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Hardy-Littlewood maximal functions, Riesz transform, and their commutators in the Dunkl setting

Shaoyong He, Lei Wang and Yun Fan

ABSTRACT. This paper investigates the boundedness of commutators associated with the Hardy-Littlewood maximal operator, the sharp maximal operator, and the Riesz transform in the Dunkl setting. We establish necessary and sufficient conditions for the boundedness of the commutators $[b, M_{\alpha,\kappa}]$ and $[b, M_\kappa^\sharp]$ on Orlicz spaces when the symbol function b belongs to a Lipschitz space, thereby obtaining new characterizations of nonnegative Lipschitz functions. Furthermore, we extend a classical result of Janson to the Dunkl framework by proving that the commutator of the Dunkl-Riesz transform, $[b, R_j]$, is bounded from $L^p_\kappa(\mathbb{R}^d)$ to $L^q_\kappa(\mathbb{R}^d)$, where $1 < p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{N}$, if and only if the symbol $b \in \mathrm{Lip}(\beta)$.

CONTENTS

1. Introduction

Let T be a classical singular integral operator, the commutator $[b, T]$ generated by T and a suitable function b is defined by

$$[b, T](f)(x) = b(x)T(f)(x) - T(bf)(x). \quad (1)$$

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Commutators play an important role in harmonic analysis and partial differential equations, particularly in the theory of non-divergence form elliptic equations with discontinuous coefficients [2, 3, 4, 10]. A foundational result by Coifman, Rochberg, and Weiss [5] shows that if T is a Calderón-Zygmund operator with a smooth homogeneous kernel, the commutator $[b, T]$ is bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$ if and only if the symbol b belongs to the space of bounded mean oscillation, $BMO(\mathbb{R}^d)$. Janson [19] later proved that for $0 < \beta < 1$, $1 < p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{d}$, the commutator $[b, T]$ is bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ if and only if b is in the homogeneous Lipschitz space $\dot{\Lambda}_\beta$. This space is defined as

$$\dot{\Lambda}_\beta = \{b : |b(x) - b(y)| \le C\|x - y\|^\beta, x, y \in \mathbb{R}^d\}.$$

Furthermore, Zhang [29, 30]established necessary and sufficient conditions for the boundedness of commutators of the Hardy-Littlewood maximal function on Lebesgue, Morrey, and more general Orlicz spaces when the symbol b is in a Lipschitz space.

For several decades, significant research has focused on the maximal function, the Riesz transform, and related topics concerning the Dunkl differential-difference operator, see [1, 7, 8, 9, 13, 16, 17, 20, 23, 25] for instance. To study differential operators associated with reflection groups, Dunkl [12] introduced the Dunkl transform, which shares many analogous properties with the classical Fourier transform. This framework provides a comprehensive generalization of Fourier analysis that encompasses most special functions related to root systems, such as spherical functions on Riemannian symmetric spaces.

Specifically, we consider the Euclidean space $\mathbb{R}^d$ equipped with the standard inner product $\langle x, y\rangle = \sum_{j=1}^d x_j y_j$ and the associated Euclidean norm $\|x\| = \{\sum_{j=1}^d |x_j|^2\}^{\frac{1}{2}}$. For a nonzero vector $\alpha \in \mathbb{R}^d$, the reflection σ_α with respect to the hyperplane orthogonal to α is defined by

$$\sigma_\alpha(x) = x - \frac{2\langle x, \alpha\rangle}{\|\alpha\|^2}\alpha.$$

A normalized root system in $\mathbb{R}^d$ is a finite set $R \subset \mathbb{R}^d \setminus \{0\}$ such that $\sigma_\alpha(R) = R$ and $\|\alpha\|^2 = 2$ for all $\alpha \in R$. The finite group G generated by the reflections $\sigma_\alpha (\alpha \in R)$ is called the reflection group, where $\sigma_\alpha(x) = x - \langle x, \alpha\rangle \alpha$ for $x \in \mathbb{R}^d$. For this reflection group G, the orbit of a point $x \in \mathbb{R}^d$ is denoted by $\mathcal{O}(x)$. The Dunkl metric d between two G-orbits $\mathcal{O}(x)$ and $\mathcal{O}(y)$ is then defined as

$$d(x, y) := \min_{\sigma \in G} \|x - \sigma(y)\|.$$

It's straightforward to observe that $d(x, y) \le \|x - y\|$. An interesting property is that $d(x, y)$ can equal zero even when $\|x - y\| > 0$. The closures of the connected components of the set

$$\{x \in \mathbb{R}^d : \langle x, \alpha\rangle \ne 0 \text{ for all } \alpha \in R\}$$

are defined as (closed) Weyl chambers. We note that the equality $\|x-y\| = d(x,y)$ holds if and only if x and y belong to the same closed Weyl chamber (see [18]).

Let R be a root system in $\mathbb{R}^d$, and let G be its associated reflection group. Given a G-invariant multiplicity function $\kappa : R \to [0,\infty)$, the Dunkl operators D_j for $1 \le j \le d$ are defined as:

$$D_j = \partial_j + \sum_{\alpha\in R^+} \kappa(\alpha)\alpha_j \frac{1-\sigma_\alpha}{\langle \alpha, x\rangle},$$

where $\sigma_\alpha f = f\circ\sigma_\alpha$ and α_j is the j-th coordinate of α. It's been shown in [11] that the Dunkl operators generate a commutative algebra and are anti-symmetry associated to the Dunkl measure

$$d\omega(x) = h_\kappa(x)dx, \text{ where } h_\kappa(x) = \prod_{\alpha\in R} |\langle x,\alpha\rangle|^{\kappa(\alpha)}.$$

Defining the parameter $\gamma_\kappa = \sum_{\alpha\in R}\kappa(\alpha)$, the homogeneous dimension associated with this setting is $N = d+\gamma_\kappa$. Furthermore, the measure of a ball $B(x,r)$ satisfies the estimate

$$\omega(B(x,r)) \approx r^d \prod_{\upsilon\in R}(|\langle x,\upsilon\rangle| + r)^{\kappa(\upsilon)},$$

which immediately implies that the measure ω is doubling. Moreover, ω satisfies a reverse doubling condition: for all $x \in \mathbb{R}^d$ and $r_1 \ge r_2 > 0$, there exists a constant $C > 0$ such that

$$C^{-1}\left(\frac{r_1}{r_2}\right)^d \le \frac{\omega(B(x,r_1))}{\omega(B(x,r_2))} \le C\left(\frac{r_1}{r_2}\right)^N. \tag{2}$$

Consequently, the space $\mathbb{R}^d$, when equipped with the Euclidean distance and the measure ω, constitutes a space of homogeneous type in the sense of Coifman and Weiss [6].

Let $0 \le \alpha < N$. For a locally integrable function $f \in L^1_{\kappa,loc}(\mathbb{R}^d) = L^1_{loc}(\mathbb{R}^d, d\omega)$, the fractional maximal operator $M_{\alpha,\kappa}$ in the Dunkl framework is defined by

$$M_{\alpha,\kappa}(f)(x) = \sup_{x\in B} \frac{1}{\omega(B)^{1-\alpha/N}} \int_B |f(y)|\,d\omega(y).$$

The sharp maximal function $M^\sharp_\kappa$ in the Dunkl setting is given by

$$M^\sharp_\kappa(f)(x) = \sup_{x\in B}\frac{1}{\omega(B)}\int_B |f(y) - f_B|\,d\omega(y),$$

where $f_B = \frac{1}{\omega(B)}\int_B f(y)\,d\omega(y)$ denotes the mean value of f over the ball B. For a function $b \in L^1_{\kappa,loc}(\mathbb{R}^d)$, the fractional maximal commutator associated with the Dunkl operator is defined as

$$M_{\alpha,b,\kappa}(f)(x) = \sup_{x\in B}\frac{1}{\omega(B)^{1-\alpha/N}}\int_B |b(x)-b(y)|\,|f(y)|\,d\omega(y).$$

In the special case $\alpha = 0$, we denote

$$M_\kappa \equiv M_{0,\kappa} \quad \text{and} \quad M_{b,\kappa} \equiv M_{0,b,\kappa},$$

which correspond respectively to the Dunkl maximal operator and the maximal commutator operator on $\mathbb{R}^d$. Similar to (1), the nonlinear commutators of $M_{\alpha,\kappa}$ and $M_\kappa^\sharp$ with a locally integrable function b are defined respectively by

$$[b, M_{\alpha,\kappa}](f)(x) = b(x)M_{\alpha,\kappa}(f)(x) - M_{\alpha,\kappa}(bf)(x),$$
$$[b, M_\kappa^\sharp](f)(x) = b(x)M_\kappa^\sharp(f)(x) - M_\kappa^\sharp(bf)(x).$$

In [21], the authors investigated the boundedness of the fractional maximal commutator $M_{b,\alpha,\kappa}$ and the commutator $[b, M_{\alpha,\kappa}]$, on Orlicz space for functions b in the space $BMO_\kappa(\mathbb{R}^d)$. The $BMO_\kappa(\mathbb{R}^d)$ space is the class of those $f \in L^1_{\kappa,loc}(\mathbb{R}^d)$ satisfying

$$\|f\|_{BMO_\kappa(\mathbb{R}^d)} := \sup_B \frac{1}{\omega(B)} \int_B |f - f_B| dw < \infty,$$

where the supremum ranges over all finite balls B in $\mathbb{R}^d$.

Following the notation of [24], a function $f \in L^1_{\kappa,loc}(\mathbb{R}^d)$ belongs to the Lipschitz space Lip(β), for $0 < \beta < \infty$, if the following norm is finite:

$$\|f\|_{\mathrm{Lip}(\beta)} := \sup_B \frac{1}{\omega(B)^{1+\frac{\beta}{N}}} \int_B |f - f_B| dw < \infty.$$

In the case where $\beta = 0$, the space Lip(β) coincides with the space BMO_κ. Separately, the authors of [22] introduced the Lipschitz space $\dot{\Lambda}^d_\beta$ associated with the Dunkl metric d. For $\beta \in (0, 1)$, it is defined as the space of functions f on $\mathbb{R}^d$ satisfying

$$\|f\|_{\dot{\Lambda}^d_\beta} := \sup_{d(x,y)\neq 0} \frac{|f(x) - f(y)|}{d(x, y)^\beta} < \infty.$$

We note that for $0 < \beta < 1$, $\dot{\Lambda}^d_\beta \subset \dot{\Lambda}_\beta \subset \mathrm{Lip}(\beta)$.

The primary objective of this paper is to determine the necessary and sufficient conditions for the boundedness of the commutators $[b, M_{\alpha,\kappa}]$ and $[b, M_\kappa^\sharp]$ on Orlicz spaces for functions $b \in \mathrm{Lip}(\beta)$. Significantly, we also obtain new characterizations of non-negative Lipschitz functions. Since the Orlicz space is defined as a generalization of the Lebesgue L^p space, and to properly establish this space in the Dunkl setting, we begin by reviewing the definition of Young functions.

Definition 1.1. A function $\Phi : [0, \infty) \to [0, \infty]$ is called a Young function if it is convex and left-continuous, satisfies $\Phi(0) = 0$ with $\lim_{r\to 0^+} \Phi(r) = 0$, and $\lim_{r\to\infty} \Phi(r) = \infty$.

From the convexity of Φ and the condition $\Phi(0) = 0$, it follows directly that any Young function is increasing. We denote the set of Young functions satisfying

$$0 < \Phi(r) < \infty \quad \text{for} \quad 0 < r < \infty$$

by $\mathcal{Y}$. For any Young function Φ and $0 \leq s \leq \infty$, let

$$\Phi^{-1}(s) := \inf\{r \geq 0 : \Phi(r) > s\}.$$

Finally, a Young function Φ is said to satisfy the ∇_2-condition, denoted by $\Phi \in \nabla_2$, if there exists some $C > 1$ such that

$$\Phi(r) \leq \frac{1}{2C}\Phi(Cr), \quad r \geq 0.$$

Definition 1.2. For a Young function Φ, the Orlicz space $L^{\Phi}_{\kappa}(\mathbb{R}^d)$ is defined as

$$L^{\Phi}_{\kappa}(\mathbb{R}^d) = \{f \in L^1_{\kappa,loc}(\mathbb{R}^d) : \int_{\mathbb{R}^d} \Phi(\lambda|f(x)|)\, d\omega(x) < \infty \text{ for some } \lambda > 0\}.$$

The Orlicz space $L^{\Phi}_{\kappa}(\mathbb{R}^d)$ is a Banach space equipped with the Luxemburg norm defined by

$$\|f\|_{L^{\Phi}_{\kappa}} := \inf\{\lambda > 0 : \int_{\mathbb{R}^d} \Phi(\frac{|f(x)|}{\lambda})\, d\omega(x) \leq 1\}.$$

When $\Phi(r) = r^p$ for all $r \in [0,\infty)$ with $1 \leq p < \infty$, we have $L^{\Phi}_{\kappa}(\mathbb{R}^d) = L^{p}_{\kappa}(\mathbb{R}^d)$. If $\Phi(r) = 0$ for all $r \in [0,1]$ and $\Phi(r) = \infty$ for all $r \in (1,\infty)$, then $L^{\Phi}_{\kappa}(\mathbb{R}^d) = L^{\infty}_{\kappa}(\mathbb{R}^d)$. For a ball $B \subset \mathbb{R}^d$, we write

$$\|f\|_{L^{\Phi}_{\kappa}(B)} := \|f\chi_B\|_{L^{\Phi}_{\kappa}(\mathbb{R}^d)}.$$

Further details on Young functions and Orlicz spaces can be found in [26].

For a given ball $B \subset \mathbb{R}^d$ and $0 \leq \alpha < N$, we define the localized Dunkl maximal function associated with B by

$$M_{\alpha,\kappa,B}f(x) := \sup_{x\in B'\subseteq B} \frac{1}{\omega(B')^{1-\alpha/N}} \int_{B'} |f(y)|\, d\omega(y),$$

where the supremum is taken over all balls B' satisfying $x \in B' \subseteq B$. In particular, when $\alpha = 0$, we write $M_{\kappa,B} := M_{0,\kappa,B}$.

Theorem 1.3. *Let $0 < \beta < \infty$, $0 \leq \alpha < N$, and $0 < \alpha + \beta < N$, and let $b \in L^1_{\kappa,loc}(\mathbb{R}^d)$. Assume that Φ and Ψ are Young functions satisfying $\Phi \in \mathcal{Y} \cap \nabla_2$ and*

$$\Psi^{-1}(t) \approx \Phi^{-1}(t)\, t^{-(\alpha+\beta)/N}.$$

Then the following statements are equivalent:

(1) *$b \in \mathrm{Lip}(\beta)$ and $b \geq 0$ almost everywhere;*
(2) *the commutator $[b, M_{\alpha,\kappa}]$ is bounded from $L^{\Phi}_{\kappa}(\mathbb{R}^d)$ to $L^{\Psi}_{\kappa}(\mathbb{R}^d)$;*
(3) *there exists a constant $C > 0$ such that*

$$\sup_B \frac{1}{\omega(B)^{\beta/N}} \Psi^{-1}(\omega(B)^{-1}) \Big\| b(\cdot) - M_{\kappa,B}(b)(\cdot) \Big\|_{L^{\Psi}_{\kappa}(B)} \leq C; \tag{3}$$

(4) *there exists a constant* $C > 0$ *such that*

$$\sup_B \frac{1}{\omega(B)^{\beta/N}\omega(B)} \int_B |b(x) - M_{\kappa,B}(b)(x)|\, d\omega(x) \le C. \tag{4}$$

Setting $\Phi(t) = t^p$ and $\Psi(t) = t^q$ in Theorem 1.3, we obtain the following result.

Corollary 1.4. *Let* $0 < \beta < \infty$, $0 \le \alpha < N$, *and* $0 < \alpha + \beta < N$, *and let* $b \in L^1_{\kappa,loc}(\mathbb{R}^d)$. *If* $1 < p < \frac{N}{\alpha+\beta}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha+\beta}{N}$, *then the following statements are equivalent:*

(1) $b \in \mathrm{Lip}(\beta)$ *and* $b \ge 0$ *almost everywhere;*
(2) *the commutator* $[b, M_{\alpha,\kappa}]$ *is bounded from* $L^p_\kappa(\mathbb{R}^d)$ *to* $L^q_\kappa(\mathbb{R}^d)$*;*
(3) *there exists a constant* $C > 0$ *such that*

$$\sup_B \frac{1}{\omega(B)^{\beta/N+1/q}} \left\| b(\cdot) - M_{\kappa,B}(b)(\cdot) \right\|_{L^q_\kappa(B)} \le C;$$

(4) *there exists a constant* $C > 0$ *such that*

$$\sup_B \frac{1}{\omega(B)^{\beta/N}\omega(B)} \int_B |b(x) - M_{\kappa,B}(b)(x)|\, d\omega(x) \le C.$$

Our next theorem characterizes the boundedness of the maximal commutator $M_{\alpha,b,\kappa}$.

Theorem 1.5. *Let* $0 < \beta < \infty$, $0 \le \alpha < N$, *and* $0 < \alpha + \beta < N$, *and let* $b \in L^1_{\kappa,loc}(\mathbb{R}^d)$. *Assume that* Φ *and* Ψ *are Young functions satisfying* $\Phi \in \mathcal{Y} \cap \nabla_2$ *and*

$$\Psi^{-1}(t) \approx \Phi^{-1}(t)\, t^{-(\alpha+\beta)/N}.$$

Then the following statements are equivalent:

(1) $b \in \mathrm{Lip}(\beta)$*;*
(2) $M_{\alpha,b,\kappa}$ *is bounded from* $L^\Phi_\kappa(\mathbb{R}^d)$ *to* $L^\Psi_\kappa(\mathbb{R}^d)$*;*
(3) *there exists a constant* $C > 0$ *such that*

$$\sup_B \frac{1}{\omega(B)^{\beta/N}} \Psi^{-1}(\omega(B)^{-1}) \left\| b(\cdot) - b_B \right\|_{L^\Psi_\kappa(B)} \le C. \tag{5}$$

We now focus on the nonlinear commutator $[b, M^\sharp_\kappa]$ and establish the following theorem.

Theorem 1.6. *Let* $0 < \beta < N$ *and* $b \in L^1_{\kappa,loc}(\mathbb{R}^d)$. *Suppose that* Φ *and* Ψ *are Young functions satisfying* $\Phi \in \mathcal{Y} \cap \nabla 2$ *and* $\Psi^{-1}(t) \approx \Phi^{-1}(t)t^{-\beta/N}$. *Then the following statements are equivalent:*

(1) $b \in \mathrm{Lip}(\beta)$ *and* $b \geqslant 0$ *almost everywhere;*
(2) *the commutator* $[b, M^\sharp_\kappa]$ *is bounded from* $L^\Phi_\kappa(\mathbb{R}^d)$ *to* $L^\Psi_\kappa(\mathbb{R}^d)$*;*
(3) *there exists a constant* $C > 0$ *such that*

$$\sup_B \frac{1}{\omega(B)^{\beta/N}} \Psi^{-1}(\omega(B)^{-1}) \left\| b(\cdot) - 2M^\sharp_\kappa(b\chi_B)(\cdot) \right\|_{L^\Psi_\kappa(B)} \leqslant C; \tag{6}$$

(4) *there exists a constant $C > 0$ such that*

$$\sup_B \frac{1}{\omega(B)^{\beta/N}\omega(B)} \int_B \Big|b(x) - 2M_\kappa^\sharp(b\chi_B)(x)\Big| d\omega(x) \leqslant C. \tag{7}$$

Setting $\Phi(t) = t^p$ and $\Phi(t) = t^q$ in Theorem 1.6 yields the following result.

Corollary 1.7. *Let $0 < \beta < N$ and $b \in L^1_{\kappa,\mathrm{loc}}(\mathbb{R}^d)$. If $1 < p < \frac{N}{\beta}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{N}$, then the following statements are equivalent:*

(1) *$b \in \mathrm{Lip}(\beta)$ and $b \geqslant 0$ almost everywhere;*
(2) *the commutator $[b, M_\kappa^\sharp]$ is bounded from $L^p_\kappa(\mathbb{R}^d)$ to $L^q_\kappa(\mathbb{R}^d)$.*
(3) *there exists a constant $C > 0$ such that*

$$\sup_B \frac{1}{\omega(B)^{\beta/N+1/q}} \Big\|b(\cdot) - 2M_\kappa^\sharp(b\chi_B)(\cdot)\Big\|_{L^q_\kappa(B)} \leqslant C;$$

(4) *there exists a constant $C > 0$ such that*

$$\sup_B \frac{1}{\omega(B)^{\beta/N}\omega(B)} \int_B \Big|b(x) - 2M_\kappa^\sharp(b\chi_B)(x)\Big| d\omega(x) \leqslant C.$$

We would like to point out that Theorem 1.3, Theorem 1.5, and Theorem 1.6 are of independent interest. The commutators in the Dunkl setting are considerably more intricate than their Euclidean counterparts. A major difficulty lies in the fact that the relation $\omega(B) \approx r^N$ is not known to hold for arbitrary balls in the Dunkl framework; as a result, many classical estimates that rely on the exact polynomial growth of the measure fail. This forces us to develop a more delicate covering and averaging argument (see the proof of Theorem 1.3 for details). Moreover, the symbol class considered in this paper is broader than those studied previously. Indeed, the space $\mathrm{Lip}(\beta)$ we adopt contains the space $\dot{\Lambda}_\beta$ introduced in [22] as a proper subset. Consequently, our results not only extend but also unify several existing characterizations in the Dunkl setting, while revealing new phenomena that do not appear in the Euclidean or classical weighted cases.

The second main result of this paper concerns the boundedness of the commutator associated with the Dunkl Riesz transform. Analogous to the classical theory of singular integrals, a natural Riesz transform can be defined in the Dunkl setting. The case $d = 1$ was first studied by Thangavelu and Xu [28], who established the L^p-boundedness of the corresponding Riesz transform. This result was later extended to the general case of arbitrary dimension d by Amri and Sifi [1]. For $j = 1, 2, \dots, d$, the Dunkl Riesz transforms R_j are defined by

$$R_j(f)(x) = c_\kappa \lim_{\epsilon\to 0} \int_{|y|>\epsilon} \tau_x(f)(-y) \frac{y_j}{\|y\|^{p_\kappa}} \, d\omega(y),$$

where $c_\kappa = 2^{\frac{p_\kappa - 1}{2}} \frac{\Gamma(\frac{p_\kappa}{2})}{\sqrt{\pi}}$ and $p_\kappa = \gamma_\kappa + d + 1$. In [1], the authors further obtained an explicit representation for the integral kernel $R_j(x, y)$, so that

$$R_j(f)(x) = \int_{\mathbb{R}^d} R_j(x, y) f(y)\, d\omega(y).$$

In a recent study [23], the authors established pointwise size and regularity estimates for the kernel $R_j(x, y)$, which incorporate both the Euclidean and Dunkl metrics. They also investigated the commutators of functions in $BMO_\kappa(\mathbb{R}^d)$ and $BMO_G(\mathbb{R}^d)$ with the Dunkl-Riesz transforms. The space $BMO_G(\mathbb{R}^d)$, associated with the metric $d(x, y)$, is defined as the set of functions $f \in L^1_{\kappa,\mathrm{loc}}(\mathbb{R}^d)$ such that

$$\|f\|_{BMO_G(\mathbb{R}^d)} := \sup_B \frac{1}{\omega(\mathcal{O}(B))} \int_{\mathcal{O}(B)} |f - f_{\mathcal{O}(B)}|\, d\omega < \infty.$$

Here, $f_{\mathcal{O}(B)} = \frac{1}{\omega(\mathcal{O}(B))} \int_{\mathcal{O}(B)} f\, d\omega$. It is worth noting that $BMO_G(\mathbb{R}^d)$ is a proper subspace of $BMO_\kappa(\mathbb{R}^d)$ (see [20]). Dziubański1 and Hejna [18] extend the upper bound in [23] by showing that it holds for $b \in BMO_k$, that is, there is a constant $C > 0$ such that

$$\|[b, R_j]\|_{L^p_\kappa(\mathbb{R}^d) \to L^p_\kappa(\mathbb{R}^d)} \le C \|b\|_{BMO_\kappa(\mathbb{R}^d)}.$$

Together with the lower bounds established in [23], this result generalizes the classical theorem of Coifman, Rochberg, and Weiss to the Dunkl setting.

The final main result of this paper is formulated as follows:

Theorem 1.8. *Let $1 < p < q < \infty$, $0 < \beta < 1$, and assume that $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{N}$. Consider the commutator of the Dunkl-Riesz transform defined by $[b, R_j](f)(x) = b(x) R_j f(x) - R_j(bf)(x)$. Then $b \in \mathrm{Lip}(\beta)$ if and only if $[b, R_j]$ is bounded from $L^p_\kappa(\mathbb{R}^d)$ to $L^q_\kappa(\mathbb{R}^d)$. Moreover,*

$$\|b\|_{\mathrm{Lip}(\beta)} \approx \|[b, R_j]\|_{L^p_\kappa(\mathbb{R}^d) \to L^q_\kappa(\mathbb{R}^d)}.$$

Remark 1.9. Theorem 1.8 extends the classical result of Janson [19] to the Dunkl setting. In the Euclidean case, Janson proved that for $0 < \beta < 1$ and $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{d}$, the commutator $[b, R_j]$ is bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ if and only if b belongs to the homogeneous Lipschitz space $\dot{\Lambda}_\beta$. Here, we establish the analogous characterization in the Dunkl framework, where the underlying geometric structure is governed not only by the standard Euclidean metric but also by the so-called Dunkl "metric" induced by the group action. Consequently, the kernel of the Riesz transform exhibits a much more complicated behavior. Notably, the symbol class $\mathrm{Lip}(\beta)$ considered in our theorem is more general than $\dot{\Lambda}_\beta$ studied in [22, 23].

This paper is structured as follows. Section 2 introduces several auxiliary lemmas. The proofs of Theorems 1.3, 1.5, and 1.6 are presented in Section 3.

Section 4 is devoted to the study of the upper and lower estimates for the commutator $[b, R_j]$.

Throughout this paper, the symbol C denotes a positive constant independent of the main variables, whose value may change from line to line. We write $A \approx B$ if there exists $C > 0$ such that $C^{-1}B \le A \le CB$.

2. Auxiliary lemmas

To prove our main Theorems, we need some Lemmas.

Lemma 2.1. [24] *Let $0 < \beta < \infty$. If f belongs to $\mathrm{Lip}(\beta)$, then there exists a function g such that*

(i) $f(x) = g(x)$ for almost everywhere $x \in \mathbb{R}^d$;

(ii)

$$|g(x) - g(y)| \le C\omega(B)^{\frac{\beta}{N}}$$

holds for any ball B containing x and y.

Lemma 2.2. [21] *Let $0 < \alpha < N$, and let Φ, Ψ be Young functions with $\Phi \in \mathcal{Y} \cap \nabla_2$. Then the condition*

$$r^{-\frac{\alpha}{N}}\Phi^{-1}(r) \le C\Psi^{-1}(r), \text{ for all } r > 0,$$

where $C > 0$ is independent of r, is necessary and sufficient for the boundedness of the operator $M_{\alpha,\varkappa}$ from $L^{\Phi}_{\varkappa}(\mathbb{R}^d)$ to $L^{\Psi}_{\varkappa}(\mathbb{R}^d)$.

Lemma 2.3. [21] *Let Φ be a Young function and B be a ball in $\mathbb{R}^d$. Then*

$$\|\chi_B\|_{L^{\Phi}_{\varkappa}} = \frac{1}{\Phi^{-1}\big(\omega(B)^{-1}\big)}.$$

Lemma 2.4. [21] *For a Young function Φ and for the ball B the following inequality is valid:*

$$\int_B |f(y)|\, d\omega(y) \le 2\omega(B)\Phi^{-1}\big(\omega(B)^{-1}\big)\|f\chi_B\|_{L^{\Phi}_{\varkappa}}.$$

Lemma 2.5. *For a given ball B and $0 \le \alpha < N$, we define the following maximal function:*

$$M_{\alpha,\varkappa,B}(f)(x) = \sup_{x\in B_0 \subseteq B} \frac{1}{\omega(B_0)^{1-\frac{\alpha}{N}}} \int_{B_0} |f(y)|\, d\omega(y),$$

where the supremum is taken over all balls B_0 such that $x \in B_0 \subseteq B$. Then

$$M_{\alpha,\varkappa}(\chi_B)(x) = M_{\alpha,\varkappa,B}(\chi_B)(x) \quad \text{for all } x \in B. \tag{8}$$

Proof. Recall the definition of the fractional maximal operator $M_{\alpha,\varkappa}$:

$$M_{\alpha,\varkappa}(f\chi_B)(x) = \sup_{x\in B_1} \frac{1}{\omega(B_1)^{1-\frac{\alpha}{N}}} \int_{B_1} |f(y)\chi_B(y)|\, d\omega(y),$$

where the supremum are taken over all balls B_1 containing x.

From the definitions, the collection of balls used to define $M_{\alpha,\kappa,B}(f)(x)$ is a subset of the collection of balls used to define $M_{\alpha,\kappa}(f\chi_B)(x)$. This immediately yields the inequality:

$$M_{\alpha,\kappa,B}(f)(x) \leq M_{\alpha,\kappa}(f\chi_B)(x).$$

To prove the identity (8), we must establish the reverse inequality,

$$M_{\alpha,\kappa}(\chi_B)(x) \leq M_{\alpha,\kappa,B}(\chi_B)(x).$$

This requires showing that for any ball B_1 containing x, there exists a ball $B_0 \subseteq B$ also containing x such that:

$$\frac{1}{\omega(B_1)^{1-\frac{\alpha}{N}}}\int_{B_1}|\chi_B(y)|\,d\omega(y) \leq \frac{1}{\omega(B_0)^{1-\frac{\alpha}{N}}}\int_{B_0}|\chi_B(y)|\,d\omega(y). \tag{3.2}$$

We proceed with three cases based on the ball B_1.

Case 1: $B_1 \cap B = \varnothing$. In this case, $\chi_B(y) = 0$ for all $y \in B_1$. The integral on the left-hand side of (3.2) is zero, so the inequality holds trivially.

Case 2: $B_1 \cap B \neq \varnothing$ and $\omega(B_1) \geq \omega(B)$. Let us choose $B_0 = B$. Since $B_1 \cap B \subseteq B$ and $\omega(B_1) \geq \omega(B)$, we can estimate the left-hand side of (3.2) as follows:

$$\begin{aligned}\frac{1}{\omega(B_1)^{1-\frac{\alpha}{N}}}\int_{B_1}|\chi_B(y)|\,d\omega(y) &= \frac{\omega(B_1\cap B)}{\omega(B_1)^{1-\frac{\alpha}{N}}}\\ &\leq \frac{\omega(B)}{\omega(B)^{1-\frac{\alpha}{N}}}\\ &= \frac{1}{\omega(B_0)^{1-\frac{\alpha}{N}}}\int_{B_0}|\chi_B(y)|\,d\omega(y).\end{aligned}$$

Thus, inequality (3.2) holds.

Case 3: $B_1 \cap B \neq \varnothing$ and $\omega(B_1) < \omega(B)$. In this case, the left-hand side of (3.2) is

$$\frac{1}{\omega(B_1)^{1-\frac{\alpha}{N}}}\int_{B_1}|\chi_B(y)|\,d\omega(y) = \frac{\omega(B_1\cap B)}{\omega(B_1)^{1-\frac{\alpha}{N}}} \leq \frac{\omega(B_1)}{\omega(B_1)^{1-\frac{\alpha}{N}}} = \omega(B_1)^{\frac{\alpha}{N}}.$$

For the right-hand side (3.2), we again choose $B_0 = B$:

$$\frac{1}{\omega(B_0)^{1-\frac{\alpha}{N}}}\int_{B_0}|\chi_B(y)|\,d\omega(y) = \frac{\omega(B_0\cap B)}{\omega(B_0)^{1-\frac{\alpha}{N}}} = \frac{\omega(B)}{\omega(B)^{1-\frac{\alpha}{N}}} = \omega(B)^{\frac{\alpha}{N}}.$$

By our assumption for this case, $\omega(B_1) < \omega(B)$, which implies $\omega(B_1)^{\frac{\alpha}{N}} \leq \omega(B)^{\frac{\alpha}{N}}$. Therefore, (3.2) holds. This completes the proof of Lemma 2.5.

□

Lemma 2.6. *Let $\beta > 0$, $0 \leq \alpha < N$, $0 < \alpha+\beta < N$, $b \in L^1_{\kappa,loc}(\mathbb{R}^d)$. Suppose that Φ and Ψ are Young functions, $\Phi \in \mathcal{Y} \cap \nabla_2$ and $\Psi^{-1}(t) \approx \Phi^{-1}(t)t^{-(\alpha+\beta)/N}$. If $[b, M_{\alpha,\kappa}]$ is bounded from $L^\Phi_\kappa(\mathbb{R}^d)$ to $L^\Psi_\kappa(\mathbb{R}^d)$, then $b \in \mathrm{Lip}(\beta)$.*

Proof. From Lemma 2.5, we know that $M_{\alpha,\kappa}(\chi_B)(x) = M_{\alpha,\kappa,B}(\chi_B)(x) = \omega(B)^{\frac{\alpha}{N}}$ for all $x \in B$. This yields

$$\begin{aligned}
&\frac{1}{\omega(B)^{\frac{\beta}{N}}}\Psi^{-1}(\omega(B)^{-1})\|b(\cdot) - \omega(B)^{-\frac{\alpha}{N}}M_{\alpha,\kappa}(b\chi_B)(\cdot)\|_{L^{\Psi}_{\kappa}(B)} \\
&< \frac{1}{\omega(B)^{\frac{\alpha+\beta}{N}}}\Psi^{-1}(\omega(B)^{-1})\|b(\cdot)M_{\alpha,\kappa}(\chi_B)(\cdot) - M_{\alpha,\kappa}(b\chi_B)(\cdot)\|_{L^{\Psi}_{\kappa}(B)} \\
&= \frac{1}{\omega(B)^{\frac{\alpha+\beta}{N}}}\Psi^{-1}(\omega(B)^{-1})\|[b, M_{\alpha,\kappa}](\chi_B)\|_{L^{\Psi}_{\kappa}(B)} \\
&\leq C\frac{1}{\omega(B)^{\frac{\alpha+\beta}{N}}}\Psi^{-1}(\omega(B)^{-1})\|\chi_B\|_{L^{\Phi}_{\kappa}(B)} \\
&\leq C\frac{1}{\omega(B)^{\frac{\alpha+\beta}{N}}}\frac{\Psi^{-1}(\omega(B))^{-1}}{\Phi^{-1}(\omega(B))^{-1}} \\
&\leq C,
\end{aligned} \tag{9}$$

where in the last step we have applied Lemma 2.3 and the hypothesis $\Psi^{-1}(t) \approx \Phi^{-1}(t)t^{-(\alpha+\beta)/N}$.

For a fixed ball B, define

$$E = \{x \in B : b(x) \leq b_B\}, \qquad F = \{x \in B : b(x) \geq b_B\}.$$

It follows that

$$\int_E |b(x) - b_B|\, d\omega(x) = \int_F |b(x) - b_B|\, d\omega(x).$$

Hence, we obtain

$$\begin{aligned}
&\frac{1}{\omega(B)^{\frac{\beta}{N}}\omega(B)}\int_B |b(x) - b_B|\, d\omega(x) \\
=&\frac{2}{\omega(B)^{\frac{\beta}{N}}\omega(B)}\int_E |b(x) - b_B|\, d\omega(x) \\
\leq&\frac{2}{\omega(B)^{\frac{\beta}{N}}\omega(B)}\int_B |b(x) - \omega(B)^{-\frac{\alpha}{N}}M_{\alpha,\kappa}(b\chi_B)(x)|\, d\omega(x),
\end{aligned}$$

where the inequality is due to the fact that $b(x) \le b_B \le \omega(B)^{-\frac{\alpha}{N}} M_{\alpha,\kappa}(b\chi_B)(x)$ for all $x \in E$. Applying Lemma 2.4, we further deduce

$$\begin{aligned}&\frac{1}{\omega(B)^{\frac{\beta}{N}}\omega(B)}\int_B |b(x)-b_B|\,d\omega(x)\\ \le&\frac{4}{\omega(B)^{\frac{\beta}{N}}}\Psi^{-1}(\omega(B)^{-1})\|b\,(\cdot)-\omega(B)^{-\frac{\alpha}{N}}M_{\alpha,\kappa}(b\chi_B)\,(\cdot)\,\|_{L^{\Psi}_{\kappa}(B)}\\ \le& C.\end{aligned}$$

Therefore, we conclude that $b \in \mathrm{Lip}(\beta)$, which completes the proof of Lemma 2.6. □

Lemma 2.7. [23] *For each $j = 1, 2, \dots, N$ and every ball $B = B(x_0, r) \subset \mathbb{R}^d$, there exists another ball $\widetilde{B} = B(y_0, r)$ such that $\|x_0 - y_0\| = 5r$. Furthermore, for all $(x, y) \in B \times \widetilde{B}$, we have*

$$|R_j(x,y)| \ge \frac{C}{\omega(B(x_0,r))}.$$

3. Proof of Theorems 1.3, 1.5 and 1.6

Proof of Theorem 1.3. (1)⇒(2). For any fixed ball B,

$$\begin{aligned}&|[M_{\alpha,\kappa},b]f(x)| = \left|M_{\alpha,\kappa}(bf)(x)-b(x)M_{\alpha,\kappa}(f)(x)\right|\\ &\le\left|\sup_{x\in B}\frac{1}{\omega(B)^{1-\frac{\alpha}{N}}}\int_B b(y)|f(y)|\,d\omega(y)-\sup_{x\in B}\frac{1}{\omega(B)^{1-\frac{\alpha}{N}}}\int_B b(x)|f(y)|\,d\omega(y)\right|\\ &\le\sup_{x\in B}\frac{1}{\omega(B)^{1-\frac{\alpha}{N}}}\int_B|b(x)-b(y)||f(y)|\,d\omega(y)\\ &=M_{\alpha,b,\kappa}(f)(x).\end{aligned}$$

Since $b \in \mathrm{Lip}(\beta)$, we have

$$\begin{aligned}M_{\alpha,b,\kappa}(f)(x) =&\sup_{x\in B}\frac{1}{\omega(B)^{1-\frac{\alpha}{N}}}\int_B|b(x)-b(y)||f(y)|\,d\omega(y)\\ \le& C\|b\|_{\mathrm{Lip}(\beta)}\sup_{x\in B}\frac{1}{\omega(B)^{1-\frac{\alpha}{N}}}\int_B\omega(B)^{\frac{\beta}{N}}|f(y)|\,d\omega(y)\\ =&C\|b\|_{\mathrm{Lip}(\beta)}M_{\alpha+\beta,\kappa}(f)(x).\end{aligned}$$

By Lemma 2.2, $M_{\alpha+\beta,\kappa}$ is bounded from $L^{\Phi}_{\kappa}(\mathbb{R}^d)$ to $L^{\Psi}_{\kappa}(\mathbb{R}^d)$ if $\alpha+\beta < N$. Hence, $[b, M_{\alpha,\kappa}]$ is bounded from $L^{\Phi}(\mathbb{R}^d, d\omega)$ to $L^{\Psi}(\mathbb{R}^d, d\omega)$.

(2) ⇒ (3). We divide the proof into two cases based on the value of α.

Case 1: $\alpha = 0$. For any fixed ball B and $x \in B$, by Lemma 2.5, we have $M(\chi_B)(x) = \chi_B(x) = 1$. Thus,

$$\left|b(x)-M_{\kappa,B}(b)(x)\right| = \left|b(x)M_{\kappa}(\chi_B)(x)-M_{\kappa,B}(b)(x)\right|$$

$$
\begin{aligned}
&= M_{\kappa,B}(b)(x) - b(x)M_\kappa(\chi_B)(x) \\
&\leq \Big|M_\kappa(b\chi_B)(x) - b(x)M_\kappa(\chi_B)(x)\Big| \\
&= \Big|[b, M_\kappa](\chi_B)(x)\Big|,
\end{aligned}
$$

where we have used the fact that $M_\kappa(b\chi_B)(x) \geq M_{\kappa,B}(b)(x) \geq b(x)M_\kappa(\chi_B)(x)$ for $x \in B$. Since $[b, M_\kappa]$ is bounded from $L^\Phi_\kappa(\mathbb{R}^d, d\omega)$ to $L^\Psi_\kappa(\mathbb{R}^d, d\omega)$ and $\Psi^{-1}(t) \approx \Phi^{-1}(t)t^{-\beta/N}$, Lemma 2.6 gives

$$
\begin{aligned}
&\frac{1}{\omega(B)^{\frac{\beta}{N}}}\Psi^{-1}(\omega(B)^{-1})\|b - M_{\kappa,B}(b)\|_{L^\Psi_\kappa(B)} \\
&\leq \frac{1}{\omega(B)^{\frac{\beta}{N}}}\Psi^{-1}(\omega(B)^{-1})\|[b, M_\kappa](\chi_B)\|_{L^\Psi_\kappa(B)} \\
&\leq C\frac{1}{\omega(B)^{\frac{\beta}{N}}}\Psi^{-1}(\omega(B)^{-1})\|\chi_B\|_{L^\Phi_\kappa(B)} \\
&\leq C\frac{1}{\omega(B)^{\frac{\beta}{N}}}\frac{\Psi^{-1}(\omega(B)^{-1})}{\Phi^{-1}(\omega(B)^{-1})} \\
&\leq C.
\end{aligned}
$$

Case 2: $0 < \alpha < N$. For any fixed ball B, we have

$$
\begin{aligned}
&\frac{1}{\omega(B)^{\frac{\beta}{N}}}\Psi^{-1}(\omega(B)^{-1})\|b - M_{\kappa,B}(b)\|_{L^\Psi_\kappa(B)} \\
&\leq \frac{1}{\omega(B)^{\frac{\beta}{N}}}\Psi^{-1}(\omega(B)^{-1})\|b - \omega(B)^{-\alpha/N}M_{\alpha,\kappa}(b\chi_B)\|_{L^\Psi_\kappa(B)} \\
&\quad + \frac{1}{\omega(B)^{\frac{\beta}{N}}}\Psi^{-1}(\omega(B)^{-1})\|\omega(B)^{-\alpha/N}M_{\alpha,\kappa}(b\chi_B) - M_{\kappa,B}(b)\|_{L^\Psi_\kappa(B)} \\
&:= I_1 + I_2.
\end{aligned}
$$

For I_1, from (9), we have

$$I_1 \leq C.$$

For I_2, note that for $x \in B$, $M_{\kappa,B}(\chi_B)(x) = \chi_B(x)$. Combining this with Lemma 2.5 yields $M(\chi_B)(x) = \chi_B(x)$ for $x \in B$. Thus for any $x \in B$, we get

$$
\begin{aligned}
&\Big|\omega(B)^{-\alpha/N}M_{\alpha,\kappa}(b\chi_B)(x) - M_{\kappa,B}(b)(x)\Big| \\
&= \omega(B)^{-\alpha/N}\Big|M_{\alpha,\kappa}(b\chi_B)(x) - \omega(B)^{\alpha/N}M_{\kappa,B}(b)(x)\Big| \\
&= \omega(B)^{-\alpha/N}\Big|M_{\alpha,\kappa}(b\chi_B)(x) - M_{\alpha,\kappa}(\chi_B)(x)M_{\kappa,B}(b)(x)\Big| \\
&\leq \omega(B)^{-\alpha/N}\Big|M_{\alpha,\kappa}(b\chi_B)(x) - b(x)M_{\alpha,\kappa}(\chi_B)(x)\Big| \\
&\quad + \omega(B)^{-\alpha/N}\Big|b(x)M_{\alpha,\kappa}(\chi_B)(x) - M_{\alpha,\kappa}(\chi_B)(x)M_{\kappa,B}(b)(x)\Big| \\
&= \omega(B)^{-\alpha/N}\Big|M_{\alpha,\kappa}(b\chi_B)(x) - b(x)M_{\alpha,\kappa}(\chi_B)(x)\Big|
\end{aligned}
$$

$$
\begin{aligned}
&+ \omega(B)^{-\alpha/N} M_{\alpha,\kappa}(\chi_B)(x)\Big|b(x)M_\kappa(\chi_B)(x) - M_{\kappa,B}(b)(x)\Big| \\
\leq\ & \omega(B)^{-\alpha/N}\Big|[b, M_{\alpha,\kappa}](\chi_B)(x)\Big| + \Big|[b, M_\kappa](\chi_B)(x)\Big|.
\end{aligned}
$$

Since $[b, M_{\alpha,\kappa}]$ is bounded from $L^\Phi(\mathbb{R}^d, d\omega)$ to $L^\Psi(\mathbb{R}^d, d\omega)$, Lemma 2.6 implies $b \in \mathrm{Lip}(\beta)$. For any $x \in B$, we have the estimate

$$
\begin{aligned}
|[b, M_\kappa](\chi_B)(x)| &= |b(x)M_\kappa(\chi_B)(x) - M_\kappa(b\chi_B)(x)| \\
&\leq \sup_{B_1 \ni x} \frac{1}{\omega(B_1)} \int_{B_1} |b(x) - b(y)||\chi_B(y)|\, d\omega(y) \\
&\leq C\omega(B)^{\frac{\beta}{N}} \sup_{B_1 \ni x} \frac{1}{\omega(B_1)} \int_{B_1} |\chi_B(y)|\, d\omega(y) \\
&= C\omega(B)^{\frac{\beta}{N}} M_\kappa(\chi_B)(x) \\
&= C\omega(B)^{\frac{\beta}{N}} \chi_B(x).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I_2 &\leq \frac{1}{\omega(B)^{\frac{\beta}{N}}} \Psi^{-1}(\omega(B)^{-1}))\big(\|\omega(B)^{-\alpha/N}[b, M_{\alpha,\kappa}](\chi_B)(x)\|_{L^\Psi_\kappa(B)} \\
&\quad + \|[b, M_\kappa](\chi_B)(x)\|_{L^\Psi_\kappa(B)}\big) \\
&\leq C\frac{1}{\omega(B)^{\frac{\beta}{N}}} \Psi^{-1}(\omega(B)^{-1})\big(\omega(B)^{-\alpha/N}\|\chi_B\|_{L^\Phi_\kappa(B)} + \omega(B)^{\frac{\beta}{N}}\|\chi_B\|_{L^\Psi_\kappa(B)}\big) \\
&\leq C.
\end{aligned}
$$

Combining the estimates for I_1 and I_2, the inequality holds.

(3)⇒(4). For any fixed ball B, by Lemma 2.4 and (3), we get

$$
\begin{aligned}
&\frac{1}{\omega(B)^{\frac{\beta}{N}}\omega(B)} \int_B |b(x) - M_{\kappa,B}(b)(x)|\, d\omega(x) \\
&\leq C\frac{1}{\omega(B)^{\frac{\beta}{N}}} \Psi^{-1}(\omega(B)^{-1})\|b\,(\cdot) - M_{\kappa,B}(b)\,(\cdot)\,\|_{L^\Psi_\kappa(B)} \\
&\leq C,
\end{aligned}
$$

which implies (4).

(4)⇒(1). By Lemma 2.1, it suffices to show that there exists a constant $C > 0$ such that for any ball $B \subset \mathbb{R}^d$,

$$
\frac{1}{\omega(B)^{\frac{\beta}{N}}\omega(B)} \int_B |b(x) - b_B|\, d\omega(x) \leq C.
$$

For a given ball B, let $E = \{x \in B : b(x) \leq b_B\}$ and $F = \{x \in B : b(x) > b_B\}$. Then

$$
\int_E |b(x) - b_B|\, d\omega(x) = \int_F |b(x) - b_B|\, d\omega(x).
$$

Thus,

$$
\begin{aligned}
&\frac{1}{\omega(B)^{\frac{\beta}{N}}\omega(B)}\int_B |b(x)-b_B|\,d\omega(x)\\
&= \frac{1}{\omega(B)^{\frac{\beta}{N}}\omega(B)}\int_{E\cup F} |b(x)-b_B|\,d\omega(x)\\
&= \frac{2}{\omega(B)^{\frac{\beta}{N}}\omega(B)}\int_{E} |b(x)-b_B|\,d\omega(x)\\
&\leq \frac{2}{\omega(B)^{\frac{\beta}{N}}\omega(B)}\int_{E} |b(x)-M_{\kappa,B}(b)(x)|\,d\omega(x)\\
&\leq \frac{2}{\omega(B)^{\frac{\beta}{N}}\omega(B)}\int_{B} |b(x)-M_{\kappa,B}(b)(x)|\,d\omega(x)\\
&\leq C.
\end{aligned}
$$

Hence, $b\in \mathrm{Lip}(\beta)$.

Next, we prove that $b\geq 0$, which is equivalent to showing $b^-=0$. For any given ball B and $x\in B$, we have

$$0\leq b^+(x)\leq |b(x)|\leq M_{\kappa,B}(b)(x).$$

Thus,

$$0\leq b^-(x)\leq M_{\kappa,B}(b)(x)-b^+(x)+b^-(x)=M_{\kappa,B}(b)(x)-b(x).$$

From (4), it follows that

$$
\begin{aligned}
\frac{1}{\omega(B)}\int_B b^-(x)\,d\omega(x) &\leq \frac{1}{\omega(B)}\int_B |M_{\kappa,B}(b)(x)-b(x)|\,d\omega(x)\\
&= \omega(B)^{\frac{\beta}{N}}\frac{1}{\omega(B)^{\frac{\beta}{N}}\omega(B)}\int_B |M_{\kappa,B}(b)(x)-b(x)|\,d\omega(x)\\
&\leq C\omega(B)^{\frac{\beta}{N}}.
\end{aligned}
$$

By the Lebesgue differentiation theorem, we consider the limit as $\omega(B)\to 0$. The inequality

$$\omega(B(x,r))\geq r^d\cdot r^{\sum\limits_{\alpha\in R}\kappa(\alpha)}=r^N$$

implies that the radius r must also tend to zero. Hence, we obtain Hence, we obtain

$$b^-(x)=0 \text{ for } a.\,e.\,x,$$

which implies $b\geq 0$ almost everywhere. This completes the proof.

Proof of Theorem 1.5. (1)⇒(2). The argument for the implication (1)⇒(2) in Theorem 1.1 yields the pointwise estimate

$$M_{\alpha,b,\kappa}(f)(x)\leq C\|b\|_{\mathrm{Lip}(\beta)}M_{\alpha+\beta,\kappa}(f)(x).$$

By Lemma 2.2, the operator $M_{\alpha+\beta,\kappa}$ is bounded from $L^{\Phi}_{\kappa}(\mathbb{R}^d)$ to $L^{\Psi}_{\kappa}(\mathbb{R}^d)$. This, combined with the pointwise estimate above, implies that $M_{\alpha,b,\kappa}$ is also bounded from $L^{\Phi}_{\kappa}(\mathbb{R}^d)$ to $L^{\Psi}_{\kappa}(\mathbb{R}^d)$.

(2)⇒(3). Let B be an arbitrary ball. For any $x \in B$, we have the pointwise estimate

$$\begin{aligned}|b(x)-b_B| \leq & \frac{1}{\omega(B)}\int_B |b(x)-b(y)|\,d\omega(y)\\ =&\frac{1}{\omega(B)^{\frac{\alpha}{N}}}\frac{1}{\omega(B)^{1-\frac{\alpha}{N}}}\int_B |b(x)-b(y)|\chi_B(y)\,d\omega(y)\\ \leq & \frac{1}{\omega(B)^{\frac{\alpha}{N}}}M_{\alpha,b,\kappa}(\chi_B)(x).\end{aligned}$$

By assumption (2), we have

$$\begin{aligned}&\frac{1}{\omega(B)^{\frac{\beta}{N}}}\Psi^{-1}(\omega(B)^{-1})\|b\,(\cdot)-b_B\|_{L^{\Psi}_{\kappa}(B)}\\ \leq & \frac{1}{\omega(B)^{\frac{\alpha+\beta}{N}}}\Psi^{-1}(\omega(B)^{-1})\|M_{\alpha,b,\kappa}(\chi_B)\,(\cdot)\,\|_{L^{\Psi}_{\kappa}(B)}\\ \leq & C\frac{1}{\omega(B)^{\frac{\alpha+\beta}{N}}}\Psi^{-1}(\omega(B)^{-1})\|\chi_B\|_{L^{\Phi}_{\kappa}(B)}\\ \leq & C\frac{1}{\omega(B)^{\frac{\alpha+\beta}{N}}}\frac{\Psi^{-1}(\omega(B)^{-1})}{\Phi^{-1}(\omega(B)^{-1})}\\ \leq & C.\end{aligned}$$

(3)⇒(1). Let B be an arbitrary ball. by Lemma 2.4 and (5), we get

$$\begin{aligned}\frac{1}{\omega(B)^{\frac{\beta}{N}}\omega(B)}\int_B |b(x)-b_B|\,d\omega(x) &\leq C\frac{1}{\omega(B)^{\frac{\beta}{N}}}\Psi^{-1}(\omega(B)^{-1})\|b\,(\cdot)-b_B\|_{L^{\Psi}_{\kappa}(B)}\\ &\leq C,\end{aligned}$$

which yields $b \in \mathrm{Lip}(\beta)$. The proof is complete.

Proof of Theorem 1.6. (1)⇒(2). Since $b \in \mathrm{Lip}(\beta)$ and $b \geqslant 0$, we can estimate the commutator $[b, M^{\sharp}_{\kappa}]$ as follows:

$$\begin{aligned}&|[b,M^{\sharp}_{\kappa}]f(x)|\\ &=\left|\sup_{x\in B}\frac{b(x)}{\omega(B)}\int_B |f(y)-f_B|\,d\omega(y)-\sup_{x\in B}\frac{1}{\omega(B)}\int_B |b(y)f(y)-(bf)_B|\,d\omega(y)\right|\\ &\leq \sup_{x\in B}\frac{1}{\omega(B)}\int_B |(b(y)-b(x))f(y)+b(x)f_B-(bf)_B|\,d\omega(y)\end{aligned}$$

$$\leq \sup_{x\in B}\left\{\frac{1}{\omega(B)}\int_B |b(y)-b(x)||f(y)|\,d\omega(y)+|b(x)f_B-(bf)_B|\right\}$$
$$\leq C\|b\|_{\mathrm{Lip}(\beta)}M_{\beta,\kappa}(f)(x)+\sup_{x\in B}|\frac{b(x)}{\omega(B)}\int_B f(z)\,d\omega(z)-\frac{1}{\omega(B)}\int_B b(z)f(z)\,d\omega(z)|$$
$$\leq C\|b\|_{\mathrm{Lip}(\beta)}M_{\beta,\kappa}(f)(x)+\sup_{x\in B}\frac{1}{\omega(B)}\int_B |b(x)-b(z)||f(z)|\,d\omega(z)$$
$$\leq C\|b\|_{\mathrm{Lip}(\beta)}M_{\beta,\kappa}(f)(x).$$

By Lemma 2.2, the operator $M_{\beta,\kappa}$ is bounded from $L^{\Phi}_{\kappa}(\mathbb{R}^d)$ to $L^{\Psi}_{\kappa}(\mathbb{R}^d)$. The pointwise estimate above thus implies that $[b, M^{\sharp}_{\kappa}]$ is also bounded from $L^{\Phi}_{\kappa}(\mathbb{R}^d)$ to $L^{\Psi}_{\kappa}(\mathbb{R}^d)$.

(2)⇒(3). We first show that for any ball B and any $x\in B$, $M^{\sharp}_{\kappa}(\chi_B)(x)=\frac{1}{2}$. From the definition of $M^{\sharp}_{\kappa}$, for any ball B_1 containing x, we have

$$\begin{aligned}
M^{\sharp}_{\kappa}(\chi_B)(x) &= \sup_{x\in B_1}\frac{1}{\omega(B_1)}\int_{B_1}|\chi_B(y)-(\chi_B)_{B_1}|\,d\omega(y)\\
&= \sup_{x\in B_1}\frac{1}{\omega(B_1)}\int_{B_1}|\chi_B(y)-\frac{1}{\omega(B_1)}\int_{B_1}\chi_B(z)\,d\omega(z)|\,d\omega(y)\\
&= \sup_{x\in B_1}\frac{1}{\omega(B_1)^2}\int_{B_1}|\omega(B_1)\chi_B(y)-\omega(B_1\cap B)|\,d\omega(y)\\
&= \sup_{x\in B_1}\frac{1}{\omega(B_1)^2}\int_{B_1\cap B}|\omega(B_1)\chi_B(y)-\omega(B_1\cap B)|\,d\omega(y)\\
&\quad+\sup_{x\in B_1}\frac{1}{\omega(B_1)^2}\int_{B_1-B}|\omega(B_1)\chi_B(y)-\omega(B_1\cap B)|\,d\omega(y)\\
&= \sup_{x\in B_1}\frac{1}{\omega(B_1)^2}\int_{B_1\cap B}|\omega(B_1-B)|\,d\omega(y)\\
&\quad+\sup_{x\in B_1}\frac{1}{\omega(B_1)^2}\int_{B_1-B}|\omega(B_1\cap B)|\,d\omega(y)\\
&= \frac{2\omega(B_1\cap B)\omega(B_1-B)}{\omega(B_1)^2}.
\end{aligned}$$

Using the inequality $4rs\leq(r+s)^2$ with $r=\omega(B_1\cap B)$ and $s=\omega(B_1\setminus B)$, we get $r+s=\omega(B_1)$, so the expression is bounded by $\frac{2}{4}=\frac{1}{2}$. Thus, $M^{\sharp}_{\kappa}(\chi_B)(x)\leq\frac{1}{2}$.

Conversely, for $x\in B$, we can choose a ball $B_1\supset B$ with $\omega(B_1)=2\omega(B)$. Then $\omega(B_1\cap B)=\omega(B)$ and $\omega(B_1\setminus B)=\omega(B)$. For this specific ball, the expression above becomes $\frac{2\omega(B)\omega(B)}{(2\omega(B))^2}=\frac{1}{2}$. Since $M^{\sharp}_{\kappa}(\chi_B)(x)$ is the supremum over all such balls B_1, it must be at least $\frac{1}{2}$. Therefore, $M^{\sharp}_{\kappa}(\chi_B)(x)=\frac{1}{2}$ for all $x\in B$.

Now, for any $x \in B$, we can write

$$\begin{aligned} b(x) - 2M_\kappa^\sharp(b\chi_B)(x) =& 2(\frac{1}{2}b(x) - M_\kappa^\sharp(b\chi_B)(x)) \\ =& 2(M_\kappa^\sharp(\chi_B)(x)b(x) - M_\kappa^\sharp(b\chi_B)(x)) \\ =& 2[b, M_\kappa^\sharp](\chi_B)(x). \end{aligned}$$

By assumption (2), $[b, M_\kappa^\sharp]$ is bounded from $L_\kappa^\Phi(\mathbb{R}^d)$ to $L_\kappa^\Psi(\mathbb{R}^d)$. Using this, Lemma 2.3, and the relation $\Psi^{-1}(t) \approx \Phi^{-1}(t)t^{-\beta/N}$, we have

$$\begin{aligned} &\frac{1}{\omega(B)^{\frac{\beta}{N}}}\Psi^{-1}(\omega(B)^{-1})\|b\,(\cdot) - 2M_\kappa^\sharp(b\chi_B)\,(\cdot)\,\|_{L_\kappa^\Psi(B)} \\ &= 2\frac{1}{\omega(B)^{\frac{\beta}{N}}}\Psi^{-1}(\omega(B)^{-1})\|[b, M_\kappa^\sharp](\chi_B)\|_{L_\kappa^\Psi(B)} \\ &\leq C\frac{1}{\omega(B)^{\frac{\beta}{N}}}\Psi^{-1}(\omega(B)^{-1})\|\chi_B\|_{L_\kappa^\Phi(\mathbb{R}^d)} \\ &\leq C\frac{1}{\omega(B)^{\frac{\beta}{N}}}\frac{\Psi^{-1}(\omega(B)^{-1})}{\Phi^{-1}(\omega(B)^{-1})} \\ &\leq C. \end{aligned}$$

(3)⇒(4). Let B be an arbitrary ball. By Lemma 2.4 and inequality (6), we have

$$\begin{aligned} &\frac{1}{\omega(B)^{\frac{\beta}{N}}\omega(B)}\int_B |b(x) - 2M_\kappa^\sharp(b\chi_B)(x)|\,d\omega(x) \\ &\leq C\frac{1}{\omega(B)^{\frac{\beta}{N}}}\Psi^{-1}(\omega(B)^{-1})\|b\,(\cdot) - 2M_\kappa^\sharp(b\chi_B)\,(\cdot)\,\|_{L_\kappa^\Psi(B)} \\ &\leq C. \end{aligned}$$

(4)⇒(1). First, we show that $2M_\kappa^\sharp(b\chi_B)(x) \geq |b_B|$ for any $x \in B$. For such an x, we choose a ball $B_1 \supset B$ with $\omega(B_1) = 2\omega(B)$. This gives

$$\begin{aligned} M_\kappa^\sharp(b\chi_B)(x) \geq& \frac{1}{\omega(B_1)}\int_{B_1} |b\chi_B(y) - (b\chi_B)_{B_1}|\,d\omega(y) \\ =& \frac{1}{2\omega(B)}\int_{B_1} |b\chi_B(y) - \frac{1}{2\omega(B)}\int_B b(z)\,d\omega(z)|\,d\omega(y) \\ =& \frac{1}{2\omega(B)}\int_B |b(y) - \frac{1}{2}b_B|\,d\omega(y) + \frac{1}{4}|b_B|. \end{aligned}$$

By the triangle inequality,

$$\begin{aligned}|b_B| \le& \frac{1}{\omega(B)}\int_B |b(y)-\frac{1}{2}b_B|\,d\omega(y)+\frac{1}{\omega(B)}\int_B |\frac{1}{2}b_B|\,d\omega(y)\\ =&\frac{1}{\omega(B)}\int_B |b(y)-\frac{1}{2}b_B|\,d\omega(y)+\frac{1}{2}|b_B|.\end{aligned}$$

Therefore

$$2M_\kappa^\sharp(b\chi_B)(x)\ge |b_B|.$$

Now, let $E=\{x\in B : b(x)\le b_B\}$. Since $\int_B(b(x)-b_B)d\omega(x)=0$, we have $\int_{B\setminus E}(b(x)-b_B)d\omega(x)=\int_E(b_B-b(x))d\omega(x)$. This implies

$$\begin{aligned}\frac{1}{\omega(B)}\int_B |b(x)-b_B|\,d\omega(x) =& \frac{2}{\omega(B)}\int_E (b_B-b(x))\,d\omega(x)\\ \le& \frac{2}{\omega(B)}\int_E (2M_\kappa^\sharp(b\chi_B)(x)-b(x))\,d\omega(x)\\ \le& \frac{2}{\omega(B)}\int_B |2M_\kappa^\sharp(b\chi_B)(x)-b(x)|\,d\omega(x).\end{aligned}$$

By (1.6), we get

$$\begin{aligned}&\frac{1}{\omega(B)^{\frac{\beta}{N}}\omega(B)}\int_B |b(x)-b_B|\,d\omega(x)\\ \le& \frac{2}{\omega(B)^{\frac{\beta}{N}}\omega(B)}\int_B |2M_\kappa^\sharp(b\chi_B)(x)-b(x)|\,d\omega(x)\\ \le& C.\end{aligned}$$

By Lemma 2.1, this implies $b\in \mathrm{Lip}(\beta)$.

Finally, we show that $b\ge 0$ for almost everywhere x. It suffices to prove that $b^-=0$ for almost every $x\in B$. Since $2M_\kappa^\sharp(b\chi_B)(x)\ge |b_B|$, we obtain that

$$2M_\kappa^\sharp(b\chi_B)(x)-b(x)\ge |b_B|-b^+(x)+b^-(x).$$

By (7), for any ball B, there is a constant $C>0$ such that

$$\begin{aligned}C \ge& \frac{1}{\omega(B)^{\frac{\beta}{N}}\omega(B)}\int_B |b(x)-2M_\kappa^\sharp(b\chi_B)(x)|\,d\omega(x)\\ \ge& \frac{1}{\omega(B)^{\frac{\beta}{N}}\omega(B)}\int_B (2M_\kappa^\sharp(b\chi_B)(x)-b(x))\,d\omega(x)\\ \ge& \frac{1}{\omega(B)^{\frac{\beta}{N}}\omega(B)}\int_B (|b_B|-b^+(x)+b^-(x))\,d\omega(x)\\ =& \frac{1}{\omega(B)^{\frac{\beta}{N}}}\Big(|b_B|-\frac{1}{\omega(B)}\int_B b^+(x)\,d\omega(x)+\frac{1}{\omega(B)}\int_B b^-(x)\,d\omega(x)\Big).\end{aligned}$$

Therefore, for any ball $B \subset \mathbb{R}^d$, we have

$$|b_B| - \frac{1}{\omega(B)} \int_B b^+(x)\, d\omega(x) + \frac{1}{\omega(B)} \int_B b^-(x)\, d\omega(x) \leq C\omega(B)^{\frac{\beta}{N}}.$$

Following a similar argument as in the proof of Theorem 1.3, by letting $\omega(B)$ tends to 0, we find that $2b^-(x) \leq 0$ for almost everywhere $x \in B$. Since $b^-(x) \geq 0$, we must have $b^-(x) = 0$ a.e. This completes the proof.

4. Proof of Theorem 1.8

Let us begin with the following lemma.

Lemma 4.1. *Let b belong to* Lip(β). *Let $x \in \mathbb{R}^d$ and let $0 < r < R$. Then,*

$$|b_{B(x,r)} - b_{B(x,R)}| \leq C\|b\|_{\mathrm{Lip}(\beta)}\omega(B(x,R))^{\frac{\beta}{N}}, \tag{10}$$

$$|b_{B(y,r)} - b_{B(x,r)}| \leq C\|b\|_{\mathrm{Lip}(\beta)}\omega(B(x,r))^{\frac{\beta}{N}} \text{ for } \|x-y\| \leq 2r, \tag{11}$$

$$|b_{B(x,r)} - b_{B(\sigma(x),r)}| \leq C\|b\|_{\mathrm{Lip}(\beta)}\omega(B(\sigma(x), \|x-\sigma(x)\| + r))^{\frac{\beta}{N}}. \tag{12}$$

Proof. Inequalities (10) and (11) follow directly from well-known results on metric spaces equipped with doubling measures (see [24]). To establish (12), observe that if $\|x - \sigma(x)\| \leq 2r$, then (12) follows immediately from (11). Assume now that $\|x - \sigma(x)\| > 2r$, and let j be the smallest positive integer such that $\|x - \sigma(x)\| \leq 2^j r$. Applying (10) and (11), we obtain

$$\begin{aligned}
&|b_{B(x,r)} - b_{B(\sigma(x),r)}| \\
&\leq |b_{B(x,r)} - b_{B(x,2^{j+2}r)}| + |b_{B(x,2^{j+2}r)} - b_{B(\sigma(x),2^{j+2}r)}| + |b_{B(\sigma(x),2^{j+2}r)} - b_{B(\sigma(x),r)}| \\
&\leq C\|b\|_{\mathrm{Lip}(\beta)}\big(\omega(B(x,2^{j+2}r))^{\frac{\beta}{N}} + \omega(B(\sigma(x),2^{j+2}r))^{\frac{\beta}{N}}\big) \\
&\leq C\|b\|_{\mathrm{Lip}(\beta)}\omega(B(\sigma(x),2^{j+2}r))^{\frac{\beta}{N}} \leq C\|b\|_{\mathrm{Lip}(\beta)}\omega(B(\sigma(x), \|x-\sigma(x)\| + r))^{\frac{\beta}{N}},
\end{aligned}$$

where the third inequality follows from the fact that

$$\omega(B(\sigma(x), 2^{j+2}r)) \approx \omega(B(x, 2^{j+2}r)).$$

This completes the proof. □

Proof of Theorem 1.8. We shall prove

$$\|[b, R_j]\|_{L^q_\kappa(\mathbb{R}^d)} \leq C\|b\|_{\mathrm{Lip}(\beta)}\|f\|_{L^p_\kappa(\mathbb{R}^d)}$$

for compactly supported functions $f \in L^p_\kappa(\mathbb{R}^d)$, which form a dense subspace in $L^p_\kappa(\mathbb{R}^d)$. To this end, by [27, p. 148, Theorem 2, and the remark below it], it is sufficient to show that

$$\|([b, R_j])^\sharp\|_{L^q_\kappa(\mathbb{R}^d)} \leq C\|b\|_{\mathrm{Lip}(\beta)}\|f\|_{L^p_\kappa(\mathbb{R}^d)}.$$

Let $x \in \mathbb{R}^d$ and let $B = B(x_0, r)$ be any ball that contains x. We enumerate the elements of $G \setminus \{\mathrm{id}\}$ as sequence $\sigma_1, \sigma_2, \dots, \sigma_{|G|-1}$. We define the sets $U_j \subseteq \mathbb{R}^d$, $j = 1, 2, \dots, |G| - 1$, inductively:

$$U_1 = \left\{z \in \mathbb{R}^d : \|z - x_0\| > 5r, \|z - \sigma_1(x_0)\| \le 5r\right\},$$

$$U_{j+1} = \left\{z \in \mathbb{R}^d : \|z - x_0\| > 5r, \|z - \sigma_{j+1}(x_0)\| \le 5r\right\} \setminus \left(\bigcup_{j_1=1}^{j} U_{j_1}\right)$$

for $1 \le j \le |G| - 1$. For a compactly supported function $f \in L^p_\kappa(\mathbb{R}^d)$, we decompose

$$f = f_1 + f_2 + \sum_{j=1}^{|G|-1} f_{\sigma_j},$$

where

$$f_1 := f \cdot \chi_{5B}, \quad f_2 := f \cdot \chi_{(\mathcal{O}(5B))^c}, \quad f_{\sigma_j} := f \cdot \chi_{U_j}.$$

Observe that $[b, R_j]f = [b - b_B, R_j]f$. For $y \in B$, we set

$$[b, R_j]f_1(y) = (b(y) - b_B)R_j f_1(y) + R_j((b_B - b)f_1)(y) =: g_{11}(y) + g_{12}(y),$$
$$[b, R_j]f_2(y) = (b(y) - b_B)R_j f_2(y) + R_j((b_B - b)f_2)(y) =: g_{21}(y) + g_{22}(y),$$
$$[b, R_j]f_{\sigma_j}(y) = (b(y) - b_B)R_j f_{\sigma_j}(y) + R_j((b_B - b)f_{\sigma_j})(y) =: g_{\sigma_j 1}(y) + g_{\sigma_j 2}(y).$$

Fix $1 < s < p$. By the fact that

$$|(g_{11})_B| \le \frac{1}{\omega(B)} \int_B |g_{11}(y)| d\omega(y),$$

by the definition of g_{11}, and by Lemma 2.1,

$$\begin{aligned}
\frac{1}{\omega(B)} \int_B |g_{11}(y) - (g_{11})_B| d\omega(y) &\le \frac{2}{\omega(B)} \int_B |g_{11}(y)| d\omega(y) \\
&\le 2\frac{1}{\omega(B)} \int_B |b(y) - b_B| \cdot |R_j f_1(y)| d\omega(y) \\
&\le C\|b\|_{\mathrm{Lip}(\beta)} \frac{1}{\omega(B)^{1-\frac{\beta}{N}}} \int_B |R_j f_1(y)| d\omega(y) \\
&\le C\|b\|_{\mathrm{Lip}(\beta)} M_{\beta,\kappa}(R_j f_1)(x).
\end{aligned}$$

Hence,

$$g_{11}^\sharp(x) \le C\|b\|_{\mathrm{Lip}(\beta)} M_{\beta,\kappa}(R_j f_1)(x).$$

The same method gives

$$g_{21}^\sharp(x) \le C\|b\|_{\mathrm{Lip}(\beta)} M_{\beta,\kappa}(R_j f_2)(x), \quad g_{\sigma_j 1}^\sharp(x) \le C\|b\|_{\mathrm{Lip}(\beta)} M_{\beta,\kappa}(R_j f_{\sigma_j})(x).$$

To deal with g_{12}, we choose $s, t \in (1, \infty)$ such that $1 < st < q < \infty$ and set $\ell = st$. Then, by Hölder's inequality and L^s_ω-boundedness of R_j,

$$\frac{1}{\omega(B)}\int_B \left|g_{12}(y)-(g_{12})_B\right| d\omega(y) \le \frac{2}{\omega(B)}\int_B \left|g_{12}(y)\right| d\omega(y)$$
$$\le \frac{2}{\omega(B)}\int_B \left|R_j\big((b(\cdot)-b_B)\cdot f_1\big)(y)\right| d\omega(y)$$
$$\le 2\left(\frac{1}{\omega(B)}\int_B \left|R_j\big((b(\cdot)-b_B)\cdot f_1\big)(y)\right|^s d\omega(y)\right)^{1/s}$$
$$\le C\left(\frac{1}{\omega(5B)}\int_{5B} \left|(b(y)-b_B)\cdot f_1(y)\right|^s d\omega(y)\right)^{1/s}$$
$$\le C\left(\frac{1}{\omega(5B)}\int_{5B} \left|b(y)-b_B\right|^{st'} d\omega(y)\right)^{1/(st')}\cdot\left(\frac{1}{\omega(5B)}\int_{5B} \left|f_1(y)\right|^{st} d\omega(y)\right)^{1/(st)}.$$

Applying (10), we get

$$\left(\frac{1}{\omega(5B)}\int_{5B} \left|b(y)-b_B\right|^{st'} d\omega(y)\right)^{1/(st')}$$
$$\le \left(\frac{1}{\omega(5B)}\int_{5B} \left|b(y)-b_{5B}\right|^{st'} d\omega(y)\right)^{1/(st')} + \left(\frac{1}{\omega(5B)}\int_{5B} \left|b_{5B}-b_B\right|^{st'} d\omega(y)\right)^{1/(st')}$$
$$\le C\|b\|_{\mathrm{Lip}(\beta)}\omega(5B)^{\frac{\beta}{N}} + C\|b\|_{\mathrm{Lip}(\beta)}\omega(5B)^{\frac{\beta}{N}}$$
$$\le C\|b\|_{\mathrm{Lip}(\beta)}\omega(5B)^{\frac{\beta}{N}}.$$

Therefore, we conclude that

$$g_{12}^{\sharp}(x) \le C\|b\|_{\mathrm{Lip}(\beta)}\left(M_{\beta,\kappa}(f_1^{\ell})(x)\right)^{1/\ell}.$$

We turn to analyze g_{22}. Observe that for $z \notin \mathcal{O}(5B)$ and $y \in B$ we have $\|x_0-y\| \le \frac{d(x_0,z)}{2}$. Let Γ be a fixed closed Weyl chamber such that $x_0 \in \Gamma$. Then,

$$|g_{22}(y)-g_{22}(x_0)|$$
$$\le \int_{\mathbb{R}^d} |R_j(y,z)-R_j(x_0,z)||b_B-b(z)||f_2(z)|dw(z)$$
$$= \sum_{\sigma\in G}\int_{z\in\sigma(\Gamma)} |R_j(y,z)-R_j(x_0,z)||b_B-b(z)||f_2(z)|dw(z)$$
$$\le C\sum_{\sigma\in G}\int_{z\in\sigma(\Gamma)} \frac{\|y-x_0\|}{\|x_0-z\|}\frac{1}{w(B(x_0,d(x_0,z)))}|b_B-b(z)||f_2(z)|dw(z)$$
$$=: \sum_{\sigma\in G} J_\sigma(y),$$

where the second inequality follows from the pointwise regularity estimates for R_j investigated in [23]. Let us note that

$$w(B(x_0, d(x_0, z))) = w(B(\sigma(x_0), \|\sigma(x_0) - z\|)) \text{ for } z \in \sigma(\Gamma),$$

since the measure dw is G-invariant and $d(\sigma(x_0), z) = \|\sigma(x_0) - z\|$ in this case. In dealing with $J_\sigma(y)$, we shall use the inequalities:

$$\|x_0 - z\| \geq \max(\|x_0 - \sigma(x_0)\|/2, r)$$

for $z \in \sigma(\Gamma)$ and

$$\|\sigma(x_0) - \sigma(x)\| \leq r < 5r \leq \|\sigma(x_0) - z\| = d(x_0, z) \leq \|x_0 - z\|$$

for $z \in \sigma(\Gamma), z \notin \mathcal{O}(5B)$. Hence,

$$\begin{aligned} J_\sigma(y) &\leq C \int_{z\in\sigma(\Gamma)} \frac{r}{\|x_0 - z\|} \cdot \frac{1}{w\big(B(\sigma(x_0), \|\sigma(x_0) - z\|)\big)} |b_B - b_{B(\sigma(x_0),r)}||f_2(z)|dw(z) \\ &\quad + C \int_{z\in\sigma(\Gamma)} \frac{r}{\|x_0 - z\|} \cdot \frac{1}{w\big(B(\sigma(x_0), \|\sigma(x_0) - z\|)\big)} |b_{B(\sigma(x_0),r)} - b(z)||f_2(z)|dw(z) \\ &=: J_{\sigma,1}(y) + J_{\sigma,2}(y). \end{aligned}$$

Further, by (12),

$$\begin{aligned} &J_{\sigma_1}(y) \qquad (13) \\ &\leq C \int_{z\in\sigma(\Gamma)} (\frac{r}{\|x_0 - z\|})^{1-\beta} \cdot (\frac{r}{\|x_0 - z\|})^{\beta} \omega(B(\sigma(x_0), \|x_0 - \sigma(x_0)\| + r))^{\frac{\beta}{N}} \|b\|_{\mathrm{Lip}(\beta)} \\ &\quad \times \frac{|f_2(z)|}{w\big(B(\sigma(x_0), \|\sigma(x_0) - z\|)\big)} dw(z) \\ &\leq C\|b\|_{\mathrm{Lip}(\beta)} \int_{z\in\sigma(\Gamma)} (\frac{r}{\|x_0 - z\|})^{1-\beta} \cdot (\frac{r}{\|x_0 - z\|})^{\beta} \Big(\frac{\omega(B(\sigma(x_0), \|x_0 - \sigma(x_0)\| + r))}{w\big(B(\sigma(x_0), \|\sigma(x_0) - z\|)\big)} \Big)^{\frac{\beta}{N}} \\ &\quad \times \frac{|f_2(z)|}{w\big(B(\sigma(x_0), \|\sigma(x_0) - z\|)\big)^{1-\frac{\beta}{N}}} dw(z) \\ &\leq C\|b\|_{\mathrm{Lip}(\beta)} \sum_{j=2}^{\infty} \int_{\substack{z\in\sigma(\Gamma), \\ \|\sigma(x_0) - z\| \sim 2^j r}} \Big(\frac{r}{\|x_0 - z\|} \Big)^{1-\beta} \cdot \frac{|f_2(z)|}{w\big(B(\sigma(x_0), \|\sigma(x_0) - z\|)\big)^{1-\frac{\beta}{N}}} dw(z) \\ &\leq C\|b\|_{\mathrm{Lip}(\beta)} M_{\beta,\kappa}(f_2)(\sigma(x)), \qquad (14) \end{aligned}$$

where the third inequality follows from (2).

We turn to considering $J_{\sigma,2}(y)$. Applying (10), we obtain

$$\begin{aligned} J_{\sigma,2}(y) &\leq C \sum_{j=2}^{\infty} \int_{\substack{z\in\sigma(\Gamma), \\ \|z - \sigma(x_0)\| \sim 2^j r}} \frac{r}{2^j r} |b_{B(\sigma(x_0),r)} - b(z)| \frac{|f_2(z)|}{w\big(B(\sigma(x_0), 2^j r)\big)} dw(z) \\ &\leq C \sum_{j=2}^{\infty} 2^{-j} \int_{\|z - \sigma(x_0)\| \sim 2^j r} |b_{B(\sigma(x_0),r)} - b_{B(\sigma(x_0),2^j r)} + b_{B(\sigma(x_0),2^j r)} - b(z)| \end{aligned}$$

$$\times \frac{|f_2(z)|dw(z)}{w\big(B(\sigma(x_0),2^jr)\big)}$$

$$\leq C\|b\|_{\mathrm{Lip}(\beta)}\sum_{j=2}^{\infty}2^{-j}\int_{\|z-\sigma(x_0)\|\sim 2^jr}\frac{|f_2(z)|}{\big(w\big(B(\sigma(x_0),2^jr)\big)\big)^{1-\frac{\beta}{N}}}dw(z)$$

$$\leq C\|b\|_{\mathrm{Lip}(\beta)}M_{\beta,\varkappa}(f_2)(\sigma(x)). \tag{15}$$

Thus, by (13) and (15), we have

$$g_{22}^{\sharp}(x)\leq C\|b\|_{\mathrm{Lip}(\beta)}\sum_{\sigma\in G}M_{\beta,\varkappa}(f_2)(\sigma(x)).$$

Finally we turn to estimate g_{σ_j2}. To this end, we note that for $z\in U_j$ and $y\in B$, we have

$$\|z-y\|\geq\|z-x_0\|-\|x_0-y\|\geq 5r-r=4r,$$

$$\|x_0-\sigma_j(x_0)\|\leq\|x_0-y\|+\|z-y\|+\|z-\sigma_j(x_0)\|\leq 6r+\|z-y\|\leq\frac{5}{2}\|z-y\|.$$

Consequently,

$$\int_B|R_j(z,y)|d\omega(y) \tag{16}$$

$$\leq C\int_B\frac{d(z,y)}{\|z-y\|}\frac{1}{w(B(z,d(z,y)))}d\omega(y)$$

$$\leq C\frac{r}{(r+\|x_0-\sigma_j(x_0)\|)}\int_B\frac{d(z,y)}{r}\frac{1}{w(B(z,d(z,y)))}d\omega(y)$$

$$\leq C\frac{r}{(r+\|x_0-\sigma_j(x_0)\|)}\int_{\mathcal{O}(B(z,16r))}\frac{d(z,y)}{r}\frac{1}{w(B(z,d(z,y)))}d\omega(y)$$

$$\leq C\frac{r}{(r+\|x_0-\sigma_j(x_0)\|)}\sum_{j=-4}^{\infty}\int_{d(z,y)\sim 2^{-j}r}\frac{d(z,y)}{r}\frac{1}{w(B(z,d(z,y)))}d\omega(y)$$

$$\leq C\frac{r}{(r+\|x_0-\sigma_j(x_0)\|)}\sum_{j=-4}^{\infty}2^{-j}\int_{d(z,y)\sim 2^{-j}r}\frac{1}{w(B(z,2^{-j}r))}d\omega(y)$$

$$\leq C\frac{r}{(r+\|x_0-\sigma_j(x_0)\|)}. \tag{17}$$

Hence, by (16) and the similar method of (13) and (15),

$$\frac{1}{\omega(B)}\int_B\Big|g_{\sigma_j2}(y)-(g_{\sigma_j2})_B\Big|d\omega(y)$$

$$\leq\frac{2}{\omega(B)}\int_B\Big|g_{\sigma_j2}(y)\Big|d\omega(y)$$

$$\leq\frac{2}{\omega(B)}\int_B\int_{U_j}|R_j(y,z)||b_B-b(z)|\cdot|f_{\sigma_j}(z)|dw(z)d\omega(y)$$

$$\leq C\frac{r}{(r+\|\sigma_j(x_0)-x_0\|)}\cdot\frac{1}{\omega(B)}\int_{U_j}\Big|b_B-b_{B(\sigma_j(x_0),r)}+b_{B(\sigma_j(x_0),r)}-b_{B(\sigma_j(x_0),5r)}$$
$$+b_{B(\sigma_j(x_0),5r)}-b(z)\Big|\cdot|f_{\sigma_j}(z)|dw(z)$$
$$\leq C\|b\|_{\mathrm{Lip}(\beta)}\frac{1}{\Big(\omega(B(\sigma_j(x_0),r))\Big)^{1-\frac{\beta}{N}}}\int_{U_j}|f_{\sigma_j}(z)|dw(z)$$
$$\leq C\|b\|_{\mathrm{Lip}(\beta)}M_{\beta,\kappa}(f_{\sigma_j})(\sigma_j(x)),$$

which implies

$$|g^{\sharp}_{\sigma_j 2}(x)|\leq C\|b\|_{\mathrm{Lip}(\beta)}M_{\beta,\kappa}(f_{\sigma_j})(\sigma_j(x)).$$

Combining the above estimates and taking the L^q_ω-norm on both sides, we invoke Lemma 2.2, the L^p_ω to L^q_ω boundedness of the fractional maximal operator $M_{\beta,\kappa}$, and the G-invariance of $d\omega$, to deduce that

$$\|[b,R_j]\|_{L^q_\kappa(\mathbb{R}^d)}\leq C\|[b,R_j]^{\sharp}\|_{L^q_\kappa(\mathbb{R}^d)}\leq C\|b\|_{\mathrm{Lip}(\beta)}\|f\|_{L^p_\kappa(\mathbb{R}^d)}$$

for $\frac{1}{q}=\frac{1}{p}-\frac{\beta}{N}$.

Conversely, we shall prove

$$\|b\|_{\mathrm{Lip}(\beta)}\|f\|_{L^p_\kappa(\mathbb{R}^d)}\leq C\|[b,R_j]\|_{L^q_\kappa(\mathbb{R}^d)}. \tag{18}$$

Here we adapted a recent real analysis approach from [14, 15].

Definition 4.2. Assume that f is finite almost everywhere on $\mathbb{R}^d$. For any subset $B\subseteq\mathbb{R}^d$ with finite measure $\omega(B)$, we define a median value $m_f(B)$ of f over B as a real number satisfying

$$\omega(x\in B: f(x)>m_f(B))\leq\frac{1}{2}\omega(B),\quad \omega(x\in B: f(x)<m_f(B))\leq\frac{1}{2}\omega(B).$$

Let $B(x_0,r)$ be any ball with center $x_0\in\mathbb{R}^d$ and radius $r>0$. Note that

$$\begin{aligned}[b,R_j]f(x)&=b(x)R_j(f)(x)-R_j(bf)(x)\\&=\int_{\mathbb{R}^d}(b(x)-b(y))R_j(x,y)f(y)\,d\omega(y).\end{aligned}$$

Let $\widetilde{B}=B(\tilde{x_0},r)$ such that $y_j-x_j\geq r$ and $\|x-y\|\approx r$ for $x\in B$ and $y\in\widetilde{B}$. Then based on Definition 4.2, we choose two measurable sets

$$E_1\subset\{y\in\widetilde{B}: b(y)<m_b(\widetilde{B})\}\quad\text{and}\quad E_2\subset\{y\in\widetilde{B}: b(y)\geq m_b(\widetilde{B})\}$$

such that $\omega(E_i)=\frac{1}{2}\omega(\widetilde{B})$, $i=1,2$, and that $E_1\cup E_2=\widetilde{B}$, $E_1\cap E_2=\varnothing$. Moreover, we define

$$B_1:=\{x\in B: b(x)\geq m_b(\widetilde{B})\}\quad\text{and}\quad B_2:=\{x\in B: b(x)\leq m_b(\widetilde{B})\}$$

Now based on the definition of E_i and B_i, for $(x,y)\in B_i\times E_i$, $i=1,2$, we have

$$\begin{aligned}|b(x)-b(y)| &= |b(x)-m_b(\widetilde{B})+m_b(\widetilde{B})-b(y)|\\ &= |b(x)-m_b(\widetilde{B})|+|m_b(\widetilde{B})-b(y)|\\ &\geq |b(x)-m_b(\widetilde{B})|.\end{aligned}$$

Hence, we have the following facts.

$$\begin{aligned}&\text{(i) } B=B_1\cup B_2, \widetilde{B}=E_1\cup E_2 \text{ and } \omega(E_i)\geq \frac{1}{2}\omega(\widetilde{B});\\ &\text{(ii) } b(x)-b(y) \text{ does not change sign for all } (x,y)\in B_i\times E_i, i=1,2;\\ &\text{(iii) } |b(x)-m_b(\widetilde{B})|\leq |b(x)-b(y)| \text{ for all } (x,y)\in B_i\times E_i, i=1,2.\end{aligned}\tag{19}$$

By Lemma 2.7, we have

$$|R_j(x,y)|\gtrsim \frac{1}{\omega(B(x_0,r))} \text{ for } (x,y)\in B_i\times E_i, i=1,2.$$

Let $f_i=\chi_{E_i}$, $i=1,2$. The facts in (19) give

$$\begin{aligned}&\frac{1}{\omega(B)^{1+\frac{\beta}{N}}}\sum_{i=1}^{2}\int_B |[b,R_j]f_i(x)|\,d\omega(x)\\ &\geq \frac{1}{\omega(B)^{1+\frac{\beta}{N}}}\sum_{i=1}^{2}\int_{B_i} |[b,R_j]f_i(x)|\,d\omega(x)\\ &= \frac{1}{\omega(B)^{1+\frac{\beta}{N}}}\sum_{i=1}^{2}\int_{B_i}\int_{E_i} |b(x)-b(y)||R_j(x,y)|\,d\omega(y)\,d\omega(x)\\ &\geq \frac{C}{\omega(B)^{1+\frac{\beta}{N}}}\sum_{i=1}^{2}\int_{B_i} |b(x)-m_b(\widetilde{B})|\frac{1}{\omega(B(x_0,r))}\int_{E_i} d\omega(y)\,d\omega(x)\\ &\geq \frac{C}{\omega(B)^{1+\frac{\beta}{N}}}\sum_{i=1}^{2}\int_{B_i} |b(x)-m_b(\widetilde{B})|\,d\omega(x)\\ &\geq \frac{C}{\omega(B)^{1+\frac{\beta}{N}}}\int_B |b(x)-b_B|\,d\omega(x).\end{aligned}$$

On the other hand, since $[b,R_j]$ maps $L^p_\kappa(\mathbb{R}^d)$ continuously into $L^q_\kappa(\mathbb{R}^d)$, we have

$$\frac{1}{\omega(B)^{1+\frac{\beta}{N}}}\sum_{i=1}^{2}\int_B |[b,R_j]f_i(x)|\,d\omega(x)$$

$$\leq \frac{C}{\omega(B)^{1+\frac{\beta}{N}}} \sum_{i=1}^{2} \left(\int_B |[b, R_j] f_i(x)|^q \, d\omega(x) \right)^{\frac{1}{q}} \omega(B)^{\frac{1}{q'}}$$

$$\leq \frac{C}{\omega(B)^{1+\frac{\beta}{N}}} \frac{1}{\omega(B)} \sum_{i=1}^{2} \|[b, R_j]\|_{L^p_\kappa(\mathbb{R}^d) \longrightarrow L^p_\kappa(\mathbb{R}^d)} \omega(E_i)^{\frac{1}{p}} \omega(B)^{\frac{1}{q'}}$$

$$\leq C \|[b, R_j]\|_{L^p_\kappa(\mathbb{R}^d) \longrightarrow L^p_\kappa(\mathbb{R}^d)},$$

where the last inequality follows from $\omega(\widetilde{B}) \leq C\omega(B)$ and $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{N}$. This establishes (18) and completes the proof.

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(Shaoyong He) DEPARTMENT OF MATHEMATICS, HUZHOU NORMAL UNIVERSITY, HUZHOU 313000, CHINA
hsyongmath@sina.com

(Lei Wang) DEPARTMENT OF MATHEMATICS, HUZHOU NORMAL UNIVERSITY, HUZHOU 313000, CHINA
wanglei112025@163.com

(Yun Fan) DEPARTMENT OF MATHEMATICS, HUZHOU NORMAL UNIVERSITY, HUZHOU 313000, CHINA
fanyun@huznu.edu.cn

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