{k=1}^{K_M} \chi_{\widetilde{A}_k^M} \chi_{\widetilde{A}^M} F_M^{(i)}(\mathcal{L}, T) \chi_{A_k^M} f \right\|_{L^s(G)}^q \\
&\qquad\qquad + \left\| \sum_{k=1}^{K_M} (1 - \chi_{\widetilde{A}_k^M}) \chi_{\widetilde{A}^M} F_M^{(i)}(\mathcal{L}, T) \chi_{A_k^M} f \right\|_{L^s(G)}^q \Bigg].
\end{aligned}$$

Regarding the second term inside the summation in (4.37), using Hölder's inequality,

$$\begin{aligned}
&\left\| \sum_{k=1}^{K_M} (1 - \chi_{\widetilde{A}_k^M}) \chi_{\widetilde{A}^M} F_M^{(i)}(\mathcal{L}, T) \chi_{A_k^M} f \right\|_{L^s(G)} \\
&\leq \sum_{k=1}^{K_M} \left(2^M 2^{\gamma i}\right)^{d_1(\frac{1}{s}-\frac{1}{2})} \, 2^{i2d_2(\frac{1}{s}-\frac{1}{2})} \|(1 - \chi_{\widetilde{A}_k^M}) \chi_{\widetilde{A}^M} F_M^{(i)}(\mathcal{L}, T) \chi_{A_k^M} f\|_{L^2(G)}.
\end{aligned}$$

To estimate this we need the following lemma.

Lemma 4.1. [39, Proposition 8.2] *Let G be a Métivier group such that $d_1 = 4$ and $d_2 = 3$. $\mathcal{K}_{F_M(\mathcal{L},T)}$ denote the convolution kernel of $F_M(\mathcal{L}, T)$. Then there exists a direction $v \in \mathfrak{g}_2$ with $|v| = 1$ such that for any $\alpha \geq 0$ we have*

$$\int_G \left| \langle u, v\rangle^\alpha \mathcal{K}_{F_{M,R}(\mathcal{L},T)}(x, u) \right|^2 \, d(x, u) \leq C \, 2^{M(2\alpha - d_2)} \|F\|_{L^2_\alpha}^2.$$

Then with the help of Lemma 4.1, similarly as in (4.20) (also see [39, Lemma 8.3]) and using the fact $A_k^M \subseteq \widetilde{A}^M$, for $s' > 1/2$ we get

$$\begin{aligned}
&\|(1 - \chi_{\widetilde{A}_k^M}) \chi_{\widetilde{A}^M} F_M^{(i)}(\mathcal{L}, T) \chi_{A_k^M} f\|_{L^2(G)} \\
&\lesssim C 2^{-2i\gamma \bar{N}(1/p-1/2)} \, 2^{2is'(1-1/p)} \, \|F^{(i)}\|_{L^2} \, \|\chi_{\widetilde{A}^M} f\|_{L^p(G)} \\
&\lesssim C 2^{-2i\gamma \bar{N}(1/p-1/2)} \, 2^{2is'(1-1/p)} \, \|F^{(i)}\|_{L^2} \, 2^{iQ(1/p-1/q)} \|\chi_{\widetilde{B}_0} f\|_{L^q(G)},
\end{aligned}$$

where in the second line we have also used the fact $\|F^{(i)}\|_{L^2_{\bar{N}}} \sim 2^{i\bar{N}} \|F^{(i)}\|_{L^2}$.

Now using upper bound of $K_M \lesssim 2^{i-M} \leq 2^i$ we have

$$\begin{aligned}
&\left\| \sum_{k=1}^{K_M} (1-\chi_{\widetilde{A}_k^M}) \chi_{\widetilde{A}^M} F_M^{(i)}(\mathcal{L},T)\chi_{A_k^M} f \right\|_{L^s(G)} \\
&\leq C 2^i \big(2^M 2^{\gamma i}\big)^{d_1(1/s-1/2)} 2^{i2d_2(1/s-1/2)} 2^{-2i\gamma\bar{N}(1/p-1/2)} 2^{2is'(1-1/p)} \\
&\qquad\qquad\qquad \times \|F^{(i)}\|_{L^2}\, 2^{iQ(1/p-1/q)} \|\chi_{\widetilde{B}_0} f\|_{L^q(G)}.
\end{aligned}$$

Therefore by choosing $\bar{N}$ large, one term in (4.37) can be estimated as follows.

$$\begin{aligned}
&2^{iQ(1/q-1/s)q}\, \|b\|^q_{BMO^\varrho(G)} 2^{i\varepsilon} 2^{C\gamma j} \\
&\qquad \times \sum_{M=-l_0}^{i} \sum_{m=1}^{N_M} \left\| \sum_{k=1}^{K_M} (1-\chi_{\widetilde{A}_k^M}) \chi_{\widetilde{A}^M} F_M^{(i)}(\mathcal{L},T)\chi_{A_k^M} f \right\|^q_{L^s(G)} \\
&\leq C 2^{iQ(1/q-1/s)q}\, \|b\|^q_{BMO^\varrho(G)} 2^{i\varepsilon} 2^{C\gamma j} \sum_{M=-l_0}^{i} \sum_{m=1}^{N_M} 2^{iq} \big(2^M\, 2^{\gamma i}\big)^{d_1(1/s-1/2)q} \\
&2^{i2d_2(1/s-1/2)q} 2^{-2i\gamma\bar{N}(1/p-1/2)q}\, 2^{2is'(1-1/p)q}\, \|F^{(i)}\|^q_{L^2}\, 2^{iQ(1/p-1/q)q} \|\chi_{\widetilde{B}_0} f\|^q_{L^q(G)} \\
&\leq C 2^{iQ(1/q-1/s)q}\, \|b\|^q_{BMO^\varrho(G)} 2^{i\varepsilon} 2^{C\gamma j} 2^{i(1+d_1)} \big(2^i\, 2^{\gamma i}\big)^{d_1(1/s-1/2)q}\, 2^{2id_2(1/s-1/2)q} \\
&\qquad \times 2^{-2i\gamma\bar{N}(1/p-1/2)q}\, 2^{2is'(1-1/p)q}\, \|F\|^q_{L^2}\, 2^{iQ(1/p-1/q)q} \|\chi_{\widetilde{B}_0} f\|^q_{L^q(G)} \\
&\leq C 2^{-i\delta}\, \|b\|^q_{BMO^\varrho(G)} \|F\|^q_{L^2}\, \|\chi_{\widetilde{B}_0} f\|^q_{L^q(G)},
\end{aligned} \tag{4.38}$$

for some $\delta > 0$.

On the other hand using the bounded overlapping property of the sets $\widetilde{A}_k^M$ we have

$$\begin{aligned}
&\left\| \sum_{k=1}^{K_M} \chi_{\widetilde{A}_k^M} \chi_{\widetilde{A}^M} F_M^{(i)}(\mathcal{L},T)\chi_{A_k^M} f \right\|^q_{L^s(G)} \\
&\leq C 2^{C\gamma i} \left(\sum_{k=1}^{K_M} \|\chi_{\widetilde{A}_k^M} F_M^{(i)}(\mathcal{L},T)\chi_{A_k^M} f\|^s_{L^s(G)} \right)^{q/s}.
\end{aligned}$$

From Hölder's inequality, Corollary 3.2 and using $q < s$ for any $\tilde{s} > 1/2$ we get

$$\begin{aligned}
&\sum_{k=1}^{K_M} \|\chi_{\widetilde{A}_k^M} F_M^{(i)}(\mathcal{L},T)\chi_{A_k^M} f\|^s_{L^s(G)} \\
&\leq C \sum_{k=1}^{K_M} \big(2^M\, 2^{\gamma i}\big)^{(d_1+1)(1/s-1/2)s}\, 2^{i(2d_2-1)(1/s-1/2)s}\, \|F_M^{(i)}(\mathcal{L},T)\chi_{A_k^M} f\|^s_{L^2(G)} \\
&\leq C \big(2^M\, 2^{\gamma i}\big)^{(d_1+1)(1/s-1/2)s}\, 2^{i(2d_2-1)(1/s-1/2)s}\, 2^{-Md_2(1/p-1/2)s}\, \|\psi F^{(i)}\|^s_{2^M,2}
\end{aligned}$$

$$
\begin{aligned}
&\qquad 2^{M(d_1+1)(1/p-1/q)s}\, 2^{i(2d_2-1)(1/p-1/q)s} \sum_{k=1}^{K_M} \|\chi_{A_k^M} f\|^s_{L^q(G)} \\
&\leq C\big(2^M\, 2^{\gamma i}\big)^{(d_1+1)(1/s-1/2)s}\, 2^{i(2d_2-1)(1/s-1/2)s}\, 2^{-Md_2(1/p-1/2)s}\, \|\psi F^{(i)}\|^s_{2^M,2} \\
&\qquad 2^{M(d_1+1)(1/p-1/q)s}\, 2^{i(2d_2-1)(1/p-1/q)s} \left(\sum_{k=1}^{K_M} \|\chi_{A_k^M} f\|^q_{L^q(G)}\right)^{s/q} \\
&\leq C\big(2^M\, 2^{\gamma i}\big)^{(d_1+1)(1/s-1/2)s}\, 2^{i(2d_2-1)(1/s-1/2)s}\, 2^{-Md_2(1/p-1/2)s} \\
&\qquad 2^{(i-M)\tilde{s}s}\|F^{(i)}\|^s_{L^2} 2^{M(d_1+1)(1/p-1/q)s}\, 2^{i(2d_2-1)(1/p-1/q)s}\, \|\chi_{A^M} f\|^s_{L^q(G)}.
\end{aligned}
$$

Therefore the remaining term in (4.37)(the one term which not estimated above) we estimate as follows.

$$
\begin{aligned}
&(2^i t)^{Q(1/q-1/s)q}\, \|b\|^q_{BMO^\varrho(G)} 2^{i\varepsilon} 2^{C\gamma j} \\
&\qquad \sum_{M=-l_0}^{i} \sum_{m=1}^{N_M} \left\| \sum_{k=1}^{K_M} \chi_{\widetilde{A}_k^M} \chi_{\widetilde{A}^M} F_M^{(i)}(\mathcal{L},T) \chi_{A_k^M} f \right\|^q_{L^s(G)} \\
&\leq C 2^{iQ(1/q-1/s)q}\, \|b\|^q_{BMO^\varrho(G)} 2^{i\varepsilon} 2^{C\gamma j} \sum_{M=-l_0}^{i} \sum_{m=1}^{N_M} \\
&\qquad \times \big(2^M\, 2^{\gamma i}\big)^{(d_1+1)(1/s-1/2)q}\, 2^{i(2d_2-1)(1/s-1/2)q}\, 2^{-Md_2(1/p-1/2)q}\, 2^{(i-M)\tilde{s}q} \\
&\qquad\qquad \times \|F^{(i)}\|^q_{L^2} 2^{M(d_1+1)(1/p-1/q)q}\, 2^{i(2d_2-1)(\frac{1}{p}-\frac{1}{q})q}\, \|\chi_{B_m^M} f\|^q_{L^q(G)} \\
&\leq C 2^{i\varepsilon}\, \|b\|^q_{BMO^\varrho(G)} 2^{i\varepsilon} 2^{C\gamma j} \sum_{M=-l_0}^{i} 2^{M(d_1+1)(1/s-1/2)q}\, 2^{i(2d_2-1)(1/s-1/2)q} \\
&\qquad \times 2^{-Md_2(1/p-1/2)q}\, 2^{(i-M)\tilde{s}q} 2^{-i\beta q} \|F^{(i)}\|^q_{L^2_\beta} \\
&\qquad\qquad \times 2^{M(d_1+1)(1/p-1/q)q}\, 2^{i(2d_2-1)(1/p-1/q)q}\, \|\chi_{\widetilde{B}_0} f\|^q_{L^q(G)} \\
&\leq C 2^{i\varepsilon}\, 2^{C\gamma j}\, \|b\|^q_{BMO^\varrho(G)} \\
&\qquad \sum_{M=-l_0}^{i} 2^{M(1/p-1/q+1/s-1/2)q + Md_1(1/p-1/q+1/s-1/2)q - Md_2(1/p-1/2)q - M\tilde{s}q} \\
&\qquad\qquad \times 2^{-i(1/p-1/q+1/s-1/2)q + i\tilde{s}q + 2id_2(1/p-1/q+1/s-1/2)q} \\
&\qquad\qquad\qquad \times 2^{-id(1/p-1/2)q} \|F^{(i)}\|^q_{L^2_{d(1/p-1/2)}}\, \|\chi_{\widetilde{B}_0} f\|^q_{L^q(G)} \\
&\leq C 2^{i\varepsilon}\, 2^{C\gamma j}\, \|b\|^q_{BMO^\varrho(G)} \|F\|^q_{L^2_{d(1/p-1/2)}}\, \|\chi_{\widetilde{B}_0} f\|^q_{L^q(G)}
\end{aligned}
$$

$$\times \sum_{M=-l_0}^{i} 2^{q(M-i)\{1+(d_1-d_2)-\tilde{s}\frac{2p}{2-p}\}(1/p-1/2)}.$$

Note that for $1 \leq p \leq p_{d_2}$ we have

$$\frac{1}{p} - \frac{1}{2} \geq \frac{d_2+3}{2(d_2+1)} - \frac{1}{2} = \frac{1}{d_2+1},$$

therefore at $(d_1, d_2) = (4, 3)$ we get

$$1 + (d_1 - d_2) - \tilde{s}\frac{2p}{2-p} \geq 1 + (d_1 - d_2) - \tilde{s}(d_2 + 1) = 2 - 4\tilde{s}.$$

Then for $\tilde{s} > 1/2$, choosing $\tilde{s}$ very close to $1/2$ one can see

$$\sum_{M=-l_0}^{i} 2^{q(M-i)\{1+(d_1-d_2)-\tilde{s}\frac{2p}{2-p}\}(1/p-1/2)} < \infty.$$

As $\beta > d(1/p - 1/2)$, choosing $\gamma > 0$ very small yields

$$\begin{aligned}(2^i t)^{Q(1/q-1/s)q} \|b\|^q_{BMO^{\varrho}(G)} 2^{i\varepsilon} 2^{C\gamma j} \qquad\qquad (4.39)\\ \times \sum_{M=-l_0}^{i} \sum_{m=1}^{N_M} \left\| \sum_{k=1}^{K_M} \chi_{\widetilde{A}_k^M} \chi_{\widetilde{A}^M} F_M^{(i)}(\mathcal{L}, T) \chi_{A_k^M} f \right\|^q_{L^s(G)}\\ \leq C 2^{-i\delta} \|b\|^q_{BMO^{\varrho}(G)} \|F\|^q_{L^2_\beta} \|\chi_{\widetilde{B}_0} f\|^q_{L^q(G)},\end{aligned}$$

for some $\delta > 0$.

Then combining the estimates of (4.38) and (4.39) from (4.37) we get

$$\begin{aligned}&\left\| \sum_{M=-l_0}^{i} \sum_{m=1}^{N_M} \chi_{4B_0}(b - b_{B_0}) \chi_{\widetilde{B}_m^M} F_M^{(i)}(\mathcal{L}, T) \chi_{B_m^M} f \right\|_{L^q(G)}\\ &\leq C 2^{-i\delta} \|b\|_{BMO^{\varrho}(G)} \|F\|_{L^2_\beta} \|\chi_{\widetilde{B}_0} f\|_{L^q(G)}.\end{aligned}$$

This completes the proof of Theorem 1.2 for the point $(d_1, d_2) = (4, 3)$. □

Proof of Theorem 1.4. As the class of Heisenberg group is smaller than the class of Métivier groups and $p_{d_1,d_2} = p_{d_2} = \frac{2(d_2+1)}{d_2+3}$ for $(d_1, d_2) \notin \{(8, 6), (8, 7)\}$, proof of Theorem (1.4) follows from the proof of Theorem (1.3) except for the point $(d_1, d_2) = \{(8, 6), (8, 7)\}$. We will prove Theorem (1.4) by proving Theorem (1.2) for the case of Heisenberg type group with $1 \leq p \leq 2(d_2+1)/(d_2+3)$. If one look carefully the proof of Theorem (1.2), one can see that the restriction for the range of p comes from the estimates (4.22)- (4.25), which occurs because the weaker type restriction estimate available for Métivier groups. Therefore it is clear that the presence of discrete norm $\|F\|_{2^M,2}$ in the right hand side of Theorem 3.1 is the main obstacle from getting desired ranges of p for the case $(d_1, d_2) \in \{(8, 6), (8, 7)\}$. If we can replace the discrete norm $\|F\|_{2^M,2}$ with

$\|F\|_{L^2(G)}$, we can improve the ranges of p. Such estimates are known to be true for Heisenberg type groups. Indeed we have the following result.

Proposition 4.2. [38, Theorem 3.2] *Let $\mathbb{H}$ be Heisenberg type group. Suppose $1 \leq p \leq 2(d_2+1)/(d_2+3)$. Then for F and F_M as defined in Section* (3)*, we have*

$$\|F_M(\mathcal{L},T)f\|_{L^2(G)} \leq C2^{-Md_2(1/p-1/2)}\,\|F\|_{L^2}\,\|f\|_{L^p(G)}.$$

For Heisenberg type groups the eigen values are of the form $[k]|\mu| = (2k + d_1/2)|\mu|$. Although the cutoff considered in [38, Theorem 3.2] (they have considered cutoff along $[k] := 2k + d_1/2$ variable) is different from the cutoff here we considered (along $|\mu|$ variable), because the support of F lies away from origin, these two cutoffs are essentially give the same result. Indeed instead of taking $\Theta(2^{-M}[k])$ if we take $\Theta(2^M|\mu|)$, then the same conclusion hold as in [38, Theorem 3.2].

Therefore having Proposition (4.2) in hand and following the arguments of the proof of Theorem 1.2, in the estimate of (4.23) instead of Proposition (3.1), here using Proposition (4.2), and choosing γ small enough for $1 \leq p \leq 2(d_2 + 1)/(d_2 + 3)$ we get

$$\begin{aligned}
&\left\|\sum_{M=0}^{i}\sum_{m=1}^{N_M}\chi_{4B_0}(b-b_{B_0})\chi_{\tilde{B}_m^M}F_M^{(i)}(\mathcal{L},T)\chi_{B_m^M}f\right\|_{L^q(G)}^q\\
&\lesssim 2^{i\varepsilon}\,2^{i2d_2(1/p-1/2)q}\,\|b\|_{BMO^\varrho(G)}^q\,\|\psi F^{(i)}\|_{L^2}^q\,\|\chi_{\widetilde{B}_0}f\|_{L^q(G)}^q\\
&\qquad\qquad\qquad\times\sum_{M=0}^{i}2^{M(d_1-d_2)(1/p-1/2)q}\\
&\lesssim 2^{i\varepsilon}\,2^{id(1/p-1/2)q}\,2^{-i\beta q}\,\|b\|_{BMO^\varrho(G)}^q\,\|F\|_{L^2_\beta}^q\,\|\chi_{\widetilde{B}_0}f\|_{L^q(G)}^q\\
&\qquad\qquad\qquad\times\sum_{M=0}^{i}2^{(M-i)(d_1-d_2)(1/p-1/2)q}\\
&\lesssim 2^{-\delta i}\,\|b\|_{BMO^\varrho(G)}^q\,\|F\|_{L^2_\beta}^q\,\|\chi_{\widetilde{B}_0}f\|_{L^q(G)}^q,
\end{aligned}$$

for some $\delta > 0$, as we have $\beta > d(1/p - 1/2)$, where in the last inequality we have used the fact that on Heisenberg type groups we always have $d_1 > d_2$.

Hence, one can conclude that Theorem (1.2) holds on Heisenberg type groups for all $d_1, d_2 \geq 1$ with $1 \leq p \leq 2(d_2 + 1)/(d_2 + 3)$. □

5. Compactness of Bochner-Riesz commutator

In this section, we give the prove Theorem (1.5). In order to prove Theorem (1.5), we need the following result which characterizes the relatively compact subsets of $L^p(G)$, which is also known as Kolmogorov-Riesz compactness theorem.

Proposition 5.1. [8, Lemma 4.3] *Let $1 < p < \infty$ and $(x_0, u_0) \in G$. The subset $\mathcal{F}$ in $L^p(G)$ is relatively compact if and only if the following conditions are satisfied:*

(1) $\sup_{f \in \mathcal{F}} \|f\|_{L^p(G)} < \infty$;

(2) $\lim_{R\to\infty} \int_{G\setminus B((x_0,u_0),R)} |f(x,u)|^p \, d(x,u) = 0$ *uniformly in* $f \in \mathcal{F}$;

(3) $\lim_{r\to 0} \int_G |f(x,u) - f_{B((x,u),r)}|^p \, d(x,u) = 0$ *uniformly in* $f \in \mathcal{F}$.

With the above proposition in hand we are ready to proof the Theorem (1.5).

Proof of Theorem (1.5). Let us set $F(\eta) = (1-\eta^2)_+^\alpha$. Notice that from (4.2) and (4.3), it follows that $\sum_{i=0}^N [b, F^{(i)}(\sqrt{\mathcal{L}})]$ converges to $[b, F(\sqrt{\mathcal{L}})]$ in the $L^q(G)$ norm. Then using the fact that compact operators are closed under the norm limit, the proof is reduced to show that $[b, F^{(i)}(\sqrt{\mathcal{L}})]$ is compact for all $i \geq 0$. Moreover, since $C_c^\infty(G)$ is dense in $CMO^\varrho(G)$, we may restrict ourselves to the case when $b \in C_c^\infty(G)$. To this end, let $\mathcal{F}$ be an arbitrary bounded set in $L^q(G)$. To prove that $[b, F^{(i)}(\sqrt{\mathcal{L}})]$ is a compact operator, we have to show that the set $\mathcal{E} = \{[b, F^{(i)}(\sqrt{\mathcal{L}})]f \, : \, f \in \mathcal{F}\}$ is relatively compact for each $i \geq 0$, that is $\mathcal{E}$ satisfies all the conditions (1), (2) and (3) of Proposition (5.1).

The condition (1) is follows readily from (4.3) and the fact that $F \in L^2_\beta$ if and only if $\alpha > \beta - 1/2$. Indeed

$$\sup_{f\in\mathcal{F}} \|[b, F^{(i)}(\sqrt{\mathcal{L}})]f\|_{L^q(G)} \leq C \sup_{f\in\mathcal{F}} \|b\|_{BMO^\varrho(G)} \|F\|_{L^2_\beta(\mathbb{R})} \|f\|_{L^q(G)} < \infty.$$

Next, we proceed to verify remaining other two conditions of Proposition (5.1). To this end, we choose a function $\psi \in C_c^\infty(-4,4)$ such that $\psi(\lambda) = 1$ on $(-2,2)$ and $\varphi \in C_c^\infty(2,8)$. Then for all $\lambda > 0$ we can decompose as follows:

$$F^{(i)}(\lambda) = (\psi F^{(i)})(\lambda) + \sum_{\iota \geq 0} (\varphi_\iota F^{(i)})(\lambda), \tag{5.1}$$

where $\phi_\iota(\lambda) = \varphi(2^{-\iota}\lambda)$ for $\iota > 0$.

Now let us verify the condition (2). First note that it is enough to verify the condition (2) for all $C_c^\infty(G)$ functions in $\mathcal{F}$. Let $\epsilon > 0$. Then for $f \in \mathcal{F}$, there exists $g \in C_c^\infty(G)$ such that $\|f - g\|_{L^q(G)} < \epsilon$. So that from (4.3) for any $R > 0$, we have

$$\begin{aligned} &\|\chi_{G\setminus B(0,R)}[b, F^{(i)}(\sqrt{\mathcal{L}})]f\|_{L^q(G)} \qquad (5.2)\\ &\leq \|\chi_{G\setminus B(0,R)}[b, F^{(i)}(\sqrt{\mathcal{L}})](f-g)\|_{L^q(G)} + \|\chi_{G\setminus B(0,R)}[b, F^{(i)}(\sqrt{\mathcal{L}})]g\|_{L^q(G)}\\ &\leq \epsilon + \|\chi_{G\setminus B(0,R)}[b, F^{(i)}(\sqrt{\mathcal{L}})]g\|_{L^q(G)}. \end{aligned}$$

Therefore to check the condition (2), enough to show as $R \to \infty$,

$$\|\chi_{G\setminus B(0,R)}[b, F^{(i)}(\sqrt{\mathcal{L}})]g\|_{L^q(G)} < \epsilon \quad \text{for} \quad g \in C_c^\infty(G).$$

As $b \in C_c^\infty(G)$, there exists $j \in \mathbb{N}$ such that $\operatorname{supp} b \subseteq B(0, 2^j)$. Take $R = 2 \cdot 2^{2j}$. Then using the expression (5.1), we decompose as follows.

$$\begin{aligned}
&\|\chi_{G\setminus B(0,\,2\cdot 2^{2j})}[b, F^{(i)}(\sqrt{\mathcal{L}})]g\|_{L^q(G)} \\
&\leq \|\chi_{G\setminus B(0,\,2\cdot 2^{2j})}[b, (\psi F^{(i)})(\sqrt{\mathcal{L}})]g\|_{L^q(G)} \\
&\qquad\qquad\qquad + \sum_{\iota\geq 0} \|\chi_{G\setminus B(0,\,2\cdot 2^{2j})}[b, (\varphi_\iota F^{(i)})(\sqrt{\mathcal{L}})]g\|_{L^q(G)}.
\end{aligned}$$

First we estimate the second term in the right hand side. Using the support of b and Young's inequality

$$\begin{aligned}
&\|\chi_{G\setminus B(0,2\cdot 2^{2j})}[b, (\varphi_\iota F^{(i)})(\sqrt{\mathcal{L}})]g\|_{L^q(G)} \\
&= \left\{\int_{G\setminus B(0,2\cdot 2^{2j})} \left| \int_{B(0,2^j)} \mathcal{K}_{(\varphi_\iota F^{(i)})(\sqrt{\mathcal{L}})}((x,u)^{-1}(y,t)) \right.\right. \\
&\qquad\qquad\qquad\qquad \left.\left. \times\, b(y,t)\, g(y,t)\, d(y,t) \right|^q d(x,u) \right\}^{1/q} \\
&\leq \left\{\int_G \left| \int_G \chi_{G\setminus B(0,2^{2j})}((x,u)^{-1}(y,t)) \mathcal{K}_{(\varphi_\iota F^{(i)})(\sqrt{\mathcal{L}})}((x,u)^{-1}(y,t)) \right.\right. \\
&\qquad\qquad\qquad \left.\left. \times\, \chi_{B(0,2^j)}(y,t) b(y,t)\, g(y,t)\, d(y,t) \right|^q d(x,u) \right\}^{1/q} \\
&\leq C\|b\|_{L^\infty}\|g\|_{L^q(G)} \int_{G\setminus B(0,2^{2j})} |\mathcal{K}_{(\varphi_\iota F^{(i)})(\sqrt{\mathcal{L}})}(x,u)|\, d(x,u).
\end{aligned}$$

Note using the definition $F^{(i)}(\lambda) = 2^i(F * \delta_{2^i}\check{\eta})(\lambda)$ as in (4.29), for any fixed $s > 0$,

$$\|\varphi_\iota F^{(i)}(2^\iota \cdot)\|_{L^\infty_s} \leq C 2^{-\bar{N}(i+\iota)}\|F\|_{L^2}, \tag{5.3}$$

for any $\bar{N} > 0$.

Then using Proposition (3.4), (5.3) and Lemma (2.4) we have

$$\begin{aligned}
&\int_{G\setminus B(0,2^{2j})} |\mathcal{K}_{(\varphi_\iota F^{(i)})(\sqrt{\mathcal{L}})}(x,u)|\, d(x,u) \\
&\leq C \int_{\|(x,u)\|>2^{2j}} \frac{(1+2^\iota\|(x,u)\|)^s |\mathcal{K}_{(\varphi_\iota F^{(i)})(\sqrt{\mathcal{L}})}(x,u)|}{(1+2^\iota\|(x,u)\|)^s}\, d(x,u) \\
&\leq C 2^{\iota Q}\, \|\varphi_\iota F^{(i)}(2^\iota\cdot)\|_{L^\infty_{s+\varepsilon}} \int_{\|(x,u)\|>2^{2j}} \frac{d(x,u)}{(1+2^\iota\|(x,u)\|)^s} \\
&\leq C 2^{\iota Q} 2^{-\bar{N}(i+\iota)}\, 2^{-\iota s} 2^{2j(-s+Q)},
\end{aligned}$$

for all $\bar{N} > 0$, $\varepsilon > 0$ and $s > Q$.

Then choosing $s > Q$ and $\bar{N} > Q$, as $j \to \infty$ we see that

$$\begin{aligned}&\sum_{\iota\geq 0} \|\chi_{G\setminus B(0,2\cdot 2^{2j})}[b,(\varphi_\iota F^{(i)})(\sqrt{\mathcal{L}})]g\|_{L^q(G)} \\ &\leq C\|b\|_{L^\infty}\|g\|_{L^q(G)} 2^{-2j(s-Q)}\sum_{\iota\geq 0} 2^{-\iota(\bar{N}-Q)} < \epsilon.\end{aligned} \tag{5.4}$$

A similar argument also shows that as $j \to \infty$,

$$\|\chi_{G\setminus B(0,2\cdot 2^{2j})}[b,(\psi F^{(i)})(\sqrt{\mathcal{L}})]g\|_{L^q(G)} < \epsilon, \tag{5.5}$$

where for each $i \geq 0$, one have to use the following fact, which is a consequence of Sobolev embedding theorem

$$\|\psi F^{(i)}\|_{L^\infty_s} \leq C 2^{i(s+1/2+\epsilon)}\|F\|_{L^2} \leq C_{i,s,\epsilon}. \tag{5.6}$$

As we are proving compactness for each $i \geq 0$, so our constants $C_{i,s,\epsilon}$ may depend on i.

Thus combining estimates (5.4), (5.5) and (5.2), as $j \to \infty$ we conclude that

$$\|\chi_{G\setminus B(0,2\cdot 2^{2j})}[b,F^{(i)}(\sqrt{\mathcal{L}})]f\|_{L^q(G)} < \epsilon.$$

Not it only remains to check the condition (3) for $\mathcal{E}$. Once again using the density of $C_c^\infty(G)$ functions in $L^q(G)$ enough to check condition (3) for $f \in C_c^\infty(G)$. First we write

$$\begin{aligned}&[b,F^{(i)}(\sqrt{\mathcal{L}})]f(x,u) - \big([b,F^{(i)}(\sqrt{\mathcal{L}})]f\big)_{B((x,u),r)} \\ &= \frac{1}{|B((x,u),r)|}\int_{B((x,u),r)} \Big\{[b,F^{(i)}(\sqrt{\mathcal{L}})]f(x,u) - [b,F^{(i)}(\sqrt{\mathcal{L}})]f(y,t)\Big\}\, d(y,t) \\ &= \frac{1}{|B(0,r)|}\int_{B(0,r)} \Big\{[b,F^{(i)}(\sqrt{\mathcal{L}})]f(x,u) - [b,F^{(i)}(\sqrt{\mathcal{L}})]f((x,u)(y,t))\Big\}\, d(y,t),\end{aligned}$$

and then for $\varepsilon > 0$, we divide it into four parts as follows.

$$\begin{aligned}&[b,F^{(i)}(\sqrt{\mathcal{L}})]f(x,u) - [b,F^{(i)}(\sqrt{\mathcal{L}})]f((x,u)(y,t)) \\ &= \int_G \mathcal{K}_{F^{(i)}(\sqrt{\mathcal{L}})}((x',u')^{-1}(x,u))(b(x,u)-b(x',u'))f(x',u')d(x',u') \\ &- \int_G \mathcal{K}_{F^{(i)}(\sqrt{\mathcal{L}})}((x',u')^{-1}(x,u)(y,t))(b((x,u)(y,t))-b(x',u'))f(x',u')d(x',u') \\ &=: J_1 + J_2 + J_3 + J_4.\end{aligned}$$

where

$$\begin{aligned}J_1 = \int_{\|((x',u')^{-1}(x,u))\|>\varepsilon^{-1}\|(y,t)\|} &\mathcal{K}_{F^{(i)}(\sqrt{\mathcal{L}})}((x',u')^{-1}(x,u)) \\ &(b(x,u)-b((x,u)(y,t)))f(x',u')\, d(x',u'),\end{aligned}$$

$$J_2 = \int_{\|((x',u')^{-1}(x,u))\|>\varepsilon^{-1}\|(y,t)\|} (b((x,u)(y,t)) - b(x',u'))$$

$$\times (\mathcal{K}_{F^{(i)}(\sqrt{\mathcal{L}})}((x',u')^{-1}(x,u)) - \mathcal{K}_{F^{(i)}(\sqrt{\mathcal{L}})}((x',u')^{-1}(x,u)(y,t))) \times f(x',u')\,d(x',u'),$$

$$J_3 = \int_{\|((x',u')^{-1}(x,u))\|\le\varepsilon^{-1}\|(y,t)\|} \mathcal{K}_{F^{(i)}(\sqrt{\mathcal{L}})}((x',u')^{-1}(x,u))(b(x,u)-b(x',u')) \times f(x',u')\,d(x',u'),$$

and

$$J_4 = -\int_{\|((x',u')^{-1}(x,u))\|\le\varepsilon^{-1}\|(y,t)\|} \mathcal{K}_{F^{(i)}(\sqrt{\mathcal{L}})}((x',u')^{-1}(x,u)(y,t)) (b((x,u)(y,t)) - b(x',u'))f(x',u')\,d(x',u').$$

We now estimate each term J_1, J_2, J_3 and J_4 separately. For the term J_1, using (5.1) we first write

$$\begin{aligned}
|J_1| &\le C\Bigg[\int_{\|((x',u')^{-1}(x,u))\|>\varepsilon^{-1}\|(y,t)\|} |\mathcal{K}_{(\psi F^{(i)})(\sqrt{\mathcal{L}})}((x',u')^{-1}(x,u))| \times |f(x',u')|\,d(x',u') \\
&\quad + \sum_{\iota\ge 0}\int_{\|((x',u')^{-1}(x,u))\|>\varepsilon^{-1}\|(y,t)\|} |\mathcal{K}_{(\varphi_\iota F^{(i)})(\sqrt{\mathcal{L}})}((x',u')^{-1}(x,u))| \times |f(x',u')|\,d(x',u')\Bigg]|b(x,u)-b((x,u)(y,t))| \\
&=: J_{11} + J_{12}.
\end{aligned}$$

Let $0<\tau<1$. Then using mean value theorem together with Proposition (3.4) and (5.3), for any $\bar{N}>0$ we see that

$$\begin{aligned}
&|J_{12}| \\
&\le C\|(y,t)\|\sum_{\iota\ge 0}\int_{\|((x',u')^{-1}(x,u))\|>\varepsilon^{-1}\|(y,t)\|} |\mathcal{K}_{(\varphi_\iota F^{(i)})(\sqrt{\mathcal{L}})}((x',u')^{-1}(x,u))| \times |f(x',u')|\,d(x',u') \\
&\le C\|(y,t)\|\sum_{\iota\ge 0}\sum_{k=0}^{\infty}\int_{2^k\varepsilon^{-1}\|(y,t)\|<\|((x',u')^{-1}(x,u))\|\le 2^{k+1}\varepsilon^{-1}\|(y,t)\|} \\
&\qquad \times \frac{(1+2^\iota\|(x',u')^{-1}(x,u)\|)^{Q+\tau}|\mathcal{K}_{(\varphi_\iota F^{(i)})(\sqrt{\mathcal{L}})}((x',u')^{-1}(x,u))|}{(1+2^\iota\|(x',u')^{-1}(x,u)\|)^{Q+\tau}} \times |f(x',u')|\,d(x',u') \\
&\le C\|(y,t)\|\sum_{\iota\ge 0}\sum_{k=0}^{\infty}\frac{2^{\iota Q}\|\phi_\iota F^{(i)}(2^\iota\cdot)\|_{L^\infty_{Q+\tau+\epsilon}}|B((x,u),2^{k+1}\varepsilon^{-1}\|(y,t)\|)|}{\{2^\iota 2^k\varepsilon^{-1}\|(y,t)\|\}^{Q+\tau}}
\end{aligned}$$

$$
\begin{aligned}
&\times \frac{1}{|B((x,u),2^{k+1}\varepsilon^{-1}\|(y,t)\|)|}\int_{\varrho((x',u'),(x,u))\le 2^{k+1}\varepsilon^{-1}\|(y,t)\|}|f(x',u')|\,d(x',u')\\
&\le C\|(y,t)\|\sum_{\iota\ge 0}\sum_{k=0}^{\infty}\frac{2^{\iota Q}2^{-\bar N(i+\iota)}\big(2^{k+1}\varepsilon^{-1}\|(y,t)\|\big)^{Q}}{\{2^{\iota}2^{k}\varepsilon^{-1}\|(y,t)\|\}^{Q+\tau}}\mathcal{M}f(x,u)\\
&\le C\varepsilon^{\tau}\|(y,t)\|^{1-\tau}\mathcal{M}f(x,u)\left(\sum_{k=1}^{\infty}\frac{1}{2^{k\tau}}\right)\left(\sum_{\iota\ge 0}\frac{1}{2^{\iota(\bar N+\tau)}}\right)\\
&\le C\varepsilon^{\tau}\|(y,t)\|^{1-\tau}\mathcal{M}f(x,u),
\end{aligned}
$$

where $\mathcal{M}$ denote the Hardy-Littlewood maximal operator defined on G relative to the distance ϱ.

On the other hand, using Proposition (3.4), (5.6) proceeding as above, we have

$$|J_{11}|\le C\varepsilon^{\tau}2^{i(Q+\tau+1+2\epsilon)}\|(y,t)\|^{1-\tau}\mathcal{M}f(x,u).$$

Therefore combining both estimates J_{11} and J_{12}, we get

$$
\begin{aligned}
|J_1| &\le C\varepsilon^{\tau}\|(y,t)\|^{1-\tau}(1+2^{i(Q+\tau+1+2\epsilon)})\mathcal{M}f(x,u)\\
&\le C\varepsilon^{\tau}\|(y,t)\|^{1-\tau}\mathcal{M}f(x,u).
\end{aligned}
$$

Similarly as in J_1, we also break J_2 into two parts J_{21} and J_{22} corresponding to ψ and $\sum_{\iota\ge 0}\phi_{\iota}$. Let γ_0 be a horizontal curve joining between two points (x,u) and $((x,u)(y,t))$. Then for J_{22}, using mean value theorem combining with Proposition (3.4) and (5.3), we get

$$
\begin{aligned}
&|J_{22}|\\
&\le C\sum_{\iota\ge 0}\int_{\|((x',u')^{-1}(x,u))\|>\varepsilon^{-1}\|(y,t)\|}|b((x,u)(y,t))-b(x',u')|\\
&\qquad |\mathcal{K}_{(\varphi_{\iota}F^{(i)})(\sqrt{\mathcal{L}})}((x',u')^{-1}(x,u))-\mathcal{K}_{(\varphi_{\iota}F^{(i)})(\sqrt{\mathcal{L}})}((x',u')^{-1}(x,u)(y,t))|\\
&\qquad\qquad \times|f(x',u')|\,d(x',u')\\
&\le C\|b\|_{L^{\infty}}\|(y,t)\|\sum_{\iota\ge 0}\int_{\|((x',u')^{-1}(x,u))\|>\varepsilon^{-1}\|(y,t)\|}\int_0^1\\
&\qquad\qquad \times|X\mathcal{K}_{(\varphi_{\iota}F^{(i)})(\sqrt{\mathcal{L}})}((x',u')^{-1}\gamma_0(s))||f(x',u')|\,d(x',u')\,ds\\
&\le C\,\|(y,t)\|\sum_{\iota\ge 0}\sum_{k\ge 0}\int_{2^{k+1}\varepsilon^{-1}\|(y,t)\|\ge\|((x',u')^{-1}(x,u))\|>2^{k}\varepsilon^{-1}\|(y,t)\|}\int_0^1\\
&\qquad \frac{(1+2^{\iota}\|((x',u')^{-1}\gamma_0(s))\|)^{Q+\tau}|X\mathcal{K}_{(\varphi_{\iota}F^{(i)})(\sqrt{\mathcal{L}})}((x',u')^{-1}\gamma_0(s))|}{(1+2^{\iota}\|((x',u')^{-1}\gamma_0(s))\|)^{Q+\tau}}\\
&\qquad\qquad \times|f(x',u')|\,d(x',u')\,ds
\end{aligned}
$$

$$\leq C\,\|(y,t)\| \sum_{\iota\geq 0}\sum_{k\geq 0} \frac{2^{\iota(Q+1)}\|\phi_\iota F^{(i)}(2^\iota\cdot)\|_{L^\infty_{Q+\tau+\epsilon}} |B((x,u),2^{k+1}\varepsilon^{-1}\|(y,t)\|)|}{(2^\iota 2^k\varepsilon^{-1}\|(y,t)\|)^{Q+\tau}}$$
$$\times \frac{1}{|B((x,u),2^{k+1}\varepsilon^{-1}\|(y,t)\|)|}\int_{\varrho((x',u'),(x,u))<2^{k+1}\varepsilon^{-1}\|(y,t)\|} |f(x',u')|\,d(x',u')$$
$$\leq C\,\|(y,t)\|^{1-\tau}\,\varepsilon^\tau \mathcal{M}f(x,u),$$

provided $\|(y,t)\|$ is very small and $\bar{N}>1$. Here in the second last inequality we have used the fact $\|((x',u')^{-1}\gamma_0(s))\| \sim \|((x',u')^{-1}(x,u))\|$ if $\|(y,t)\|$ is very small. (Actually later we will estimate for $\|(y,t)\|\leq r$ and $r\to 0$.)

In a similar fashion, we also get

$$|J_{21}| \leq C\varepsilon^\tau 2^{i(Q+\tau+1+2\epsilon)}\|(y,t)\|^{1-\tau}\mathcal{M}f(x,u).$$

Then combining both the estimates for J_{21} and J_{22} we get,

$$\begin{aligned}|J_2| &\leq C\varepsilon^\tau\|(y,t)\|^{1-\tau}(1+2^{i(Q+\tau+1+2\epsilon)})\mathcal{M}f(x,u)\\ &\leq C_i\varepsilon^\tau\|(y,t)\|^{1-\tau}\mathcal{M}f(x,u).\end{aligned}$$

In case of J_3, similarly as above, we again write J_3 as a sum of J_{31} and J_{32}. Using mean value theorem, Proposition 3.4 and (5.6) we see that

$$|J_{31}| \leq C\int_{\|((x',u')^{-1}(x,u))\|\leq\varepsilon^{-1}\|(y,t)\|} \|((x',u')^{-1}(x,u))\|$$
$$|K_{(\psi F^{(i)})(\sqrt{\mathcal{L}})}((x',u')^{-1}(x,u))|\,|f(x',u')|\,d(x',u')$$
$$\leq C\varepsilon^{-1}\|(y,t)\|\|\psi F^{(i)}\|_{L^\infty_1}\,|B((x,u),\varepsilon^{-1}\|(y,t)\|)|$$
$$\times\frac{1}{|B((x,u),\varepsilon^{-1}\|(y,t)\|)|}\int_{B((x,u),\varepsilon^{-1}\|(y,t)\|)} |f(x',u')|\,d(x',u')$$
$$\leq C2^{i(3/2+\epsilon)}\varepsilon^{-1}\|(y,t)\|\,(\varepsilon^{-1}\|(y,t)\|)^Q\,\mathcal{M}f(x,u)$$
$$\leq C2^{i(3/2+\epsilon)}(\varepsilon^{-1}\|(y,t)\|)^{Q+1}\mathcal{M}f(x,u).$$

For J_{32}, proceeding similarly as in J_{31}, we obtain that

$$|J_{32}| \leq C(\varepsilon^{-1}\|(y,t)\|)^{Q+1}\,\mathcal{M}f(x,u).$$

Thus, combining both J_{31} and J_{32} we get

$$\begin{aligned}|J_3| &\leq C(1+2^{i(1+2\epsilon)})(\varepsilon^{-1}\|(y,t)\|)^{Q+1}\,\mathcal{M}f(x,u)\\ &\leq C_i(\varepsilon^{-1}\|(y,t)\|)^{Q+1}\,\mathcal{M}f(x,u).\end{aligned}$$

For J_4, again we decompose as in J_1 into two parts. Using Proposition 3.4, (5.6), mean value theorem, and left invariance of ϱ, we see that

$$|J_{41}| \leq \int_{\|((x',u')^{-1}(x,u))\|\leq\varepsilon^{-1}\|(y,t)\|} |\mathcal{K}_{(\psi F^{(i)})(\sqrt{\mathcal{L}})}((x',u')^{-1}(x,u)(y,t))|$$
$$\times\,|b((x,u)(y,t))-b(x',u')|\,|f(x',u')|\,d(x',u')$$

$$\leq C \int_{\varrho((x',u'),(x,u))\leq \varepsilon^{-1}\|(y,t)\|} |\mathcal{K}_{(\psi F^{(i)})(\sqrt{\mathcal{L}})}((x',u')^{-1}(x,u)(y,t))| \times \varrho((x,u)(y,t),(x',u'))\, |f(x',u')|\, d(x',u')$$
$$\leq C(\varepsilon^{-1}\|(y,t)\| + \varrho((x,u)(y,t),(x,u))\|\psi F^{(i)}\|_{L^\infty_1} \times \int_{\varrho((x',u'),(x,u))\leq \varepsilon^{-1}\|(y,t)\|} |f(x',u')|\, d(x',u')$$
$$\leq C(\varepsilon^{-1}\|(y,t)\|)^{Q+1} 2^{i(3/2+\epsilon)}\, \mathcal{M}f(x,u).$$

A similar calculation also shows that

$$|J_{42}| \leq (\varepsilon^{-1}\|(y,t)\|)^{Q+1}\, \mathcal{M}f(x,u).$$

Thus, combining both J_{41} and J_{42}, we get

$$|J_4| \leq C_i(\varepsilon^{-1}\|(y,t)\|)^{Q+1}\, \mathcal{M}f(x,u).$$

Now, gathering all the estimates of J_1, J_2, J_3 and J_4, and using the L^p $(p > 1)$ boundedness of $\mathcal{M}$, we see that

$$\int_G \left|[b, F^{(i)}(\sqrt{\mathcal{L}})]f(x,u) - ([b, F^{(i)}(\sqrt{\mathcal{L}})]f)_{B((x,u,)r)}\right|^p d(x,u)$$
$$\leq \int_G \frac{1}{|B(0,r)|^p}\Big(\int_{B(0,r)} \big(|J_1| + |J_2| + |J_3| + |J_4|\big)\, d(y,t)\Big)^p d(x,u)$$
$$\leq C_i\Big(\varepsilon^\tau r^{1-\tau} + (\varepsilon^{-1}r)^{Q+1}\Big)^p \int_G \mathcal{M}f(x,u)^p d(x,u)$$
$$\leq C\Big(\varepsilon^\tau r^{1-\tau} + (\varepsilon^{-1}r)^{Q+1}\Big)^p \int_G |f(x,u)|^p d(x,u).$$

Finally, taking $r < \varepsilon^2$, as $r \to 0$ we conclude that

$$\int_G \left|[b, F^{(i)}(\sqrt{\mathcal{L}})]f(x,u) - ([b, F^{(i)}(\sqrt{\mathcal{L}})]f)_{B((x,u),r)}\right|^p d(x,u) < \epsilon.$$

This shows that $[b, F^{(i)}(\sqrt{\mathcal{L}})]$ satisfies condition (3) of Proposition (5.1). This completes the proof of Theorem 1.5. □

Data availability

No data was used for the research described in this article.

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Conflict of Interest Statement

The authors declare no conflict of interest.

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