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Wold-type decomposition for doubly commuting two-isometries

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ABSTRACT. In this article, we prove that any pair of doubly commuting 2-isometries on a Hilbert space has a Wold-type decomposition. Moreover, the analytic part of the pair is unitary equivalent to the pair of multiplication by the coordinate functions on a Dirichlet-type space on the bidisc.

CONTENTS

1. Introduction

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ denote the set of all bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *concave* if for every $h \in \mathcal{H}$,

$$\|T^2h\|^2 - 2\|Th\|^2 + \|h\|^2 \leq 0. \tag{1}$$

An operator $T \in \mathcal{B}(\mathcal{H})$ is called a 2-*isometry* if for each $h \in \mathcal{H}$, (1) holds with equality. A closed subspace $\mathcal{W} \subseteq \mathcal{H}$ is said to be a *wandering subspace* for T if $\mathcal{W} \perp T^n\mathcal{W}$ for all $n \geq 1$ and an operator $T \in \mathcal{B}(\mathcal{H})$ is said to have *wandering subspace property* if $\mathcal{H} = \bigvee_{n\geq 0} T^n\mathcal{W}$ for some wandering subspace $\mathcal{W}$ of T. In fact, if $T \in \mathcal{B}(\mathcal{H})$ has the wandering subspace property then the wandering subspace $\mathcal{W}$ is uniquely determined and is given by $\mathcal{H} \ominus T\mathcal{H}$. Throughout the article, the subspace $\mathcal{H} \ominus T\mathcal{H}$ shall be denoted by $\mathcal{E}_T$.

A key and highly useful result in the study of bounded linear operators on a Hilbert space is the Wold decomposition theorem. The classical version states

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that any isometry $V \in \mathcal{B}(\mathcal{H})$ (i.e., $V^*V = I_{\mathcal{H}}$) is a direct sum of a unitary and an unilateral shift (see [22] and also see [21, page 3]). The Wold decomposition and results analogous to it have profound impact in Operator theory, Operator algebra and also invariant subspace problem for Hilbert space of holomorphic functions (see [6, 9, 7, 19]). Wold-type decompositions of 2-isometric operators also have been studied by many authors, see [15, 18, 11]. For a 2-isometry $T \in \mathcal{B}(\mathcal{H})$, it is well-known that the subspace $\cap_{n\geq 0} T^n\mathcal{H}$ $(:= \mathcal{H}_0)$ is T-reducing and $T|_{\mathcal{H}_0}$ is unitary (see [11, Proposition 1.1]). The operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *analytic* provided $\mathcal{H}_0 = \{0\}$. In [14], the author proves the following Wold-type decomposition for the class of analytic concave operators.

Theorem 1.1 ([14]). *Let $T \in \mathcal{B}(\mathcal{H})$ be an analytic concave operator. Then $\mathcal{H} \ominus T\mathcal{H}$ is a wandering subspace for T.*

Later, Shimorin extended this result for general bounded left-invertible operators (see [18, Theorem 3.6]). In [11], Olofsson reproves the Wold-type decomposition for the general class of 2-isometries and reveals the connection between analytic 2-isometries and the Dirichlet shift which is a generalization of [15, Theorem 5.1]. For the sake of completeness of this article, we state this result in Theorem 1.2.

Theorem 1.2 ([11]). *Let T be a 2-isometry on $\mathcal{H}$. Then there exists a positive $\mathcal{B}(\mathcal{E}_T)$-valued operator measure μ on the unit circle $\mathbb{T}$ such that*

$$(T, \mathcal{H}) \cong \left(\begin{bmatrix} U & 0 \\ 0 & M_z \end{bmatrix}, \mathcal{H}_0 \oplus \mathcal{D}_{\mathcal{E}_T}(\mu) \right),$$

where $\mathcal{H}_0$ is T-reducing subspace, U is unitary and M_z on $\mathcal{D}_{\mathcal{E}_T}(\mu)$ is an analytic 2-isometry.

For the definition and properties of $\mathcal{D}_{\mathcal{E}_T}(\mu)$ space, we refer the reader to [15, 11, 5].

A natural question which has been of interest to many operator theorists is that 'Can an analogous result be obtained in a multivariate setting?' There are many interesting results for commuting isometries as well as for the class of pairs of commuting analytic 2-isometries with some additional assumptions on the pair (see [20, 9, 17, 3, 4, 1] and references therein). It is presumable that some extra assumptions on pair of commuting 2-isometries are necessary to achieve results analogous to Theorem 1.1 and Theorem 1.2. In this article, we shall consider the class of pairs of doubly commuting 2-isometries. A pair $T = (T_1, T_2)$ of commuting bounded linear operators acting on a Hilbert space $\mathcal{H}$ is said to be *doubly commuting* if $T_i^*T_j = T_jT_i^*$ for $i, j = 1, 2$ with $i \neq j$. The natural examples of doubly commuting pairs are the pairs of the co-ordinate wise multiplication operators on the Hardy space, the Bergman space, and the Dirichlet space over the bidisc $\mathbb{D}^2$ (defined in Section 2 and also see [1]).

A Wold-type decomposition for a pair of doubly commuting isometries is derived in [20] (see also [2, 8, 13, 16]). More precisely,

Theorem 1.3. *(M. Slocinski) Let $V = (V_1, V_2)$ be a pair of doubly commuting isometries on a Hilbert space $\mathcal{H}$. Then there exists a unique decomposition*

$$\mathcal{H} = \mathcal{H}_{ss} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{uu},$$

such that $\mathcal{H}_{ij}$ is joint V-reducing subspace of $\mathcal{H}$ for all $i, j = s, u$, and V_1 on $\mathcal{H}_{ij}$ is a shift if $i = s$ and unitary if $i = u$, and that V_2 is a shift if $j = s$ and unitary if $j = u$.

In Theorem 4.3, we establish a Wold-type decomposition for any pair of doubly commuting 2-isometries acting on a Hilbert space as a generalization of Theorem 1.2. The essence of analytic structure in it also gives a more finer version of Theorem 5.1 in [10]. The novelty of the work, in this article, is uncovering a link between the shifts on Dirichlet spaces and general pairs of doubly commuting 2-isometries that need not be cyclic or analytic. As a corollary of Theorem 4.3, one can derive Theorem 1.3. Before we turn our attention to Theorem 4.3, in Section 2, we note a few notations and preliminaries.

2. Preliminaries

Let μ_1 and μ_2 be two positive $\mathcal{B}(\mathcal{E})$-valued operator measures defined on the unit circle $\mathbb{T}$, where $\mathcal{E}$ is a Hilbert space. For $i = 1, 2$, the Poisson integral of μ_i is denoted by $P[\mu_i]$ and is given by

$$P[\mu_i](z_i) = \int_{\mathbb{T}} P(z_i, e^{i\theta}) d\mu_i(e^{i\theta})$$

where $P(z_i, e^{i\theta}) = \frac{(1-|z_i|^2)}{|e^{i\theta}-z_i|^2}$ for any $z_i \in \mathbb{D}$ and $\theta \in \mathbb{R}$. Note that $P[\mu_i](z_i)$ is a positive operator in $\mathcal{B}(\mathcal{E})$. Let the space of all $\mathcal{E}$-valued analytic functions on the bidisc be denoted by $\mathcal{O}(\mathbb{D}^2, \mathcal{E})$. Now, for any $f \in \mathcal{O}(\mathbb{D}^2, \mathcal{E})$, we define the scalars $D^{(2)}_{\mu_1,\mu_2,i}(f)$ for $i = 1, 2, 3$,

$$D^{(2)}_{\mu_1,\mu_2,1}(f) := \lim_{r\to 1^-} \iint_{\mathbb{D}\times\mathbb{T}} \left\langle P[\mu_1](z_1)\frac{\partial f}{\partial z_1}(z_1, re^{it}), \frac{\partial f}{\partial z_1}(z_1, re^{it})\right\rangle_{\mathcal{E}} \, dt\, dA(z_1)$$

$$D^{(2)}_{\mu_1,\mu_2,2}(f) := \lim_{r\to 1^-} \iint_{\mathbb{T}\times\mathbb{D}} \left\langle P[\mu_2](z_2)\frac{\partial f}{\partial z_2}(re^{it}, z_2), \frac{\partial f}{\partial z_2}(re^{it}, z_2)\right\rangle_{\mathcal{E}} \, dA(z_2)\, dt$$

$$D^{(2)}_{\mu_1,\mu_2,3}(f) := \iint_{\mathbb{D}\times\mathbb{D}} \left\langle P[\mu_1](z_1)P[\mu_2](z_2)\frac{\partial^2 f}{\partial z_1 \partial z_2}(z_1, z_2), \frac{\partial^2 f}{\partial z_1 \partial z_2}(z_1, z_2)\right\rangle_{\mathcal{E}} \, dA(z_1)\, dA(z_2),$$

where dt denotes the normalized arc length measure and dA denotes the normalized area measure. Furthermore, we define the Dirichlet-type integral of a function f on bidisc as

$$D^{(2)}_{\mu_1,\mu_2}(f) := D^{(2)}_{\mu_1,\mu_2,1}(f) + D^{(2)}_{\mu_1,\mu_2,2}(f) + D^{(2)}_{\mu_1,\mu_2,3}(f).$$

Combining all these, the *Dirichlet-type space over bidisc for the operator-valued measures* μ_1, μ_2 on $\mathbb{T}$, denoted by $\mathcal{D}^2_{\mathcal{E}}(\mu_1, \mu_2)$, is defined to be the space of all

$\mathcal{E}$-valued analytic functions f such that $D^{(2)}_{\mu_1,\mu_2}(f)$ is finite. The norm of $f \in \mathcal{D}^2_{\mathcal{E}}(\mu_1, \mu_2)$ is denoted by $\|f\|_{\mu_1,\mu_2}$ and is given by

$$\|f\|^2_{\mu_1,\mu_2} := \|f\|^2_{H^2(\mathbb{D}^2)} + D^{(2)}_{\mu_1,\mu_2,1}(f) + D^{(2)}_{\mu_1,\mu_2,2}(f) + D^{(2)}_{\mu_1,\mu_2,3}(f). \tag{2}$$

The space $\mathcal{D}^2_{\mathcal{E}}(\mu_1, \mu_2)$ turns out to be a Hilbert space with respect to the norm given in (2). In [1], the scalar-valued Dirichlet-type space (that is, when $\mathcal{E} = \mathbb{C}$) is studied in some detail.

Let $\mathbb{Z}_{\geq 0}$ denote the set of all non-negative integers. Throughout the article, for $m, p \in \mathbb{Z}_{\geq 0}$, we adopt $(m \wedge p)$ as a concise notation for $\min\{m, p\}$. For any $f \in \mathcal{D}^2_{\mathcal{E}}(\mu_1, \mu_2)$ with $f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{m,n} z_1^m z_2^n$, the following identities can easily be derived (see [1] for scalar-valued case).

$$\|f\|^2_{H^2(\mathbb{D}^2)} = \sum_{m,n=0}^{\infty} \|a_{m,n}\|^2$$

$$D^{(2)}_{\mu_1,\mu_2,1}(f) = \lim_{r\to 1^-} \sum_{m,p=1}^{\infty} \sum_{n=0}^{\infty} (m \wedge p)\langle \widehat{\mu_1}(p-m) a_{m,n}, a_{p,n}\rangle r^{2n} \tag{3}$$

$$D^{(2)}_{\mu_1,\mu_2,2}(f) = \lim_{r\to 1^-} \sum_{m=0}^{\infty} \sum_{n,q=1}^{\infty} (n \wedge q)\langle \widehat{\mu_2}(n-q) a_{m,n}, a_{m,q}\rangle r^{2m} \tag{4}$$

$$D^{(2)}_{\mu_1,\mu_2,3}(f) = \sum_{m,n=1}^{\infty} \sum_{p,q=1}^{\infty} (m \wedge p)(n \wedge q)\langle \widehat{\mu_2}(q-n)\widehat{\mu_1}(p-m) a_{m,n}, a_{p,q}\rangle, \tag{5}$$

where n^{th}-Fourier coefficient of the measure μ is defined by

$$\widehat{\mu}(n) := \int_{\mathbb{T}} \overline{x}^n \, d\mu(x).$$

Let $M_z = (M_{z_1}, M_{z_2})$, where M_{z_i} corresponds to the multiplication by the co-ordinate function z_i for $i = 1, 2$. Note that, the operators M_{z_1} and M_{z_2} are bounded linear operators on $\mathcal{D}^2_{\mathcal{E}}(\mu_1, \mu_2)$. Moreover, we have the following theorem. Since the proof of the theorem is similar to that of the scalar-valued case, which is available in [1, Theorem 3.12], we omit it.

Theorem 2.1. *For any $f \in \mathcal{O}(\mathbb{D}^2, \mathcal{E})$, the following statements are equivalent.*

(1) $f \in \mathcal{D}^2_{\mathcal{E}}(\mu_1, \mu_2)$
(2) $z_1 f \in \mathcal{D}^2_{\mathcal{E}}(\mu_1, \mu_2)$
(3) $z_2 f \in \mathcal{D}^2_{\mathcal{E}}(\mu_1, \mu_2)$.

We also list out some of the properties of the vector-valued Dirichlet-type spaces in the following theorem. We do not provide the proofs since it is verbatim with the scalar-valued case. For more details, we refer the reader to [1].

Theorem 2.2. *Suppose μ_1 and μ_2 are two positive $\mathcal{B}(\mathcal{E})$-valued operator measures with $\mathcal{E}$ being a Hilbert space. Then*

(a) *the set of all polynomials in two variables with coefficients from $\mathcal{E}$ is a dense set in $\mathcal{D}^2_{\mathcal{E}}(\mu_1, \mu_2)$.*
(b) *the multiplication operators M_{z_1}, M_{z_2} are analytic 2-isometries on $\mathcal{D}^2_{\mathcal{E}}(\mu_1, \mu_2)$.*

3. von Neumann Wold-type decomposition for analytic pair

In this section, we deal with a pair of doubly commuting analytic 2-isometries on a Hilbert space and completely characterize them in terms of shifts on Dirichlet-type spaces over the bidisc.

We begin with a pair of commuting analytic 2-isometries (T_1, T_2) defined on a Hilbert space $\mathcal{H}$. Using the definition of 2-isometry, it is straightforward to show that T_1 and T_2 are left-invertible operators and we consider the left-inverses $L_i := (T_i^*T_i)^{-1}T_i^*$ for $i = 1, 2$. Also, note that for each $i = 1, 2$, the operator $P_i := I - T_iL_i$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{E}_i := \mathcal{H} \ominus T_i(\mathcal{H}) = \ker T_i^*$. The defect operator corresponding to T_i is $D_i := (T_i^*T_i - I)^{1/2}$, the corresponding defect space is $\mathcal{D}_i := \overline{D_i(\mathcal{H})}$ and $P_{\mathcal{D}_i}$ is the orthogonal projection onto $\mathcal{D}_i$. For this pair (T_1, T_2) on $\mathcal{H}$, we introduce the map $\tilde{T}_i : \mathcal{D}_i \to \mathcal{D}_i$, defined by $D_ix \mapsto D_iT_ix$, for each $i = 1, 2$. Note that, $D_iT_ix \in \mathcal{D}_i$ and using the 2-isometric property of each T_i on $\mathcal{H}$ for each $x \in \mathcal{H}$ we have

$$||D_iT_ix||^2 = ||T_i^2x||^2 - ||T_ix||^2 = ||T_ix||^2 - ||x||^2 = ||D_ix||^2.$$

It not only shows that the maps are well-defined but also proves that they are isometries. Therefore it has a unitary extension on some Hilbert space $\mathcal{K}$. Moreover, using spectral theorem, for each integer $m \geq 0$, we have

$$\tilde{T}_i^{\,m} = \int_{\mathbb{T}} e^{im\theta} dE_i(e^{i\theta}),$$

where $E_i : \Omega \to \mathcal{B}(\mathcal{K})$ denotes the spectral measure corresponding to the unitary extension of $\tilde{T}_i$. A direct application of Theorem 1.2 (see also [11, Theorem 4.1]) gives the existence of positive $\mathcal{B}(\mathcal{E}_i)$-valued operator measures μ_i defined on $\mathbb{T}$ by

$$\mu_i(\sigma) = P_iD_iP_{\mathcal{D}_i}E_i(\sigma)D_i|_{\mathcal{E}_i} \tag{6}$$

such that T_i is unitarily equivalent to M_z on $\mathcal{D}_{\mathcal{E}_i}(\mu_i)$. For more details, we refer [11] to the reader.

If the pair (T_1, T_2) is doubly commuting, then we have the following identity

$$(I - T_1L_1)(I - T_2L_2) = (I - T_2L_2)(I - T_1L_1).$$

That is, the orthogonal projections $P_1 = I - T_1L_1$ and $P_2 = I - T_2L_2$ are commutative. Hence, the product $P := P_1P_2$ is also an orthogonal projection onto $\mathcal{E} := \ker T_1^* \cap \ker T_2^*$ and due to [1, Lemma 4.1], the subspace $\mathcal{E}$ is a non-trivial subspace of $\mathcal{H}$. We consider for $i = 1, 2$, $\tilde{\mu}_i(\sigma) := PD_iP_{\mathcal{D}_i}E_i(\sigma)D_i|_{\mathcal{E}}$ which turns out to be positive $\mathcal{B}(\mathcal{E})$-valued operator measures defined on $\mathbb{T}$.

In the following lemma, we recall an important operator identity for analytic 2-isometry, proved in [14].

Lemma 3.1. *Let $S \in \mathcal{B}(\mathcal{H})$ be an analytic 2-isometry. Then the operators $L_S := (S^*S)^{-1}S^*$, $P_S = I - SL_S$ and $D_S = (S^*S - I)^{\frac{1}{2}}$ satisfy*

$$\|x\|^2 = \sum_{k=0}^{\infty} \|P_S L_S^k x\|^2 + \sum_{k=1}^{\infty} \|D_S L_S^k x\|^2 \qquad \text{for all } x \in \mathcal{H}. \tag{7}$$

The map V in the following proposition is motivated by [11] and serves as an isometry from $\mathcal{H}$ onto $\mathcal{D}^2_{\mathcal{E}}(\tilde{\mu_1}, \tilde{\mu_2})$. This proposition plays a crucial role in this article.

Proposition 3.2. *Let (T_1, T_2) be a pair of analytic doubly commuting 2-isometries on a Hilbert space $\mathcal{H}$. Then the maps $\tilde{\mu_1}$ and $\tilde{\mu_2}$, defined by*

$$\tilde{\mu_i}(\sigma) = PD_iP_{\mathcal{D}_i}E_i(\sigma)D_iP$$

for $i = 1, 2$, are positive $\mathcal{B}(\mathcal{E})$-valued operator measures on $\mathbb{T}$ and $V : \mathcal{H} \to \mathcal{D}^2_{\mathcal{E}}(\tilde{\mu_1}, \tilde{\mu_2})$ is a unitary, where

$$Vx(z_1, z_2) := \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(PL_1^mL_2^nx)z_1^mz_2^n. \tag{8}$$

Proof. By hypothesis for $i = 1, 2$, each T_i is analytic 2-isometry on $\mathcal{H}$, and therefore by Theorem 1.2, we get positive $\mathcal{B}(\mathcal{E}_i)$-valued operator measures say μ_1, μ_2 and unitary maps V_i from $\mathcal{H}$ onto $\mathcal{D}(\mu_i)$ defined by

$$(V_ix)(z) = \sum_{m\geq 0}(P_iL_i^mx)z^m$$

such that $V_iT_i = M_{z_i}V_i$ for $i = 1, 2$. From (7), for each $i = 1, 2$ and $x \in \mathcal{H}$, we have

$$\|x\|^2 = \sum_{k=0}^{\infty}\|P_iL_i^kx\|^2 + \sum_{k=1}^{\infty}\|D_iL_i^kx\|^2. \tag{9}$$

Consider the equation (9) for $i = 1$, replace x by $P_2L_2^nx$ and then by taking sum over $n \in \mathbb{Z}_{\geq 0}$, we get

$$\sum_{n=0}^{\infty}\|P_2L_2^nx\|^2 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\|PL_1^mL_2^nx\|^2 + \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\|D_1L_1^mP_2L_2^nx\|^2. \tag{10}$$

Similarly, x is replaced with $D_2L_2^nx$ in the equation (9) for $i = 1$ and then taking sum over $n \in \mathbb{N}$, we get

$$\sum_{n=1}^{\infty}\|D_2L_2^nx\|^2 = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\|P_1L_1^mD_2L_2^nx\|^2 + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\|D_1L_1^mD_2L_2^nx\|^2. \tag{11}$$

Let D denote the operator D_1D_2 and $L^{m,n}$ represent the operator $L_1^mL_2^n$. Using (10) and (11) in (9), we obtain

$$\|x\|^2 = \sum_{\substack{m=0\\n=0}}^{\infty} \|PL^{m,n}x\|^2 + \sum_{\substack{m=1\\n=0}}^{\infty} \|D_1P_2L^{m,n}x\|^2 + \sum_{\substack{m=0\\n=1}}^{\infty} \|P_1D_2L^{m,n}x\|^2 + \sum_{\substack{m=1\\n=1}}^{\infty} \|DL^{m,n}x\|^2. \tag{12}$$

Moreover, by hypothesis, the pair (T_1, T_2) is doubly commuting and hence by [1, Theorem 4.3], the pair has the wandering subspace property. Therefore, the given Hilbert space can be written as

$$\mathcal{H} = \bigvee_{m,n\geq 0} T_1^mT_2^n(\mathcal{E}).$$

With the help of two positive measures μ_1 on $\mathcal{B}(\mathcal{E}_1)$ and μ_2 on $\mathcal{B}(\mathcal{E}_2)$ given in (6), we define two $\mathcal{B}(\mathcal{E})$-valued measures $\tilde{\mu_1}, \tilde{\mu_2}$ on $\mathbb{T}$ by the rule

$$\tilde{\mu_i}(\sigma) = PD_iP_{\mathcal{D}_i}E_i(\sigma)D_iP, \qquad i = 1, 2.$$

It is easy to deduce that $\tilde{\mu_1}(\sigma) = P_2\mu_1(\sigma)P_2$ and $\tilde{\mu_2}(\sigma) = P_1\mu_2(\sigma)P_1$. Here, for each $i = 1, 2$, $\mu_i(\sigma)$ is a positive operator and $\tilde{\mu_i}(\sigma)$ is a compression of $\mu_i(\sigma)$ for every σ. Thus both the measures $\tilde{\mu_1}$ and $\tilde{\mu_2}$ are positive. We consider the Dirichlet-type space $\mathcal{D}^2_{\mathcal{E}}(\tilde{\mu_1}, \tilde{\mu_2})$ corresponding to the measures $\tilde{\mu_1}, \tilde{\mu_2}$ and as well as the map defined in (8).

Then for any x in $\mathcal{H}$ of the form $\sum_{m,n=0}^{k} T_1^mT_2^na_{m,n} \in \mathcal{H}$ with $a_{m,n} \in \mathcal{E}$ and for all $(z_1, z_2) \in \mathbb{D}^2$, we have

$$(Vx)(z_1, z_2) = \sum_{m,n=0}^{k} \sum_{k_1,k_2=0}^{\infty} PL_1^{k_1}L_2^{k_2}T_1^mT_2^na_{m,n}z_1^{k_1}z_2^{k_2} = \sum_{m,n=0}^{k} a_{m,n}z_1^mz_2^n.$$

Note that, $\|Vx\|^2_{H^2(\mathbb{D}^2)} = \sum_{m,n=0}^{\infty} \|PL^{m,n}x\|^2$ for each $x \in \mathcal{H}$. Also, using the doubly commuting property of the pair (T_1, T_2) on $\mathcal{H}$ and representation of isometry $\tilde{T_1}$, we derive for each $x \in \mathcal{H}$ and $n_1, n_2 \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned} D_1P_2L^{n_1,n_2}x &= \sum_{m,n=0}^{\infty} D_1L_1^{n_1}P_2L_2^{n_2}T_1^mT_2^nx_{m,n} \\ &= \sum_{m=n_1}^{\infty} D_1T_1^{m-n_1}x_{m,n_2} \\ &= \sum_{m=0}^{\infty} \int_{\mathbb{T}} e^{im\theta_1}dE_1(e^{i\theta_1})D_1x_{m+n_1,n_2}. \end{aligned}$$

Using the above identity, we compute the norm

$$\|D_1P_2L^{n_1,n_2}x\|^2 = \sum_{j_1,k_1=0}^{\infty} \langle P_1D_1P_{\mathcal{D}_1}\int_{\mathbb{T}} e^{i(j_1-k_1)\theta_1} dE_1(e^{i\theta_1})D_1x_{j_1+n_1,n_2}, x_{k_1+n_1,n_2}\rangle$$
$$= \sum_{j_1,k_1=n_1}^{\infty} \langle \widehat{\mu_1}(k_1-j_1)x_{j_1,n_2}, x_{k_1,n_2}\rangle.$$

Combining all the above identities, we derive

$$\sum_{n_2=0}^{\infty}\sum_{n_1=1}^{\infty} \|D_1P_2L^{n_1,n_2}x\|^2 = \sum_{n_2=0}^{\infty}\sum_{j_1,k_1=1}^{\infty}(j_1\wedge k_1)\langle \widehat{\mu_1}(k_1-j_1)x_{j_1,n_2}, x_{k_1,n_2}\rangle. \quad (13)$$

Similarly, we can derive

$$\sum_{n_1=0}^{\infty}\sum_{n_2=1}^{\infty} \|D_2P_1L^{n_1,n_2}x\|^2 = \sum_{n_1=0}^{\infty}\sum_{j_2,k_2=1}^{\infty}(j_2\wedge k_2)\langle \widehat{\mu_2}(k_2-j_2)x_{n_1,j_2}, x_{n_1,k_2}\rangle. \quad (14)$$

Following a similar calculation, we also get

$$DL^{n_1,n_2}x = \sum_{m,n=0}^{\infty} D_1L_1^{n_1}D_2L_2^{n_2}T_1^mT_2^nx_{m,n}$$
$$= \sum_{m,n=0}^{\infty}\Big(\int_{\mathbb{T}} e^{im\theta_1}dE_1(e^{i\theta_1})\Big)D_1\Big(\int_{\mathbb{T}} e^{in\theta_2}dE_2(e^{i\theta_2})\Big)D_2x_{m+n_1,n+n_2}.$$

Using the above identity, we calculate the norm

$$\|DL^{n_1,n_2}x\|^2 = \sum_{j_1,k_1=n_1}^{\infty}\sum_{j_2,k_2=n_2}^{\infty}\langle \widehat{\mu_1}(k_1-j_1)\widehat{\mu_2}(k_2-j_2)x_{j_1,j_2}, x_{k_1,k_2}\rangle.$$

Therefore, we have

$$\sum_{n_1,n_2=1}^{\infty}\|DL^{n_1,n_2}x\|^2 =$$
$$\sum_{j_1,k_1=1}^{\infty}\sum_{j_2,k_2=1}^{\infty}(j_1\wedge k_1)(j_2\wedge k_2)\langle \widehat{\mu_1}(k_1-j_1)\widehat{\mu_2}(k_2-j_2)x_{j_1,j_2}, x_{k_1,k_2}\rangle. \quad (15)$$

Following the description of the term $D^{(2)}_{\tilde{\mu}_1,\tilde{\mu}_2,1}(Vx)$ in (3) and comparing it with the equation (13), we deduce that

$$D^{(2)}_{\tilde{\mu}_1,\tilde{\mu}_2,1}(Vx) = \lim_{r\to 1^-}\sum_{n=0}^{\infty}\sum_{m,p=1}^{\infty}(m\wedge p)\langle \widehat{\mu_1}(p-m)x_{m,n}, x_{p,n}\rangle r^{2n}$$
$$= \lim_{r\to 1^-}\sum_{n_2=0}^{\infty}\sum_{n_1=1}^{\infty}\|D_1P_2L^{n_1,n_2}x\|^2r^{2n_2},$$

and note that for each fixed $r \in (0, 1)$,

$$\sum_{n_1=1}^{\infty} \|D_1 P_2 L^{n_1,n_2} x\|^2 r^{2n_2} < \sum_{n_1=1}^{\infty} \|D_1 P_2 L^{n_1,n_2} x\|^2.$$

Thus, with the help of the Dominated Convergence Theorem, we can interchange the limit and the sum. Therefore, it shows that

$$D^{(2)}_{\tilde{\mu}_1,\tilde{\mu}_2,1}(Vx) = \sum_{n_2=0}^{\infty} \sum_{n_1=1}^{\infty} \|D_1 P_2 L^{n_1,n_2} x\|^2. \tag{16}$$

Similarly, we obtain the following identities by comparing the equations (4) with (14), and (5) with (15)

$$D^{(2)}_{\tilde{\mu}_1,\tilde{\mu}_2,2}(Vx) = \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{\infty} \|P_1 D_2 L^{n_1,n_2} x\|^2 \text{ and} \tag{17}$$

$$D^{(2)}_{\tilde{\mu}_1,\tilde{\mu}_2,3}(Vx) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \|D L^{n_1,n_2} x\|^2. \tag{18}$$

The equations (16), (17), (18) put together in (12) establishes the fact that V is an isometry.

Since, by [1, Lemma 3.14], the set of polynomials is dense in the space $\mathcal{D}^2_{\mathcal{E}}(\tilde{\mu}_1, \tilde{\mu}_2)$, it is enough to show that for any polynomial $p \in \mathcal{D}^2_{\mathcal{E}}(\mu_1, \mu_2)$, with the expression $p(z_1, z_2) = \sum_{m,n=0}^{k} a_{m,n} z_1^m z_2^n$, where $a_{m,n} \in \mathcal{E}$, there exists an element $x \in \mathcal{H}$ such that $Vx = p$. Using the hypothesis of this proposition, for any $(z_1, z_2) \in \mathbb{D}^2$, we get that

$$\begin{aligned} V\Big(\sum_{m,n=0}^{k} T_1^m T_2^n a_{m,n}\Big)(z_1, z_2) &= \sum_{m,n=0}^{k} \sum_{k_1,k_2=0}^{\infty} P L_1^{k_1} L_2^{k_2} T_1^m T_2^n a_{m,n} z_1^{k_1} z_2^{k_2} \\ &= \sum_{m,n=0}^{k} a_{m,n} z_1^m z_2^n. \end{aligned}$$

Therefore, $x = \sum_{m,n=0}^{k} T_1^m T_2^n a_{m,n}$ satisfies the equation $Vx = p$. Since the map V is an isometry, it has closed range. Hence V maps $\mathcal{H}$ onto $\mathcal{D}^2_{\mathcal{E}}(\tilde{\mu}_1, \tilde{\mu}_2)$. This completes the proof. $\square$

The following theorem gives a complete characterization of the class of pair of doubly commuting analytic 2-isometries acting on a Hilbert space.

Theorem 3.3. *Let (T_1, T_2) be a pair of analytic doubly commuting 2-isometries on a Hilbert space $\mathcal{H}$. Then (T_1, T_2) on $\mathcal{H}$ is unitarily equivalent with (M_{z_1}, M_{z_2}) on $\mathcal{D}^2_{\mathcal{E}}(\tilde{\mu}_1, \tilde{\mu}_2)$, where $\tilde{\mu}_i(\sigma) = PD_i P_{\mathcal{D}_i} E_i(\sigma) D_i P$ for $i = 1, 2$, are positive $\mathcal{B}(\mathcal{E})$-valued operator measures, where $\mathcal{E} = \ker T_1^* \cap \ker T_2^*$.*

Proof. In the previous proposition, we proved the map $V : \mathcal{H} \to \mathcal{D}^2_{\mathcal{E}}(\tilde{\mu_1}, \tilde{\mu_2})$, defined above, is unitary. Therefore, it is enough to prove that $VT_i = M_{z_i}V$, for $i = 1, 2$. For $z_1, z_2 \in \mathbb{D}$, we obtain

$$\begin{aligned}(VT_1x)(z_1, z_2) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(PL_1^mL_2^nT_1x)z_1^mz_2^n = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}(PL_1^{m-1}L_2^nx)z_1^mz_2^n \\ &= z_1\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}(PL_1^{m-1}L_2^nx)z_1^{m-1}z_2^n \\ &= M_{z_1}(Vx)(z_1, z_2).\end{aligned}$$

Similarly, it can be verified that $VT_2 = M_{z_2}V$.

This guarantees that (T_1, T_2) on $\mathcal{H}$ is unitarily equivalent to (M_{z_1}, M_{z_2}) on $\mathcal{D}^2_{\mathcal{E}}(\tilde{\mu_1}, \tilde{\mu_2})$. □

Remark 3.4. Following [1, Remark 5.2], it can be shown that if the pair (M_{z_1}, M_{z_2}) on $\mathcal{D}^2_{\mathcal{E}_\eta}(\eta_1, \eta_2)$ is unitarily equivalent to the pair (M_{z_1}, M_{z_2}) on $\mathcal{D}^2_{\mathcal{E}_\beta}(\beta_1, \beta_2)$ for some finite positive operator-valued Borel measures η_1, η_2, β_1 and β_2 on $\mathbb{T}$ with the joint kernels $\mathcal{E}_\eta$ and $\mathcal{E}_\beta$ of $(M^*_{z_1}, M^*_{z_2})$ on $\mathcal{D}^2_{\mathcal{E}_\eta}(\eta_1, \eta_2)$ and $\mathcal{D}^2_{\mathcal{E}_\beta}(\beta_1, \beta_2)$ respectively, then there exists a unitary operator $U : \mathcal{E}_\eta \to \mathcal{E}_\beta$ such that $\eta_i(\sigma) = U^*\beta_i(\sigma)U$ for all Borel subsets σ of the unit circle $\mathbb{T}$.

4. von Neumann Wold type decomposition for general pair

The main purpose of this section is to develop a von Neumann Wold-type decomposition for the class of doubly commuting 2-isometries (T_1, T_2) acting on a Hilbert space $\mathcal{H}$. The key idea is to use the Wold-type decomposition for a single 2-isometry which we recall first for the convenience of the reader. Given a 2-isometry T on a Hilbert space $\mathcal{H}$, in [18] Shimorin shows that the operator T can be decomposed in a direct sum of unitary operator and an analytic 2-isometry. More precisely,

Theorem 4.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be a 2-isometry. Then the space $\mathcal{H}$ admits an unique decomposition of the form $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, where the subspaces $\mathcal{H}_0$ is the maximal T-reducing subspace of $\mathcal{H}$ such that $T|_{\mathcal{H}_0}$ is unitary. Moreover,*

$$\mathcal{H}_0 = \bigcap_{n\geq 0} T^n(\mathcal{H}) \quad \text{and} \quad \mathcal{H}_1 = \bigvee_{n\geq 0} T^n(\mathcal{E}_T)$$

where $\mathcal{E}_T = \mathcal{H} \ominus T(\mathcal{H})$ is the wandering subspace for T.

Before proceeding further, we note down some important properties for the pair of doubly commuting 2-isometric operators. We skip the proof, as these are straightforward properties to verify.

Lemma 4.2. *Let (T_1, T_2) be a pair of doubly commuting 2-isometries on $\mathcal{H}$. Then for $i, j \in \{1, 2\}$ with $i \neq j$,*

(a) *$kerT_i^*$ is T_j-reducing subspace of $\mathcal{H}$.*

(b) *For each $m \geq 0$, $T_i^m\Big(\bigcap_{n\geq 0} T_j^n \mathcal{E}_i\Big) = \Big(T_i^m(\mathcal{E}_i)\Big) \cap \Big(\bigcap_{n\geq 0} T_j^n \mathcal{H}\Big)$.*

In the following theorem, we get an analogue of Theorem 4.1. More precisely, we present the decomposition of the Hilbert space $\mathcal{H}$ corresponding to a pair of doubly commuting 2-isometries acting on $\mathcal{H}$. The presence of analytic structure in this result can be seen as an improved version of [10, Theorem 5.1] in this set-up.

Theorem 4.3. *Let (T_1, T_2) be a pair of doubly commuting 2-isometries on a Hilbert space $\mathcal{H}$. Then there exist positive Borel measures ν_1, ν_2, η_1 and η_2 on $\mathbb{T}$ such that*

$$\mathcal{H} \cong \mathcal{H}_{00} \oplus \mathcal{D}_{\mathcal{E}_{10}}(\nu_1) \oplus \mathcal{D}_{\mathcal{E}_{01}}(\nu_2) \oplus \mathcal{D}^2_{\mathcal{E}}(\eta_1, \eta_2)$$

where $\mathcal{E}_{10} = \cap_{n\geq 0} T_2^n\, kerT_1^*$, $\mathcal{E}_{01} = \cap_{m\geq 0} T_1^m\, kerT_2^*$ and $\mathcal{E} = kerT_1^* \cap kerT_2^*$. Moreover,

$$T_1 \cong \begin{bmatrix} U_0 & 0 & 0 & 0 \\ 0 & M_z & 0 & 0 \\ 0 & 0 & U_1 & 0 \\ 0 & 0 & 0 & M_{z_1} \end{bmatrix} \quad and \quad T_2 \cong \begin{bmatrix} V_0 & 0 & 0 & 0 \\ 0 & V_1 & 0 & 0 \\ 0 & 0 & M_z & 0 \\ 0 & 0 & 0 & M_{z_2} \end{bmatrix}$$

where U_0 and V_0 are unitary on $\mathcal{H}_{00} = \cap_{m,n\geq 0} T_1^m T_2^n \mathcal{H}$, U_1 and V_1 are unitary on $\mathcal{D}_{\mathcal{E}_{01}}(\nu_2)$ and $\mathcal{D}_{\mathcal{E}_{10}}(\nu_1)$, respectively.

Proof. By hypothesis, T_1 is a 2-isometry on $\mathcal{H}$. Then, by applying Theorem 4.1, the Hilbert space $\mathcal{H}$ can be decomposed into two T_1-reducing subspaces, that is

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$$

where $\mathcal{H}_0 = \bigcap_{m\geq 0} T_1^m(\mathcal{H})$ and $\mathcal{H}_1 = \bigvee_{m\geq 0} T_1^m(\mathcal{E}_1)$ with $\mathcal{E}_1 = kerT_1^*$. Since the pair (T_1, T_2) on $\mathcal{H}$ is doubly commutative, so by Lemma 4.2, both the subspaces $\mathcal{H}_0$ and $\mathcal{H}_1$ are T_2-reducing. Thus an application of Theorem 4.1 to the operator $T_2|_{\mathcal{H}_0}$ on $\mathcal{H}_0$ yields the decomposition of $\mathcal{H}_0$ into two T_2-reducing subspaces. That is,

$$\begin{aligned} \mathcal{H}_0 &= \bigcap_{n\geq 0} T_2^n(\mathcal{H}_0) \oplus \bigvee_{n\geq 0} T_2^n(\mathcal{H}_0 \ominus T_2\mathcal{H}_0) \\ &= \bigcap_{m,n\geq 0} T_1^m T_2^n \mathcal{H} \oplus \bigvee_{n\geq 0} T_2^n\Big(\bigcap_{m\geq 0} T_1^m(\mathcal{H} \ominus T_2\mathcal{H})\Big), \end{aligned}$$

where the last equality follows from the doubly commuting property of the pair (T_1, T_2). We write

$$\mathcal{H}_{00} := \bigcap_{m,n\geq 0} T_1^m T_2^n \mathcal{H}, \qquad \mathcal{H}_{01} := \bigvee_{n\geq 0} T_2^n\Big(\bigcap_{m\geq 0} T_1^m \mathcal{E}_2\Big).$$

Note that, for $i = 1, 2$, following Theorem 4.1, the subspace $\cap_{m\geq 0} T_i^m \mathcal{H}$ is T_i-reducing and $T_i|_{\cap_{m\geq 0} T_i^m \mathcal{H}}$ is unitary. Since $\mathcal{H}_{00} \subseteq \cap_{m\geq 0} T_i^m \mathcal{H}$ is a T_i-reducing subspace for any $i = 1, 2$, therefore $T_1|_{\mathcal{H}_{00}}$ and $T_2|_{\mathcal{H}_{00}}$ both are unitary on $\mathcal{H}_{00}$. Also, by Lemma 4.2, we know that the space

$$\mathcal{H}_{01} = \Big(\bigvee_{n\geq 0} T_2^n \mathcal{E}_2\Big) \cap \Big(\bigcap_{m\geq 0} T_1^m \mathcal{H}\Big).$$

That is, $\mathcal{H}_{01}$ is the intersection of two joint (T_1, T_2)-reducing subspaces of $\mathcal{H}$, therefore it is also a joint (T_1, T_2)-reducing subspace. Along with that, since $\mathcal{H}_{01} \subseteq \bigcap_{m\geq 0} T_1^m \mathcal{H} = \mathcal{H}_0$ and $T_1|_{\mathcal{H}_0}$ is a unitary operator, so $T_1|_{\mathcal{H}_{01}}$ is also a unitary operator.

Now we apply Theorem 4.1 for the operator $T_2|_{\mathcal{H}_1}$ on $\mathcal{H}_1$. That yields the decomposition of $\mathcal{H}_1$ into two T_2-reducing subspaces. That is, by using the hypothesis on the pair we have

$$\begin{aligned}\mathcal{H}_1 &= \bigcap_{n\geq 0} T_2^n(\mathcal{H}_1) \oplus \bigvee_{n\geq 0} T_2^n(\mathcal{H}_1 \ominus T_2\mathcal{H}_1)\\ &= \bigvee_{m\geq 0} T_1^m\Big(\bigcap_{n\geq 0} T_2^n \mathcal{E}_1\Big) \oplus \bigvee_{m,n\geq 0} T_1^m T_2^n(\mathcal{E}_1 \cap \mathcal{E}_2).\end{aligned}$$

We denote $\mathcal{H}_{10} = \bigvee_{m\geq 0} T_1^m\big(\bigcap_{n\geq 0} T_2^n \mathcal{E}_1\big)$ and $\mathcal{H}_{11} = \bigvee_{m,n\geq 0} T_1^m T_2^n(\mathcal{E}_1 \cap \mathcal{E}_2)$. Again by using Lemma 4.2, we have $\mathcal{H}_{10} = \big(\bigvee_{m\geq 0} T_1^m \mathcal{E}_1\big) \cap \big(\bigcap_{n\geq 0} T_2^n \mathcal{H}\big)$. Being intersection of two joint (T_1, T_2)-reducing subspace, the space $\mathcal{H}_{10}$ is also a joint (T_1, T_2)-reducing subspace of $\mathcal{H}$. Since $\mathcal{H}_{10} \subseteq \bigcap_{n\geq 0} T_2^n \mathcal{H}$, so $T_2|_{\mathcal{H}_{10}}$ is an unitary. Since $\mathcal{H}_{11} = \mathcal{H}_1 \ominus \mathcal{H}_{10}$, so $\mathcal{H}_{11}$ is also a joint (T_1, T_2)-reducing subspace. Moreover, the pair $(T_1|_{\mathcal{H}_{11}}, T_2|_{\mathcal{H}_{11}})$ is a doubly commuting pair of analytic 2-isometries on $\mathcal{H}_{11}$. Combining all the above decompositions we have

$$\mathcal{H} = \mathcal{H}_{00} \oplus \mathcal{H}_{10} \oplus \mathcal{H}_{01} \oplus \mathcal{H}_{11}$$

and for $\alpha_1, \alpha_2 \in \{0, 1\}$, T_i on $\mathcal{H}_{\alpha_1,\alpha_2}$ is unitary if $\alpha_i = 0$ for each $i = 1, 2$. From the above decomposition of the Hilbert space $\mathcal{H}$, we know $T_1|_{\mathcal{H}_{10}}$ is also a 2-isometry on $\mathcal{H}_{10}$. And, by Lemma 4.2(b), we note that

$$T_1^m\Big(\bigcap_{n\geq 0} T_2^n \mathcal{E}_1\Big) = \Big(T_1^m(\mathcal{E}_1)\Big) \cap \Big(\bigcap_{n\geq 0} T_2^n \mathcal{H}\Big)$$

for each $m \geq 0$. Now, using the fact that $\mathcal{E}_1$ is a wandering subspace for T_1 on $\mathcal{H}$, we conclude that $\cap_{n\geq 0} T_2^n \mathcal{E}_1$ is a wandering subspace for $T_1|_{\mathcal{H}_{10}}$ on $\mathcal{H}_{10}$ and the description of the subspace $\mathcal{H}_{10}$ helps to conclude that $T_1|_{\mathcal{H}_{10}}$ is an analytic 2-isometric operator on $\mathcal{H}_{10}$. Hence, by applying Theorem 1.2, we can immediately conclude that

$$(T_1|_{\mathcal{H}_{10}}, \mathcal{H}_{10}) \cong (M_z, \mathcal{D}_{\mathcal{E}_{10}}(\nu_1))$$

for some positive Borel measure ν_1 on $\mathbb{T}$ with $\mathcal{E}_{10} = \cap_{n\geq 0} T_2^n \mathcal{E}_1$. Analogously, we can also prove that the subspace $\mathcal{E}_{01} = \cap_{m\geq 0} T_1^m \, ker T_2^*$ is a wandering subspace for $T_2|_{\mathcal{H}_{01}}$ and $(T_2|_{\mathcal{H}_{01}}, \mathcal{H}_{01}) \cong (M_z, \mathcal{D}_{\mathcal{E}_{01}}(\nu_2))$ for some positive $\mathcal{B}(\mathcal{E}_{01})$-valued measure ν_2 on $\mathbb{T}$. Moreover, as shown above, the pair $(T_1|_{\mathcal{H}_{11}}, T_2|_{\mathcal{H}_{11}})$ is doubly commuting analytic 2-isometries on $\mathcal{H}_{11}$, therefore, by Theorem 3.3, we have

$(T_1|_{\mathcal{H}_{11}}, T_2|_{\mathcal{H}_{11}})$ on $\mathcal{H}_{11}$ is unitary equivalent with (M_{z_1}, M_{z_2}) on $\mathcal{D}^2_{\mathcal{E}}(\eta_1, \eta_2)$ for some positive measures η_1 and η_2 on $\mathbb{T}$ with $\mathcal{E} = \mathcal{E}_1 \cap \mathcal{E}_2$. □

Remark 4.4. For a pair of doubly commuting 2-isometries (T_1, T_2) acting on a Hilbert space $\mathcal{H}$, the decomposition in Theorem 4.3 is unique in the following sense: for $\alpha_1, \alpha_2, i, j \in \{0, 1\}$, if the subspace $\mathcal{K}_{\alpha_i,\alpha_j}$ is reducing for T_i such that $\mathcal{H} = \mathcal{K}_{00} \oplus \mathcal{K}_{10} \oplus \mathcal{K}_{01} \oplus \mathcal{K}_{11}$ with the properties that the restriction of T_i to $\mathcal{K}_{\alpha_i,\alpha_j}$ is unitary and analytic for $\alpha_i = 0$ and $\alpha_i = 1$ respectively, then $\mathcal{K}_{00} = \mathcal{H}_{00}, \mathcal{K}_{10} = \mathcal{H}_{10}, \mathcal{K}_{01} = \mathcal{H}_{01}$ and $\mathcal{K}_{11} = \mathcal{H}_{11}$. Moreover the measures ν_1, ν_2, η_1 and η_2 in Theorem 4.3 are determined uniquely upto a unitary (see Remark 3.4).

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