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Non-weight representations for the Drinfeld double of the Jordan plane

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Abstract. This paper studies non-weight representations of the Drinfeld double $\mathcal{D}$ of the bosonization of the Jordan plane. We give a complete classification of simple Whittaker, twisted Whittaker, and quasi-Whittaker $\mathcal{D}$-modules and determine their annihilators explicitly.

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1. Introduction

The representation theory of Hopf algebras lies at the intersection of quantum algebra, category theory, and noncommutative geometry. Among the central constructions in this area is the Drinfeld double, which produces new quasi-triangular Hopf algebras from existing ones and captures deep symmetries in quantum groups. The Drinfeld double $\mathcal{D}$ of the bosonization of the Jordan plane is a Hopf algebra first introduced by Andruskiewitsch and Peña Pollastri in [AP21]. This algebra exhibits several striking properties that make it an object of interest from multiple perspectives. It serves as a nontrivial and illuminating example of a pointed Hopf algebra with rich algebraic structure and representation theory. The Drinfeld doubles of the super Jordan plane and the restricted Jordan plane were studied in [AP22] and [ABDF], respectively. Both the Jordan plane and the super Jordan plane serve as pivotal examples in the study of Nichols algebras over abelian groups with finite Gelfand–Kirillov dimension [AAH].

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Finite-dimensional simple $\mathcal{D}$-modules were classified, and the centre of $\mathcal{D}$ was computed in [ADP]. A remarkable feature of the representation theory of $\mathcal{D}$, established in [ADP, Theorem 3.11], is that the finite-dimensional simple $\mathcal{D}$-modules are in bijective correspondence with those of $U(\mathfrak{sl}_2)$. Furthermore, certain indecomposable modules and the Gabriel quiver of $\mathcal{D}$ were studied in [AP25]. The prime and primitive spectra of $\mathcal{D}$ were determined in [BS], where it was also shown that $\mathcal{D}$ satisfies the Dixmier–Moeglin equivalence. Weight $\mathcal{D}$-modules were investigated in [Lu], where a classification was obtained for simple weight modules possessing at least one finite-dimensional weight space. Naturally, the next step is to study simple weight modules with infinite-dimensional weight spaces and simple non-weight modules. In this paper, we study non-weight representations of the Drinfeld double of the Jordan plane, with a particular focus on Whittaker, twisted Whittaker, and quasi-Whittaker modules. Our aim is to provide a complete classification of simple non-weight modules of these types and to determine their annihilators explicitly.

A particularly important class of non-weight modules is given by Whittaker modules, originally introduced in the context of complex semisimple Lie algebras by Kostant [Kos]. The definition of Whittaker modules is closely tied to the triangular decomposition of a Lie algebra $\mathfrak{g}$, where one typically imposes a character condition on the action of the nilpotent (or 'positive') part. Assuming the character $\eta \in (\mathfrak{n}/[\mathfrak{n},\mathfrak{n}])^*$ of the nilradical of a Borel subalgebra of $\mathfrak{g}$ is nondegenerate, Kostant [Kos] established a bijection between the isomorphism classes of Whittaker modules and the ideals of the centre of $U(\mathfrak{g})$. In the degenerate case, the structure of Whittaker modules becomes more intricate. This situation was further explored by McDowell [Mc85, Mc93]. A geometric approach to the study of the category of Whittaker modules was developed by Miličić and Soergel [MS97, MS14]. Whittaker modules break the semisimplicity imposed by weight decompositions and have since been extended to a wide array of algebras that admit triangular decompositions, including non-semisimple Lie algebras [Chr, OW], quantum groups [Ond, Sev], Lie superalgebras [BCW, Ch], and generalized Weyl algebras [BO]. A general categorical framework for Whittaker modules was proposed in [BM].

For the algebra $\mathcal{D}$, see Definition 2.1, we exploit two distinct triangular decompositions, each giving rise to a corresponding class of Whittaker-type modules. The first leads to the classical notion of Whittaker modules, while the second gives rise to a broader class, which we refer to as twisted Whittaker modules. These twisted Whittaker modules strictly contain the Whittaker modules as special cases and offer a more flexible framework capable of capturing a broader range of non-weight representations. From a structural perspective, the concept of local finiteness provides a natural unifying framework for understanding Whittaker-type modules. While the definition of (twisted) Whittaker modules typically involves the existence of a cyclic vector satisfying certain eigenvalue conditions with respect to a positive subalgebra, for simple $\mathcal{D}$-modules, such properties are guaranteed by the local finiteness of the module over the corresponding subalgebra. This perspective not only streamlines the identification of Whittaker-type modules but also motivates natural generalizations. In particular, by requiring local finiteness over other well-chosen subalgebras (such as the commutative polynomial subalgebra $\Bbbk[x,u]$), we are led to the notion of quasi-Whittaker modules, which extend the scope of Whittaker-type theory beyond the setting of triangular decompositions.

Primitive ideals, as the annihilators of simple modules, serve as a crucial link between the ring-theoretic structure of an algebra and its representation theory. Brown and Stafford [BS] provided a complete description of the prime and primitive spectra of $\mathcal{D}$. A natural next step is to associate each primitive ideal P of $\mathcal{D}$ with a canonical simple module whose annihilator is precisely P. It was shown in [Lu] that a distinguished primitive ideal, denoted $\mathfrak{m}_+\mathcal{D}$, does not appear as the annihilator of any simple weight $\mathcal{D}$-module with a finite-dimensional weight space. In fact, if a simple weight module has annihilator equal to the ideal $\mathfrak{m}_+\mathcal{D}$, then all of its weight spaces must be infinite-dimensional. In this paper, we show that every primitive ideal of $\mathcal{D}$ arises as the annihilator of a simple Whittaker module. In particular, the distinguished primitive ideal $\mathfrak{m}_+\mathcal{D}$ naturally occurs as the annihilator of certain such modules.

Let us now briefly describe the content of the paper. In Section 2, we recall the definition and basic structural properties of the Hopf algebra $\mathcal{D}$. We also review the classification of simple Whittaker modules for the universal enveloping algebra $U(\mathfrak{sl}_2)$, which appears naturally as a quotient of $\mathcal{D}$. In Section 3, we classify all simple Whittaker modules over $\mathcal{D}$ and explicitly determine their annihilators. The classification distinguishes between the cases $\alpha = 0$ and $\alpha \neq 0$, where $(0,\alpha)$ is the type of the Whittaker module. In Section 4, we define and classify simple twisted Whittaker modules, based on a different triangular decomposition of $\mathcal{D}$. This broader class of non-weight modules includes all Whittaker modules as special cases. In Section 5, we introduce quasi-Whittaker modules, defined as cyclic modules generated by vectors satisfying eigenvalue conditions with respect to the commutative subalgebra $\Bbbk[x,u] \subset \mathcal{D}$. We prove that a simple $\mathcal{D}$-module is quasi-Whittaker if and only if it is locally finite over $\Bbbk[x,u]$. A complete classification of simple quasi-Whittaker modules is given, along with an explicit description of their annihilators.

In this paper, a module means a left module, $\Bbbk$ is an algebraically closed field of characteristic zero, $\Bbbk^* := \Bbbk \setminus \{0\}$. We denote by $\mathbb{Z}$ and $\mathbb{N}$ the set of integers and non-negative integers, respectively.

2. Preliminaries

In this section, we recall some basic facts about the Drinfeld double $\mathcal{D}$ of the bosonization of the Jordan plane. We also review the classification of simple Whittaker modules over the universal enveloping algebra of $\mathfrak{sl}_2$, which will be used in the sections that follow.

2.1. The Hopf algebra $\mathcal{D}$. The quadratic algebra

$$\mathtt{R} = \Bbbk\langle x,\ y \mid yx - xy = -\frac{1}{2}x^2\rangle,$$

commonly known as the Jordan plane, is a braided Hopf algebra where x and y are primitive. The Hopf algebra $\mathcal{D}$ is the Drinfeld double of the bosonization of the Jordan plane. It was first introduced in [AP21] and further studied in [ADP, AP25, BS, Lu]. The algebra $\mathcal{D}$ is a Noetherian domain with Gelfand–Kirillov dimension 6, and its definition is recalled below.

Definition 2.1. The Hopf algebra $\mathcal{D}$ is generated by $h, u, v, g^{\pm 1}, x$, and y subject to the following defining relations

$$[h,u] = -u, \quad [h,v] = -v, \quad [v,u] = -\frac{1}{2}u^2;$$

$$gg^{-1} = g^{-1}g = 1, \quad gxg^{-1} = x, \quad gyg^{-1} = y + x, \quad [y,x] = -\frac{1}{2}x^2;$$

$$\begin{aligned} &[h,g] = 0, && [u,g] = 0, && [v,g] = gu, \\ &[h,x] = x, && [u,x] = 0, && [v,x] = (1-g) + xu, \\ &[h,y] = y, && [u,y] = 1-g, && [v,y] = -gh + yu. \end{aligned}$$

The coalgebra structure is determined by specifying that g is grouplike, u and h are primitive, x and y are $(g,1)$-primitive, and $\Delta(v) = v \otimes 1 + 1 \otimes v + h \otimes u$.

In Definition 2.1, we have replaced the generator ζ with h, differing from the notation used in [ADP, AP21, BS]. Notice that both the subalgebras generated by $\{u,\ v\}$ and $\{x,\ y\}$ are isomorphic to the Jordan plane. Let J denote the (Hopf) subalgebra of $\mathcal{D}$ generated by h, u, and v. Then J is a bosonization of the Jordan plane, and $\mathcal{D}$ can be regarded as the Drinfeld double of J. In the literature, J is also known as the κ-deformed quantum Galilei group, see e.g. [GGKM]. The quantum Galilei group, constructed by F. Bonechi, et al. [BGST], is a central extension of J. Let K denote the (Hopf) subalgebra of $\mathcal{D}$ generated by $g^{\pm 1}$, x, and y. As noted in [AP21, Lemma 2.2], there is a non-degenerate skew Hopf pairing between J and K, which determines the multiplication relations between these subalgebras. The set $\{u^i v^j g^k h^\ell x^m y^n \mid i, j, \ell, m, n \in \mathbb{N},\ k \in \mathbb{Z}\}$ is a PBW basis of $\mathcal{D}$. In fact, $\mathcal{D}$ admits an ascending filtration with associated graded algebra $\operatorname{gr}\mathcal{D} \cong \Bbbk[x_1, \dots, x_5, x_6^{\pm 1}]$, a commutative polynomial algebra in 6 variables with one variable inverted. The adjoint action of h on $\mathcal{D}$ induces a $\mathbb{Z}$-grading on $\mathcal{D}$. Specifically, we have the decomposition $\mathcal{D} = \bigoplus_{i \in \mathbb{Z}} \mathcal{D}_i$, where $\mathcal{D}_i = \{d \in \mathcal{D} \mid [h,d] = id\}$. By [ADP, Remark 2.2], the Hopf algebra $\mathcal{D}$ is pointed.

The following map $\tau : \mathcal{D} \to \mathcal{D}$ defines an automorphism of the algebra $\mathcal{D}$:

$$\begin{aligned} \tau(h) &= -h, \quad \tau(u) = x, \quad \tau(v) = -hx - y, \\ \tau(g) &= g, \qquad \tau(x) = u, \quad \tau(y) = hu - v. \end{aligned} \tag{1}$$

Furthermore, we have $\tau^2 = \mathrm{id}_{\mathcal{D}}$.

2.2. The centre of $\mathcal{D}$. We recall the description of the centre of $\mathcal{D}$ from [ADP, Theorem 4.10]. Define the elements

$$q := ux + 2(1+g), \qquad s := xv + uy + (-\frac{1}{2}ux + g - 1)h - 2(g+1). \tag{2}$$

Recall that an element r in a ring R is normal if $rR = Rr$. The elements q and s are normal in the algebra $\mathcal{D}$. In fact, they commute with the generators u, x, g and h, and obey the following relations: $qy = (y + \frac{1}{2}x)q$, $qv = (v - \frac{1}{2}u)q$; $sy = (y + \frac{1}{2}x)s$, and $sv = (v - \frac{1}{2}u)s$. By [ADP, Theorem 4.10], the following elements are central in $\mathcal{D}$:

$$z := q^2 g^{-1}, \quad \omega := qsg^{-1}, \quad \theta := s^2 g^{-1}.$$

Furthermore, the centre of $\mathcal{D}$ is isomorphic to the commutative algebra $Z(\mathcal{D}) \cong \Bbbk[z,\, \omega,\, \theta]/\langle z\theta - \omega^2\rangle$. Applying the automorphism τ defined in (1), we find that $\tau(q) = q$ and $\tau(s) = -s - 2q$. It follows that

$$\tau(z) = z, \quad \tau(\omega) = -\omega - 2z, \quad \tau(\theta) = \theta + 4z + 4\omega.$$

We label the maximal ideals of $Z(\mathcal{D})$ following the notation in [BS]. Specifically, the maximal spectrum of $Z(\mathcal{D})$ is given by

$$\begin{aligned} &\{\mathfrak{m}_{(\dot\omega,\dot\theta)} := \langle z - \dot\omega^2/\dot\theta,\, \omega - \dot\omega,\, \theta - \dot\theta\rangle \mid \dot\omega \in \Bbbk,\, \dot\theta \in \Bbbk^*\} \\ &\cup \{\mathfrak{m}_{\dot z} := \langle z - \dot z,\, \omega,\, \theta\rangle \mid \dot z \in \Bbbk\}. \end{aligned}$$

The singular locus of $Z(\mathcal{D})$ is $\{\mathfrak{m}_0 = \langle z,\, \omega,\, \theta\rangle\}$. One particularly noteworthy maximal ideal of $Z(\mathcal{D})$ is $\mathfrak{m}_+ := \mathfrak{m}_{(-16,16)} = \langle z - 16,\, \omega + 16,\, \theta - 16\rangle$, which requires special attention in the study of $\mathcal{D}$-modules. Note that both $\mathfrak{m}_0$ and $\mathfrak{m}_+$ are τ-invariant.

2.3. A quotient isomorphic to $U(\mathfrak{sl}_2)$. The subalgebra $\mathcal{O} := \Bbbk[g^{\pm 1}, x, u]$ is a commutative Hopf subalgebra of $\mathcal{D}$. Note that $\mathcal{O}$ is isomorphic to the algebra $\mathcal{O}(\mathbf{G})$ of regular functions on the algebraic group $\mathbf{G} := (\mathbf{G}_a \times \mathbf{G}_a) \rtimes \mathbf{G}_m$. Let $U(\mathfrak{sl}_2)$ be the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2 = \Bbbk H \oplus \Bbbk E \oplus \Bbbk F$, with Lie brackets $[H,E] = 2E$, $[H,F] = -2F$, and $[E,F] = H$. As shown in [ADP, AP21], there is an exact sequence of Hopf algebras: $\mathcal{O} \hookrightarrow \mathcal{D} \twoheadrightarrow U(\mathfrak{sl}_2)$, so that $U(\mathfrak{sl}_2)$ appears as a quotient Hopf algebra of $\mathcal{D}$. More precisely, by [AP21, Proposition 2.7], the ideal $\langle x,\, 1-g,\, u\rangle$ of $\mathcal{D}$ is a Hopf ideal, and the quotient algebra $\mathcal{U} := \mathcal{D}/\langle x,\, 1-g,\, u\rangle$ is isomorphic to $U(\mathfrak{sl}_2)$. From the defining relations of $\mathcal{D}$, it follows that $\mathcal{U} \cong \Bbbk\langle h,\, y,\, v \mid [h,y] = y,\, [h,v] = -v,\, [y,v] = h\rangle$. It is convenient to fix the following isomorphism of algebras:

$$\mathcal{U} \xrightarrow{\sim} U(\mathfrak{sl}_2), \quad h \mapsto \frac{1}{2}H, \quad y \mapsto \frac{1}{2}E, \quad v \mapsto F. \tag{3}$$

According to [Lu, Lemma 2.2], the ideal $\langle x,\, 1-g,\, u\rangle$ can also be presented as a left ideal:

$$\langle x,\, 1-g,\, u\rangle = \mathcal{D}x + \mathcal{D}(1-g) + \mathcal{D}u \tag{4}$$

The centre of $\mathcal{U}$ is given by $Z(\mathcal{U}) = \Bbbk[\Omega]$, where $\Omega := 2vy + h^2 + h$ is the Casimir element.

2.4. Simple Whittaker modules over $\mathfrak{sl}_2$. We will need the classification of simple Whittaker modules over $\mathfrak{sl}_2$. Let M be a $\mathcal{U}$-module. The module M is called a Whittaker module of type $\alpha \in \Bbbk$ if it is generated by an element $w \in M$ satisfying $yw = \alpha w$. The module M is called a highest weight module if it is generated by an element $w \in M$ satisfying $hw = \lambda w$ and $yw = 0$ for some $\lambda \in \Bbbk$.

Theorem 2.2. *Let M be a simple Whittaker $\mathcal{U}$-module of type α.*

(1) *If $\alpha \in \Bbbk^*$, then M is isomorphic to $M_{\alpha,\chi} := \mathcal{U}/\mathcal{U}(y-\alpha,\, \Omega-\chi)$ for a unique $\chi \in \Bbbk$. Moreover, for any $\chi \in \Bbbk$, the module $M_{\alpha,\chi}$ is simple, whose annihilator is given by $\mathrm{ann}_{\mathcal{U}} M_{\alpha,\chi} = \langle \Omega - \chi\rangle$.*
(2) *If $\alpha = 0$, then M is a simple highest weight module.*

Proof. (1) The result can be found in, e.g., [ArP, Theorem 2.5, Theorem 2.6], or [Kos, Theorem 3.6.1].

(2) The statement follows from [ArP, Proposition 2.11]. □

For an algebra R, let $\widehat{R}$ denote the set of isomorphism classes of simple R-modules. Given an R-module M, we use $[M]$ to denote its isomorphism class. If $\mathcal{P}$ is a property of modules that is invariant under module isomorphisms, we define $\widehat{R}(\mathcal{P})$ to be the subset of $\widehat{R}$ consisting of all isomorphism classes of simple R-modules satisfying $\mathcal{P}$. For instance, we denote by $\widehat{\mathcal{U}}(\text{Whittaker-}\alpha)$ the set of isomorphism classes of simple Whittaker $\mathcal{U}$-modules of type α, and by $\widehat{\mathcal{U}}(\text{highest weight})$ the set of isomorphism classes of simple highest weight $\mathcal{U}$-modules. By Theorem 2.2, we have

$$\widehat{\mathcal{U}}(\text{Whittaker-}\alpha) = \begin{cases} \{[M_{\alpha,\chi}] \mid \chi \in \Bbbk\}, & \text{if } \alpha \in \Bbbk^*, \\ \widehat{\mathcal{U}}(\text{highest weight}), & \text{if } \alpha = 0. \end{cases} \tag{5}$$

3. Whittaker modules

In this section, we classify all simple Whittaker $\mathcal{D}$-modules and explicitly determine their annihilators. Remarkably, every primitive ideal of $\mathcal{D}$ arises as the annihilator of some simple Whittaker module.

3.1. Whittaker modules and local finiteness. Let $\mathcal{D}^+$ be the subalgebra of $\mathcal{D}$ generated by x and y, and let $\mathcal{D}^-$ be the subalgebra of $\mathcal{D}$ generated by u and v. Additionally, let $\mathcal{D}^0$ denote the subalgebra of $\mathcal{D}$ generated by $g^{\pm 1}$ and h. Then $\mathcal{D}$ admits the following triangular decomposition

$$\mathcal{D} = \mathcal{D}^- \otimes \mathcal{D}^0 \otimes \mathcal{D}^+.$$

Notice that both $\mathcal{D}^+$ and $\mathcal{D}^-$ are isomorphic to the Jordan plane. An algebra homomorphism $\eta : \mathcal{D}^+ \to \Bbbk$ is called a character of $\mathcal{D}^+$. Note that the characters of $\mathcal{D}^+$ are in bijection with the maximal ideals of $\mathcal{D}^+$ of codimension one. It is easy to see that if η is a character of $\mathcal{D}^+$, then $\eta(x) = 0$ and $\eta(y) = \alpha$ for some $\alpha \in \Bbbk$. Therefore, there is a natural bijection:

$$\{\text{characters of } \mathcal{D}^+\} \longrightarrow \Bbbk, \quad \eta \mapsto \eta(y).$$

Let M be a $\mathcal{D}$-module, and let $\eta : \mathcal{D}^+ \to \Bbbk$ be a character of $\mathcal{D}^+$. An element $w \in M$ is called a Whittaker vector of type η if it satisfies $ew = \eta(e)w$ for all $e \in \mathcal{D}^+$. If η is given by $\eta(x) = 0$ and $\eta(y) = \alpha$, we also refer to w as a Whittaker vector of type $(0, \alpha)$, by abuse of language. A module M is called a Whittaker module of type $(0, \alpha)$ if it is generated by a Whittaker vector of type $(0, \alpha)$. Recall that an R-module M is called locally finite if every element $m \in M$ is contained in a finite-dimensional R-submodule. The following lemma provides a sufficient condition for a simple $\mathcal{D}$-module to be a Whittaker module.

Lemma 3.1. *Let M be a simple $\mathcal{D}$-module. If M is $\mathcal{D}^+$-locally finite then M is a Whittaker module.*

Proof. Let $w_0 \in M$ be a nonzero vector. Then there exists a finite-dimensional $\mathcal{D}^+$-submodule $N \subseteq M$ such that $w_0 \in N$. Since $\Bbbk$ is algebraically closed, the linear operator $y : N \to N$ has an eigenvalue $\alpha \in \Bbbk$, and hence there exists a nonzero vector $w \in N$ such that $(y - \alpha)w = 0$. From the relation $[y, x] = -1/2x^2$, it follows that x acts nilpotently on the finite-dimensional space N. Therefore, there exists a smallest $n \in \mathbb{N}$ such that $\xi := x^n w \neq 0$ but $x^{n+1}w = 0$. Then we have $x\xi = 0$ and $(y-\alpha)\xi = 0$. Therefore, ξ is a Whittaker vector of type $(0, \alpha)$. Since M is simple and $\xi \neq 0$, we conclude that $M = \mathcal{D}\xi$, and hence M is a Whittaker module. □

3.2. The Whittaker modules $Y_{\alpha,\beta}$. For any $\alpha \in \Bbbk$, consider the associated one-dimensional $\mathcal{D}^+$-module $\Bbbk_\alpha := \mathcal{D}^+/\mathcal{D}^+(x,\ y - \alpha)$. Define the induced module $Y_\alpha := \mathcal{D} \otimes_{\mathcal{D}^+} \Bbbk_\alpha$. Then Y_α is a Whittaker module of type $(0, \alpha)$. It is clear that $Y_\alpha \cong \mathcal{D}/\mathcal{D}(x,\ y-\alpha)$. Moreover, Y_α has the following universal property: If V is a Whittaker $\mathcal{D}$-module of type $(0, \alpha)$, then V is an epimorphic image of Y_α. To analyze the structure of simple Whittaker modules, we introduce an auxiliary family of modules: For $\alpha \in \Bbbk$ and $\beta \in \Bbbk^*$, define

$$Y_{\alpha,\beta} := \mathcal{D}/\mathcal{D}(x,\ y - \alpha,\ g - \beta). \tag{6}$$

The following lemma gives the motivation for introducing the modules $Y_{\alpha,\beta}$.

Lemma 3.2. *If M is a simple Whittaker $\mathcal{D}$-module of type $(0, \alpha)$, then M is an epimorphic image of $Y_{\alpha,\beta}$ for some $\beta \in \Bbbk^*$.*

Proof. Suppose M is generated by a Whittaker vector w of type $(0, \alpha)$. Then $xw = 0$ and $(y - \alpha)w = 0$. By the expression of q given in (2), we have $qw = 2(1 + g)w$. Since $\mathcal{D}$ is a Jacobson ring that satisfies the Nullstellensatz (see [BS, Theorem 4.3]), any central element of $\mathcal{D}$ acts on M via scalar multiplication. In particular, the central element $z = q^2 g^{-1}$ acts as a scalar, say μ, on M. Therefore, $0 = (z-\mu)w = [4(1+g)^2g^{-1}-\mu]w$. Since $\Bbbk$ is algebraically closed, this implies that there is a Whittaker vector $\xi \in M$ of type $(0,\alpha)$ such that ξ is also an eigenvector of g, say $g\xi = \beta\xi$ for some $\beta \in \Bbbk^*$. The simplicity of M then implies that M is an epimorphic image of the module $Y_{\alpha,\beta}$. □

Let $\bar{1} = 1 + \mathcal{D}(x,\ y-\alpha,\ g-\beta)$ (the coset of 1) be the canonical generator of $Y_{\alpha,\beta}$. Then from the PBW basis of $\mathcal{D}$, we have $Y_{\alpha,\beta} = J\,\bar{1}$, and $Y_{\alpha,\beta}$ is a free module over J of rank one. As observed in [LX], the subalgebra $J = \Bbbk\langle h,\ u,\ v\rangle \subset \mathcal{D}$ is isomorphic to the universal enveloping algebra of a three-dimensional solvable Lie algebra. Specifically, define

$$v' := hu - 2v.$$

Then J is isomorphic to the universal enveloping algebra of the Lie algebra $\mathfrak{a} = \Bbbk h \oplus \Bbbk u \oplus \Bbbk v'$ with Lie bracket relations: $[h,u] = -u$, $[h,v'] = -v'$ and $[u,v'] = 0$. Since $Y_{\alpha,\beta}$ is a free module over J, we have the explicit basis

$$Y_{\alpha,\beta} = \bigoplus_{i,j,k\in\mathbb{N}} \Bbbk v'^i u^j h^k\,\bar{1}. \tag{7}$$

3.3. The Whittaker modules $Y_{\alpha,\beta,\gamma}$ where $\beta \neq 1$. The structure of $Y_{\alpha,\beta}$ differs depending on whether $\beta = 1$ or $\beta \neq 1$. We intend to determine the simple quotients of $Y_{\alpha,\beta}$ when $\beta \neq 1$. First, it can be verified by induction that, for all integers $n \geq 1$,

$$[x, v'^n] = \sum_{i=1}^{n} \binom{n}{i} u^{i-1} v'^{n-i}(q - 2^{i+1}g); \tag{8}$$

$$[y, u^n] = n(g-1)u^{n-1}. \tag{9}$$

The following lemma provides a description of the submodules of $Y_{\alpha,\beta}$ for $\beta \in \Bbbk^*$ and $\beta \neq 1$, and further characterizes its simple quotients and their annihilators.

Lemma 3.3. *Let $\alpha \in \Bbbk$, $\beta \in \Bbbk^*$ and $\beta \neq 1$. Then any submodule M of $Y_{\alpha,\beta}$ is generated by an element of the form $p(s)\,\bar{1}$ for some polynomial $p(s) \in \Bbbk[s]$. Moreover, M is maximal if and only if it is generated by $(s-\gamma)\,\bar{1}$ for some $\gamma \in \Bbbk$. The corresponding simple quotient $Y_{\alpha,\beta}/\mathcal{D}(s-\gamma)\,\bar{1}$ is isomorphic to the module*

$$Y_{\alpha,\beta,\gamma} := \mathcal{D}/\mathcal{D}(x,\ y-\alpha,\ g-\beta,\ s-\gamma).$$

Define $\dot{z} := 4(1+\beta)^2\beta^{-1}$, $\dot{\omega} := 2(1+\beta)\gamma\beta^{-1}$, and $\dot{\theta} := \gamma^2\beta^{-1}$. The annihilator of $Y_{\alpha,\beta,\gamma}$ is given by

$$\operatorname{ann}_{\mathcal{D}} Y_{\alpha,\beta,\gamma} = \begin{cases} \mathfrak{m}_{(\dot{\omega},\dot{\theta})}\mathcal{D}, & \text{if } \gamma \neq 0, \\ \mathfrak{m}_{\dot{z}}\mathcal{D}, & \text{if } \beta \neq -1 \text{ and } \gamma = 0, \\ q\mathcal{D} + s\mathcal{D}, & \text{if } \beta = -1 \text{ and } \gamma = 0. \end{cases}$$

Proof. By the expressions of q and s, we have $q\,\bar{1} = 2(1+\beta)\,\bar{1}$ and $s\,\bar{1} = [\alpha u + (\beta-1)h-(\beta+3)]\,\bar{1}$. Since $\beta \neq 1$, it follows that $h\,\bar{1} = (\beta-1)^{-1}(s-\alpha u+\beta+3)\,\bar{1}$. Then from (7) we obtain

$$Y_{\alpha,\beta} = \bigoplus_{i,j,k\in\mathbb{N}} \Bbbk v'^i u^j s^k\,\bar{1}. \tag{10}$$

Let w be a nonzero element in M. By (10), we can express w as

$$w = \sum_{i=0}^{n} v'^i f_i(u,s)\,\bar{1},$$

where $f_i(u,s) \in \Bbbk[u,s]$. We claim that $f_i(u,s)\,\bar{1} \in M$ for all $0 \le i \le n$. Suppose not, we may assume that $f_n(u,s)\,\bar{1} \notin M$. Define the v'-degree of w as $\deg_{v'}(w) := n$. Applying (8), we obtain

$$xw = \sum_{i=1}^{n}\sum_{j=1}^{i} \tbinom{i}{j} v'^{i-j}u^{j-1}(q-2^{j+1}g)f_i(u,s)\,\bar{1} = 2(1-\beta)nv'^{n-1}f_n(u,s)\,\bar{1} + \cdots$$

where the three dots represent terms of lower v'-degree. Since $\beta \neq 1$, if $n > 0$ then xw is a nonzero element in M with v'-degree $n-1$. Repeating this process, we obtain $x^n w = 2^n(1-\beta)^n n! f_n(u,s)\,\bar{1}$. Thus $f_n(u,s)\,\bar{1} \in M$, contradicting our assumption. Similarly, for any $\mathtt{w} \in \Bbbk[u,s]\,\bar{1} \cap M$, we can write $\mathtt{w} = \sum_{i=0}^{n} u^i c_i(s)\bar{1}$, where $c_i(s) \in \Bbbk[s]$. We claim that $c_i(s)\,\bar{1} \in M$ for all $0 \le i \le n$. Suppose not, we may assume $c_n(s)\,\bar{1} \notin M$, and define $\deg_u(\mathtt{w}) := n$. Using (9), we compute:

$$(y-\alpha)\mathtt{w} = (\beta-1)\sum_{i=1}^{n} iu^{i-1}c_i(s)\,\bar{1}.$$

Since $\beta \neq 1$, if $n > 0$ then $(y-\alpha)\mathtt{w}$ is a nonzero element in M with u-degree $n-1$. Repeating this process, we obtain $(y-\alpha)^n\mathtt{w} = (\beta-1)^n n! c_n(s)\,\bar{1}$. Thus $c_n(s)\,\bar{1} \in M$, contradicting our assumption. In particular, if M is a nonzero submodule of $Y_{\alpha,\beta}$, then $M \cap \Bbbk[s]\,\bar{1} \neq 0$.

Now, define $k = \min\{\deg\psi(s) \mid \psi(s) \in \Bbbk[s]\setminus\{0\},\ \text{and } \psi(s)\,\bar{1} \in M\}$. Choose a monic polynomial $p(s) \in \Bbbk[s]$ such that $\deg p(s) = k$ and $p(s)\,\bar{1} \in M$. The above argument shows that for any $w = \sum_{i,j} v'^i u^j \psi_{i,j}(s)\,\bar{1} \in M$ one has $\psi_{i,j}(s)\,\bar{1} \in M$ for all i,j. By the definition of $p(s)$, it follows that $p(s) \mid \psi_{i,j}(s)$ for all i,j. Therefore, M is a cyclic module generated by $p(s)\,\bar{1}$.

Clearly, if M is a maximal submodule of $Y_{\alpha,\beta}$, then $p(s)$ must be irreducible. Since $\Bbbk$ is algebraically closed, we have $p(s) = s-\gamma$ for some $\gamma \in \Bbbk$. Conversely, let M be the submodule of $Y_{\alpha,\beta}$ generate by $(s-\gamma)\,\bar{1}$ where $\gamma \in \Bbbk$. Suppose that M is contained in some proper submodule N. By the above argument, N is generated by $p(s)\,\bar{1}$ for some monic polynomial $p(s) \in \Bbbk[s]$. Furthermore, the inclusion $(s-\gamma)\,\bar{1} \in N$ implies that $p(s) \mid s-\gamma$. Hence $p(s) = s-\gamma$, and thus $N = M$, proving that M is maximal in $Y_{\alpha,\beta}$. Clearly, the quotient $Y_{\alpha,\beta}/M$ is isomorphic to the module $Y_{\alpha,\beta,\gamma}$.

It remains to determine the annihilator of $Y_{\alpha,\beta,\gamma}$. First, it is straightforward to verify that the central elements $z - \dot{z}$, $\omega - \dot{\omega}$, and $\theta - \dot{\theta}$ annihilate $Y_{\alpha,\beta,\gamma}$. Therefore, the ideal $\mathfrak{p} := (z - \dot{z})\mathcal{D} + (\omega - \dot{\omega})\mathcal{D} + (\theta - \dot{\theta})\mathcal{D}$ is contained in $\mathrm{ann}_{\mathcal{D}} Y_{\alpha,\beta,\gamma}$. Moreover, since $\beta \neq 1$, we have $\dot{z} \neq 16$, ensuring that $\mathfrak{p} \neq \mathfrak{m}_+\mathcal{D}$. Next, consider the following three cases: (i) If $\gamma \neq 0$, then $\dot{\theta} \neq 0$, and by [BS, Theorem 4.1(i)], the ideal $\mathfrak{p}$ is maximal in $\mathcal{D}$. This forces $\mathrm{ann}_{\mathcal{D}} Y_{\alpha,\beta,\gamma} = \mathfrak{p}$. (ii) If $\beta \neq -1$ and $\gamma = 0$, then $\dot{z} \in \Bbbk^*$ and $\dot{\omega} = \dot{\theta} = 0$. Again by [BS, Theorem 4.1(i)], the ideal $\mathfrak{p}$ is maximal in $\mathcal{D}$, implying $\mathrm{ann}_{\mathcal{D}} Y_{\alpha,\beta,\gamma} = \mathfrak{p}$. (iii) If $\beta = -1$ and $\gamma = 0$, then q and s annihilate the canonical generator of $Y_{\alpha,\beta,\gamma}$. Since q and s are normal in $\mathcal{D}$, the ideal $P_0 := q\mathcal{D} + s\mathcal{D}$ annihilates $Y_{\alpha,\beta,\gamma}$. By [BS, Theorem 4.13], the ideal P_0 is maximal in $\mathcal{D}$, and therefore $\mathrm{ann}_{\mathcal{D}} Y_{\alpha,\beta,\gamma} = P_0$. □

Remark 3.4. Let $\beta \in \Bbbk^*$ with $\beta \neq 1$. The module $Y_{\alpha,\beta,\gamma}$ is a free $\Bbbk[u, v']$-module of rank one whose annihilator is independent of α. When $\alpha = 0$, it becomes isomorphic to the simple weight module $\mathcal{D}/\mathcal{D}(x, y, g - \beta, h - \lambda)$ where $\lambda = \frac{\gamma+\beta+3}{\beta-1}$, corresponding to the reduced Verma module in [Lu, Proposition 3.2]. The family $Y_{\alpha,\beta,\gamma}$ thus interpolates between weight and non-weight $\mathcal{D}$-modules.

Proposition 3.5. *Let $\alpha, \alpha' \in \Bbbk$, $\beta, \beta' \in \Bbbk^* \setminus \{1\}$, and $\gamma, \gamma' \in \Bbbk$. Then $Y_{\alpha,\beta,\gamma} \cong Y_{\alpha',\beta',\gamma'}$ if and only if $\alpha' = \alpha$, $\beta' = \beta$, and $\gamma' = \gamma$.*

Proof. The sufficiency is immediate. We prove the necessity. Let $\bar{\mathbf{1}} = 1 + \mathcal{D}(x, y - \alpha, g - \beta, s - \gamma)$ be the canonical generator of $Y_{\alpha,\beta,\gamma}$. Then $Y_{\alpha,\beta,\gamma} = \Bbbk[u, v']\bar{\mathbf{1}}$ is a free $\Bbbk[u, v']$-module of rank one. Set $Y := Y_{\alpha,\beta,\gamma}$ and $Y' := Y_{\alpha',\beta',\gamma'}$.

For a $\mathcal{D}$-module M, write $\ker_M(x) := \{m \in M \mid xm = 0\}$. Since g and s commute with x, and $xyw = (yx + \frac{1}{2}x^2)w = 0$ for every $w \in \ker_Y(x)$, the subspace $\ker_Y(x)$ is stable under g, s, and y.

We first determine $\ker_Y(x)$. Clearly $\Bbbk[u]\bar{\mathbf{1}} \subseteq \ker_Y(x)$. Conversely, let $w = \sum_{i=0}^{n} f_i(u)v'^i\bar{\mathbf{1}} \in \ker_Y(x)$ where $f_i(u) \in \Bbbk[u]$ and $f_n(u) \neq 0$. If $n > 0$, then by (8),

$$\begin{aligned} 0 = xw =& \sum_{i=1}^{n}\sum_{j=1}^{i} \tbinom{i}{j}(2(1+\beta) - 2^{j+1}\beta) f_i(u)u^{j-1}v'^{i-j}\,\bar{\mathbf{1}} \\ =& 2(1-\beta)nf_n(u)v'^{n-1}\,\bar{\mathbf{1}} + \cdots \end{aligned}$$

where the remaining terms have smaller v'-degree. Since $\beta \neq 1$, the leading coefficient is nonzero, a contradiction. Hence $n = 0$, and thus $\ker_Y(x) = \Bbbk[u]\bar{\mathbf{1}}$.

Now suppose that $\varphi : Y \to Y'$ is an isomorphism of $\mathcal{D}$-modules. Since φ commutes with the action of x, we have $\varphi(\ker_Y(x)) = \ker_{Y'}(x)$. The operators $g-\beta$ and $s-\gamma$ vanish on $\ker_Y(x)$, and hence also on $\ker_{Y'}(x)$. On the other hand, by the above description of $\ker_{Y'}(x)$, the operators g and s act on this space as the scalars β' and γ', respectively. Therefore, $(g - \beta)|_{\ker_{Y'}(x)} = (\beta' - \beta)\,\mathrm{id}$, and $(s - \gamma)|_{\ker_{Y'}(x)} = (\gamma' - \gamma)\,\mathrm{id}$. Since both operators vanish, it follows that $\beta' = \beta$, and $\gamma' = \gamma$.

It remains to determine α. Let $w = \sum_{i=0}^{n} c_i u^i \bar{\mathbf{1}} \in \ker_Y(x)$ with $c_i \in \Bbbk$. By (9),

$$(y-\alpha)w = (\beta - 1)\sum_{i=1}^{n} i c_i u^{i-1}\, \bar{\mathbf{1}}.$$

Hence $y-\alpha$ acts locally nilpotently on $\ker_Y(x)$. By the isomorphism φ, the operator $y-\alpha$ also acts locally nilpotently on $\ker_{Y'}(x)$. Since $y-\alpha'$ is locally nilpotent there as well, their difference $(y-\alpha)-(y-\alpha') = \alpha'-\alpha$ is locally nilpotent on the nonzero space $\ker_{Y'}(x)$. Therefore $\alpha' = \alpha$, and the proof is complete. □

3.4. The Whittaker modules $\mathcal{Y}_{\alpha,1,\mu}$. In this subsection, we consider a class of Whittaker modules $\mathcal{Y}_{\alpha,1,\mu}$ of type $(0,\alpha)$, which are quotients of $Y_{\alpha,1}$. Let R be the subalgebra of $\mathcal{D}$ generated by $g^{\pm 1}$, x, y, and u. Then R can be presented as an Ore extension $R = \Bbbk[g^{\pm 1}, x, u][y;\delta]$ where the derivation δ is given by $\delta(g) = -gx$, $\delta(x) = -1/2x^2$ and $\delta(u) = g-1$. Notice that x is normal in R, and we have the following quotient algebra

$$R/Rx \cong \Bbbk\langle g^{\pm 1},\, u,\, y \mid [g,u]=0,\ [g,y]=0,\ [y,u]=g-1\rangle.$$

In particular, g is central in R/Rx. Additionally, R/Rx is isomorphic to a localization of the universal enveloping algebra of the three-dimensional Heisenberg algebra. It is straightforward to verify that any finite-dimensional simple R-module must be isomorphic to one of the following modules:

$$S_{\alpha,\mu} := R/R(x,\ y-\alpha,\ g-1,\ u-\mu) \quad \text{where} \quad \alpha,\mu \in \Bbbk.$$

For any α, $\mu \in \Bbbk$, define

$$\mathcal{Y}_{\alpha,1,\mu} := \mathcal{D} \otimes_R S_{\alpha,\mu}. \tag{11}$$

Notably, $\mathcal{Y}_{\alpha,1,\mu}$ is a Whittaker module of type $(0,\alpha)$, and it admits an equivalent description as

$$\mathcal{Y}_{\alpha,1,\mu} \cong \mathcal{D}/\mathcal{D}(x,\ y-\alpha,\ g-1,\ u-\mu). \tag{12}$$

The next lemma shows that the module $\mathcal{Y}_{\alpha,1,\mu}$ is simple if and only if $\mu \in \Bbbk^*$. Furthermore, when $\mu = 0$, the simple quotients of $\mathcal{Y}_{\alpha,1,0}$ correspond precisely to the simple Whittaker $\mathcal{U}$-modules.

Lemma 3.6. *Let α, $\mu \in \Bbbk$.*

(1) *If $\mu \in \Bbbk^*$, then $\mathcal{Y}_{\alpha,1,\mu}$ is a simple $\mathcal{D}$-module, and*

$$\operatorname{ann}_{\mathcal{D}} \mathcal{Y}_{\alpha,1,\mu} = \langle z-16,\ \omega - 4(\alpha\mu-4),\ \theta - (\alpha\mu-4)^2\rangle.$$

Moreover, for $\alpha, \alpha' \in \Bbbk$ and $\mu, \mu' \in \Bbbk^*$, we have $\mathcal{Y}_{\alpha,1,\mu} \cong \mathcal{Y}_{\alpha',1,\mu'}$ if and only if $\alpha' = \alpha$ and $\mu' = \mu$.

(2) *If $\mu = 0$, then $\mathcal{Y}_{\alpha,1,0} \cong \mathcal{U}/\mathcal{U}(y-\alpha)$ is naturally a $\mathcal{U}$-module.*
   (a) *If $\alpha \in \Bbbk^*$, its simple quotients are Whittaker modules $M_{\alpha,\chi} = \mathcal{U}/\mathcal{U}(y-\alpha,\ \Omega-\chi)$, with $\operatorname{ann}_{\mathcal{D}} M_{\alpha,\chi} = \langle x,\ \Omega-\chi\rangle$.*
   (b) *If $\alpha = 0$, its simple quotients are highest weight $\mathcal{U}$-modules.*

Proof. (1) Note that $\mathcal{D}$ is a free right R-module with basis $\{v'^i h^j \mid i, j \in \mathbb{N}\}$. Letting $\mathsf{e} := 1 \otimes 1$, it follows from (11) that $\mathcal{Y}_{\alpha,1,\mu} = \oplus_{i,j\in\mathbb{N}} \Bbbk v'^i h^j \mathsf{e}$. Let M be a nonzero submodule of $\mathcal{Y}_{\alpha,1,\mu}$, and take a nonzero element $w \in M$. We can write $w = \sum_{j=0}^{n} f_j(v')h^j \mathsf{e}$, where $f_j(v') \in \Bbbk[v']$ and $f_n(v') \neq 0$. Define the h-degree of w as $\deg_h(w) := n$. Since u commutes with v', applying $(u-\mu)$ to w gives, for $n > 0$,

$$(u-\mu)w = \mu \sum_{j=0}^{n} f_j(v')\left((h+1)^j - h^j\right)\mathsf{e} = \mu \sum_{j=1}^{n}\sum_{k=1}^{j} \tbinom{j}{k} f_j(v')h^{j-k}\mathsf{e}. \quad (13)$$

Since $\mu \in \Bbbk^*$, it follows that if $n > 0$ then $(u-\mu)w$ is a nonzero element in M with h-degree $n-1$. Repeating this process, we eventually obtain a nonzero element $w' \in \Bbbk[v']\mathsf{e}$ contained in M. Write $w' = \sum_{i=0}^{m} \gamma_i v'^i \mathsf{e}$ where $\gamma_i \in \Bbbk$ and $\gamma_m \neq 0$, and define $\deg_{v'}(w') := m$. Using the relation $gv' = (v' + 2u)g$, we compute

$$(g-1)w' = \sum_{i=0}^{m} \gamma_i \left((v'+2u)^i - v'^i\right)\mathsf{e} = \sum_{i=1}^{m}\sum_{j=1}^{i} \gamma_i (2\mu)^j \tbinom{i}{j} v'^{i-j}\mathsf{e}.$$

Since $\mu \in \Bbbk^*$, if $m > 0$ then $(g-1)w'$ is a nonzero element of M with v'-degree $m-1$. Repeating this process, we obtain $(g-1)^m w' \in \Bbbk^*\mathsf{e}$, so $\mathsf{e} \in M$, and thus $M = \mathcal{Y}_{\alpha,1,\mu}$. Therefore, $\mathcal{Y}_{\alpha,1,\mu}$ is a simple module.

It is easily verified that $z\mathsf{e} = 16\mathsf{e}$, $\omega\mathsf{e} = 4(\alpha\mu-4)\mathsf{e}$, and $\theta\mathsf{e} = (\alpha\mu-4)^2\mathsf{e}$. Thus the ideal $\mathfrak{p} = \langle z-16,\ \omega-4(\alpha\mu-4),\ \theta-(\alpha\mu-4)^2\rangle$ is contained in $\operatorname{ann}_{\mathcal{D}}\mathcal{Y}_{\alpha,1,\mu}$. If $\alpha \in \Bbbk^*$, then by [BS, Theorem 4.1(i)], the ideal $\mathfrak{p}$ is maximal in $\mathcal{D}$, which forces $\operatorname{ann}_{\mathcal{D}}\mathcal{Y}_{\alpha,1,\mu} = \mathfrak{p}$. In the case where $\alpha = 0$, we have $\mathfrak{p} = \mathfrak{m}_+\mathcal{D}$. Suppose, for contradiction, that $\operatorname{ann}_{\mathcal{D}}\mathcal{Y}_{0,1,\mu}$ strictly contains $\mathfrak{m}_+\mathcal{D}$. Then by [BS, Theorem 4.4(v)], it must hold that $\langle x,\ g-1,\ u\rangle \subseteq \operatorname{ann}_{\mathcal{D}}\mathcal{Y}_{0,1,\mu}$. However, since $\mu \in \Bbbk^*$, it is evident that u does not annihilate $\mathcal{Y}_{0,1,\mu}$. This contradiction implies that $\operatorname{ann}_{\mathcal{D}}\mathcal{Y}_{0,1,\mu} = \mathfrak{m}_+\mathcal{D}$.

By (13), the element $u-\mu$ acts locally nilpotently on $\mathcal{Y}_{\alpha,1,\mu}$. Similarly, $u-\mu'$ acts locally nilpotently on $\mathcal{Y}_{\alpha',1,\mu'}$. Suppose that $\varphi : \mathcal{Y}_{\alpha,1,\mu} \to \mathcal{Y}_{\alpha',1,\mu'}$ is an isomorphism of $\mathcal{D}$-modules. Then, via φ, the operator $u-\mu$ acts locally nilpotently on $\mathcal{Y}_{\alpha',1,\mu'}$. Thus $(u-\mu)-(u-\mu') = \mu'-\mu$ is locally nilpotent on $\mathcal{Y}_{\alpha',1,\mu'}$. This is possible only when $\mu' = \mu$. It remains to determine α. Since isomorphic modules have the same annihilator, we have $\operatorname{ann}_{\mathcal{D}}\mathcal{Y}_{\alpha,1,\mu} = \operatorname{ann}_{\mathcal{D}}\mathcal{Y}_{\alpha',1,\mu'}$. Using the description of these annihilators and the equality $\mu' = \mu$, we obtain $4(\alpha\mu-4) = 4(\alpha'\mu-4)$. Because $\mu \in \Bbbk^*$, it follows that $\alpha' = \alpha$.

(2) When $\mu = 0$, by (4), the ideal $\langle x,\ g-1,\ u\rangle$ is contained in the left ideal $\mathcal{D}(x,\ y-\alpha,\ g-1,\ u)$. Then from (3) and (12), it follows that $\mathcal{Y}_{\alpha,1,0} \cong \mathcal{U}/\mathcal{U}(y-\alpha)$. Thus $\mathcal{Y}_{\alpha,1,0}$ is the universal Whittaker $\mathcal{U}$-module. The statement then follows from Theorem 2.2. $\square$

3.5. Classification of simple Whittaker modules of type $(0, \alpha)$ where $\alpha \in \Bbbk^*$. The following theorem gives a classification of simple Whittaker $\mathcal{D}$-modules of type $(0, \alpha)$ where $\alpha \in \Bbbk^*$.

Theorem 3.7. *Let η be the character of $\mathcal{D}^+$ such that $\eta(x) = 0$ and $\eta(y) = \alpha$ where $\alpha \in \Bbbk^*$. Then the set of isomorphism classes of simple Whittaker $\mathcal{D}$-modules of type $(0, \alpha)$ is given by*

$$\widehat{\mathcal{D}}(\text{Whittaker-}(0,\alpha)) = \{[Y_{\alpha,\beta,\gamma}] \mid 1 \neq \beta \in \Bbbk^*, \gamma \in \Bbbk\} \cup \{[\mathcal{Y}_{\alpha,1,\mu}] \mid \mu \in \Bbbk^*\} \cup \widehat{\mathcal{U}}(\text{Whittaker-}\alpha).$$

Proof. By Lemma 3.3, for any $1 \neq \beta \in \Bbbk^*$ and $\gamma \in \Bbbk$, $Y_{\alpha,\beta,\gamma}$ is a simple Whittaker module of type $(0, \alpha)$. It has been established in Lemma 3.6(1) that, for any $\mu \in \Bbbk^*$, $\mathcal{Y}_{\alpha,1,\mu}$ is a simple Whittaker module of type $(0, \alpha)$. It is evident that, any simple Whittaker $\mathcal{U}$-module of type α is naturally a simple Whittaker $\mathcal{D}$-module of type $(0, \alpha)$.

Conversely, let M be a simple Whittaker $\mathcal{D}$-module of type $(0, \alpha)$. By Lemma 3.2, M is a quotient of $Y_{\alpha,\beta}$ for some $\beta \in \Bbbk^*$. If $\beta \neq 1$, then Lemma 3.3 implies $M \cong Y_{\alpha,\beta,\gamma}$ for some $\gamma \in \Bbbk$. Now suppose $\beta = 1$. Then there is a Whittaker vector $\xi \in M$ of type $(0, \alpha)$ such that $(g-1)\xi = 0$. It can be verified that $q\xi = 4\xi$ and $s\xi = (\alpha u - 4)\xi$. The central element $\omega = qsg^{-1}$ acts on M as a scalar, say ν. It follows that $0 = (\omega - \nu)\xi = [4(\alpha u - 4) - \nu]\xi$, implying ξ is an eigenvector of u with eigenvalue $\mu = (16 + \nu)/(4\alpha)$. Thus M is an epimorphic image of the module $\mathcal{Y}_{\alpha,1,\mu}$, see (12). If $\mu \in \Bbbk^*$, then by Lemma 3.6(1), $\mathcal{Y}_{\alpha,1,\mu}$ is simple, so $M \cong \mathcal{Y}_{\alpha,1,\mu}$. If $\mu = 0$, then by Lemma 3.6(2), any simple quotient of $\mathcal{Y}_{\alpha,1,0}$ is a simple $\mathcal{U}$-module, and in this case $M \cong M_{\alpha,\chi}$ for a unique $\chi \in \Bbbk$. □

3.6. Singular Whittaker modules. Let M be a $\mathcal{D}$-module. An element $w \in M$ is called a singular Whittaker vector if it satisfies $xw = 0$, $yw = 0$, and $(g - 1)w = 0$. A module M is called a singular Whittaker module if it is generated by a singular Whittaker vector. In particular, any singular Whittaker module is a Whittaker module of type $(0, 0)$. We denote by $\mathrm{Sing}(M)$ the set of all singular vectors in M. Notably, $\mathcal{Y}_{0,1,\mu}$ for $\mu \in \Bbbk^*$ are simple singular Whittaker modules with annihilator $\mathfrak{m}_+\mathcal{D}$; see Lemma 3.6(1).

Let Γ be the subalgebra of $\mathcal{D}$ generated by x, y, $g^{\pm 1}$, h, and u. Then Γ can be presented as an iterated Ore extension over $\mathcal{O} = \Bbbk[g^{\pm 1}, x, u]$ in the form

$$\Gamma = \Bbbk[g^{\pm 1}, x, u][y; \delta_1][h; \delta_2]$$

where δ_1 is the derivation of $\mathcal{O}$ determined by $\delta_1(g) = -gx$, $\delta_1(x) = -1/2x^2$, $\delta_1(u) = g - 1$, and δ_2 is the derivation such that $\delta_2(g) = 0$, $\delta_2(x) = x$, $\delta_2(u) = -u$, and $\delta_2(y) = y$. In particular, Γ is a Noetherian domain of Gelfand–Kirillov dimension 5. Note that x is normal in Γ, g is central in $\Gamma/\Gamma x$, and y is normal in the factor algebra $\Gamma/\Gamma(x, g-1)$. It follows that the left ideal $\Gamma(x, g-1, y)$ is actually an ideal of Γ. It is easily checked that

$$\Gamma/\Gamma(x, g-1, y) \cong B := \Bbbk[u][h; \delta] \quad \text{where} \quad \delta(u) = -u. \tag{14}$$

Clearly, B is isomorphic to the universal enveloping algebra of the Borel subalgebra of $\mathfrak{sl}_2$. If M is a Γ-module that is annihilated by the ideal $\Gamma(x, g-1, y)$, then M can be naturally viewed as a B-module via (14). A classification of simple B-modules was obtained by Block in [Bl, Proposition 6.1, Theorem 6.2]. Note that a simple B-module M is infinite-dimensional if and only if $u \in B$ acts injectively on M.

If S is a left Ore set of a ring R and M is a left R-module, then the set

$$\mathrm{tor}_S(M) := \{m \in M \mid sm = 0 \text{ for some } s \in S\}$$

is a submodule of M, called the S-torsion submodule of M. The module M is said to be S-torsion if $\mathrm{tor}_S(M) = M$, and S-torsionfree if $\mathrm{tor}_S(M) = 0$. Recall that, a $\mathcal{D}$-module M is called a weight module if it decomposes into a direct sum of eigenspaces for the Cartan element $h \in \mathcal{D}$, that is, $M = \bigoplus_{\mu\in\Bbbk} M_\mu$, where $M_\mu = \{m \in M \mid hm = \mu m\}$. The next proposition establishes a connection between the structure of a singular Whittaker $\mathcal{D}$-module and the space of its singular Whittaker vectors.

Proposition 3.8. *Let M be a singular Whittaker $\mathcal{D}$-module. Then:*

(1) $\mathrm{Sing}(M)$ *is a Γ-module annihilated by $\Gamma(x, y, g-1)$. In particular,* $\mathrm{Sing}(M)$ *naturally inherits the structure of a B-module by (14).*
(2) *M is a simple module that is not a weight module if and only if* $\mathrm{Sing}(M)$ *is an infinite-dimensional simple B-module.*

Proof. (1) Clearly, x, y and $g-1$ annihilate $\mathrm{Sing}(M)$. It is easily verified that $h\mathrm{Sing}(M) \subseteq \mathrm{Sing}(M)$ and $u\mathrm{Sing}(M) \subseteq \mathrm{Sing}(M)$.

(2) Suppose M is a simple module that is not a weight module. We claim that u acts injectively on M. Note that u acts ad-locally nilpotently on $\mathcal{D}$. Consequently, the set $S_u := \{u^i \mid i \in \mathbb{N}\}$ is an Ore set in $\mathcal{D}$. Assume, for contradiction, that u acts non-injectively on M. Since M is simple, it must be S_u-torsion. This implies that there is a singular Whittaker vector in M that is annihilated by u. Consequently, M is an epimorphic image of the module $\mathcal{W} := \mathcal{D}/\mathcal{D}(x, y, g-1, u)$. By (4), we have $\mathcal{W} \cong \mathcal{U}/\mathcal{U}y$. Then by Theorem 2.2(2), it follows that M is a simple highest weight module, contradicting our assumption.

To show that $\mathrm{Sing}(M)$ is a simple B-module, it suffices to prove that for any nonzero vector $\xi \in \mathrm{Sing}(M)$, we have $B\xi = \mathrm{Sing}(M)$. Clearly, $\mathcal{D}$ is a free left/right Γ-module and $\mathcal{D} = \bigoplus_{i\in\mathbb{N}} v'^i\Gamma$. Since M is simple, we have

$$M = \mathcal{D}\xi = \sum_{i\in\mathbb{N}} v'^i\Gamma\xi = \sum_{i\in\mathbb{N}} v'^i B\xi.$$

For any nonzero $w \in \mathrm{Sing}(M)$, we can express $w = \sum_{i=0}^n v'^i b_i\xi$, where $b_i \in B$ and $b_n\xi \neq 0$. Suppose $n > 0$, we seek a contradiction. Using the relation $gv' = (v' + 2u)g$ and the fact that g commutes with b_i, we compute

$$
\begin{aligned}
0 = (g-1)w &= \sum_{i=1}^{n}\Big((v'+2u)^i - v'^i\Big)b_i\xi \\
&= \sum_{i=1}^{n}\sum_{j=1}^{i}\tbinom{i}{j}v'^{i-j}(2u)^j b_i\xi = nv'^{n-1}(2u)b_n\xi + \cdots .
\end{aligned}
\tag{15}
$$

Repeating this process, we obtain $0 = (g-1)^n w = n!(2u)^n b_n\xi$. Since u acts injectively on M, it follows that $b_n\xi = 0$, which leads to a contradiction. This proves that $\mathrm{Sing}(M)$ is a simple B-module. Furthermore, since $u \in B$ acts injectively on $\mathrm{Sing}(M)$, it follows that $\mathrm{Sing}(M)$ is infinite-dimensional.

Conversely, assume that $\mathrm{Sing}(M)$ is a simple infinite-dimensional B-module, then u acts injectively on $\mathrm{Sing}(M)$. Let N be a nonzero submodule of M. We claim that $N \cap \mathrm{Sing}(M) \neq 0$. In fact, since M is generated by a singular Whittaker vector ξ, as above we have $M = \mathcal{D}\xi = \sum_{i\in\mathbb{N}} v'^i B\xi$. Let w be a nonzero element in N, we can express $w = \sum_{i=0}^{n} v'^i b_i\xi$, where $b_i \in B$ and $b_n\xi \neq 0$. If $n = 0$ then $w = b_0\xi \in \mathrm{Sing}(M)$. If $n > 0$, then as calculated in (15), we obtain $(g-1)^n w = n!(2u)^n b_n\xi$ is a nonzero element in $\mathrm{Sing}(M)$, since u acts injectively on $\mathrm{Sing}(M)$. This proves that $N \cap \mathrm{Sing}(M) \neq 0$. Consequently, $\mathrm{Sing}(M) \subseteq N$, since $\mathrm{Sing}(M)$ is a simple B-module. This implies that $M \subseteq N$, since M is generated by $\mathrm{Sing}(M)$. Therefore, we have $N = M$, proving that M is simple. It remains to show that M is not a weight module. Suppose, for contradiction, that M is a weight module. Then M is an epimorphic image of the Verma module $M(\lambda, 1) := \mathcal{D}/\mathcal{D}(h-\lambda, x, y, g-1)$ for some $\lambda \in \Bbbk$. By [Lu, Proposition 3.2(2)], the unique simple quotient of $M(\lambda, 1)$ is annihilated by the ideal $\langle x, g-1, u\rangle$ of $\mathcal{D}$. This implies that u annihilates $\mathrm{Sing}(M)$, leads to a contradiction. □

Let M be an infinite-dimensional simple B-module, which is regarded as a Γ-module via (14). Define the induced module

$$M^\dagger := \mathcal{D} \otimes_\Gamma M.$$

For any $\mu \in \Bbbk^*$, the module $M_\mu := B/B(u-\mu)$ is an infinite-dimensional simple B-module. It is easily seen that $M_\mu^\dagger \cong \mathcal{Y}_{0,1,\mu}$; see (11) and (12). Thus, by Lemma 3.6(1), $M_\mu^\dagger$ is a simple singular Whittaker module with $\mathrm{ann}_{\mathcal{D}} M_\mu^\dagger = \mathfrak{m}_+\mathcal{D}$. The following lemma gives a classification of all simple singular Whittaker $\mathcal{D}$-modules that are not weight modules, and shows that the annihilator of any such module equals $\mathfrak{m}_+\mathcal{D}$.

Lemma 3.9. (1) *If M is an infinite-dimensional simple B-module, then $M^\dagger$ is a simple singular Whittaker $\mathcal{D}$-module that is not a weight module. Moreover,* $\mathrm{ann}_{\mathcal{D}} M^\dagger = \mathfrak{m}_+\mathcal{D}$.

(2) *If W is a simple singular Whittaker $\mathcal{D}$-module that is not a weight module, then $W \cong M^\dagger$ for some infinite-dimensional simple B-module M.*

(3) *If M and M' are infinite-dimensional simple B-modules, then $M^\dagger \cong M'^\dagger$ if and only if $M \cong M'$.*

Proof. (1) Since $\mathcal{D}$ is a free right Γ-module and $\mathcal{D} = \oplus_{i\in\mathbb{N}} v'^i\Gamma$, it follows that
$$M^\dagger = \bigoplus_{i\in\mathbb{N}} v'^i \otimes M.$$
We claim that $\mathrm{Sing}(M^\dagger) = 1 \otimes M \cong M$. Let w be a nonzero singular Whittaker vector in $M^\dagger$, which we can express as $w = \sum_{i=0}^n v'^i \otimes \xi_i$, where $\xi_i \in M$ and $\xi_n \neq 0$. Note that $u \in B$ acts injectively on M, since M is infinite-dimensional. Suppose that $n > 0$, then as calculated in (15), we have
$$0 = (g-1)w = \sum_{i=1}^{n}\sum_{j=1}^{i} v'^{i-j} \otimes \tbinom{i}{j}(2u)^j\xi_i = v'^{n-1} \otimes n(2u)\xi_n + \cdots.$$
This is a contradiction, since $n(2u)\xi_n \neq 0$. Therefore, $\mathrm{Sing}(M^\dagger) = 1 \otimes M$. Then we conclude from Proposition 3.8(2) that $M^\dagger$ is a simple singular Whittaker module that is not a weight module. Next, for any nonzero element $w = 1 \otimes \xi \in 1 \otimes M$, we have $qw = 4w$ and $sw = -4w$. It follows that $(z-16)w = 0$, $(\omega + 16)w = 0$ and $(\theta - 16)w = 0$. Since $M^\dagger$ is generated by w, we have $\mathfrak{m}_+\mathcal{D} \subseteq \mathrm{ann}_{\mathcal{D}} M^\dagger$. Suppose, for contradiction, that $\mathrm{ann}_{\mathcal{D}} M^\dagger$ strictly contains $\mathfrak{m}_+\mathcal{D}$. Then by [BS, Theorem 4.4(v)], it must hold that $\langle x,\ g-1,\ u\rangle \subseteq \mathrm{ann}_{\mathcal{D}} M^\dagger$. In particular, this would imply u annihilates M, contradicting the injectivity of u on M. Hence, $\mathrm{ann}_{\mathcal{D}} M^\dagger = \mathfrak{m}_+\mathcal{D}$.

(2) By Proposition 3.8(2), $M := \mathrm{Sing}(W)$ is an infinite-dimensional simple B-module. Then from statement (1) it follows that $M^\dagger$ is a simple singular Whittaker module. The following map $\varrho : M^\dagger \to W$, $\sum v'^i \otimes \xi_i \mapsto \sum v'^i \xi_i$, where $\xi_i \in M$, defines a homomorphism of $\mathcal{D}$-modules. Since both $M^\dagger$ and W are simple modules, ϱ is an isomorphism.

(3) Suppose that $\varphi : M^\dagger \to M'^\dagger$ is an isomorphism of $\mathcal{D}$-modules. Then φ preserves singular Whittaker vectors, and hence restricts to an isomorphism $1 \otimes M \to 1 \otimes M'$. Identifying $1 \otimes M$ and $1 \otimes M'$ with M and M', respectively, we obtain an isomorphism of B-modules $M \cong M'$. The converse is immediate: an isomorphism $M \cong M'$ induces an isomorphism $M^\dagger \cong M'^\dagger$. □

3.7. Classification of simple Whittaker modules of type (0, 0). The following theorem gives a classification of simple Whittaker modules of type $(0,0)$.

Theorem 3.10. *The set of isomorphism classes of simple Whittaker $\mathcal{D}$-modules of type* $(0,0)$ *is given by*
$$\begin{aligned}\widehat{\mathcal{D}}(\text{Whittaker-}(0,0)) = &\{[Y_{0,\beta,\gamma}] \mid 1 \neq \beta \in \Bbbk^*, \gamma \in \Bbbk\} \\ &\cup \{[M^\dagger] \mid [M] \in \widehat{B}(\text{inf. dim.})\} \cup \widehat{\mathcal{U}}(\text{highest weight}).\end{aligned}$$

Proof. By Lemma 3.3, the modules $Y_{0,\beta,\gamma}$ for $1 \neq \beta \in \Bbbk^*$ and $\gamma \in \Bbbk$ are simple Whittaker modules of type $(0,0)$. It follows from Lemma 3.9(1) that $M^\dagger$ is a simple Whittaker module of type $(0,0)$ for any infinite-dimensional simple B-modules. Clearly, any simple highest weight $\mathcal{U}$-module is a Whittaker $\mathcal{D}$-module of type $(0,0)$.

Conversely, let W be a simple Whittaker $\mathcal{D}$-module of type $(0,0)$. Then by Lemma 3.2, W is an epimorphic image of $Y_{0,\beta}$ for some $\beta \in \mathbb{k}^*$. If $\beta \neq 1$, then by Lemma 3.3, W is isomorphic to $Y_{0,\beta,\gamma}$ for some $\gamma \in \mathbb{k}$. Suppose now that $\beta = 1$. Then there is a Whittaker vector $\xi \in W$ of type $(0,0)$ such that $(g-1)\xi = 0$, i.e., ξ is a singular Whittaker vector. Thus, W is a simple singular Whittaker module. If W is not a weight module, then by Lemma 3.9(2), W is isomorphic to $M^\dagger$ for some infinite-dimensional simple B-module. If W is a weight module, then M is an epimorphic image of the Verma module $M(\lambda,1) = \mathcal{D}/\mathcal{D}(h-\lambda,\ x,\ y,\ g-1)$ for some $\lambda \in \mathbb{k}$. By [Lu, Proposition 3.2(2)], the unique simple quotient of $M(\lambda,1)$ is annihilated by the ideal $\langle x,\ g-1,\ u\rangle$ of $\mathcal{D}$. Therefore, W is a simple highest weight $\mathcal{U}$-module. This completes the proof. □

Remark 3.11. As noted in Remark 3.4, for $\beta \in \mathbb{k}^*$ with $\beta \neq 1$, the module $Y_{0,\beta,\gamma}$ is isomorphic to a simple reduced Verma $\mathcal{D}$-module. According to [Lu, Theorem 3.3], the set $\{[Y_{0,\beta,\gamma}] \mid 1 \neq \beta \in \mathbb{k}^*, \gamma \in \mathbb{k}\} \cup \widehat{\mathcal{U}}(\text{highest weight})$ comprises all isomorphism classes of simple highest weight $\mathcal{D}$-modules. Combining this with Theorem 3.10, we have

$$\widehat{\mathcal{D}}(\text{Whittaker-}(0,0)) = \left\{[M^\dagger] \mid [M] \in \widehat{B}(\text{inf. dim.})\right\} \cup \widehat{\mathcal{D}}(\text{highest weight}).$$

Theorem 3.7 and Theorem 3.10 give a complete classification of simple Whittaker $\mathcal{D}$-modules.

Remark 3.12. The annihilators of all simple Whittaker $\mathcal{D}$-modules of type $(0,0)$ have been explicitly determined. In particular, Lemma 3.3 describes the annihilator of $Y_{0,\beta,\gamma}$, while Lemma 3.9 shows that $\mathrm{ann}_{\mathcal{D}} M^\dagger = \mathfrak{m}_+\mathcal{D}$ for any infinite-dimensional simple B-module M. It is a classical result that all primitive ideals of $\mathcal{U}$ arise as annihilators of simple highest weight $\mathcal{U}$-modules. In light of the classification of primitive ideals of $\mathcal{D}$ in [BS, Theorem 1.1], it follows that every primitive ideal of $\mathcal{D}$ arises as the annihilator of a simple Whittaker modules of type $(0,0)$. In other words, the map $\widehat{\mathcal{D}}(\text{Whittaker-}(0,0)) \twoheadrightarrow \mathrm{Prim}(\mathcal{D})$, $[M] \mapsto \mathrm{ann}_{\mathcal{D}} M$, is surjective.

4. Twisted Whittaker modules

In this section, we introduce and study twisted Whittaker modules over $\mathcal{D}$. This class of non-weight modules encompasses all Whittaker modules as special cases. We provide a classification of simple twisted Whittaker $\mathcal{D}$-modules and explicitly determine their annihilators.

4.1. Twisted Whittaker modules and local finiteness. Recall that $v' = hu - 2v$. We now introduce the element $y' := hx + 2y$. As noted in [Lu], replacing the generators y and v with y' and v', respectively, yields a more symmetric presentation of $\mathcal{D}$. Applying the automorphism τ (see (1)), we obtain $\tau(y') = v'$ and $\tau(v') = y'$. The new generators satisfy the following commutation relations

$$[y', x] = 0, \quad [v', u] = 0, \quad gv' = (v' + 2u)g, \quad gy' = (y' + 2x)g.$$

Furthermore, applying τ to the identity (8) yields that, for all $n \geq 1$,

$$[u, y'^n] = \sum_{i=1}^{n} \binom{n}{i} x^{i-1} y'^{n-i} (q - 2^{i+1} g). \tag{16}$$

Define the subalgebras $\mathcal{D}^{<0} := \Bbbk[u, v']$, $\mathcal{D}^0 := \Bbbk[g^{\pm 1}, h]$, and $\mathcal{D}^{>0} := \Bbbk[x, y']$. Then $\mathcal{D}$ admits the following triangular decomposition:

$$\mathcal{D} = \mathcal{D}^{<0} \otimes \mathcal{D}^0 \otimes \mathcal{D}^{>0}. \tag{17}$$

Moreover, the automorphism τ interchanges the subalgebras $\mathcal{D}^{<0}$ and $\mathcal{D}^{>0}$, while leaving $\mathcal{D}^0$ fixed.

An algebra homomorphism $\eta : \mathcal{D}^{>0} \to \Bbbk$ is called a character of $\mathcal{D}^{>0}$. Since $\mathcal{D}^{>0}$ is a polynomial algebra, the set of characters of $\mathcal{D}$ is in bijection with $\Bbbk^2$. Let η be a character of $\mathcal{D}^{>0}$ determined by $\eta(x) = \alpha$ and $\eta(y') = \beta$, where $\alpha, \beta \in \Bbbk$, and let M be a $\mathcal{D}$-module. An element $w \in M$ is called a twisted Whittaker vector of type (α, β) if $xw = \alpha w$ and $y'w = \beta w$. We say that M is a twisted Whittaker module of type (α, β) if it is generated by a twisted Whittaker vector of type (α, β).

Lemma 4.1. *Let M be a simple $\mathcal{D}$-module. If M is $\mathcal{D}^{>0}$-locally finite then M is a twisted Whittaker module.*

Proof. Let $w_0 \in M$ be a nonzero vector. Then there exists a finite-dimensional $\mathcal{D}^{>0}$-submodule $N \subseteq M$ such that $w_0 \in N$. Since x and y' commute, they have a common eigenvector $w \in N$. Thus, w is a twisted Whittaker vector. Since M is simple, we have $M = \mathcal{D}w$, and hence M is a twisted Whittaker module. $\square$

Let M be a twisted Whittaker module generated by a twisted Whittaker vector $w \in M$ of type (α, β). If $\alpha = 0$, then $xw = 0$ and $y'w = \beta w$. It follows that $yw = \frac{1}{2}(y' - hx)w = \frac{1}{2}\beta w$. Thus, w is a Whittaker vector, and M is a Whittaker module. Since a classification of simple Whittaker modules has been established in the previous section, we will henceforth assume that $\alpha \neq 0$.

4.2. The universal twisted Whittaker module $T_{\alpha,\beta}$. For $\alpha \in \Bbbk^*$ and $\beta \in \Bbbk$, define

$$T_{\alpha,\beta} := \mathcal{D}/\mathcal{D}(x - \alpha,\ y' - \beta).$$

The module $T_{\alpha,\beta}$ satisfies the following universal property: If M is a twisted Whittaker $\mathcal{D}$-module of type (α, β), then M is an epimorphic image of $T_{\alpha,\beta}$. In analogy with Verma modules in Lie theory, $T_{\alpha,\beta}$ serves as a standard object from which all simple twisted Whittaker modules of type (α, β) are obtained via quotienting. Let $\bar{1} = 1 + \mathcal{D}(x - \alpha,\ y' - \beta)$ denote the canonical generator of $T_{\alpha,\beta}$. In view of the triangular decomposition (17), it follows that $T_{\alpha,\beta}$ is isomorphic, as a vector space, to the tensor product $\mathcal{D}^{<0} \otimes \mathcal{D}^0$. Moreover, we have

$$T_{\alpha,\beta} = \bigoplus_{i \in \mathbb{Z},\ j,k,\ell \in \mathbb{N}} \Bbbk g^i h^j u^k v'^{\ell}\ \bar{1}. \tag{18}$$

Note that the element s (see (2)) can be expressed as

$$s = -\frac{1}{2}v'x + \frac{1}{2}uy' - \frac{3}{2}ux - \frac{1}{2}hux + h(g-1) - g - 3. \tag{19}$$

From the expression of q given in (2), we obtain $q\,\bar{1} = (\alpha u + 2(1+g))\,\bar{1}$. Since $\alpha \in \Bbbk^*$, it follows that

$$u\,\bar{1} = \alpha^{-1}(q - 2g - 2)\,\bar{1}. \tag{20}$$

Using the expression for s in (19), we obtain

$$s\,\bar{1} = -\frac{1}{2}\alpha v'\,\bar{1} - \frac{1}{2}(3\alpha - \beta)u\,\bar{1} - \frac{1}{2}\alpha hu\,\bar{1} + (h(g-1) - g - 3)\,\bar{1}.$$

Substituting (20) into this expression yields,

$$v'\,\bar{1} = -2\alpha^{-1}s\,\bar{1} - \alpha^{-2}(3\alpha - \beta + \alpha h)(q - 2g - 2)\,\bar{1} + 2\alpha^{-1}(h(g-1) - g - 3)\,\bar{1}. \tag{21}$$

Since $\alpha \in \Bbbk^*$, it follows from (18), (20), and (21) that the elements $g^i h^j q^k s^\ell\,\bar{1}$ ($i \in \mathbb{Z}$, $j, k, \ell \in \mathbb{N}$) span $T_{\alpha,\beta}$. Hence,

$$T_{\alpha,\beta} = \Bbbk[g^{\pm 1}, h, q, s]\,\bar{1}. \tag{22}$$

4.3. Some simple twisted Whittaker modules. In this subsection, we introduce four families of simple twisted Whittaker $\mathcal{D}$-modules. As we will show later in Theorem 4.6, every simple twisted Whittaker module of type (α, β), where $\alpha \in \Bbbk^*$, is isomorphic to one of these modules.

To begin, for $\alpha \in \Bbbk^*$ and $\beta \in \Bbbk$, define

$$W(\alpha, \beta) := \mathcal{D}/\mathcal{D}(x - \alpha,\ y' - \beta,\ q,\ s).$$

This is clearly a twisted Whittaker module of type (α, β). Since the elements q and s are normal in $\mathcal{D}$, it follows that $qT_{\alpha,\beta}$ and $sT_{\alpha,\beta}$ are submodules of $T_{\alpha,\beta}$. Consequently, we have an isomorphism of $\mathcal{D}$-modules: $W(\alpha,\beta) \cong T_{\alpha,\beta}/(qT_{\alpha,\beta} + sT_{\alpha,\beta})$. Let $\bar{\mathbf{1}} = 1 + \mathcal{D}(x - \alpha,\ y' - \beta,\ q,\ s)$ denote the canonical generator of $W(\alpha,\beta)$. Then $W(\alpha,\beta)$ is spanned by the elements $g^i h^j\,\bar{\mathbf{1}}$ ($i \in \mathbb{Z}$, $j \in \mathbb{N}$). The following proposition establishes the simplicity of $W(\alpha,\beta)$ and determines its annihilator.

Proposition 4.2. *For any $\alpha \in \Bbbk^*$ and $\beta \in \Bbbk$, the module $W(\alpha,\beta)$ is simple, and*

$$\operatorname{ann}_{\mathcal{D}} W(\alpha,\beta) = q\mathcal{D} + s\mathcal{D}.$$

Moreover, for $\alpha, \alpha' \in \Bbbk^*$ and $\beta, \beta' \in \Bbbk$, we have $W(\alpha,\beta) \cong W(\alpha',\beta')$ if and only if $\alpha' = \alpha$, and $\beta' = \beta - 2i\alpha$ for some $i \in \mathbb{Z}$.

Proof. Let M be a nonzero submodule of $W(\alpha,\beta)$, and let $w \in M$ be a nonzero element in M. We can write $w = \sum_{i=0}^{n} \gamma_i(g) h^i\,\bar{\mathbf{1}}$, where $\gamma_i(g) \in \Bbbk[g^{\pm 1}]$ and $\gamma_n(g) \neq 0$. If $n > 0$ then

$$(x - \alpha)w = \alpha \sum_{i=1}^{n} \sum_{k=1}^{i} \gamma_i(g) \binom{i}{k} (-1)^k h^{i-k}\,\bar{\mathbf{1}}. \tag{23}$$

Repeated application of $(x - \alpha)$ successively lowers the h-degree. Hence, we eventually obtain an element $w' = \gamma(g)\,\bar{\mathbf{1}} \in M$, for some nonzero polynomial

$\gamma(g) \in \Bbbk[g^{\pm 1}]$. Write $w' = \sum_{i=-m}^{n} c_i g^i \,\bar{\mathbf{1}}$, where $c_i \in \Bbbk$. Since $y'g^i = g^i(y' - 2ix)$, the terms $g^i \,\bar{\mathbf{1}}$ are eigenvectors for y' with distinct eigenvalues: $y'g^i \,\bar{\mathbf{1}} = (\beta - 2i\alpha)g^i \,\bar{\mathbf{1}}$. Thus, the summands of w' are linearly independent, and some $g^i \,\bar{\mathbf{1}} \in M$. It follows that $\bar{\mathbf{1}} \in M$, so $M = W(\alpha, \beta)$. This proves that $W(\alpha, \beta)$ is simple. For the annihilator, it is clear that $P_0 := q\mathcal{D} + s\mathcal{D} \subseteq \operatorname{ann}_{\mathcal{D}} W(\alpha, \beta)$. This must be an equality, since P_0 is a maximal ideal of $\mathcal{D}$ by [BS, Theorem 4.13].

We next prove the isomorphism criterion. Observe that $g^i \,\bar{\mathbf{1}}$ is a twisted Whittaker vector of type $(\alpha, \beta - 2i\alpha)$. Since $W(\alpha, \beta)$ is simple, $g^i \bar{\mathbf{1}}$ generates the whole module. Hence $W(\alpha, \beta) \cong W(\alpha, \beta - 2i\alpha)$ for every $i \in \mathbb{Z}$..

Conversely, suppose that $\varphi : W(\alpha, \beta) \to W(\alpha', \beta')$ is an isomorphism of $\mathcal{D}$-modules. By (23), the element $x - \alpha$ acts locally nilpotently on $W(\alpha, \beta)$. Similarly, $x - \alpha'$ acts locally nilpotently on $W(\alpha', \beta')$. On the other hand, via φ, the operator $x - \alpha$ is locally nilpotent on $W(\alpha', \beta')$. Hence, the element $(x - \alpha) - (x - \alpha') = \alpha' - \alpha$ acts locally nilpotently on the module $W(\alpha', \beta')$. Thus $\alpha' = \alpha$. Now set $W := W(\alpha, \beta)$. By (23), $\ker_W(x - \alpha) = \Bbbk[g^{\pm 1}]\bar{\mathbf{1}}$. The inverse image under φ of the canonical generator of $W(\alpha, \beta')$ belongs to $\ker_W(x - \alpha)$ and is an eigenvector of y' with eigenvalue β'. Since $y'g^i \,\bar{\mathbf{1}} = (\beta - 2i\alpha)g^i \,\bar{\mathbf{1}}$, the eigenspaces of y' in $\ker_W(x - \alpha)$ are precisely the spaces spanned by $g^i \bar{\mathbf{1}}$, with eigenvalues $\beta - 2i\alpha$. Therefore, $\beta' = \beta - 2i\alpha$ for some $i \in \mathbb{Z}$. □

Next, recall that $\theta = s^2 g^{-1}$ is central in $\mathcal{D}$. For $\alpha, \gamma \in \Bbbk^*$ and $\beta \in \Bbbk$, define

$$Q(\alpha, \beta, \gamma) := \mathcal{D}/\mathcal{D}(x - \alpha,\ y' - \beta,\ q,\ \theta - \gamma).$$

This is a twisted Whittaker module of type (α, β). By construction, it is isomorphic to the quotient module $T_{\alpha,\beta}/(qT_{\alpha,\beta} + (\theta - \gamma)T_{\alpha,\beta})$. The following proposition proves that $Q(\alpha, \beta, \gamma)$ is simple and determines its annihilator.

Proposition 4.3. *For any α, $\gamma \in \Bbbk^*$ and $\beta \in \Bbbk$, the module $Q(\alpha, \beta, \gamma)$ is simple, and*

$$\operatorname{ann}_{\mathcal{D}} Q(\alpha, \beta, \gamma) = \langle q,\ \theta - \gamma \rangle = \mathfrak{m}_{(0,\gamma)}\mathcal{D}.$$

Moreover, for $\alpha, \alpha' \in \Bbbk^*$, $\beta, \beta' \in \Bbbk$, and $\gamma, \gamma' \in \Bbbk^*$, we have $Q(\alpha, \beta, \gamma) \cong Q(\alpha', \beta', \gamma')$ if and only if $\alpha' = \alpha$, $\beta' = \beta - i\alpha$ for some $i \in \mathbb{Z}$, and $\gamma' = \gamma$.

Proof. Let $\tilde{1} = 1 + \mathcal{D}(x - \alpha,\ y' - \beta,\ q,\ \theta - \gamma)$ be the canonical generator of $Q(\alpha, \beta, \gamma)$. Then clearly, $q\,\tilde{1} = 0$ and $s^2\,\tilde{1} = \gamma g\,\tilde{1}$. From (22), it follows that

$$Q(\alpha, \beta, \gamma) = \Bbbk[g^{\pm 1}, h]\,\tilde{1} + \Bbbk[g^{\pm 1}, h]s\,\tilde{1}.$$

Let N be a nonzero submodule of $Q(\alpha, \beta, \gamma)$, and let $w \in N$ be a nonzero element. Then we can write $w = \sum_{i=0}^{n} c_i h^i\,\tilde{1}$, with $c_i \in \Lambda := \Bbbk[g^{\pm 1}] + \Bbbk[g^{\pm 1}]s$ and $c_n \neq 0$. If $n > 0$ then

$$(x - \alpha)w = \alpha \sum_{i=1}^{n} \sum_{k=1}^{i} c_i \tbinom{i}{k} (-1)^k h^{i-k}\,\tilde{1}. \tag{24}$$

Repeated application of this identity yields an element $w' = p\,\tilde{1} \in N$ for some nonzero $p \in \Lambda$. We can write $w' = \sum \lambda_i g^i\,\tilde{1} + \sum \mu_j g^j s\,\tilde{1}$, where $\lambda_i,\ \mu_j \in \Bbbk$. Note that the nonzero terms of w' are eigenvectors for y' with distinct eigenvalues:

$$y'g^i\,\tilde{1} = (\beta - 2i\alpha)g^i\,\tilde{1}, \qquad y'g^js\,\tilde{1} = (\beta - (2j+1)\alpha)g^js\,\tilde{1}. \tag{25}$$

It follows that either $g^i\,\tilde{1} \in N$ for some $i \in \mathbb{Z}$, or $g^js\,\tilde{1} \in N$ for some $j \in \mathbb{Z}$. Since g is invertible, we conclude that either $\tilde{1} \in N$ or $s\,\tilde{1} \in N$. In the latter case, since $s^2\,\tilde{1} = \gamma g\,\tilde{1} \in N$, we also have $\tilde{1} \in N$. Hence $N = Q(\alpha,\beta,\gamma)$, and the module $Q(\alpha,\beta,\gamma)$ is simple. It is clear that $\mathfrak{m}_{(0,\gamma)}\mathcal{D} \subseteq \operatorname{ann}_{\mathcal{D}}Q(\alpha,\beta,\gamma)$. Equality holds because $\mathfrak{m}_{(0,\gamma)}\mathcal{D}$ is a maximal ideal of $\mathcal{D}$ by [BS, Theorem 4.1(1)].

We next prove the isomorphism criterion. By (25), the vectors $g^i\,\tilde{1}$ and $g^js\,\tilde{1}$ are twisted Whittaker vectors of types $(\alpha, \beta-2i\alpha)$ and $(\alpha, \beta-(2j+1)\alpha)$, respectively. Since the module is simple, each of these vectors generates $Q(\alpha,\beta,\gamma)$. Hence $Q(\alpha,\beta,\gamma) \cong Q(\alpha,\beta-i\alpha,\gamma)$ for every $i \in \mathbb{Z}$.

Conversely, suppose that $\varphi : Q(\alpha,\beta,\gamma) \to Q(\alpha',\beta',\gamma')$ is an isomorphism of $\mathcal{D}$-modules. Set $Q := Q(\alpha,\beta,\gamma)$ and $Q' := Q(\alpha',\beta',\gamma')$. By (24), the element $x-\alpha$ acts locally nilpotently on Q, while $x-\alpha'$ acts locally nilpotently on Q'. Via φ, the operator $x-\alpha$ is locally nilpotent on Q'. Therefore $(x-\alpha)-(x-\alpha') = \alpha'-\alpha$ is locally nilpotent on Q'. Hence $\alpha' = \alpha$. Since isomorphic modules have the same annihilator, $\operatorname{ann}_{\mathcal{D}}Q = \operatorname{ann}_{\mathcal{D}}Q'$, and therefore $\gamma' = \gamma$. Finally, by (24), $\ker_Q(x-\alpha) = \Lambda\tilde{1}$. The inverse image under φ of the canonical generator of Q' is a nonzero vector in $\ker_Q(x-\alpha)$ on which y' acts by the scalar β'. By (25), the eigenspaces of y' in $\ker_Q(x-\alpha)$ are precisely the spaces spanned by $g^i\,\tilde{1}$ and $g^is\,\tilde{1}$ with eigenvalues $\beta-2i\alpha$ and $\beta-(2i+1)\alpha$, respectively. Therefore, $\beta' = \beta - i\alpha$ for some $i \in \mathbb{Z}$. □

Recall that $z = q^2g^{-1}$ is central in $\mathcal{D}$. For α, $\gamma \in \Bbbk^*$ and $\beta \in \Bbbk$, define

$$S(\alpha,\beta,\gamma) := \mathcal{D}/\mathcal{D}(x-\alpha,\ y'-\beta,\ s,\ z-\gamma).$$

It is clear that $S(\alpha,\beta,\gamma)$ is a twisted Whittaker module of type (α,β). Moreover, there is an isomorphism of $\mathcal{D}$-modules: $S(\alpha,\beta,\gamma) \cong T_{\alpha,\beta}/(sT_{\alpha,\beta} + (z-\gamma)T_{\alpha,\beta})$. The following proposition establishes the simplicity of $S(\alpha,\beta,\gamma)$ and describes its annihilator.

Proposition 4.4. *For any α, $\gamma \in \Bbbk^*$ and $\beta \in \Bbbk$, the module $S(\alpha,\beta,\gamma)$ is simple, and $\operatorname{ann}_{\mathcal{D}}S(\alpha,\beta,\gamma) = \langle s,\ z-\gamma\rangle = \mathfrak{m}_\gamma\mathcal{D}$. Moreover, for $\alpha,\alpha' \in \Bbbk^*$, $\beta,\beta' \in \Bbbk$, and $\gamma,\gamma' \in \Bbbk^*$, we have $S(\alpha,\beta,\gamma) \cong S(\alpha',\beta',\gamma')$ if and only if $\alpha' = \alpha$, $\beta' = \beta - i\alpha$ for some $i \in \mathbb{Z}$, and $\gamma' = \gamma$.*

Proof. The argument is analogous to that of Proposition 4.3. We omit the details. □

Lastly, for α, μ, $\nu \in \Bbbk^*$ and $\beta \in \Bbbk$, define the module

$$F(\alpha,\beta,\mu,\nu) := \mathcal{D}/\mathcal{D}(x-\alpha,\ y'-\beta,\ \omega-\mu,\ \theta-\nu).$$

Recall that $\mathfrak{m}_{(\mu,\nu)} = \langle z-\mu^2\nu^{-1},\ \omega-\mu,\ \theta-\nu\rangle = \langle \omega-\mu,\ \theta-\nu\rangle$ is a maximal ideal of the centre $Z(\mathcal{D})$. Clearly, $F(\alpha,\beta,\mu,\nu)$ is isomorphic to the quotient module $T_{\alpha,\beta}/\mathfrak{m}_{(\mu,\nu)}T_{\alpha,\beta}$.

Proposition 4.5. *For any α, μ, $\nu \in \Bbbk^*$ and $\beta \in \Bbbk$, the module $F(\alpha,\beta,\mu,\nu)$ is simple, and*

$$\operatorname{ann}_{\mathcal{D}} F(\alpha,\beta,\mu,\nu) = \mathfrak{m}_{(\mu,\nu)}\mathcal{D}.$$

Moreover, for $\alpha,\alpha' \in \Bbbk^*$, $\beta,\beta' \in \Bbbk$, $\mu,\mu',\nu,\nu' \in \Bbbk^*$, we have $F(\alpha,\beta,\mu,\nu) \cong F(\alpha',\beta',\mu',\nu')$ if and only if $\alpha' = \alpha$, $\beta' = \beta - i\alpha$ for some $i \in \mathbb{Z}$, $\mu' = \mu$, and $\nu' = \nu$.

Proof. Let $\bar{\mathbf{1}} = 1+\mathcal{D}(x-\alpha,\ y'-\beta,\ \omega-\mu,\ \theta-\nu)$ be the canonical generator of the module $F := F(\alpha,\beta,\mu,\nu)$. Then $q^2\,\bar{\mathbf{1}} = \mu^2\nu^{-1}g\,\bar{\mathbf{1}}$, $qs\,\bar{\mathbf{1}} = \mu g\,\bar{\mathbf{1}}$ and $s^2\,\bar{\mathbf{1}} = \nu g\,\bar{\mathbf{1}}$. Moreover, $\nu q\bar{\mathbf{1}} = qs^2g^{-1}\,\bar{\mathbf{1}} = sqsg^{-1}\,\bar{\mathbf{1}} = \mu s\,\bar{\mathbf{1}}$. From these identities and (22) it follows that

$$F = \Bbbk[g^{\pm 1}, h]\,\bar{\mathbf{1}} + \Bbbk[g^{\pm 1}, h]s\,\bar{\mathbf{1}}.$$

Let M be a nonzero submodule of F, and let w be a nonzero element in M. Then we can write $w = \sum_{i=0}^{n} c_i h^i\,\bar{\mathbf{1}}$, with $c_i \in \Lambda := \Bbbk[g^{\pm 1}] + \Bbbk[g^{\pm 1}]s$, and $c_n \neq 0$. If $n > 0$, repeated application of $(x-\alpha)$ (as in (24)) reduces the h-degree, eventually yielding an element $w' = p\,\bar{\mathbf{1}} \in M$ for some nonzero $p \in \Lambda$. Write $w' = \sum \lambda_i g^i\,\bar{\mathbf{1}} + \sum \gamma_j g^j s\,\bar{\mathbf{1}}$, where $\lambda_i, \gamma_j \in \Bbbk$. The nonzero terms of w' are y'-eigenvectors with distinct eigenvalues: $y'g^i\,\bar{\mathbf{1}} = (\beta - 2i\alpha)g^i\,\bar{\mathbf{1}}$ and $y'g^js\,\bar{\mathbf{1}} = (\beta - (2j+1)\alpha)g^js\,\bar{\mathbf{1}}$. Thus, either $g^i\,\bar{\mathbf{1}} \in M$ for some $i \in \mathbb{Z}$, or $g^js\,\bar{\mathbf{1}} \in M$ for some $j \in \mathbb{Z}$. Since g is invertible, we conclude that either $\bar{\mathbf{1}} \in M$ or $s\,\bar{\mathbf{1}} \in M$. In the latter case, $s^2\,\bar{\mathbf{1}} = \nu g\,\bar{\mathbf{1}} \in M$, which again implies $\bar{\mathbf{1}} \in M$. Consequently, $M = F$, proving that F is simple. To determine the annihilator of F, note first that $\mathfrak{m}_{(\mu,\nu)}\mathcal{D} \subseteq \operatorname{ann}_{\mathcal{D}}F$ by construction. If $(\mu,\nu) \neq (-16,16)$, then the ideal $\mathfrak{m}_{(\mu,\nu)}\mathcal{D}$ is maximal in $\mathcal{D}$ by [BS, Theorem 4.1(i)], and hence $\operatorname{ann}_{\mathcal{D}}F = \mathfrak{m}_{(\mu,\nu)}\mathcal{D}$. Now assume $(\mu,\nu) = (-16,16)$, and suppose for contradiction that $\operatorname{ann}_{\mathcal{D}}F$ strictly contains $\mathfrak{m}_{(-16,16)}\mathcal{D} = \mathfrak{m}_+\mathcal{D}$. Then by [BS, Theorem 4.4(v)], the primitive ideal $\operatorname{ann}_{\mathcal{D}}F$ must contain the ideal $\langle x,\ g-1,\ u\rangle$. However, by construction x does not annihilate F, since $x\,\bar{\mathbf{1}} = \alpha\,\bar{\mathbf{1}} \neq 0$. This contradiction shows that the assumption $\operatorname{ann}_{\mathcal{D}}F \supsetneq \mathfrak{m}_+\mathcal{D}$ is false. Hence, $\operatorname{ann}_{\mathcal{D}}F = \mathfrak{m}_+\mathcal{D}$.

It remains to prove the isomorphism criterion. The vectors $g^i\bar{\mathbf{1}}$ and $g^js\bar{\mathbf{1}}$ are twisted Whittaker vectors of types $(\alpha, \beta-2i\alpha)$ and $(\alpha, \beta-(2j+1)\alpha)$, respectively. Since $F(\alpha,\beta,\mu,\nu)$ is simple, each of these vectors generates the whole module. Hence $F(\alpha,\beta,\mu,\nu) \cong F(\alpha,\beta-i\alpha,\mu,\nu)$ for every $i \in \mathbb{Z}$.

Conversely, suppose that $F(\alpha,\beta,\mu,\nu) \cong F(\alpha',\beta',\mu',\nu')$. As in Proposition 4.3, the local nilpotence of $x-\alpha$ and $x-\alpha'$ implies that $\alpha' = \alpha$. The equality of annihilators gives $\mu' = \mu$ and $\nu' = \nu$. Finally, comparing the y'-eigenvalues of the vectors in $\ker_F(x-\alpha)$, as in Proposition 4.3, yields $\beta' = \beta - i\alpha$ for some $i \in \mathbb{Z}$. □

4.4. Classification of simple twisted Whittaker modules. The following theorem provides a classification of simple twisted Whittaker $\mathcal{D}$-modules of type (α,β), where $\alpha \in \Bbbk^*$ and $\beta \in \Bbbk$.

Theorem 4.6. *Let $\alpha \in \Bbbk^*$ and $\beta \in \Bbbk$, and let M be a $\mathcal{D}$-module. Then M is a simple twisted Whittaker module of type (α,β) if and only if M is isomorphic to one of the following modules:*

- *$W(\alpha, \beta)$;*
- *$Q(\alpha, \beta, \gamma)$ where $\gamma \in \Bbbk^*$;*
- *$S(\alpha, \beta, \gamma)$ where $\gamma \in \Bbbk^*$;*
- *$F(\alpha, \beta, \mu, \nu)$ where $\mu,\ \nu \in \Bbbk^*$.*

Proof. It has been established in Propositions 4.2–4.5, respectively, that the modules $W(\alpha, \beta)$, $Q(\alpha, \beta, \gamma)$ (where $\gamma \in \Bbbk^*$), $S(\alpha, \beta, \gamma)$ (where $\gamma \in \Bbbk^*$) and $F(\alpha, \beta, \mu, \nu)$ (where $\mu,\ \nu \in \Bbbk^*$) are simple twisted Whittaker modules of type (α, β).

For the converse, let M be a simple twisted Whittaker module of type (α, β). Then M is isomorphic to a simple quotient of the universal module $T_{\alpha,\beta}$. Since both q and s are normal elements in $\mathcal{D}$, the kernels and images of the operators $q,\ s : M \to M$ are submodules of M. We distinguish the following four cases:

Case (i): $\ker(q) \neq 0$ and $\ker(s) \neq 0$. Since M is simple, we have $\ker(q) = \ker(s) = M$, implying that M is annihilated by the ideal $q\mathcal{D} + s\mathcal{D}$. Therefore, M is a quotient of the module $W(\alpha, \beta)$, which is simple by Proposition 4.2. Hence, $M \cong W(\alpha, \beta)$.

Case (ii): $\ker(q) \neq 0$ and $\ker(s) = 0$. Here, we have $\ker(q) = M$ and $\mathrm{im}(s) = M$. Thus, M is annihilated by $q\mathcal{D}$, and s acts bijectively on M. Since $\mathcal{D}$ satisfies the Nullstellensatz ([BS, Theorem 4.3]), all central elements act on M by scalars. In particular, the central element $\theta = s^2 g^{-1}$ acts on M as a nonzero scalar $\gamma \in \Bbbk^*$. It follows that M is an epimorphic image of the module $Q(\alpha, \beta, \gamma)$, which is simple by Proposition 4.3. Therefore, $M \cong Q(\alpha, \beta, \gamma)$.

Case (iii): $\ker(q) = 0$ and $\ker(s) \neq 0$. Since M is simple, we have $\mathrm{im}(q) = M$ and $\ker(s) = M$. Therefore, q acts bijectively on M, and M is annihilated by $s\mathcal{D}$. The central element $z = q^2 g^{-1}$ thus acts on M as a nonzero scalar $\gamma \in \Bbbk^*$. Hence, M is a quotient of the module $S(\alpha, \beta, \gamma)$, which is simple by Proposition 4.4, implying $M \cong S(\alpha, \beta, \gamma)$.

Case (iv): $\ker(q) = 0$ and $\ker(s) = 0$. In this case, we have $\mathrm{im}(q) = M$ and $\mathrm{im}(s) = M$. Thus, both q and s act bijectively on M. The central elements $\omega = qsg^{-1}$ and $\theta = s^2 g^{-1}$ act on M as nonzero scalars μ and ν, respectively. Therefore, M is a quotient of the module $F(\alpha, \beta, \mu, \nu)$, which is simple by Proposition 4.5. It follows that $M \cong F(\alpha, \beta, \mu, \nu)$. □

Remark 4.7. Let M be a simple twisted Whittaker $\mathcal{D}$-module of type (α, β) where $\alpha, \beta \in \Bbbk$. If $\alpha = 0$, then M is in fact a Whittaker module of type $(0, \frac{1}{2}\beta)$, and a classification of such modules is given in Section 3 (Theorem 3.7 and Theorem 3.10). On the other hand, if $\alpha \in \Bbbk^*$, then Theorem 4.6 provides a complete classification of simple twisted Whittaker modules of type (α, β). In both cases, the annihilator of each simple twisted Whittaker module has been determined explicitly.

5. Quasi-Whittaker modules

In this section, we classify all simple $\mathcal{D}$-modules that are locally finite over the polynomial subalgebra $\Bbbk[x,u]$. For each such module, we also explicitly determine its annihilator.

5.1. Quasi-Whittaker modules and local finiteness. Define the subalgebra $\mathcal{P} := \Bbbk[x,u]$. Let $\eta : \mathcal{P} \to \Bbbk$ be the algebra homomorphism determined by $\eta(x) = \alpha$ and $\eta(u) = \beta$, where $\alpha, \beta \in \Bbbk$. Let M be a $\mathcal{D}$-module. An element $w \in M$ is called a quasi-Whittaker vector of type (α, β) if it satisfies $xw = \alpha w$ and $uw = \beta w$. We say that M is a quasi-Whittaker module of type (α, β) if it is generated by a quasi-Whittaker vector of type (α, β). The following lemma provides a characterization of simple quasi-Whittaker module in terms of a local finiteness condition.

Lemma 5.1. *Let M be a simple $\mathcal{D}$-module. Then M is locally finite as a $\mathcal{P}$-module if and only if it is a quasi-Whittaker $\mathcal{D}$-module.*

Proof. Suppose M is a locally finite $\mathcal{P}$-module. Then, for any nonzero element $w_0 \in M$, there exists a finite-dimensional $\mathcal{P}$-submodule $N \subseteq M$ such that $w_0 \in N$. Since x and u commute, they have a common eigenvector $w \in N$. Since M is simple, we have $M = \mathcal{D}w$, showing that M is a quasi-Whittaker module.

Conversely, suppose M is a quasi-Whittaker module generated by a quasi-Whittaker vector w of type (α, β). Then for any $\xi \in M$, we can write $\xi = dw$ for some $d \in \mathcal{D}$. Since the elements x and u act ad-locally nilpotently on $\mathcal{D}$, there exists n_1, $n_2 \in \mathbb{N}$ such that $\mathrm{ad}_x^{n_1}(d) = 0$ and $\mathrm{ad}_u^{n_2}(d) = 0$. It follows that $(x-\alpha)^{n_1}\xi = (x-\alpha)^{n_1} dw = \mathrm{ad}_x^{n_1}(d)w = 0$, and similarly, $(u-\beta)^{n_2}\xi = (u-\beta)^{n_2} dw = \mathrm{ad}_u^{n_2}(d)w = 0$. Therefore, the $\mathcal{P}$-submodule $\mathcal{P}\xi$ is finite-dimensional, and M is locally finite as a $\mathcal{P}$-module. □

Let M be a quasi-Whittaker module of type (α, β). Then every nonzero quasi-Whittaker vector $w \in M$ must also be of type (α, β). Indeed, suppose w is a quasi-Whittaker vector of type (α', β'), so that $(x-\alpha')w = 0$ and $(u-\beta')w = 0$. Since M is a quasi-Whittaker module of type (α, β), the operators $x-\alpha$ and $u-\beta$ act locally nilpotently on M. Hence, for the same w, there exists $n \in \mathbb{N}$ such that $(x-\alpha)^n w = 0$ and $(u-\beta)^n w = 0$. These two facts together imply that the difference $(x-\alpha') - (x-\alpha) = \alpha - \alpha'$ must act nilpotently on w. Since w is nonzero, this forces $\alpha' = \alpha$, and similarly $\beta' = \beta$. Therefore, all quasi-Whittaker vectors in M are necessarily of the same type (α, β). Let $\mathcal{K}(M)$ denote the set of all quasi-Whittaker vectors in M. Then

$$\mathcal{K}(M) = \ker(x-\alpha) \cap \ker(u-\beta).$$

Lemma 5.2. *Let M be a quasi-Whittaker $\mathcal{D}$-module of type (α, β). If N is a nonzero submodule of M then $N \cap \mathcal{K}(M) \neq 0$. Furthermore, if $\dim \mathcal{K}(M) = 1$, then M is a simple module.*

Proof. Let $\xi \in N$ be a nonzero element. Since $x-\alpha$ and $u-\beta$ act locally nilpotently on M, the subspace $\mathcal{P}\xi$ is finite-dimensional. Since x and u commute, they have a common eigenvector $w \in \mathcal{P}\xi \subseteq N$. Thus $w \in N$ is a quasi-Whittaker vector, and so $N \cap \mathcal{K}(M) \neq 0$.

Suppose now $\dim \mathcal{K}(M) = 1$. Then for any nonzero element $w \in \mathcal{K}(M)$, we have $\mathcal{K}(M) = \Bbbk w$. Note that $M = \mathcal{D}w$, and any nonzero submodule $N \subseteq M$ contains a nonzero quasi-Whittaker vector by the argument above. Consequently, $w \in N$, which implies $N = M$. Therefore, M is simple. $\square$

5.2. The quasi-Whittaker modules $L_{\alpha,\beta,\gamma}$. For $\alpha, \beta \in \Bbbk$ and $\gamma \in \Bbbk^*$, define

$$L_{\alpha,\beta,\gamma} := \mathcal{D}/\mathbf{I}_{\alpha,\beta,\gamma}, \quad \text{where} \quad \mathbf{I}_{\alpha,\beta,\gamma} = \mathcal{D}(x-\alpha,\ u-\beta,\ g-\gamma).$$

This is clearly a quasi-Whittaker module of type (α, β). Recall that

$$\mathcal{O} = \Bbbk[g^{\pm 1}, x, u]$$

is a commutative Hopf subalgebra of $\mathcal{D}$. Thus any simple $\mathcal{O}$-module is isomorphic to $K_{\alpha,\beta,\gamma} := \mathcal{O}/\mathcal{O}(x-\alpha,\ u-\beta,\ g-\gamma)$ for some $\alpha, \beta \in \Bbbk$ and $\gamma \in \Bbbk^*$. There is an isomorphism of $\mathcal{D}$-modules: $L_{\alpha,\beta,\gamma} \cong \mathcal{D} \otimes_{\mathcal{O}} K_{\alpha,\beta,\gamma}$. Let $\bar{1} = 1 + \mathbf{I}_{\alpha,\beta,\gamma}$ be the canonical generator of $L_{\alpha,\beta,\gamma}$. Then from the PBW basis of $\mathcal{D}$, the elements $h^i y^j v^k\, \bar{1}$ $(i, j, k \in \mathbb{N})$ form a basis of $L_{\alpha,\beta,\gamma}$. Since x and u commute with s, it is evident that $\Bbbk[s]\,\bar{1} \subseteq \mathcal{K}(L_{\alpha,\beta,\gamma})$.

Lemma 5.3. *If M is a simple quasi-Whittaker module of type (α, β), then M is an epimorphic image of $L_{\alpha,\beta,\gamma}$ for some $\gamma \in \Bbbk^*$.*

Proof. Since M is a quasi-Whittaker module, it is generated by a quasi-Whittaker vector w of type (α, β). From the expression of q, we have $qw = (2(1+g)+\alpha\beta)w$. As the central element $z = q^2 g^{-1}$ acts on M as scalar multiplication by some $\mu \in \Bbbk$, it follows that $0 = (z-\mu)w = [(2(1+g)+\alpha\beta)^2 g^{-1} - \mu]w$. Since $\Bbbk$ is algebraically closed, this implies that there exists a nonzero quasi-Whittaker vector $\xi \in M$ of type (α, β) that is also an eigenvector of g, say $g\xi = \gamma\xi$ for some $\gamma \in \Bbbk^*$. By the simplicity of M, we must have $M = \mathcal{D}\xi$, and hence M is an epimorphic image of the module $L_{\alpha,\beta,\gamma}$. $\square$

Remark 5.4. To classify simple quasi-Whittaker modules, it suffices, by Lemma 5.3, to determine the simple quotients of $L_{\alpha,\beta,\gamma}$. Notice that when $(\alpha, \beta, \gamma) = (0, 0, 1)$, the left ideal $\mathbf{I}_{0,0,1}$ is identical to the ideal $\langle x,\ u,\ g-1\rangle$ by (4). Thus, $L_{0,0,1} \cong \mathcal{U}$, and any simple quotient of $L_{0,0,1}$ is a simple $\mathcal{U}$-module. In this case, we have $q\,\bar{1} = 4\,\bar{1}$, $s\,\bar{1} = -4\,\bar{1}$, and the annihilator of $L_{0,0,1}$ is given by $\operatorname{ann}_{\mathcal{D}} L_{0,0,1} = \mathfrak{m}_+\mathcal{D}$. In the remainder of this section, we consider the case where $(\alpha, \beta, \gamma) \neq (0, 0, 1)$.

Let $\alpha, \beta, \varsigma \in \Bbbk$ and $\gamma \in \Bbbk^*$. If $(\alpha, \beta, \gamma) \neq (0, 0, 1)$, we define

$$\mathcal{V}(\alpha, \beta, \gamma, \varsigma) := \mathcal{D}/\mathbf{J}_{\alpha,\beta,\gamma,\varsigma}, \quad \text{where } \mathbf{J}_{\alpha,\beta,\gamma,\varsigma} := \mathcal{D}(x-\alpha,\ u-\beta,\ g-\gamma,\ s-\varsigma).$$

This is clearly a quasi-Whittaker $\mathcal{D}$-module of type (α, β). Moreover, it is evident that $\mathcal{V}(\alpha, \beta, \gamma, \varsigma)$ is isomorphic to the quotient module $L_{\alpha,\beta,\gamma}/\mathcal{D}(s-\varsigma)\,\bar{1}$. As noted in Remark 5.4, if $(\alpha,\beta,\gamma)=(0,0,1)$, then it must be that $\varsigma=-4$; otherwise, the module $\mathcal{V}(0,0,1,\varsigma)$ would collapse to zero. The following proposition shows that $\mathcal{V}(\alpha,\beta,\gamma,\varsigma)$ is simple and also determines its annihilator. Furthermore, the module $\mathcal{V}(\alpha,\beta,\gamma,\varsigma)$ is genuinely non-weight when $(\alpha,\beta)\neq(0,0)$, and weight when $(\alpha,\beta)=(0,0)$. Thus, in some sense weight and non-weight $\mathcal{D}$-modules are rejoined in the family $\mathcal{V}(\alpha,\beta,\gamma,\varsigma)$.

Proposition 5.5.

(1) *If $(\alpha,\beta)\neq(0,0)$, then for any $\gamma\in\Bbbk^*$ and $\varsigma\in\Bbbk$, the module $\mathcal{V}(\alpha,\beta,\gamma,\varsigma)$ is simple. Moreover, define $\dot z=(\alpha\beta+2(1+\gamma))^2\gamma^{-1}$, $\dot\omega:=(\alpha\beta+2(1+\gamma))\varsigma\gamma^{-1}$, and $\dot\theta:=\varsigma^2\gamma^{-1}$. Then the annihilator of $\mathcal{V}(\alpha,\beta,\gamma,\varsigma)$ is given by*

$$\operatorname{ann}_{\mathcal{D}}\mathcal{V}(\alpha,\beta,\gamma,\varsigma)=\begin{cases}\mathfrak{m}_{(\dot\omega,\dot\theta)}\mathcal{D}, & \text{if } \varsigma\neq0,\\ \mathfrak{m}_{\dot z}\mathcal{D}, & \text{if } \varsigma=0 \text{ and } \alpha\beta+2(1+\gamma)\neq0,\\ q\mathcal{D}+s\mathcal{D}, & \text{if } \varsigma=0 \text{ and } \alpha\beta+2(1+\gamma)=0.\end{cases}$$

(2) *If $(\alpha,\beta)=(0,0)$ and $\gamma\neq1$, then $\mathcal{V}(0,0,\gamma,\varsigma)$ is a simple weight module, and each of its weight spaces is infinite-dimensional. Moreover, define $\dot z=4(1+\gamma)^2\gamma^{-1}$, $\dot\omega:=2(1+\gamma)\varsigma\gamma^{-1}$, and $\dot\theta:=\varsigma^2\gamma^{-1}$. Then the annihilator of $\mathcal{V}(0,0,\gamma,\varsigma)$ is given by*

$$\operatorname{ann}_{\mathcal{D}}\mathcal{V}(0,0,\gamma,\varsigma)=\begin{cases}\mathfrak{m}_{(\dot\omega,\dot\theta)}\mathcal{D}, & \text{if } \varsigma\neq0,\\ \mathfrak{m}_{\dot z}\mathcal{D}, & \text{if } \varsigma=0 \text{ and } \gamma\neq-1,\\ q\mathcal{D}+s\mathcal{D}, & \text{if } \varsigma=0 \text{ and } \gamma=-1.\end{cases}$$

(3) *If $\alpha,\alpha',\beta,\beta',\varsigma,\varsigma'\in\Bbbk$, and $\gamma,\gamma'\in\Bbbk^*$, then $\mathcal{V}(\alpha,\beta,\gamma,\varsigma)\cong\mathcal{V}(\alpha',\beta',\gamma',\varsigma')$ if and only if $\alpha'=\alpha$, $\beta'=\beta$, $\gamma'=\gamma$, and $\varsigma'=\varsigma$.*

Proof. (1) Let $\tilde 1:=1+\mathbf{J}_{\alpha,\beta,\gamma,\varsigma}$ be the canonical generator of $\mathcal{V}:=\mathcal{V}(\alpha,\beta,\gamma,\varsigma)$. Note that the element s can alternatively be expressed as $s=vx-ux+yu+(-\frac{1}{2}ux+g-1)h-2(g+1)$. It follows that

$$\varsigma\,\tilde 1=s\,\tilde 1=\Big(\alpha v+\beta y+(-\frac{1}{2}\alpha\beta+\gamma-1)h-\alpha\beta-2(\gamma+1)\Big)\,\tilde 1. \tag{26}$$

Recall that $\mathcal{V}\cong L_{\alpha,\beta,\gamma}/\mathcal{D}(s-\varsigma)\,\bar 1$, and the elements $h^iy^jv^k\,\bar 1$ $(i,j,k\in\mathbb{N})$ form a basis of $L_{\alpha,\beta,\gamma}$.

If $\alpha\in\Bbbk^*$, then from (26) it follows that $\mathcal{V}$ is spanned by the elements $h^iy^j\,\tilde 1$ $(i,j\in\mathbb{N})$. Since $y'\,\tilde 1=(\alpha h+2y)\,\tilde 1$, we also have $\mathcal{V}=\sum_{i,j\in\mathbb{N}}\Bbbk h^iy'^j\,\tilde 1$. Now, let N be a nonzero submodule of $\mathcal{V}$, and let w be a nonzero element in N. Write $w=\sum_{i=0}^n c_i(h)y'^i\,\tilde 1$, where $c_i(h)\in\Bbbk[h]$ and $c_n(h)\neq0$. Using the relation $gy'=(y'+2x)g$, we deduce that, for $n>0$,

$$(g-\gamma)w=\gamma\sum_{i=1}^{n}\sum_{k=1}^{i}c_i(h)\tbinom{i}{k}(2\alpha)^ky'^{i-k}\,\tilde 1.$$

Since $\alpha,\gamma\in\Bbbk^*$, repeated application of this identity yields an element $w'=c(h)\,\tilde 1\in N$, where $c(h)\in\Bbbk[h]$ is nonzero. Write $w'=\sum_{i=0}^m k_ih^i\,\tilde 1$, where

$k_i \in \Bbbk$ and $k_m \neq 0$. If $m > 0$, then

$$(x-\alpha)w' = \alpha \sum_{i=1}^{m} \sum_{j=1}^{i} k_i \tbinom{i}{j} (-1)^j h^{i-j}\, \tilde{1}.$$

Applying this identity repeatedly, we eventually obtain $\tilde{1} \in N$. Thus, $N = \mathcal{V}$, and so $\mathcal{V}$ is simple.

If $\beta \in \Bbbk^*$, then from (26) it follows that $\mathcal{V}$ is spanned by the elements $h^i v^j\, \tilde{1}$ ($i, j \in \mathbb{N}$). Since $v'\, \tilde{1} = (\beta h - 2v)\, \tilde{1}$, we also have $\mathcal{V} = \sum_{i,j\in\mathbb{N}} h^i v'^j\, \tilde{1}$. Now, let N be a nonzero submodule of $\mathcal{V}$, and let w be a nonzero element in N. Then we can write $w = \sum_{i=0}^{n} c_i(h) v'^i\, \tilde{1}$, where $c_i(h) \in \Bbbk[h]$ and $c_n(h) \neq 0$. Using the relation $gv' = (v' + 2u)g$, we deduce that if $n > 0$ then

$$(g-\gamma)w = \gamma \sum_{i=1}^{n} \sum_{k=1}^{i} c_i(h) \tbinom{i}{k} (2\beta)^k v'^{i-k}\, \tilde{1}.$$

Since $\beta, \gamma \in \Bbbk^*$, repeated application of this identity yields an element $w' = c(h)\, \tilde{1} \in N$, where $c(h) \in \Bbbk[h]$ is nonzero. Write $w' = \sum_{i=0}^{m} k_i h^i\, \tilde{1}$, where $k_i \in \Bbbk$ and $k_m \neq 0$. If $m > 0$, then

$$(u-\beta)w' = \beta \sum_{i=1}^{m} \sum_{j=1}^{i} k_i \tbinom{i}{j} h^{i-j}\, \tilde{1}.$$

Applying this identity repeatedly, we eventually obtain $\tilde{1} \in N$. Thus, $N = \mathcal{V}$, and so $\mathcal{V}$ is simple.

We now determine the annihilator of $\mathcal{V}$. Since $q\, \tilde{1} = (\alpha\beta + 2(1+\gamma))\, \tilde{1}$ and $s\, \tilde{1} = \varsigma\, \tilde{1}$, it follows immediately that the ideal $\mathfrak{p} := (z - \dot{z})\mathcal{D} + (\omega - \dot{\omega})\mathcal{D} + (\theta - \dot{\theta})\mathcal{D}$ is contained in $\operatorname{ann}_{\mathcal{D}} \mathcal{V}$. We consider three cases: (i) If $\varsigma \neq 0$, then $\dot{\theta} \neq 0$. If $\mathfrak{p} \neq \mathfrak{m}_+\mathcal{D}$, then by [BS, Theorem 4.1(i)], the ideal $\mathfrak{p} = \mathfrak{m}_{(\dot{\omega},\dot{\theta})}\mathcal{D}$ is maximal in $\mathcal{D}$, and hence $\operatorname{ann}_{\mathcal{D}} \mathcal{V} = \mathfrak{m}_{(\dot{\omega},\dot{\theta})}\mathcal{D}$. Assume now that $\mathfrak{p} = \mathfrak{m}_+\mathcal{D}$ and suppose, for contradiction, that $\operatorname{ann}_{\mathcal{D}} \mathcal{V} \supsetneq \mathfrak{m}_+\mathcal{D}$. Then by [BS, Theorem 4.4(v)], the primitive ideal $\operatorname{ann}_{\mathcal{D}} \mathcal{V}$ must contain $\langle x, g-1, u\rangle$. However, by construction, at least one of x or u does not annihilate $\mathcal{V}$, since $(\alpha, \beta) \neq (0,0)$. This contradiction shows that the assumption $\operatorname{ann}_{\mathcal{D}} \mathcal{V} \supsetneq \mathfrak{m}_+\mathcal{D}$ is false. Hence, $\operatorname{ann}_{\mathcal{D}} \mathcal{V} = \mathfrak{m}_+\mathcal{D}$. (ii) If $\varsigma = 0$ and $\alpha\beta + 2(1+\gamma) \neq 0$, then $\dot{z} \in \Bbbk^*$ and $\dot{\omega} = \dot{\theta} = 0$. By [BS, Theorem 4.1(i)], the ideal $\mathfrak{p} = \mathfrak{m}_{\dot{z}}\mathcal{D}$ is maximal in $\mathcal{D}$, and hence $\operatorname{ann}_{\mathcal{D}} \mathcal{V} = \mathfrak{m}_{\dot{z}}\mathcal{D}$. (iii) If $\varsigma = 0$ and $\alpha\beta + 2(1+\gamma) = 0$, then $P_0 = q\mathcal{D} + s\mathcal{D}$ annihilates $\mathcal{V}$. By [BS, Theorem 4.13], the ideal P_0 is maximal, and so $\operatorname{ann}_{\mathcal{D}} \mathcal{V} = P_0$.

(2) If $(\alpha, \beta) = (0,0)$ and $\gamma \neq 1$, then from equation (26), we obtain $h\, \tilde{1} = \lambda\, \tilde{1}$ where $\lambda := \frac{2(\gamma+1)+\varsigma}{\gamma-1}$. This implies that the left ideal $\mathcal{D}(x, u, g-\gamma, s-\varsigma)$ coincides with $\mathcal{D}(x, u, g-\gamma, h-\lambda)$. Therefore, the module $\mathcal{V} := \mathcal{V}(0,0,\gamma,\varsigma) = \mathcal{D}/\mathcal{D}(x, u, g-\gamma, h-\lambda)$ is a weight module. From the PBW basis of $\mathcal{D}$, it follows that $\mathcal{V} = \bigoplus_{i,j\in\mathbb{N}} \Bbbk y^i v^j\, \tilde{1}$, so each weight space of $\mathcal{V}$ is infinite-dimensional. Moreover, we also have the alternative basis $\mathcal{V} = \bigoplus_{i,j\in\mathbb{N}} \Bbbk y'^i v'^j\, \tilde{1}$. Let N be a

nonzero submodule of $\mathcal{V}$, and let w be a nonzero element in N. We can write $w = \sum_{i=0}^{n} c_i(y')v'^i\,\tilde{1}$, where $c_i(y') \in \Bbbk[y']$ and $c_n(y') \neq 0$. If $n > 0$, then using the relations $[x, y'] = 0$ and $[u, v'] = 0$, together with identity (8), we compute

$$xw = \sum_{i=1}^{n} c_i(y') \sum_{k=1}^{i} \binom{i}{k} u^{k-1} v'^{i-k}(q - 2^{k+1}g)\,\tilde{1} = 2(1-\gamma)\sum_{i=0}^{n} ic_i(y')v'^{i-1}\,\tilde{1}.$$

Since $\gamma \neq 1$, repeated application of this identity yields an element $w' = p(y')\,\tilde{1} \in N$, where $p(y')$ is a nonzero polynomial in $\Bbbk[y']$. We can express $w' = \sum_{i=0}^{m} c_i y'^i\,\tilde{1}$, where $c_i \in \Bbbk$ and $c_m \neq 0$. Using the identity (16), we compute:

$$uw' = 2(1-\gamma)\sum_{i=0}^{m} ic_i y'^{i-1}\,\tilde{1}.$$

Again, by repeated application, we obtain $\tilde{1} \in N$, implying that $N = \mathcal{V}$. Hence, the module $\mathcal{V}$ is simple. The annihilator of $\mathcal{V}$ can be determined analogously to the previous cases. Notably, $\mathfrak{m}_+\mathcal{D}$ does not appear as the annihilator of $\mathcal{V}(0, 0, \gamma, \varsigma)$.

(3) By [BS, Lemma 3.5(1)], the elements x and u act ad-locally nilpotently on $\mathcal{D}$. Moreover, using the defining relations, one verifies that $\mathrm{ad}_g(x) = \mathrm{ad}_g(u) = \mathrm{ad}_g(h) = 0$, $\mathrm{ad}_g^2(y) = 0$, $\mathrm{ad}_g^2(v) = 0$; and $\mathrm{ad}_s(x) = \mathrm{ad}_s(u) = \mathrm{ad}_s(g) = \mathrm{ad}_s(h) = 0$, $\mathrm{ad}_s^2(y) = 0$, $\mathrm{ad}_s^2(v) = 0$. Hence, by the Leibniz rule, the elements g and s also act ad-locally nilpotently on $\mathcal{D}$.

Let $\mathcal{V} = \mathcal{V}(\alpha, \beta, \gamma, \varsigma)$. Since $\mathcal{V}$ is cyclically generated by $\tilde{1}$, for any $w = d\tilde{1}$ with $d \in \mathcal{D}$, ad-local nilpotence gives $(x-\alpha)^n w = \mathrm{ad}_x^n(d)\tilde{1} = 0$ for some $n \in \mathbb{N}$. The same argument applies to $u - \beta$, $g - \gamma$, and $s - \varsigma$. Thus these four elements act locally nilpotently on $\mathcal{V}$.

Now suppose that $\mathcal{V}(\alpha, \beta, \gamma, \varsigma) \cong \mathcal{V}(\alpha', \beta', \gamma', \varsigma')$. Then, on $\mathcal{V}(\alpha', \beta', \gamma', \varsigma')$, both $x - \alpha$ and $x - \alpha'$ act locally nilpotently. Hence $(x-\alpha) - (x-\alpha') = \alpha' - \alpha$ is locally nilpotent, which implies $\alpha' = \alpha$. Similarly, $\beta' = \beta$, $\gamma' = \gamma$, and $\varsigma' = \varsigma$. □

5.3. Classification of simple quasi-Whittaker modules. We denote by $\widehat{\mathcal{D}}(\text{qWh-}(\alpha, \beta))$ the set of isomorphism classes of simple quasi-Whittaker $\mathcal{D}$-modules of type (α, β). The following theorem provides a classification of such modules.

Theorem 5.6. *Let M be a simple quasi-Whittaker $\mathcal{D}$-module of type (α, β), where $(\alpha, \beta) \in \Bbbk^2$.*

(1) *If $(\alpha, \beta) \neq (0, 0)$ then M is isomorphic to $\mathcal{V}(\alpha, \beta, \gamma, \varsigma)$ for some $\gamma \in \Bbbk^*$ and $\varsigma \in \Bbbk$. Hence,*

$$\widehat{\mathcal{D}}(\text{qWh-}(\alpha, \beta)) = \big\{[\mathcal{V}(\alpha, \beta, \gamma, \varsigma)] \mid \gamma \in \Bbbk^*,\ \varsigma \in \Bbbk\big\}.$$

(2) *If $(\alpha, \beta) = (0, 0)$ then M is either isomorphic to $\mathcal{V}(0, 0, \gamma, \varsigma)$ for some $\gamma \in \Bbbk^* \setminus \{1\}$ and $\varsigma \in \Bbbk$, or isomorphic to a simple $\mathcal{U}$-module. Thus,*

$$\widehat{\mathcal{D}}(\text{qWh-}(0, 0)) = \big\{[\mathcal{V}(0, 0, \gamma, \varsigma)] \mid \gamma \in \Bbbk^* \setminus \{1\},\ \varsigma \in \Bbbk\big\} \cup \widehat{\mathcal{U}}.$$

Proof. By Lemma 5.3, the module M is isomorphic to a simple quotient of $L_{\alpha,\beta,\gamma}$ for some $\gamma \in \Bbbk^*$. In particular, M is generated by a quasi-Whittaker vector w of type (α,β) satisfying $gw = \gamma w$. Since the central element $\theta = s^2 g^{-1}$ acts as a scalar μ on M, it follows that $(s^2 - \mu\gamma)w = 0$. As $\Bbbk$ is algebraically closed, this implies the existence of a quasi-Whittaker vector $w' \in M$ such that $xw' = \alpha w'$, $uw' = \beta w'$, $gw' = \gamma w'$, and $sw' = \varsigma w'$ for some $\varsigma \in \Bbbk$. Thus, M is a quotient of $\mathcal{V}(\alpha,\beta,\gamma,\varsigma)$.

(1) If $(\alpha,\beta) \neq (0,0)$, then by Proposition 5.5(1), the module $\mathcal{V}(\alpha,\beta,\gamma,\varsigma)$ is simple for all $\gamma \in \Bbbk^*$ and $\varsigma \in \Bbbk$. Hence, we have $M \cong \mathcal{V}(\alpha,\beta,\gamma,\varsigma)$.

(2) If $(\alpha,\beta) = (0,0)$, we distinguish two cases: (i) If $\gamma = 1$, then since M is a simple quotient of $L_{0,0,1}$, it follows from Remark 5.4 that M is in fact a simple $\mathcal{U}$-module. Conversely, any simple $\mathcal{U}$-module naturally admits the structure of a simple quasi-Whittaker $\mathcal{D}$-module of type $(0,0)$. (ii) If $\gamma \neq 1$, then by Proposition 5.5(2), the module $\mathcal{V}(0,0,\gamma,\varsigma)$ is simple for all $\varsigma \in \Bbbk$. Therefore, $M \cong \mathcal{V}(0,0,\gamma,\varsigma)$. □

Remark 5.7. A classification of simple $\mathcal{U}$-modules was obtained by Block in [Bl, Corollary 5.5]. Let $(\alpha,\beta) \in \Bbbk^2$. When $(\alpha,\beta) \neq (0,0)$, the family $\mathcal{V}(\alpha,\beta,\gamma,\varsigma)$ contains many simple modules whose annihilator is $\mathfrak{m}_+\mathcal{D}$. In contrast, $\mathfrak{m}_+\mathcal{D}$ does not appear as the annihilator of any $\mathcal{V}(0,0,\gamma,\varsigma)$ where $\gamma \in \Bbbk^* \setminus \{1\}$. Note that a simple quasi-Whittaker $\mathcal{D}$-module of type (α,β) is a weight module only when $(\alpha,\beta) = (0,0)$. In that case, it is either isomorphic to a module of the form $\mathcal{V}(0,0,\gamma,\varsigma)$ with $\gamma \in \Bbbk^* \setminus \{1\}$ and $\varsigma \in \Bbbk$, or to a simple weight $\mathcal{U}$-module.

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