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Reductions and cores of ideals in trivial ring extensions over integral domains

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ABSTRACT. This paper deals with reductions and core of ideals in commutative rings. We examine trivial ring extensions arising from integral domains, particularly valuation and Prüfer domains. This study builds on previous findings related to Prüfer domains and earlier work on trivial ring extensions of special modules.

CONTENTS

1. Introduction

All rings considered in this paper are commutative with identity. Let R be a ring, and I an ideal of R. An ideal J is termed a reduction of I if $J \subseteq I$ and $JI^n = I^{n+1}$ for some integer $n \geq 1$. An ideal lacking a proper reduction is termed basic. The concept of reduction was introduced by Northcott and Rees to contribute to the analytic theory of ideals in local Noetherian rings through minimal reductions. In the context of a local Noetherian ring (R, M), [29] asserts that a non-basic ideal I has at least one minimal reduction. If the residue field is infinite, a reduction $J \subseteq I$ of I is minimal if and only if $\mu(J) = l(I)$, where $\mu(J)$ denotes the number of a minimal generating set of J, and $l(I)$ denotes the analytic spread of I.

In [10, 11], Hays explored reductions of ideals in more general settings of commutative rings, not necessarily local or Noetherian. He provided a new characterization for the Prüfer condition, stating that a domain is Prüfer if and only if all its finitely generated ideals are basic. The main result of his papers is that in a domain, every ideal is basic if and only if it is a Prüfer domain with

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dimension one. For more details on this topic, refer to Huneke and Swanson's book [15] and also [12, 40].

The core of the ideal I, denoted by core(I), is defined as the intersection of all its reductions. Initially introduced by Judith Sally and mentioned in Rees and Sally's 1988 paper [36], the core arose in the context of Briancon-Skoda's Theorem. If R is a regular local ring of dimension d and I is an ideal of R, then core(I) contains the integral closure of I^d [15, Chapter 13]. Most works on the core focus on the Noetherian case, particularly Cohen-Macaulay rings and regular rings (cf. [6, 8, 14, 16, 17, 35]).

In [22], the notion of core is studied beyond Noetherian settings. The main results provide an explicit formula for the core of an ideal in a valuation domain or a pseudo-valuation domain, using multiplicative ideal theory (e.g., [18, 20, 21]). The impact of stability on the core's shape is considered, and a section is devoted to the problem of the existence of minimal reductions beyond the context of Noetherian rings. In [23], the study contributes to the cores of ideals in Noetherian rings, determining explicit formulas for the core in various classes of one-dimensional Noetherian domains, with illustrated examples. In [24], reductions of ideals in pullback constructions are investigated, covering various types of ideals in both generic and classical pullbacks. The paper [25] explores minimal reductions and cores of ideals in pullback constructions, providing explicit computations of the core in original examples. In [27], the authors derive explicit formulas for the core in several classes of Prüfer domains, with particular emphasis on ideal-theoretic properties such as stability, invertibility, and the h-local property. They also establish decomposition results for the core of an ideal in integral domains, with meaningful implications for the structure of Prüfer domains.

The Nagata idealization of an A-module E over a ring A, also known as the trivial ring extension, is represented by the ring $R := A \ltimes E$. In this construction, the underlying group is $A \times E$, and the multiplication is defined as $(a, e)(a', e') = (aa', ae' + a'e)$. Nagata introduced this concept [28] to facilitate the interaction between rings and their modules, offering diverse contexts for rings with zero-divisors. Numerous papers in the literature delve into the transfer of ring-theoretic notions in various trivial extension settings (see, for example, [1, 4, 32, 33, 34, 37, 38]). For further insights into commutative trivial extensions, readers are encouraged to consult Huckaba's book [13], Glaz's book [9], and the survey paper by D. D. Anderson and Winders [3].

In [26], we study reductions and core in generic trivial ring extensions issued over arbitrary rings (with zero-divisors) and investigate reductions and core of arbitrary ideals (i.e., not necessarily saturated) in trivial extensions over special categories of modules. This paper represents a continuation and deepening of our investigation into reductions and cores of ideals in commutative rings. Our expanded study broadens its scope to examine trivial extensions arising from integral domains, specifically from valuation and Prüfer domains. This extension builds upon our previous findings related to Prüfer domains and our initial work on trivial ring extensions emanating from special modules.

2. Reductions

For a domain A with quotient field k and a nonzero ideal I, we denote by I^{-1} the dual of I given by $(A : I) = \{x \in k \mid xI \subseteq A\}$ and by $(I : I)$ the endomorphism ring $\{x \in k \mid xI \subseteq I\}$. The following containments

$$A \subseteq (I : I) \subseteq I^{-1} \subseteq k$$

always hold. It is pertinent to recall that if $A := V$ is a valuation domain, then each nonzero ideal I is either invertible (i.e., $II^{-1} = V$) or $II^{-1} = P \in \mathrm{Spec}(V)$ with $(I : I) = V_P$ [2, Theorem 2.8]. Also, recall that a nonzero ideal I of a domain is *stable* (resp., *strongly stable*) if it is invertible (resp., principal) in its endomorphism ring $(I : I)$ (cf. [2, 7, 20, 21, 31, 32, 33, 39]). A stable ideal is always locally stable, and the converse holds under the finite character condition [30, Lemma 4.3].

We commence with a basic result that establishes a connection between the non-basic condition and strong stability in valuation domains, which will be employed consistently throughout. Recall an ideal is non-basic if it admits a proper reduction.

Lemma 2.1. *In a valuation domain, an ideal is non-basic if and only if it is strongly stable and not principal.*

Proof. Let V be a valuation domain, I a non-zero ideal of V, and $T := (I : I)$. Suppose that I is not basic. Then, I is not principal, since invertible ideals are basic [22, Lemma 2.1]. Let J be a proper reduction of I and $0 \neq a \in I \setminus J$. Since $J \subsetneqq aV \subseteq I$ and J is a reduction of I, aV is a principal reduction of I. That is, $aI = I^2$. Hence $I \subseteq (aI : I) = aT$, which yields $I = aT$, as desired. Conversely, suppose that I is strongly stable and not principal. Then, $I = aT$, for some nonzero $a \in I$, and so $aI = a^2T = I^2$. It follows that aV is a proper reduction of I and thus I is not basic. □

If J is a reduction of I, the smallest positive integer n satisfying the equality $JI^n = I^{n+1}$ is called the *reduction number* of J. It is denoted by $\mathbf{r}_I(J)$ or simply $\mathbf{r}(J)$ when no confusion is likely. In this vein, recall the well-known result that, in Prüfer domains, J is a reduction of I if and only if $\mathbf{r}(J) = 1$ (i.e., $JI = I^2$).

Consider a ring A, an A-module E, and the ring $R := A \ltimes E$. If I is a nonzero ideal of A and F is a submodule of E, then $I \ltimes F$ is an ideal of R if and only if $IE \subseteq F$. However, it's important to recall that not all ideals of R adhere to this structure [19, Example 2.5]. Throughout, given an ideal H of R, let

$$A_H := \{a \in A \mid (a, e) \in H, \text{ for some } e \in E\}, \text{ an ideal of } A$$

$$E_H := \{e \in E \mid (a, e) \in H, \text{ for some } a \in A\}, \text{ a submodule of } E.$$

Observe that $A_H \ltimes E_H$ is an ideal of R with

$$H \subseteq A_H \ltimes E_H$$

and if $H := I \ltimes F$, then $A_H = I$ and $E_H = F$. The ideals of R of the form $I \ltimes F$ are called *saturated*, and H being non-saturated indicates that $H \subsetneqq A_H \ltimes E_H$. Finally, note that two of the most significant families of trivial extensions emerge from localizations (i.e., $E = A_p$, $p \in \operatorname{Spec}(A)$) and when E is a module over the endomorphism ring.

The next result unveils a dual inequality for the reduction number of *saturated* reductions of ideals in the form of $I \ltimes E$ within the broader context, with particular implications for trivial ring extensions arising from valuation domains.

Theorem 2.2. *Let A be a ring, E an A-module, $R = A \ltimes E$, I a nonzero ideal of A, and $T := (I : I)$. Let J be a reduction of I and F a submodule of E with $JE \subseteq F$. Then:*

1. *$J \ltimes F$ is a reduction of $I \ltimes E$ with* $\mathbf{r}(J) \leq \mathbf{r}_{I\ltimes E}(J \ltimes F) \leq \mathbf{r}(J) + 1$.
2. *Assume $A := V$ is a valuation domain and then $1 \leq \mathbf{r}_{I\ltimes E}(J \ltimes F) \leq 2$. If E is a T-module or $E = V_p$, for some height one prime ideal p of V, then $\mathbf{r}_{I\ltimes E}(J \ltimes F) = 1$. If $E = V_p$ with* $\operatorname{ht}(p) > 1$, *then $\mathbf{r}_{I\ltimes E}(J_o \ltimes F_o) = 2$, for some reduction J_o of I and submodule F_o of E.*

Proof. If $J = I$, then $(I \ltimes F)(I \ltimes E) = I^2 \ltimes (IE + IF) = I^2 \ltimes IE = (I \ltimes E)^2$ and so $\mathbf{r}_{I\ltimes E}(I \ltimes F) = 1$. Next, we assume that J is a proper reduction of I (i.e, I is not basic).

(1) Let $r := \mathbf{r}(J)$. Then $JI^r = I^{r+1}$ and so

$$\begin{array}{rcl}(J \ltimes F)(I \ltimes E)^{r+1} & = & (J \ltimes F)(I^{r+1} \ltimes I^rE) \\ & = & JI^{r+1} \ltimes (JI^rE + I^{r+1}F) \\ & = & I^{r+2} \ltimes (I^{r+1}E + I^{r+1}F) \\ & = & I^{r+2} \ltimes I^{r+1}E \\ & = & (I \ltimes E)^{r+2}.\end{array}$$

It follows that $J \ltimes F$ is a reduction of $I \ltimes E$ and $\mathbf{r}_{I\ltimes E}(J \ltimes F) \leq r+1$. On the other hand, let $s := \mathbf{r}_{I\ltimes E}(J \ltimes F)$. Then, the equality $(J \ltimes F)(I \ltimes E)^s = (I \ltimes E)^{s+1}$ forces $JI^s = I^{s+1}$ and hence $r \leq s$, as desired.

(2) Assume $A := V$ is a valuation domain and let $T = V_P$, for some prime ideal P of V. Since I is non-basic, then $I = aT$, for some $0 \neq a \in I$, by Lemma 2.1. Further, the equality $JI = I^2$ yields $JT = I$. If E is a T-module, then $TE = E$ and hence $JE = JTE = IE$. If $E = V_p$, for some $p \in \operatorname{Spec}(V)$, with $\operatorname{ht}(p) = 1$, then $p \subseteq P$. Therefore, the equality $JV_P = I$ implies that $JV_p = IV_p$; that is, $JE = IE$. For both cases, we obtain

$$\begin{array}{rcl}(J \ltimes F)(I \ltimes E) & = & JI \ltimes (JE + IF) \\ & = & JI \ltimes (IE + IF) \\ & = & JI \ltimes IE \\ & = & I^2 \ltimes IE \\ & = & (I \ltimes E)^2\end{array}$$

Hence, $\mathbf{r}_{I\ltimes E}(J\ltimes F)=1$.

Next, assume $E=V_p$, for some $p\in \mathrm{Spec}(V)$ with $\mathrm{ht}(p)\geq 2$, and let $p_o\in \mathrm{Spec}(V)$ such that $0\subsetneqq p_o\subsetneqq p$. Let $0\neq a\in p_o$, $I:=aV_{p_o}\subseteq p_oV_{p_o}=p_o$, and $J=F:=ap$. Clearly, $JI=a^2pV_{p_o}=a^2V_{p_o}=I^2$ so that J is proper reduction of I. Also, $JE=apV_p=ap=F$, as desired. Further, we have

$$\begin{array}{rcl} JE+IF & = & ap+a^2V_{p_o}\\ & = & a(p+I)\\ & = & ap\\ & = & J\\ & \subsetneqq & I\\ & = & IE. \end{array}$$

It follows that $(J\ltimes F)(I\ltimes E)\subsetneqq (I\ltimes E)^2$ and hence $\mathbf{r}_{I\ltimes E}(J\ltimes F)=2$. □

Remark 2.3. Let V be a valuation domain, $E=V_p$ for some $p\in\mathrm{Spec}(V)$, $R=V\ltimes E$, I a nonzero ideal of V, and $T:=(I:I)$. It is evident from the proof of the second statement in Theorem 2.2 that: "$\mathbf{r}_{I\ltimes E}(J\ltimes F)=1$, *for every reduction J of I and submodule F of E with $JE\subseteq F$, if and only if* $\mathrm{ht}(p)\leq 1$". Notice that if $\mathrm{ht}(p)=0$, then $p=(0)$ and $E=V_{(0)}$ is the quotient field of V, hence a T-module and so the first case of Theorem 2.2(2) applies.

Follow explicit Illustrative examples for Theorem 2.2.

Example 2.4. Let k be a field and let X,Y be indeterminates over k.

(1) Let $A:=k[X^3,X^4]$, $I:=(X^3,X^4)A$, and $J:=X^3A$. Then, $T:=(I:I)=k[X^3,X^4,X^5]$. Next, let $E=T$ and $F=A$. Clearly, $JE=X^3k[X^3,X^4,X^5]\subseteq A=F$ and so $J\ltimes F$ is a subideal of $I\ltimes E$. Since $JI=(X^6,X^7)A\subsetneqq(X^6,X^7,X^8)=I^2$ and $JI^2=(X^9,X^{10},X^{11})A=I^3$. Further, $T\subseteq(I^2:I^2)$ and so I^2 is an ideal of T. Hence, $I^2F=I^2A=I^2=I^2E$ and whence

$$\begin{array}{rcl} (J\ltimes F)(I\ltimes E)^2 & = & (J\ltimes F)(I^2\ltimes IE)\\ & = & JI^2\ltimes(JIE+I^2F)\\ & = & JI^2\ltimes(JI+I^2)\\ & = & JI^2\ltimes I^2\\ & = & JI^2\ltimes I^2E\\ & = & I^3\ltimes I^2E\\ & = & (I\ltimes E)^3. \end{array}$$

Therefore, $\mathbf{r}_{I\ltimes E}(J\ltimes F)=\mathbf{r}_I(J)=2$ by Theorem 2.2.

(2) Let A, I, and J be as above. Next, let $E=A$ and $F=I$. Clearly, $JE=J\subsetneqq I=F$ and so $J\ltimes F$ is a subideal of $I\ltimes E$. Further, $JIE+I^2F=JI+I^3=JI+JI^2=JI\subsetneqq I^2=I^2E$ so that $(J\ltimes F)(I\ltimes E)^2\subsetneqq(I\ltimes E)^3$. Therefore, $2=\mathbf{r}_I(J)\lneqq\mathbf{r}_{I\ltimes E}(J\ltimes F)=3$ by Theorem 2.2.

(3) Let $V:=k[[X]]+Yk((X))[[Y]]$, $P:=Yk((X))[[Y]]$, $E:=V_P$, $I:=P^2=Y^2V_P$, and $J:=Y^2V$. Clearly, J is a reduction of I and so, by Theorem 2.2, $\mathbf{r}_{I\ltimes E}(J\ltimes F)=\mathbf{r}_I(J)=1$ for every submodule F of E with $JE\subseteq F$.

The next result investigates conditions under which the reduction number of reductions (not necessarily saturated) of $I \ltimes E$ equals 1, within trivial ring extensions of valuation domains over torsion-free modules.

Theorem 2.5. *Let V be a valuation domain, E a V-module, $R = V \ltimes E$, I a nonzero ideal of V, and $T := (I : I)$.*

(1) *Assume I is basic. Then, $r(H) = 1$ for every reduction H of $I \ltimes E$.*
(2) *Assume I is not basic and E is a torsion-free V-module. Then, $r(H) = 1$ for every reduction H of $I \ltimes E$ if and only if E is a T-module.*

Proof. (1) Assume I is basic and let H be a reduction of $I \ltimes E$. As $H \subseteq A_H \ltimes E_H \subseteq I \ltimes E$, $A_H \ltimes E_H$ is a reduction of $I \ltimes E$ and hence A_H is a reduction of I. Whence, $A_H = I$. Therefore, for every (x,e) and (y,f) in $I \ltimes E = A_H \ltimes E$, there are g,h in E such that (x,g) and (y,h) belong to H. It follows that

$$(x,e)(y,f) = (xy, xf + ye) = (x,g)(y,f) + (y,h)(0,e-g) \in H(I \ltimes E).$$

Consequently, $(I \ltimes E)^2 \subseteq H(I \ltimes E)$ and thus $H(I \ltimes E) = (I \ltimes E)^2$. That is, $r(H) = 1$.

(2) Assume I is not basic and E is a torsion-free V-module. Let $T = V_P$, for some (non-maximal) prime ideal P of V. By Lemma 2.1, I is a non-principal strongly stable ideal of V. So, $I = aT$, for some $0 \neq a \in I$.

Next, suppose that $r(H) = 1$, for every reduction H of $I \ltimes E$. Consider the principal reduction $J := aV$ of I and the submodule $F := aE$ of E. Clearly, $JE = F$ and so $J \ltimes F$ is a reduction of $I \ltimes E$ by Theorem 2.2(1). Further, by hypothesis, $r_{I \ltimes E}(J \ltimes F) = 1$, that is, $(J \ltimes F)(I \ltimes E) = (I \ltimes E)^2$ and hence

$$\begin{array}{rcl} aE & = & F + IF \\ & = & JE + IF \\ & = & IE \\ & = & aV_P E. \end{array}$$

But E is torsion-free. It follows that $TE = V_P E = E$ and thus E is a T-module.

Conversely, assume that E is a T-module and let H be a reduction of $I \ltimes E$. Clearly, $A_H \ltimes E_H$ is a reduction of $I \ltimes E$ and A_H is a reduction of I. We claim that $r(A_H \ltimes E_H) = 1$. Indeed, the fact that $I = aT$ yields $A_H T = I$. Then, $A_H E = A_H TE = IE$. So, we get

$$\begin{array}{rcl} (A_H \ltimes E_H)(I \ltimes E) & = & A_H I \ltimes (A_H E + IE_H) \\ & = & I^2 \ltimes (IE + IE_H) \\ & = & I^2 \ltimes IE \\ & = & (I \ltimes E)^2, \text{ as claimed.} \end{array}$$

Next, let $(x,e) \in A_H \ltimes E_H$ and $(y,f) \in I \ltimes E$. Then, there is $g \in E$ such that $(x,g) \in H$ and so

$$(0, xf) = (x,g)(0,f) \in H(I \ltimes E).$$

Moreover, since $A_H T = A_H V_P = I$, then $y = \dfrac{m}{t}$, for some $m \in A_H$ and $t \in V \setminus P$. Let $h \in E$ such that $(m,h) \in H$. Since E is a V_P-module, then

$\left(\frac{x}{t}, \frac{e}{t}\right) \in I \ltimes E$ and so

$$(xy, ye) = (x\frac{m}{t}, \frac{m}{t}e) = (m, h)(\frac{x}{t}, \frac{e}{t}) - (x, g)(0, \frac{h}{t}) \in H(I \ltimes E).$$

Therefore, $(x, e)(y, f) = (xy, ye) + (0, xf) \in H(I \ltimes E)$. It follows that

$$(I \ltimes E)^2 = (A_H \ltimes E_H)(I \ltimes E) \subseteq H(I \ltimes E).$$

That is, $r(H) = 1$, completing the proof of the theorem. □

Follows an illustrative example for Theorem 2.5.

Example 2.6. Let (V, M) be a valuation domain and $R = V \ltimes M$. Then, either M is idempotent or principal, and so basic. If M is idempotent, then so is $M \ltimes M$, and hence basic. If $M = aV$ is principal, then $r(H) = 1$, for every reduction H of $M \ltimes M$ by Theorem 2.5(1). Moreover, $(a, 0)R$ is a proper minimal (principal) reduction of $M \ltimes M$ with reduction number 1. Indeed, by [26, Theorem 2.3], $(a, 0)R$ is a minimal reduction of $M \ltimes M$. Further, if $(a, a) = (a, 0)(x, f)$ for some $(x, f) \in R$, then $a = af$ and so $1 = f \in M$, which is absurd. Hence $(a, a) \in (M \ltimes M) \setminus (a, 0)R$, as desired.

Additional examples related to Theorem 2.5(2) will be provided immediately following Theorem 2.7.

In the broader class of trivial extensions over Prüfer domains, the next result asserts that every nonzero ideal is a reduction with reduction number 1.

Theorem 2.7. *Let A be a Prüfer domain, E an A-module, and $R = A \ltimes E$. Then, every nonzero ideal H of R is a reduction of $A_H \ltimes E_H$ with reduction number 1. Moreover, for each nonzero ideal I of A with $IE = 0$, every reduction of $I \ltimes E$ has reduction number 1.*

Proof. As an initial step, we establish the result for valuations. Next, suppose that A is a valuation domain. Consider $(a, e), (b, f) \in A_H \ltimes E_H$ and let $g, h \in E_H$ such that $(a, g), (b, h) \in H$. If $a = 0$, then

$$(a, e)(b, f) = (0, be) = (b, h)(0, e) \in H(A_H \ltimes E_H).$$

Likewise, for $b = 0$. Therefore, we may assume that $ab \neq 0$ and hence either $a^{-1}b \in A$ or $ab^{-1} \in A$. If $a^{-1}b \in A$, then

$$\begin{aligned}(a, e)(b, f) &= (ab, af + be)\\ &= (ab, af + be - bg + bg)\\ &= (ab, a(f + a^{-1}be - a^{-1}bg) + bg))\\ &= (a, g)(b, f + a^{-1}be - a^{-1}bg) \in H(A_H \ltimes E_H).\end{aligned}$$

Likewise, for $ab^{-1} \in A$. It follows that $(A_H \ltimes E_H)^2 \subseteq H(A_H \ltimes E_H)$, as desired.

Now, assume A is a Prüfer domain. Consider a nonzero ideal H of R and let Q be any maximal ideal of R containing H. Then, $Q = M \ltimes E$, for some maximal ideal M of A. Notice that $A_H \subseteq M$ and so $A_H \ltimes E_H \subseteq Q$. To proceed further, let us recall the following fact: if S is a multiplicatively closed subset of the trivial ring extension $R := A \ltimes E$ and $S_o := S \cap A$, then the universal property of localization yields $S^{-1}R \cong S_o^{-1}A \ltimes S_o^{-1}E$. So, next, we may identify R_Q with $A_M \ltimes E_M$. Hence, by the first step, the nonzero ideal HR_Q of R_Q is a reduction of $A_{HR_Q} \ltimes E_{HR_Q}$ with reduction number 1. Therefore, we obtain

$$\begin{array}{rcl}\big(H(A_H \ltimes E_H)\big)R_Q & = & HR_Q\big((A_H)_M \ltimes (E_H)_M\big)\\ & = & HR_Q\left((A_M)_{HR_Q} \ltimes (E_M)_{HR_Q}\right)\\ & = & \left((A_M)_{HR_Q} \ltimes (E_M)_{HR_Q}\right)^2\\ & = & \big((A_H)_M \ltimes (E_H)_M\big)^2\\ & = & (A_H \ltimes E_H)^2 R_Q.\end{array}$$

It follows that $H(A_H \ltimes E_H) = (A_H \ltimes E_H)^2$, completing the proof of the first statement.

Now, let I be a nonzero ideal of A with $IE = 0$ and let H be a reduction of $I \ltimes E$. Since $H \subseteq A_H \ltimes E_H \subseteq I \ltimes E$, $A_H \ltimes E_H$ is a reduction of $I \ltimes E$ and so A_H is a reduction of I. That is, $A_H I = I^2$. It follows that

$$\begin{array}{rcl}(A_H \ltimes E_H)(I \ltimes E) & = & A_H I \ltimes (A_H E + IE_H)\\ & = & I^2 \ltimes 0\\ & = & (I \ltimes E)^2.\end{array}$$

Next, let $(x,e),(y,f) \in I \ltimes E$. Then, $xy \in I^2 = A_H I$ and so

$$xy = \sum_{1\le i\le n} a_i b_i$$

for some $a_i \in A_H$ and $b_i \in I$. For each $i = 1,\dots,n$, let $e_i \in E$ such that $(a_i,e_i) \in H$. Then

$$\begin{array}{rcl}(x,e)(y,f) & = & (xy,0)\\ & = & \Big(\sum_{1\le i\le n} a_i b_i, 0\Big)\\ & = & \sum_{1\le i\le n} (a_i,e_i)(b_i,0).\end{array}$$

Hence $(x,e)(y,f) \in H(I\ltimes E)$ and so $(I\ltimes E)^2 \subseteq H(I\ltimes E)$. Therefore, $H(I\ltimes E) = (I \ltimes E)^2$, completing the proof of the theorem. □

The following examples demonstrate that the assumption "E is a torsion-free V-module" in Theorem 2.5(2) is neither superfluous nor essential.

Example 2.8. Let V be a strongly discrete valuation domain (i.e., V has no nonzero idempotent prime ideal) with $\dim(V) \ge 2$, e.g.,

$$V = k[[X]] + Yk((X))[[Y]],$$

where X, Y are indeterminates over a field k. Let P be a nonzero non-maximal prime ideal of V. Note that $(P : P) = V_P$. Further, by [22, Theorem 2.6], $\mathrm{core}(P) = P^2 \subsetneqq P$ and so P is not a basic ideal of V, as desired.

(1) Let $E := V/P$. By Theorem 2.7, $r(H) = 1$ for every reduction H of $P \ltimes E$. However, E is a torsion V-module which is not a V_P-module. So, the torsion-free assumption is not superfluous.

(2) Let $E := V_P/P$. Obviously, E is a V_P-module with nonzero torsion, and $r(H) = 1$ for every reduction H of $P \ltimes E$ by Theorem 2.7. So, the torsion-free assumption is not necessary.

The next result investigates (minimal) reductions under stability conditions in the general context of trivial ring extensions issued over domains.

Theorem 2.9. *Let A be a domain, I a nonzero ideal of A, and $T := (I : I)$. Let E be a T-module and $R = A \ltimes E$.*

1. *Suppose I is stable and $(T : I)E = E$. Then, for every reduction H of $I \ltimes E$, $r(H) = 1$ and so $H(T \ltimes E) = I \ltimes E$.*
2. *Suppose I is strongly stable; that is, $I = aT$ for some $a \in I$. Then, for each $e \in E$, $(a, e)R$ is a reduction of $I \ltimes E$. Moreover, if the reduction aA of I is minimal, then: $aA \ltimes E$ is a minimal reduction of $I \ltimes E$ if and only if E is divisible by a.*

Proof. (1) Let H be a reduction of $I \ltimes E$. So, there is a positive integer n such that

$$\begin{aligned} H(I^n \ltimes I^{n-1}E) &= H(I \ltimes E)^n \\ &= (I \ltimes E)^{n+1} \\ &= I^{n+1} \ltimes I^n E. \end{aligned}$$

Multiplying both sides by $(T : I) \ltimes E$, and using the fact that I is stable, we obtain

$$\begin{aligned} H(I \ltimes E)^{n-1} &= H(I^{n-1} \ltimes I^{n-2}E) \\ &= I^n \ltimes I^{n-1}E \\ &= (I \ltimes E)^n. \end{aligned}$$

Iterating this process, we obtain $H(I \ltimes E) = (I \ltimes E)^2 = I^2 \ltimes IE$. Under the assumption $(T : I)E = E$, we get

$$\begin{aligned} H(I \ltimes E)((T : I) \ltimes E) &= (I^2 \ltimes IE)((T : I) \ltimes E) \\ H((I(T : I) \ltimes \big(IE + (T : I)E\big)) &= (I^2(T : I)) \ltimes \big(I^2E + I(T : I)E\big) \\ H(T \ltimes E) &= I \ltimes E. \end{aligned}$$

(2) Let $e \in E$. For every $(a\alpha, f), (a\beta, g) \in I \ltimes E$ with $\alpha, \beta \in T$ and $f, g \in E$, we have

$$\begin{aligned} (a\alpha, f)(a\beta, g) &= (a^2\alpha\beta, a(g\alpha + f\beta)) \\ &= (a, e)(a\alpha\beta, g\alpha + f\beta - e\alpha\beta) \\ &\in (a, e)(I \ltimes E). \end{aligned}$$

Therefore, $(I \ltimes E)^2 \subseteq (a, e)(I \ltimes E)$ and thus $(a, e)(I \ltimes E) = (I \ltimes E)^2$, as desired.

Now, notice that $I = aT$ implies that aA is a (principal) reduction of I and so $aA \ltimes E$ is a reduction of $I \ltimes E$ by Theorem 2.2(1). Next, assume aA is a minimal reduction of I. Suppose that E is divisible by a and let H be a reduction of $I \ltimes E$ such that $H \subseteq aA \ltimes E$. Let $J := \{x \in aA \mid (x, e) \in H \text{ for some } e \in E\}$. One can easily check that J is a subideal of aA with $H \subseteq J \ltimes E \subseteq aA \ltimes E \subseteq I \ltimes E$.

Hence $J \ltimes E$ is a reduction of $I \ltimes E$ and, by [26, Theorem 2.2], J is a reduction of I. By minimality, $J = aA$ and therefore $(a, f) \in H$ for some $f \in E$. Next, let $\alpha \in A$ and $e \in E$. By divisibility, $\alpha f = ae'$ and $e = ae''$, for some $e', e'' \in E$. It follows that

$$\begin{aligned} (a\alpha, e) &= (a\alpha, 0) + (0, e) \\ &= (a, f)(\alpha, -e') + (a, f)(0, e'') \\ &\in H. \end{aligned}$$

That is, $aA \ltimes E \subseteq H$. Hence, $H = aA \ltimes E$ and so $aA \ltimes E$ is a minimal reduction of $I \ltimes E$.

Conversely, assume that $aA \ltimes E$ is a minimal reduction of $I \ltimes E$ and let $e \in E$. By the first statement of (2), $(a, 0)R$ is a reduction of $I \ltimes E$. Further, the containment $(a, 0)R \subseteq aA \ltimes E$ forces $(a, 0)R = aA \ltimes E$. Hence, $(a, e) \in (a, 0)R$ and so $(a, e) = (a, 0)(x, f)$ for some $(x, f) \in R$. Thus $e = af$, completing the proof of the theorem. □

The following example shows that in Theorem 2.9(2), if I is a strongly stable ideal of A (i.e., $I = a(I : I)$), the principal reduction aA of I is not necessarily minimal.

Example 2.10. Let X be an indeterminate over $\mathbb{Q}$. Consider the Prüfer domain $A := \mathbb{Z} + X\mathbb{Q}[[X]]$ and the two ideals of A given by $I := X(\mathbb{Z}_{(2)} + X\mathbb{Q}[[X]])$ and $J := 3XA$. Clearly, $T := (I : I) = \mathbb{Z}_{(2)} + X\mathbb{Q}[[X]]$ and so I is a strongly stable ideal of A. Moreover, XA and J are reductions of I since $XI = JI = I^2$. However, $J \subsetneqq XA$ and so XA is not a minimal reduction of I.

Here is an illustrative example for Theorem 2.9(1).

Example 2.11. Let us borrow a construction from [27]. Let D_1 be a non-Bézout Prüfer domain and let M_1 be a maximal ideal of D_1 with infinite residue field F (e.g., $D_1 := \text{Int}(\mathbb{Z})$ and M_1 is a height-two maximal ideal of D_1 [5, Remark VI.1.8, pages 126–127]). Let V_1 be a valuation domain with quotient field F and let D be the pullback issued from the following diagram of canonical homomorphisms:

$$\begin{array}{ccc} D & \longrightarrow & V_1 \\ \downarrow & & \downarrow \\ D_1 & \xrightarrow{\varphi} & F. \end{array}$$

Then, D is a Prüfer domain and D_1 is a non-Bézout overing of D. Next, let L be the quotient field of D, X an indeterminate over L, $V = L[[X]] = L + M$, and $A = D + M$. Clearly, A is a Prüfer domain. Let J be any finitely generated nonprincipal ideal of D_1. The ideal of A given by $I := X(J + M)$ is stable but not strongly stable [27, Example 6.1]. Let $T := (I : I)$. Now, for a trivial example, let E be the quotient field of A and, obviously, $(T : I)E = E$. For a nontrivial example, consider $E := \bigcup_{n\geq 0}(T : I^n)$, the Nagata transform of I with respect to T, and one can easily check that $(T : I)E = E$.

Here is an illustrative example for Theorem 2.9(2).

Example 2.12. First, recall from the proof of [22, Theorem 2.12] that for a strongly stable ideal $I = a(I : I)$ in an integral domain A, every reduction of I contains a reduction of the form aF, where F is a finitely generated A-module satisfying $F(I : I) = (I : I)$. Next, let k be a field, X an indeterminate over k, $A := k[X^2, X^3]$, $I := (X^2, X^3)A = X^2k[X]$, $E := k[X, X^{-1}]$, and $R := A \ltimes E$. Then, $T = (I : I) = k[X]$ and so I is a strongly stable ideal of A and, clearly, X^2A is a reduction of I. We claim that X^2A is minimal. Indeed, let $J := X^2F$ be a reduction of I, where F is a finitely generated A-module with $FT = T$, such that $J \subseteq X^2A$. Then, $F \subseteq A$ and so F is an integral ideal of A. Suppose that F is proper and let $F \subseteq M$, for some maximal ideal M of A. Now, notice that the integral closure of A is equal to T, and so there is a maximal ideal N of T with $N \cap A = M$. It follows that $T = FT \subseteq MT \subseteq N$, which is absurd. By consequence, $F = A$ and so $J = X^2A$. That is, X^2A is a minimal reduction of I. Moreover, E is divisible by X^2 (since X^2 is invertible in E) and so $X^2A \ltimes E$ is a minimal reduction of $I \ltimes E$ in R by Theorem 2.9(2).

3. Core

Recall from [23] that for an ideal I, the *principal core* of I, denoted $\mathrm{pcore}(I)$, is the intersection of all principal reductions of I. In general, $\mathrm{core}(I) \subsetneqq \mathrm{pcore}(I)$, as shown in [23, Example 3.7]. Also, let $\mathrm{Red}(I)$ and $\mathrm{PRed}(I)$) denote the set of reductions and principal reduction of I, respectively. In particular, $\mathrm{PRed}_R(I) = \emptyset$ means that I has no principal reduction in R. This section studies the core and principal core of ideals in trivial ring extensions arising from domains. The main theorem focuses on the case of trivial extensions over valuation domains.

The first result provides necessary and sufficient conditions for the core of $I \ltimes E$, with $I \neq 0$, to be saturated within the general context of trivial extensions over domains. In this vein, it's worth recalling that for the case $I = 0$, it is known, by [26, Lemma 2.1], that $\mathrm{Red}(0 \ltimes E) = \{0 \ltimes E' \mid E' \text{ submodule of } E\}$ and then $\mathrm{core}(0 \ltimes E) = 0 \ltimes E_0$, where E_0 is the intersection of all nonzero submodules of E.

Theorem 3.1. *Let A be a domain, E an A-module, and $R = A \ltimes E$. Then, the following assertions are equivalent:*

(1) $\mathrm{core}_R(I \ltimes E) = \mathrm{core}_A(I) \ltimes E$*, for every nonzero ideal I of A;*
(2) *$I \ltimes E$ is basic, for every nonzero basic ideal I of A;*
(3) *$I \ltimes E$ is basic, for every nonzero invertible ideal I of A;*
(4) *$I \ltimes E$ is basic, for every nonzero principal ideal I of A;*
(5) *E is a divisible A-module.*

Proof. (1) $\Longrightarrow$ (2) Let I be a nonzero basic ideal of A. By (1), we have $\mathrm{core}_R(I \ltimes E) = \mathrm{core}_A(I) \ltimes E = I \ltimes E$, and so $I \ltimes E$ is basic.

(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) Trivial since every invertible ideal is basic by [22, Lemma 2.1], and every principal ideal is invertible.

(4) $\Longrightarrow$ (5) Let a be a non-invertible element of A and let $I := aA$ and $F := aE$. Since $IE = F$, then $I \ltimes F$ is a subideal of $I \ltimes E$ which is a reduction

of $I \ltimes E$. By hypothesis, $I \ltimes F = I \ltimes E$ and thus $E = F = aE$. Now, if a is invertible, then obviously $E = aE$. So, for both cases, E is a divisible A-module.

(5) $\Longrightarrow$ (1) Let I be a nonzero ideal of A. For any reduction J of I, $J \ltimes E$ is a reduction of $I \ltimes E$ by Theorem 2.2(1) and so we have

$$\begin{array}{rcl} \mathrm{core}_R(I \ltimes E) & \subseteq & \displaystyle\bigcap_{J \in \mathrm{Red}(I)} J \ltimes E \\ & = & \left(\displaystyle\bigcap_{J \in \mathrm{Red}(I)} J \right) \ltimes E \\ & = & \mathrm{core}_A(I) \ltimes E. \end{array}$$

For the reverse inclusion, let H be a reduction of $I \ltimes E$. Since E is divisible, by [3, Corollary 3.4], either $H = 0 \ltimes F$, for some submodule F of E, or $H = A_H \ltimes E$ with $A_H \neq 0$. In the first case, an equality of the form $H(I \ltimes E)^n = (I \ltimes E)^{n+1}$, for some positive integer n, yields

$$\begin{array}{rcl} (I^{n+1} \ltimes I^n E)^2 & = & ((I \ltimes E)^{n+1})^2 \\ & = & H^2 (I \ltimes E)^{2n} \\ & \subseteq & H^2 \\ & = & 0. \end{array}$$

Therefore, $I^{2n+2} = 0$, which implies $I = 0$, a contradiction. Thus, we are left with the case $H = A_H \ltimes E$. Then, A_H is a reduction of I, leading to $\mathrm{core}_A(I) \subseteq A_H$ and $\mathrm{core}_A(I) \ltimes E \subseteq H$. It follows that $\mathrm{core}_A(I) \ltimes E \subseteq \mathrm{core}(I \ltimes E)$ and thus $\mathrm{core}_R(I \ltimes E) = \mathrm{core}_A(I) \ltimes E$, completing the proof of the theorem. $\square$

The following corollary provides families of explicit and illustrative examples for Theorem 3.1. For this purpose, recall that, in a Prüfer domain of finite character, $\mathrm{core}(I) = I^2 I^{-1}$, for every nonzero ideal I [27, Theorem 4.3].

Corollary 3.2. *Let A be a domain, K its quotient field, and $R := A \ltimes K$. Then,* $\mathrm{core}(I \ltimes K) = \mathrm{core}(I) \ltimes K$, *for every nonzero ideal I of A. In particular, if A is a Prüfer domain of finite character, then* $\mathrm{core}(I \ltimes K) = I^2 I^{-1} \ltimes K$.

A domain possesses the finite basic ideal property (or, basic ideal property) if every finitely generated ideal (or, every ideal) is basic. Two well-known results, attributed to Hays, establish that a domain is Prüfer if and only if it has the finite basic ideal property [10, Theorem 6.5]; and it is one-dimensional Prüfer if and only if it has the basic ideal property [11, Theorem 10].

Recall that an ideal is *nonnil* if it contains a non-nilpotent element. In general, the category of non-nilpotent ideals strictly contains the category of nonnil ideals. Also, for convenience, note that a nonzero nilpotent ideal is not basic since it admits the null ideal as a proper reduction.

Corollary 3.3. *Let A be a domain, E an A-module, and $R = A \ltimes E$. Then, the following assertions are equivalent:*

(1) *Every non-nilpotent ideal of R is basic;*
(2) *Every nonnil ideal of R is basic;*

(3) *A has the basic ideal property and E is divisible;*
(4) *A is a one-dimensional Prüfer domain and E is divisible.*

Proof. We only need to prove (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1). Assume that every nonnil ideal of R is basic and let I be a nonzero ideal of A. Clearly, $I \ltimes E$ is a nonnil ideal of R and hence basic by hypothesis. It follows that I is basic in A via [26, Theorem 2.2]. Hence, A has the basic ideal property. Moreover, $\operatorname{core}_A(I) \ltimes E = I \ltimes E = \operatorname{core}_R(I \ltimes E)$. By Theorem 3.1, E is a divisible A-module.

Next, suppose that A has the basic ideal property, and E is divisible. Let H be a non-nilpotent ideal of R. Once again, according to [3, Corollary 3.4], H is either $0 \ltimes F$ for some submodule F of E, or $H = A_H \ltimes E$ with $A_H \neq 0$. However, the first case cannot occur since $(0 \ltimes F)^2 = 0$, implying that $H = A_H \ltimes E$, and A_H is basic by hypothesis. By Theorem 3.1, H is basic, concluding the proof of the corollary. □

The upcoming result investigates the principal reductions and principal core of ideals of the form $I \ltimes E$ for a strongly stable ideal I within a specific class of trivial ring extensions. Throughout, given a ring A and a module E over A, $\mathcal{U}(A)$ denotes the set of units of A and $\operatorname{Ann}_A(E) := \{a \in A \mid aE = 0\}$.

Theorem 3.4. *Let A be a domain, I a strongly stable ideal of A, and $T := (I : I)$. Let E be a (T/I)-module and $R = A \ltimes E$. Then,* $\operatorname{pcore}(I \ltimes E) \subseteq I \operatorname{Ann}_A(E) \ltimes 0$, *and equality holds if and only if $\mathcal{U}(T) \operatorname{Ann}_A(E) \subseteq A$. In particular, if $E = T/I$, then* $\operatorname{pcore}(I \ltimes E) = I^2 \ltimes 0$.

Proof. Let $I = aT$, for some nonzero $a \in I$, and for each $u \in \mathcal{U}(T)$ and $e \in E$, let

$$J_{u,e} := (ua, e)R \subseteq I \ltimes E.$$

Claim 1. $\operatorname{PRed}(I \ltimes E) = \{J_{u,e} \mid u \in \mathcal{U}(T), e \in E\}$.

Indeed, observe first that E is a module over T and, a fortiori, over A with $IE = 0$. So, $(I \ltimes E)^n = I^n \ltimes 0$, for any integer $n \geq 2$. Next, let $u \in \mathcal{U}(T)$, $e \in E$ and $y \in T$. Hence, $(a^2y, 0) = (ua, e)(au^{-1}y, 0)$. Whence

$$\begin{array}{rcl}(I \ltimes E)^2 & = & I^2 \ltimes 0 \\ & \subseteq & (au, e)(I \ltimes 0) \\ & \subseteq & J_{u,e}(I \ltimes E)\end{array}$$

So, $(I \ltimes E)^2 = J_{u,e}(I \ltimes E)$ and hence $J_{u,e}$ is a principal reduction of $I \ltimes E$. Conversely, let $J := (b, e)R$ be a principal reduction of $I \ltimes E$. That is, $J(I^n \ltimes 0) = I^{n+1} \ltimes 0$, for some integer $n \geq 2$. Write $b = ay$, for some $y \in T$. There exist $y_i \in T$ and $(a_i, e_i) \in J$, with $a_i = c_i b$ for some $c_i \in A$, such that

$$(a^{n+1}, 0) = \sum_{1 \leq i \leq s} (a_i, e_i)(a^n y_i, 0)$$

yielding

$$\begin{aligned} a^{n+1} &= a^n \sum_{1\le i\le s} a_i y_i \\ &= a^n b \sum_{1\le i\le s} c_i y_i \\ &= a^{n+1} y \sum_{1\le i\le s} c_i y_i. \end{aligned}$$

It follows that $y \in \mathcal{U}(T)$ and thus $J = J_{y,e}$ proving the claim.

Next, we prove that $\mathrm{pcore}(I \ltimes E) \subseteq I\,\mathrm{Ann}_A(E) \ltimes 0$. Indeed, by Claim 1, we have

$$\mathrm{pcore}(I \ltimes E) = \bigcap_{\substack{u\,\in\,\mathcal{U}(T)\\ e\,\in\,E}} J_{u,e} \subseteq J_{1,0} = (a,0)R.$$

Let $(x,0) \in \mathrm{pcore}(I \ltimes E)$ with $x = ay$, for some $y \in A$. For each $e \in E$, $(x,0) \in J_{1,e}$ and so $(x,0) = (a,e)(b,f) = (ab,eb)$, for some $(b,f) \in R$. Hence, $y = b$ and $ye = be = 0$. It follows that $y \in \mathrm{Ann}_A(E)$ and thus $\mathrm{pcore}(I \ltimes E) \subseteq I\,\mathrm{Ann}_A(E) \ltimes 0$, as desired.

Now, suppose that $\mathcal{U}(T)\,\mathrm{Ann}_A(E) \subseteq A$ and let $z \in \mathrm{Ann}_A(E)$, $u \in \mathcal{U}(T)$, and $e \in E$. Then, $(az,0) = (ua,e)(u^{-1}z,0)$. However, since $u^{-1}z \in \mathcal{U}(T)\,\mathrm{Ann}_A(E) \subseteq A$, $(u^{-1}z,0) \in R$, and so $(az,0) \in (au,e)R = J_{u,e}$. It follows that $(az,0) \in \mathrm{pcore}_R(I\ltimes E)$, and thus $I\,\mathrm{Ann}_A(E)\ltimes 0 \subseteq \mathrm{pcore}_R(I\ltimes E)$. Equality holds by Claim 2. Conversely, assume that $\mathrm{pcore}_R(I \ltimes E) = I\,\mathrm{Ann}_A(E) \ltimes 0$ and let $u \in \mathcal{U}(T)$ and $z \in \mathrm{Ann}_A(E)$. Then, $(az,0) \in \mathrm{pcore}_R(I\ltimes E) \subseteq J_{u^{-1},0} = (au^{-1},0)R$. Hence, $(az,0) = (au^{-1},0)(b,e)$, for some $(b,e) \in R$. Whence, $az = au^{-1}b$ and so $uz = b \in A$. Thus, $\mathcal{U}(T)\,\mathrm{Ann}_A(E) \subseteq A$.

If $E = T/I$, then $\mathrm{Ann}_A(E) = I$ and so $\mathrm{pcore}(I \ltimes E) = I^2 \ltimes 0$, completing the proof of the theorem. □

Follow illustrative examples for Theorem 3.4. In the first example, I is strongly stable, $E = T/I$, and $\mathrm{pcore}(I \ltimes E) = I^2 \ltimes 0 \subsetneqq \mathrm{pcore}(I) \ltimes 0 \subsetneqq \mathrm{pcore}(I) \ltimes E$.

Example 3.5. Let k be a field and X an indeterminate over k. Let $A := k[X^2,X^3]$ and $I := (X^2,X^3)A$. Clearly, $T := (I : I) = k[X]$ and so $I = X^2T$. So, for $E := T/I = k[X]/(X^2)$ and $R = A \ltimes E$, we obtain $\mathrm{pcore}(I \ltimes E) = I^2 \ltimes 0 = X^4k[X] \ltimes 0$ by Theorem 3.4. Next, let $J := fA$ be a principal reduction of I in A and, a fortiori, in $T = k[X]$. Then, $JI = I^2$ which yields $fk[X] = X^2k[X]$ and hence $J = X^2A$. Therefore, $\mathrm{pcore}(I) = X^2A$ and thus $\mathrm{pcore}(I \ltimes E) = X^4k[X] \ltimes 0 \subsetneqq X^2k[X^2,X^3] \ltimes 0 = \mathrm{pcore}(I) \ltimes 0$, as desired.

In the next example, I is strongly stable, E is a module over T/I, and $\mathrm{pcore}(I\ltimes E) \subsetneqq I\,\mathrm{Ann}_A(E) \ltimes 0$.

Example 3.6. Let k be a field and X an indeterminate over k. Let

$$A := k[[X^3,X^4]], \quad Q := (X^3,X^4)A, \quad I := X^6k[[X]], \quad M := Xk[[X]].$$

Clearly, $T := (I : I) = k[[X]]$ and so $I = X^6T$ is a strongly stable ideal of A. Let $E = k[[X]]/(X)$ and $R = A \ltimes E$. First, notice that $\mathrm{Ann}_A(E) = M \cap A =$

Q. Further, $X^4 + X^5 = (1 + X)X^4 \in \mathcal{U}(T)\,\mathrm{Ann}_A(E) \setminus A$. By Theorem 3.4, $\mathrm{pcore}(I \ltimes E) \subsetneqq I\,\mathrm{Ann}_A(E) \ltimes 0 = (X^9, X^{10})A \ltimes 0$.

The next result explores the core and principal core of ideals in the form of $I \ltimes M$ within trivial ring extensions arising from valuation domains (V, M), considering both cases when I is basic or non-basic.

Theorem 3.7. *Let (V, M) be a valuation domain and $R = V \ltimes M$. Let I be a nonzero ideal of V and $T := (I : I)$.*

(1) *Suppose I is non-basic. Then, $T = V_P$ for some prime ideal P of V and hence* $\mathrm{core}(I \ltimes M) = \mathrm{pcore}(I \ltimes M) = I^2P \ltimes IP$.
(2) *Suppose I is basic. If I is principal, then* $\mathrm{core}(I \ltimes M) = \mathrm{pcore}(I \ltimes M) = I^2 \ltimes IM$. *Otherwise,* $I^2 \ltimes I^2 \subseteq \mathrm{core}(I \ltimes M) \subseteq I \ltimes I$. *In particular, if I is idempotent, then* $\mathrm{core}(I \ltimes M) = I \ltimes I$.

Proof. (1) Since I is non-basic, by Lemma 2.1, I is a strongly stable non-principal ideal of V. Then, $T = V_P$ for some *non-maximal* prime ideal P of V, and $I = aT$ for some nonzero $a \in I$. Hence, $P \subsetneqq M$ and so $IM = aMT = aMV_P = aV_P = I$. Whence, $(I \ltimes M)^{n+1} = I^{n+1} \ltimes I^n$, for each integer $n \geq 1$.

For each $u \in V \setminus P$ and $m \in M$, let $J_{u,m} := (ua, m)R$.

Claim 2. *$J_{u,m}$ is a principal reduction of $I \ltimes M$.*

Indeed, for every $x, y \in T$, we have

$$(xa^3, ya^2) = (ua, m)(u^{-1}xa^2, u^{-1}(ya - mxau^{-1})).$$

It follows that

$$\begin{array}{rcl} (I \ltimes M)^3 & = & I^3 \ltimes I^2 \\ & \subseteq & (ua, m)\, I^2 \ltimes I \\ & = & (ua, m)(I \ltimes M)^2. \end{array}$$

and thus $(ua, m)(I \ltimes M)^2 = (I \ltimes M)^3$, completing the proof of the claim.

Claim 3. *Every reduction of $I \ltimes M$ contains a principal reduction of the form $J_{u,m}$, where $u \in V \setminus P$ and $m \in M$.*

Indeed, let H be a reduction of $I \ltimes M$ with $I^{n+1} \ltimes I^n = (I \ltimes M)^{n+1} = H(I \ltimes M)^n = H(I^n \ltimes I^{n-1}M)$, for some integer $n \geq 1$. Note that, for $n \geq 2$, $I^{n-1}M = I^{n-1}$ and, for $n = 1$, $I^0M = M$. Then, there exist $(a_i, m_i) \in H$, $x_i \in T$, and $y_i \in I^{n-1}M$, for $i = 1, \dots, s$ such that $(a^{n+1}, a^n) = \sum_{i=1}^{s}(a_i, m_i)(x_ia^n, y_i)$ and so $a^{n+1} = \sum_{i=1}^{s} a_ix_ia^n$. Therefore, $a = \sum_{i=1}^{s} a_ix_i$. Next, let $u \in V \setminus P$ such that $ux_i \in V$, for each i, and let $m := \sum_{i=1}^{s} ux_im_i$. It follows that $(ua, m) = \sum_{i=1}^{s}(a_i, m_i)(ux_i, 0) \in H$ and so $H \supseteq (ua, m)R$. Hence, we have established that each reduction of

$I \ltimes M$ contains a principal reduction in the form $J_{u,m}$, where $u \in V \setminus P$ and $m \in M$. As a result, the claim holds, and by consequence, we get

$$\operatorname{core}(I \ltimes M) = \operatorname{pcore}(I \ltimes M) = \bigcap_{\substack{u \in V \setminus P \\ m \in M}} J_{u,m}.$$

Next, we prove that $\operatorname{core}(I \ltimes M) \subseteq I^2P \ltimes IP$. Let $(x, f) \in \operatorname{core}(I \ltimes M)$. Then, $(x, f) \in J_{1,0} = (a, 0)R$ and so $(x, f) = (a, 0)(y, g)$, for some $(y, g) \in R$. That is, $x = ay$ and $f = ag$. If $g \in V \setminus P$, then $(x, f) \in J_{g,0} = (ga, 0)R$ and so $ag = gah$, for some $h \in M$. Hence, $1 = h \in M$, which is absurd. So, necessarily, $g \in P$ and hence $f \in IP$.

Next, we claim that $y \in aP$. We first show that $y \in P$. Deny and assume $y \in V \setminus P$. Let $m \in M \setminus P$. Then, $(x, f) \in J_{y,m} = (ya, m)R$ and so $ay = yaz$ and $ag = yah + mz$, for some $(z, h) \in R$. Therefore, $z = 1$ and hence $m = ag - yah \in I \subseteq P$, the desired contradiction. So, $y \in P$. Now, let $m \in M \setminus P$. Then, $(x, f) \in J_{m,m} = (ma, m)R$ and so $x = ay = maz$ and $f = ag = mah + mz$, for some $(z, h) \in R$. Hence, $y = mz = a(g - mh)$. Next, set $d := g - mh \in M$ since $g \in P \subset M$. Suppose, for a contradiction, that $d \in M \setminus P$. Then, $(x, f) \in J_{d,d} = (da, d)R$ and so $x = ay = dat$ and $f = ag = dak + dt$, for some $(t, k) \in R$. Hence, $y = ad = dt = a(g - dk)$ and whence $t = a$ and $d = g - dk$. It follows that $d = g(1 + k)^{-1} \in P$ as $k \in M$, absurd. So, $y = ad \in aP$, as claimed. Consequently, $(x, f) \in I^2P \ltimes IP$ and thus $\operatorname{core}(I \ltimes M) \subseteq I^2P \ltimes IP$.

For the reverse inclusion, notice first that $IP = aTP = aPV_P = aP$ and let $x, y \in P$, $u \in V \setminus P$, and $m \in M$. Then, $(a^2x, ay) = (ua, m)(u^{-1}ax, u^{-1}(y - mu^{-1}x))$, where clearly $u^{-1}(y - mu^{-1}x) \in P \subset M$, and hence $(a^2x, ay) \in J_{u,m}$. It follows that $I^2P \ltimes IP \subseteq J_{u,m}$, for every $u \in V \setminus P$ and $m \in M$, completing the proof of Assertion (1).

(2) Assume I is basic. If $I = aV$, $0 \neq a \in V$, then by [26, Theorem 2.3], we obtain

$$\begin{aligned} \operatorname{core}(I \ltimes M) &= \operatorname{pcore}(I \ltimes M) \\ &= a(aM : M) \ltimes aM \\ &= a^2(M : M) \ltimes aM \\ &= a^2V \ltimes aM \\ &= I^2 \ltimes IM. \end{aligned}$$

Next, suppose I is not principal. Then $T = (I : I) = V_P$ for some prime ideal P of V. Since I is basic (by hypothesis), I is not strongly stable by Lemma 2.1. We claim that $IP = IM = I$. Indeed, obviously, $IP \subseteq IM \subseteq I$. Suppose, for a contradiction, that $IP \subsetneqq I$, and let $a \in I \setminus IP$. Then, $IP \subsetneqq aV_P$ and so $a^{-1}IP \subsetneqq V_P$. Hence, $a^{-1}IP \subseteq P$ and whence $a^{-1}I \subseteq (P : P) = V_P$. It follows that $I \subseteq aV_P = aT \subseteq I$ and so $I = aT$, that is, I is strongly stable, absurd. Therefore, $IP = IM = I$. So, $I \ltimes I$ is a subideal of $I \ltimes M$ with $(I \ltimes I)(I \ltimes M) = I^2 \ltimes I = (I \ltimes M)^2$. Therefore, $\operatorname{core}(I \ltimes M) \subsetneqq I \ltimes I$.

Next, let H be a reduction of $I \ltimes M$. Since $H \subseteq V_H \ltimes M_H \subseteq I \ltimes M$, then $V_H \ltimes M_H$ is a reduction of $I \ltimes M$ and then V_H is a reduction of I. So, $V_H = I$.

By Theorem 2.7, H is a reduction of $I \ltimes M_H$ with $\mathbf{r}_{I\ltimes M_H}(H) = 1$. Further, $I = IM = V_H M \subseteq M_H$. Then

$$I^2 \ltimes I^2 = (I \ltimes I)^2 \subseteq (I \ltimes M_H)^2 = H(I \ltimes M_H).$$

Consequently, $I^2 \ltimes I^2 \subseteq H$, and thus $I^2 \ltimes I^2 \subseteq \operatorname{core}(I \ltimes M)$, as desired. Finally, recall that proper idempotent ideals in domains are basic [22, Lemma 2.1] and, if nonzero, are non-principal. Then, $\operatorname{core}(I \ltimes M) = I \ltimes I$, completing the proof of the theorem. □

It is known that the maximal ideal of any valuation domain is either principal or idempotent. This fact leads to the following corollary of Theorem 3.7.

Corollary 3.8. *Let (V, M) be a valuation domain and $R = V \ltimes M$. If M is not principal, then $M \ltimes M$ is basic. In general, for any prime ideal P of V, if P is not principal in V_P, then* $\operatorname{core}(P \ltimes M) = P \ltimes P$.

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