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Non-standard bi-orders on punctured torus bundles

Jonathan Johnson and **Henry Segerman**

ABSTRACT. Results of Perron and Rolfsen imply that untwisted hyperbolic once-punctured torus bundles over the circle have bi-orderable fundamental groups. They do this by showing that the action of the monodromy preserves a "standard" bi-ordering formed using the lower central series of the free group. Here we investigate other bi-orderings that punctured torus bundle groups can have. We show that for every such bi-ordering, the largest and second largest proper convex subgroups match the corresponding convex subgroups for a standard bi-ordering. Moreover, if there exists a third largest convex subgroup, it must also match the third largest convex subgroup for a standard bi-ordering. However, we also show that these groups admit non-standard bi-orderings.

CONTENTS

1. Introduction

Much progress has been made on determining when a three-manifold group is *left-orderable*, that is, having a strict total order invariant under left (but not necessarily right) multiplication. In fact, there is a conjectured answer to when a three-manifold group is left-orderable [BGW13]. However, comparatively little is known about the symmetrized version of this property.

Definition 1.1. A group is *bi-orderable* if there is a strict total order of its elements invariant under left and right multiplication. ◊

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Most of the work on connections between bi-orderability and low-dimensional topology to date has focused on the question of which three-manifolds have bi-orderable fundamental groups. Boyer, Rolfsen, and Wiest [BRW05] classified all Seifert fibered spaces with bi-orderable fundamental groups. Perron and Rolfsen [PR03] proved a sufficient condition for bi-orderability of the groups of fiber bundles over the circle. Later, Clay and Rolfsen [CR12] proved an obstruction for bi-orderability of these fiber bundle groups. Linnell, Rhemtulla, and Rolfsen [LRR08] generalized Perron and Rolfsen's result, in particular to address some non-fibered manifolds. These results have been used to show that many link complements have bi-orderable groups [CDN16, John23a, John23b, KR18, Yam17].

In this paper we turn from asking which three-manifolds have bi-orderable fundamental groups to the question of classifying the bi-orders on a bi-orderable fundamental group. In particular, we look at punctured torus bundles. Clay, Perron, and Rolfsen's results [CR12, PR03] together determine that the punctured torus bundles with bi-orderable fundamental groups are the untwisted hyperbolic punctured torus bundles (Proposition 3.2). Thus for the rest of the paper we will assume that our punctured torus bundles are of this form.

The following notions to do with *convexity* are key to describing the structure of bi-ordered groups.

Definition 1.2. A subgroup C of a bi-ordered group $(G, <)$ is *convex with respect to* $<$ if for every $a, b \in C$ and $g \in G$, when $a < g < b$ then $g \in C$. We call a convex subgroup C *maximal* if C is a proper subgroup and the only convex subgroups containing C are C and G. $\diamondsuit$

Our first result tells us that there is no choice in the maximal convex subgroup for a bi-orderable hyperbolic punctured torus group. We can then take that maximal convex subgroup and ask what *its* maximal convex subgroup is. Again there is no choice.

Theorem 1.3. *Let M be an untwisted hyperbolic punctured torus bundle with fundamental group*

$$\pi_1(M) \cong G \rtimes \mathbb{Z}$$

where G is the fundamental group of the punctured torus. Let $<$ be a bi-ordering of $\pi_1(M)$.

(1) *The maximal convex subgroup of $\pi_1(M)$ with respect to $<$ is G.*
(2) *Let $<_G$ be the induced bi-order of $<$ on G. The maximal convex subgroup of G with respect to $<_G$ is $G_2 = [G, G]$.*

A maximal convex subgroup is normal (Proposition 2.14), and it follows from a theorem of Hölder that the quotient is abelian [Hol1901]. Thus, the "most significant digit" of any such bi-order is given by a bi-ordering of this finitely generated free abelian group; such bi-orders are well understood.

One might ask if the pattern of maximal convex subgroups being determined continues. Our next result tells us that if G_2 has a maximal convex subgroup then, again, there is no choice.

Theorem 1.4. *Let M be an untwisted hyperbolic punctured torus bundle with fundamental group*

$$\pi_1(M) \cong G \rtimes \mathbb{Z}$$

where G is the fundamental group of the punctured torus. Let $<$ be a bi-ordering of $\pi_1(M)$. Let $<_2$ be the induced bi-order of $<$ on $G_2 = [G, G]$. Suppose that C is a maximal convex subgroup of G_2 with respect to $<_2$. Then C is $G_3 = [G_2, G]$.

In Theorem 1.3 the groups $\pi_1(M)$ and G must have maximal convex subgroups because they are finitely generated (Proposition 2.14). However, G_2 is not finitely generated, leaving open the possibility that it fails to have a maximal convex subgroup.

In Section 5 we build bi-orders on $\pi_1(M)$ for which G_2 does not have a maximal convex subgroup (Corollary 5.15). Our examples are not obtainable by existing construction methods in the literature. Perron and Rolfsen [PR03] show that untwisted hyperbolic punctured torus bundles are bi-orderable by showing that for each such bundle, the monodromy h induces a map $h_* : G \to G$ that preserves at least one bi-ordering of the free group G. These bi-orderings are *standard*, meaning that every term of the lower central series is convex (although each term may not be maximal in the previous term); see Definition 2.25 and Proposition 2.27 for the details. In a natural extension of this terminology, we say that the fundamental group of a punctured torus bundle is *standard* if h_* preserves a standard bi-ordering on the free group G.

Theorem 1.5. *Let M be an untwisted hyperbolic punctured torus bundle. Then $\pi_1(M)$ admits a non-standard bi-ordering.*

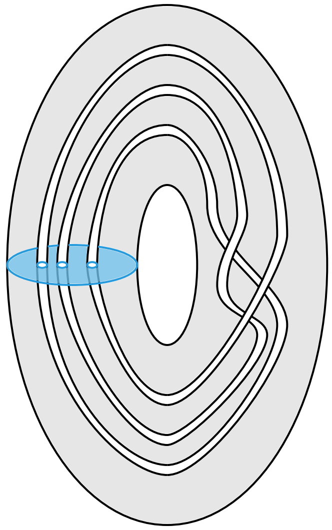
1.6. Bi-orders and eigenvalues. Here we give some relevant context to our results and discuss possible future directions.

In this section, let M be a fiber bundle over the circle whose fiber is a once punctured surface S (not necessarily a genus one surface), and let h be the monodromy of this bundle. As in the genus one case, $\pi_1(M)$ factors into a semidirect product $\pi_1(M) \cong G \rtimes_{h_*} \mathbb{Z}$ where $G \cong \pi_1(S)$ is a free group of rank twice the genus of S. Thus there is a natural map $\pi_1(M) \to \mathbb{Z}$ obtained by projecting $G \rtimes_{h_*} \mathbb{Z}$ onto the $\mathbb{Z}$ factor. Therefore a bundle over the circle has a canonical Alexander polynomial $\Delta_M(t)$. Perron and Rolfsen, and Clay and Rolfsen prove the following.

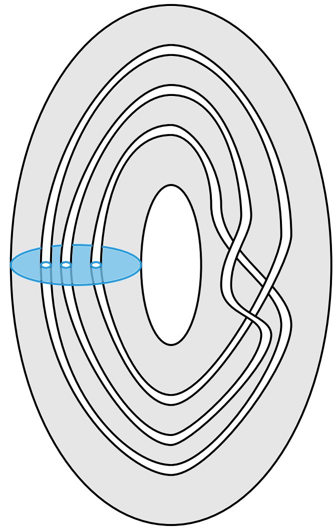
Theorem 1.7 ([PR03], Theorem 1.1)**.** *If all the roots of $\Delta_M(t)$ are real and positive then $\pi_1(M)$ admits a standard bi-ordering.*

Theorem 1.8 ([CR12], Theorem 1.1)**.** *If $\pi_1(M)$ is bi-orderable then $\Delta_M(t)$ has at least one real and positive root.*

Remark 1.9. These results are stated for fibered knot complements but the proofs can be applied to arbitrary fiber bundles over the circle without any alterations. ◊



(A) M



(B) M'

FIGURE 1.12. Bundles over the circle

Theorem 1.7 is a sufficient but not necessary condition for $\pi_1(M)$ to admit a standard bi-ordering. However, to know whether or not $\pi_1(M)$ admits a standard bi-ordering we only need to investigate the induced map $h_+ : H_1(S;\mathbb{Z}) \to H_1(S;\mathbb{Z})$. (See Proposition 2.28 for more details.)

Question 1.10. Suppose that S is a once-punctured surface, and suppose that M and M' are S bundles over the circle with monodromies h and h' respectively. Is it possible that $\pi_1(M')$ is bi-orderable, $\pi_1(M)$ is not bi-orderable, but in homology we have that $h_+ = h'_+$? ◊

Remark 1.11. If we allow S to have multiple punctures in Question 1.10 then the answer is "yes". In particular, let $\mathrm{br}(\beta)$ be the link in S^3 formed by the closure of a braid β along with the braid axis. Let M and M' be the exteriors of $\mathrm{br}(\sigma_1\sigma_2\sigma_1)$ and $\mathrm{br}(\sigma_1\sigma_2\sigma_1^{-1})$ respectively. The manifolds M and M', pictured in Figure 1.12, are S bundles over the circle where S is a 4-punctured sphere. For both bundles, the induced maps $h_+, h'_+ : H_1(S;\mathbb{Z}) \to H_1(S;\mathbb{Z})$ are described by the following matrix:

$$h_+ = h'_+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Kin and Rolfsen showed that $\pi_1(M)$ is bi-orderable [KR18, Theorem 4.10] but $\pi_1(M')$ is not bi-orderable [KR18, Proposition 4.4, Corollary 4.15]. ◊

By Proposition 2.28, an example answering Question 1.10 in the affirmative would have to involve a non-standard bi-ordering. Theorem 1.5 confirms that non-standard bi-orderings exist, although in our examples M is hyperbolic and S is of genus one, a setting in which the answer to Question 1.10 is known to be "no". (See Proposition 3.2: h_+ determines whether or not M is untwisted.)

A negative answer to Question 1.10 would have to arise from restrictions that the map on homology h_+ places on the non-standard bi-orderings of $\pi_1(M)$. In our examples, h_+ puts severe restrictions on what is possible. One might expect the same to be true in higher genus.

Question 1.13. What other non-standard bi-orderings of a hyperbolic punctured torus (or other surface) bundle group are there beyond those generated by our construction? $\diamondsuit$

1.14. Structure of the paper. We begin with an exposition of the interplay between bi-orderings of groups and their convex subgroups in Section 2. In Section 3 we recall the definition and some results about punctured torus bundles. Section 4 focuses on the restrictions that the monodromy imposes on the types of bi-orderings which can be put on the rank 2 free group. Theorems 1.3 and 1.4 are proved in Section 4.1 and Section 4.16 respectively. We describe non-standard bi-ordering of a hyperbolic punctured torus bundle group in Section 5 and so prove Theorem 1.5.

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2. Bi-orders and convex subgroups

In this section, we state some preliminary facts about bi-orderings and convex subgroups of bi-ordered groups. In particular, we explore the interplay between the bi-orderings of groups G and H related by a surjective homomorphism $f : G \to H$. For a general reference for many of these results, see Section 2.2 of Clay and Rolfsen's book [CR16]. We start by stating an alternative way to describe a bi-ordering of a group.

2.1. Positive cones and induced orderings. While the most natural method to define a bi-ordering of a group is to specify a total ordering, a bi-ordering is also specified by its set of positive elements.

Definition 2.2. A subset $P \subset G$ is a *(conjugate invariant) positive cone* if it satisfies the following:

(1) $P \cdot P \subset P$
(2) $G = P \sqcup P^{-1} \sqcup \{1\}$
(3) $g^{-1}Pg = P$ for all g in G. $\diamondsuit$

A bi-order $<$ of G corresponds to a unique positive cone by defining $P_< = \{g \in G \mid 1 < g\}$. In the other direction, a bi-order can be obtained from a (conjugate invariant) positive cone P by defining $a <_P b$ if and only if $a^{-1}b \in P$.

We use $(G, <)$, (G, P), or $(G, <, P)$ to mean a group with bi-order $<$ and/or its corresponding positive cone P.

Remark 2.3. The term "positive cone" is often used to define the positive elements of a left-ordering of a group. In this case, condition (3) is not required. The conjugate invariant positive cones of a group are precisely the positive cones associated to left-orders which also happen to be bi-orders. Since we only deal with positive cones of bi-orders, all positive cones in this paper are assumed to be conjugate invariant. ◊

Definition 2.4. Let $(G, <_G, P_G)$ and $(H, <_H, P_H)$ be bi-ordered groups, and let $f : G \to H$ be a surjective homomorphism. The bi-order of H is *induced by* the bi-order of G if either of the following equivalent conditions is true:

(1) For all $a, b \in G$, $a <_G b$ implies $f(a) \leq_H f(b)$,
(2) $f(P_G - \ker f) = P_H$. ◊

Remark 2.5. Given a surjective homomorphism $f : G \to H$, each bi-ordering of G can induce either zero or one bi-ordering of H. However, a given bi-ordering of H may be induced by many distinct bi-orderings of G. The next lemma constructs one such bi-order of G. ◊

Lemma 2.6. *Suppose $f : G \to H$ is a surjective homomorphism of groups. Let P_H be a positive cone of H and let P_K be a positive cone of $K = \ker f$ invariant under conjugation by elements of G. The set*

$$P = \{g \in G \mid f(g) \in P_H \text{ or } g \in P_K\}$$

is a positive cone of G which induces P_H.

Proof. By construction, $f(P - \ker f) \subset P_H$. Since f is surjective and $1 \notin P_H$, we have $P_H \subset f(P - \ker f)$. Therefore $f(P - \ker f) = P_H$. It remains to show that P is a positive cone. We check the properties in Definition 2.2 as follows.

(1) Consider $x, y \in P$. When $x, y \in P_K$ then $xy \in P_K$ so $xy \in P$. When $f(x) \in P_H$ then either $f(xy) = f(x)f(y)$ is a product of elements in P_H or $f(xy) = f(x)$. Therefore, $xy \in P$. Similarly, if $f(y) \in P_H$ then $xy \in P$. This proves Definition 2.2(1).
(2) Since P_H is a positive cone, $f(1) = 1 \notin P_H$. Since P_K is a positive cone, $1 \notin P_K$. Therefore, $1 \notin P$.
Consider $x \in P$. We will show that $x^{-1} \notin P$. Suppose $f(x) \in P_H$. Then, $f(x^{-1}) \notin P_H$, and since $f(x) \neq 1$, x is not in $\ker f$ so $x^{-1} \notin P_K$. Therefore, $x^{-1} \notin P$.
Suppose $x \in P_K$. Then $x^{-1} \notin P_K$ since P_K is a positive cone. Since $x \in K$ and P_H is a positive cone, $f(x^{-1}) = 1 \notin P_H$. Therefore, $x^{-1} \notin P$.
Since $1 \notin P$ and $P \cap P^{-1} = \emptyset$, it follows that $G = P \sqcup P^{-1} \sqcup \{1\}$. This proves Definition 2.2(2).
(3) Consider $x \in P$, and suppose $g \in G$. If $f(x) \in P_H$, then $f(g^{-1}xg) = f(g)^{-1}f(x)g(x) \in P_H$ since P_H is a positive cone. If $x \in P_K$, then $g^{-1}xg \in P_K$ since P_K is invariant under conjugation by elements of G. Therefore, $g^{-1}xg \in P$ and we have Definition 2.2(3). □

An application of Lemma 2.6 is the following result on semidirect products of groups with the integers.

Proposition 2.7. *Let G be a group and let $\phi \in \mathrm{Aut}(G)$. A semidirect product $G \rtimes_\phi \mathbb{Z}$ is bi-orderable if and only if there is a bi-order of G invariant under ϕ.*

Proof. First suppose that $G \rtimes_\phi \mathbb{Z}$ has a bi-order $<_G$ on G. Since ϕ is conjugation in $G \rtimes_\phi \mathbb{Z}$, it preserves $<_G$.

In the other direction, suppose that there is a bi-order of G with positive cone P_K invariant under ϕ. By an abuse of notation, we also denote the corresponding subset of $G \rtimes_\phi \mathbb{Z}$ as P_K. We apply Lemma 2.6 with $G \rtimes_\phi \mathbb{Z}$ playing the role of G, the integers $\mathbb{Z}$ playing the role of H, and the projection map $f : G \rtimes_\phi \mathbb{Z} \to \mathbb{Z}$ being the surjective homomorphism. For the positive cone P_H on $\mathbb{Z}$ we take the set of positive integers. We check that P_K is invariant under conjugation by elements of $G \rtimes_\phi \mathbb{Z}$ as follows. Consider an element $(x, 0) \in P_K$ and an arbitrary element $(g, k) \in G \rtimes_\phi \mathbb{Z}$. We calculate

$$(g, k) \cdot (x, 0) \cdot (g, k)^{-1} = (g\phi^k(x)g^{-1}, 0)$$

Since P_K is invariant under ϕ and conjugation in G, we have $(g\phi^k(x)g^{-1}, 0) \in P_K$. Thus P_K is invariant under conjugation by elements of $G \rtimes_\phi \mathbb{Z}$. Applying Lemma 2.6, we get that

$$P = \{(g, n) \in G \rtimes_\phi \mathbb{Z} \mid 0 < n \text{ or } (n = 0 \text{ and } g \in P_K)\}$$

is a positive cone of $G \rtimes_\phi \mathbb{Z}$. Therefore, $G \rtimes_\phi \mathbb{Z}$ is bi-orderable. □

2.8. Convex Subgroups. Given a surjective homomorphism of bi-orderable groups $f : G \to H$, we now consider when a bi-ordering of G induces a bi-ordering of H. The answer is related to the convex subgroups of G; see Definition 1.2. We first restate the definition of convexity in terms of positive cones.

Proposition 2.9. *Let $(G, <, P)$ be a bi-ordered group. A subgroup $C \subset G$ is convex with respect to $<$ if and only if for each $g \in G$, either $gC \subset P$, $gC \subset P^{-1}$, or $gC = C$. The analogous statement for right cosets also holds.*

Proof. Suppose that C is convex with respect to $<$. Let gC be a left coset of C such that $gC \neq C$. Thus, $g \notin C$. Let $h = gc$ where $c \in C$. For a contradiction, assume that $g \in P$ and $h \in P^{-1}$. Therefore $h = gc < 1 < g$. Multiplying by g^{-1} on the left we get $c < g^{-1} < 1$. Since C is convex, this implies that $g \in C$ and we have reached a contradiction. A similar contradiction arises if we assume $g \in P^{-1}$ and $h \in P$. Therefore, either $gC \subset P$ or $gC \subset P^{-1}$.

Now suppose that for each $g \in G$, either $gC \subset P$, $gC \subset P^{-1}$, or $gC = C$. Let a and b be in C, and let $a < g < b$ for some $g \in G$. These inequalities imply that $1 < ga^{-1}$ and $gb^{-1} < 1$. Thus, gC is not a subset of P or P^{-1}. Therefore we have that $gC = C$ and so $g \in C$.

The proof for right cosets is similar. □

As a consequence of Proposition 2.9, we have the following.

Corollary 2.10. *Let $(G, <, P)$ be a bi-ordered group. If C is a normal convex subgroup of G then there is a bi-ordering on G/C whose positive cone is the set of cosets gC such that $gC \subset P$. This bi-ordering of G/C is the one induced by G using the quotient map.* □

We now state two useful results about convex subgroups. We refer the reader to a paper by Conrad [Con59] for proofs of these facts in the context of right-orders. However, the arguments work equally well for bi-orders.

Proposition 2.11 (Conrad [Con59], Section 3.3)**.** *Let G be a bi-ordered group. The set of all convex subgroups of G is totally ordered by inclusion.* □

Remark 2.12. Recall that a convex subgroup C of a bi-ordered group $(G, <)$ is *maximal* if C is proper and the only convex subgroups containing C are G and C. By Proposition 2.11, maximal convex subgroups are unique when they exist. ◊

Proposition 2.13 (Conrad [Con59], Section 3.6)**.** *Let $(G, <)$ be a bi-ordered group. Suppose that $f : G \to H$ is a surjective homomorphism of groups. If the bi-ordering $<$ induces a bi-ordering on H then* $\ker f$ *is convex with respect to $<$. In particular, H is bi-orderable if and only if* $\ker f$ *is convex with respect to some bi-ordering of G.* □

Proposition 2.14 (Clay and Rolfsen [CR12], Lemma 2.4)**.** *If $(G, <)$ is a finitely-generated non-trivial bi-ordered group, then there exists a unique maximal convex subgroup C of G satisfying $C \neq G$. Moreover, C is normal in G and G/C is abelian. If an automorphism $\phi : G \to G$ preserves $<$ then $\phi(C) = C$.* □

Given a bi-ordered group $(G, <)$, knowing information about the set of subgroups that are convex with respect to $<$ can provide important information about the bi-ordering $<$. The following theorem is a classical result of Hölder. (See Clay and Rolfsen [CR16, Theorem 2.6] for a more recent exposition of this result.)

Theorem 2.15 (Hölder [Hol1901])**.** *Suppose that $(G, <)$ is a bi-ordered group with no proper non-trivial convex subgroups. Then there is an injective homomorphism $f : G \to \mathbb{R}$ such that the bi-order of the image of f induced by $<$ is the same as the one induced by the usual ordering of $\mathbb{R}$.*

2.16. Constructing bi-orders. In this section, we describe a general procedure for building bi-orders. Let G be a group, and consider a surjective homomorphism $f : G \to H$. Let C be the kernel of f. We can define a positive cone on G by specifying a positive cone of H and a positive cone on C invariant under conjugation by elements of G as in Lemma 2.6. By Proposition 2.13, the subgroup C will be convex with respect to this bi-order.

Abelian groups are a convenient choice for H since bi-orderability is well understood for these groups. In particular, Levi proves the following.

Theorem 2.17 (Levi [Levi42], Section 3)**.** *An abelian group is bi-orderable if and only if it is torsion-free. Thus any free abelian group is bi-orderable.* □

Example 2.18. Suppose that $A \cong \mathbb{Z}^2$.

(1) One way to define a bi-ordering of A is to compare elements one factor at a time. In other words, an element $(m, n) \in A$ is positive when $m > 0$ or when $m = 0$ and $n > 0$. The set $\{0\} \times \mathbb{Z}$ is convex with respect to this bi-ordering. This type of bi-ordering is referred to as a *lexicographical bi-ordering*.

(2) Another option is to let $v \in \mathbb{R} \oplus \mathbb{R}$ be any vector with irrational slope. Then (with the dot product defined as usual) the set $P = \{u \in A \mid u \cdot v > 0\}$ is a positive cone on A. This bi-ordering does not have any proper non-trivial convex subgroups. ♢

Example 2.19. Suppose that A is a free abelian group with a countably infinite basis $\mathcal{B}$. Choose a total ordering of $\mathcal{B}$. For example, we could index the elements of $\mathcal{B}$ by the integers or rational numbers. Every element $x \in A$ can be represented uniquely as a linear sum of elements

$$x = \sum_{v \in \mathcal{B}} k_v v$$

where $\{k_v\}_{v \in \mathcal{B}}$ is a set of integer coefficients, all but finite many of which are zero. For each a, let $v(a)$ be the maximal basis element in $\mathcal{B}$ such that a has a non-zero coefficient. Define P to be the set of all elements $a \in A$ such that the coefficient of $v(a)$ is positive. ♢

We will now construct bi-orderings of a group by choosing a sequence of nested normal subgroups in which every quotient of consecutive terms is a free abelian group. Theorem 2.17 ensures that our quotients have bi-orders. We can then use the bi-orders of the quotients to define a bi-ordering of the entire group.

Construction 2.20. Let G be a group. Suppose that we have a nested sequence of normal subgroups of G

$$G = C_1 \triangleright C_2 \triangleright \cdots \triangleright C_n$$

where C_i/C_{i+1} is a free abelian group for $1 \leq i < n$. For each $1 \leq i < n$, let $p_i : C_i \to C_i/C_{i+1}$ be the quotient map. Choose a positive cone P_n of C_n that is invariant under conjugation by all elements of G. For every $0 \leq i < n$, choose a positive cone P_i for the quotient C_i/C_{i+1} such that for all $g \in C_i$ and $x \in G$, we have that $p_i(g) \in P_i$ if and only if $p_i(xgx^{-1}) \in P_i$. Define an element $g \in G$ to be positive when either of the following occurs:

(1) $g \in C_i - C_{i+1}$ and $p_i(g) \in P_i$ for some $1 \leq i < n$
(2) $g \in P_n$ ♢

We can extend this construction to an infinite nested sequence of subgroups when their intersection is trivial.

Construction 2.21. Let G be a group. Suppose that we have a infinite nested sequence of normal subgroups of G

$$G = C_1 \triangleright C_2 \triangleright C_3 \triangleright \cdots$$

where C_i/C_{i+1} is a free abelian group and $\bigcap C_i = \{1\}$. For each positive integer i, let $p_i : C_i \to C_i/C_{i+1}$ be the quotient map. Choose a positive cone P_i for the quotient C_i/C_{i+1} invariant under conjugation by elements of G. Define an element $g \in G$ to be positive when $g \in C_i - C_{i+1}$ and $p_i(g) \in P_i$ for some i. $\diamondsuit$

A key example of the use of this general strategy is in constructing bi-orders on free groups.

Definition 2.22. Let G be a finite rank free group. Let $G_1 = G$ and for $n > 1$ let $G_n = [G_{n-1}, G]$. The sequence $\{G_n\}_{n\geq 1}$ is the *lower central series* of G. For $n \geq 1$, let $A_n = G_n/G_{n+1}$ be the *nth lower central series quotient*. $\diamondsuit$

Remark 2.23. Each A_n is a free abelian group. Hall's work [Hall34] implies that the rank of A_n is less than or equal to 2^n. $\diamondsuit$

Theorem 2.24 (Magnus [Mag35])**.** *The set of terms of the lower central series of a free group has trivial intersection.* □

The following definition is due to Rolfsen [KR18, Appendix A].

Definition 2.25. A *standard bi-order* on a free group G is one constructed as follows. Applying Theorem 2.17, choose a positive cone P_n on each lower central series quotient A_n. By Theorem 2.24, for each nontrivial element $g \in G$ there is a unique positive integer $n(g)$ such that $g \in G_{n(g)} - G_{n(g)+1}$. Let $[g]_n$ be the class of g in A_n. We define a positive cone P on G by:

$$P := \{g \in G \mid g \neq 1 \text{ and } [g]_{n(g)} \in P_{n(g)}\}. \tag{2.26}$$

$\diamondsuit$

Proposition 2.27. *A bi-ordering of the free group is standard if and only if every term of the group's lower central series is convex with respect to that bi-order.*

Proof. Let $(G, <, P)$ be a bi-ordered free group. If P is standard, then by Lemma 2.6 the restriction of the bi-order to G_{n-1} induces the bi-ordering of A_n given by P_n. Thus by Proposition 2.13, the next term G_n is convex in G_{n-1}. It follows by induction that G_n is convex in G.

Suppose that G_n is convex for all n. Then the bi-order of G induces a bi-order $<_n$ on $A_n := G_n/G_{n+1}$ for each n. Define P_n to be the positive cone of $(A_n, <_n)$. P can be described as the standard bi-order of G determined by the positive cones P_n. □

Proposition 2.28. *Suppose that G is a finite rank free group and $\phi, \phi' \in \mathsf{Aut}(G)$ are automorphisms such that the induced maps on the abelianization of G are the same. Then, ϕ preserves a standard bi-ordering of G if and only if ϕ' does.*

Proof. Consider a standard positive cone P of G determined by a collection of positive cones P_n of $A_n = G_n/G_{n+1}$. Given an automorphism $\psi \in \mathsf{Aut}(G)$, denote by ψ_n the induced automorphism on A_n. Then ψ preserves P if and only if $\psi_n(P_n) = P_n$ for all positive integers n. By a lemma of Perron and Rolfsen

[PR03, Lemma 4.5], there are canonical embeddings $\iota_n : A_n \hookrightarrow A_1^{\otimes n}$ such that the following diagram commutes.

$$\begin{array}{ccc} A_n & \xhookrightarrow{\iota_n} & A_1^{\otimes n} \\ {\scriptstyle \psi_n}\downarrow & & \downarrow{\scriptstyle \psi_1^{\otimes n}} \\ A_n & \xhookrightarrow{\iota_n} & A_1^{\otimes n} \end{array}$$

Consider two automorphisms ϕ and ϕ' of G that induce the same automorphism of the abelianization of G. In other words, we have that $\phi_1 = \phi'_1$. Suppose that $\phi(P) = P$. Then for each n, we have that $\phi_n(P_n) = P_n$. From the above commuting diagram it follows that

$$\phi_1^{\otimes n}(\iota_n(P_n)) = \iota_n(P_n).$$

Since ϕ_1 is equal to ϕ'_1, it follows that

$$\phi_1'^{\otimes n}(\iota_n(P_n)) = \iota_n(P_n).$$

Since ι_n is an embedding, the above commuting diagram gives us that $\phi'_n(P_n) = P_n$ for each n. Therefore, $\phi'(P) = P$. □

3. Punctured torus bundles

Definition 3.1. We define the *(untwisted) punctured torus bundle M* over the circle with monodromy h as

$$M \cong \frac{T \times [0,1]}{(x,1) \sim (h(x),0)}$$

where T is a torus with a boundary component and h is a homeomorphism of T which fixes ∂T pointwise. ◊

Proposition 3.2. *Suppose that M is a hyperbolic punctured torus bundle. Then $\pi_1(M)$ is bi-orderable if and only if M is untwisted.*

Proof. For hyperbolic punctured torus bundles the Alexander polynomial has two real roots. These roots have the same sign, which is positive in the untwisted case and negative in the twisted case. For untwisted punctured torus bundles then, $\pi_1(M)$ is bi-orderable by Theorem 1.7. Conversely, if $\pi_1(M)$ is bi-orderable then by Theorem 1.8 at least one of the roots is positive, and therefore the bundle is untwisted. □

Up to isotopy, the monodromy h is a product of Dehn twists about two simple closed curves on T which intersect once. By choosing a base point x on ∂T, h induces a well-defined automorphism h_* of $\pi_1(T,x)$ which is a rank 2 free group. The fundamental group of M is given by

$$\pi_1(M) \cong G \rtimes_{h_*} \mathbb{Z}$$

Here $G \cong \langle \alpha, \beta \rangle$ is the fundamental group of the punctured torus, generated by loops α and β as shown in Figure 3.3. Fixing notation, let $\mathbb{Z} \cong \langle \tau \rangle$.

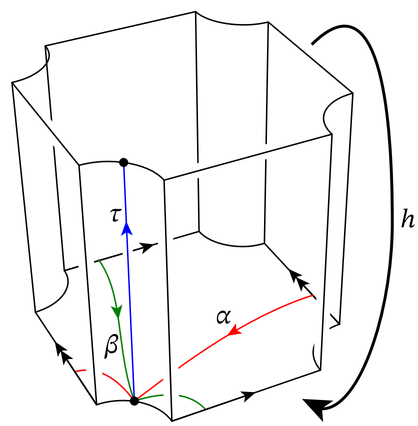


FIGURE 3.3. Generators of the fundamental group for a punctured torus bundle.

Remark 3.4. Consider the figure eight knot complement, with monodromy given by a single Dehn twist around a curve parallel to β (as in Figure 3.3), followed by a single Dehn twist around a curve parallel to α. One can check that the images of α and β under the monodromy are $\tau^{-1}\alpha\tau = h(\alpha) = \alpha\beta\alpha$ and $\tau^{-1}\beta\tau = h(\beta) = \beta\alpha$. ♢

We will use the following well-known fact a few times. See Farb-Margalit [FM12, Section 13.1] for details.

Lemma 3.5. *A pseudo-Anosov map $A \in \mathrm{SL}(2,\mathbb{Z})$ cannot have eigenvalues of either 1 or -1.* □

Lemma 3.6. *Suppose that M is a hyperbolic punctured torus bundle. Then the rank $\mathsf{rk}\, H_1(M;\mathbb{Z})$ is one.*

Proof. Let $[\alpha]$ and $[\beta]$ be the integral homology classes of the curves α and β, as in Figure 3.3. Let $A \in \mathrm{SL}(2,\mathbb{Z})$ be the map h_+ on homology with respect to the basis $[\alpha],[\beta]$. Since $\pi_1(M) \cong G \rtimes_{h_*} \mathbb{Z}$, the homology group $H_1(M;\mathbb{Z})$ decomposes as $H_1(M;\mathbb{Z}) \cong M_T \oplus \langle t\rangle$ where

$$M_T = \langle [\alpha],[\beta] \mid [\alpha] = h_+[\alpha], [\beta] = h_+[\beta]\rangle.$$

M_T is torsion if and only if $\det(I - A) \neq 0$. Since h is pseudo-Anosov, 1 cannot be an eigenvalue of A by Lemma 3.5. Therefore $\det(I - A) = \mathsf{ch}_A(1) \neq 0$. Therefore, M_T is torsion and $\mathsf{rk}\, H_1(M;\mathbb{Z}) = 1$. □

4. Maximal convex subgroups

4.1. Proof of Theorem 1.3. Recall that G is the free group on two elements. The following proposition tells us that any pair of bi-orders on a group $\pi_1(M) \cong G \rtimes_{h_*} \mathbb{Z}$ either agree or precisely disagree at the level of the $\mathbb{Z}$ term. (This is not the case for a general semidirect product of a group with the integers.)

Proposition 4.2. *The maximal convex subgroup of any bi-ordering of $\pi_1(M)$ is G, the fundamental group of the punctured torus.*

Proof. Let $<$ be a bi-order of $\pi_1(M)$. Since $\pi_1(M)$ is finitely generated, it has a maximal convex subgroup C by Proposition 2.14.

By Corollary 2.10, $H := \pi_1(M)/C$ is a nontrivial bi-orderable group. By Theorem 2.15, H is abelian. Thus, H is isomorphic to a nontrivial bi-orderable quotient of $H_1(M;\mathbb{Z})$. By Lemma 3.6, we have that $\operatorname{rk} H_1(M;\mathbb{Z}) = 1$ and $H \cong \mathbb{Z}$. Thus $\pi_1(M)/C \cong \mathbb{Z}$. Also $\pi_1(M) \cong G \rtimes_{h_*} \mathbb{Z}$, so both C and G are kernels of surjections from $\pi_1(M)$ to $\mathbb{Z}$. But because $\operatorname{rk} H_1(M;\mathbb{Z}) = 1$, there is a unique (up to sign) surjection from $\pi_1(M)$ to $\mathbb{Z}$. Therefore $C = G$. □

Thus a bi-ordering of $\pi_1(M) \cong G \rtimes_{h_*} \mathbb{Z}$ is determined by a bi-ordering of the free group G (together with a choice of which way we order $\mathbb{Z}$). The next proposition tells us that when bi-ordering the free group G subject to invariance under h_*, we initially have no choice but to follow the standard construction, as given in Definition 2.25.

Proposition 4.3. *Consider any bi-order $<$ on $\pi_1(M)$ and the induced bi-order $<_G$ on G. The maximal convex subgroup of G with respect to $<_G$ is $G_2 = [G,G]$.*

Proof. Let $<$ be a bi-order of $\pi_1(M)$. By Proposition 2.7, this induces a bi-order $<_G$ on G which is invariant under h_*. Since G is finitely generated, it has a maximal convex subgroup C by Proposition 2.14.

By Theorem 2.15, $H := G/C$ is a nontrivial abelian bi-orderable group. Thus, H is a nontrivial bi-orderable quotient of $H_1(T;\mathbb{Z})$ so either $H \cong \mathbb{Z}$ or $H \cong \mathbb{Z}^2$.

Assume for a contradiction that $H \cong \mathbb{Z}$. Let $p : G \to H_1(T;\mathbb{Z})$ be the abelianization map. The monodromy h induces a map h_+ on $H_1(T;\mathbb{Z})$. By Proposition 2.14, $h_*(C) = C$ so $h_+(p(C)) = p(h_*(C)) = p(C)$. Since $H \cong \mathbb{Z}$ then $p(C) \cong \mathbb{Z}$, and since $h_+(p(C)) = p(C)$ the eigenvalues are either 1 or -1. This is then a contradiction to Lemma 3.5. Therefore, $H \cong \mathbb{Z}^2$. It follows that $C = G_2$. □

Proof of Theorem 1.3. This follows from Propositions 4.2 and 4.3. □

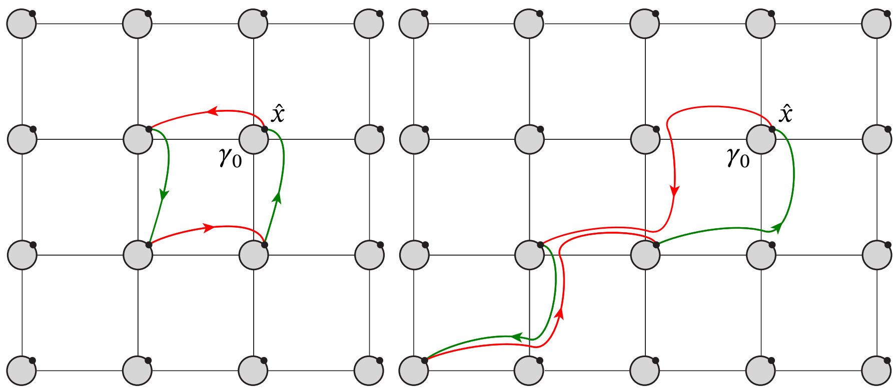
4.4. Monodromy action on G_2. Consider the abelian cover $\pi : \hat{T} \to T$ corresponding to the subgroup G_2. We orient the components of $\partial\hat{T}$ counterclockwise. Choose a component of $\partial\hat{T}$ and call it γ_0. Let $\hat{x}$ be the lift of the basepoint x on γ_0. Elements of G can be lifted to homotopy classes of paths in $\hat{T}$ based at the point $\hat{x}$.

Definition 4.5. Let $\hat{h} : \hat{T} \to \hat{T}$ be the lift of h that preserves γ_0. Let $\hat{h}_+$ be the map induced on $H_1(\hat{T};\mathbb{Z})$ by $\hat{h}$. ♢

Figure 4.6 illustrates $\hat{h}$ in the example of the figure eight knot complement.

Definition 4.7. Let $\mathcal{B}$ be the countable basis of $H_1(\hat{T};\mathbb{Z})$ consisting of the homology classes corresponding to the components of $\partial\hat{T}$. ♢

Notation 4.8. To lighten the notation, we will use the symbol γ for a generic component of $\partial\hat{T}$ and also for its homology class. ♢



(A) The lift of the commutator $\alpha\beta\alpha^{-1}\beta^{-1}$ is homotopic to a loop γ_0 in $\hat{T}$.

(B) The image $\hat{h}(\gamma_0)$ is homotopic to γ_0.

FIGURE 4.6. The map $\hat{h}$ acts on the abelian cover $\hat{T}$, and preserves γ_0. Here we show the image for the figure eight knot complement. By Remark 3.4, $h(\alpha) = \alpha\beta\alpha$ and $h(\beta) = \beta\alpha$. Note that γ_0 is fixed by $\hat{h}$.

Recall that π is the covering map from $\hat{T}$ to T.

Lemma 4.9. *Suppose that γ is a component of $\partial\hat{T}$. Let η be a path in $\hat{T}$ from γ_0 to γ. Then the class $[\pi(\eta)] \in H_1(T;\mathbb{Z})$ is independent of the choice of η.*

Proof. Suppose that η and η' are two such paths. Then the concatenation $\eta' \cdot \eta^{-1}$ is a loop in $\hat{T}$. Thus $\pi(\eta' \cdot \eta^{-1}) \in [G,G]$. Therefore $0 = [\pi(\eta' \cdot \eta^{-1})] = [\pi(\eta')\cdot\pi(\eta^{-1})] = [\pi(\eta')\cdot\pi(\eta)^{-1}] = [\pi(\eta')]-[\pi(\eta)]$. Thus $[\pi(\eta)] = [\pi(\eta')]$. □

We may therefore make the following definition.

Definition 4.10. For any choice of η as in Lemma 4.9, we define the function $q : \mathcal{B} \to H_1(T;\mathbb{Z}) \cong \mathbb{Z}^2 \subset \mathbb{R}^2$ by

$$q(\gamma) = [\pi(\eta)]. \qquad \Diamond$$

Lemma 4.11. *The map q is a bijection.*

Proof. Recall that the curves α and β are our generators for G, the fundamental group of the punctured torus T, as shown in Figure 3.3. For any element $m[\alpha]+n[\beta]$ of $H_1(T;\mathbb{Z})$, if we lift the loop $\alpha^m \cdot \beta^n$ to $\hat{T}$ we find a path based at $\hat{x}$ which ends on a loop γ such that $q(\gamma) = m[\alpha] + n[\beta]$. Thus q is surjective.

Now suppose that γ and γ' are elements of $\partial\hat{T}$ with $q(\gamma) = q(\gamma')$. Let η and η' be paths for γ and γ' as in Lemma 4.9. So $[\pi(\eta)] = [\pi(\eta')]$. Following the calculation in the proof of Lemma 4.9 backwards, we deduce that $\pi(\eta' \cdot \eta^{-1}) \in [G,G]$. This implies that $\eta' \cdot \eta^{-1}$ is a loop in $\hat{T}$, and so $\gamma = \gamma'$. Thus q is injective. □

Lemma 4.12. *We have the following.*

(1) $\hat{h}_+(\mathcal{B}) \subset \mathcal{B}$
(2) $q \circ \hat{h}_+ = h_+ \circ q$

Proof. Since $\hat{h}$ is a self-homeomorphism of $\hat{T}$, it sends boundary loops to boundary loops. Thus $\gamma \in \mathcal{B}$ implies that $\hat{h}_+(\gamma) \in \mathcal{B}$ and we have (1).

Recall that $\pi : \hat{T} \to T$ is the covering map. Then by definition, $h \circ \pi = \pi \circ \hat{h}$. Let $\gamma \in \mathcal{B}$ and let η be a path in $\hat{T}$ from γ_0 to γ. Then

$$h_+ \circ q(\gamma) = h_+([\pi(\eta)]) = [h \circ \pi(\eta)] = [\pi(\hat{h}(\eta))]$$

Note that $\hat{h}(\eta)$ is a path based at $\hat{x}$ which ends on $\hat{h}(\gamma)$. So we get that $[\pi(\hat{h}(\eta))] = q([\hat{h}(\gamma)]) = q \circ \hat{h}_+[(\gamma)]$ and we have proved (2). □

Definition 4.13. Let $\gamma \in \mathcal{B}$. The *shift* of γ by (m, n) is

$$s_{(m,n)}(\gamma) = q^{-1}\big((m, n) + q(\gamma)\big). \qquad \lozenge$$

Notation 4.14. We denote the abelianization map by $p_1 : G \to \mathbb{Z}^2$. We denote the quotient map by $p_2 : G_2 \to G_2/[G_2, G_2]$. ◊

Lemma 4.15. *Suppose that $g \in G_2$ and $x \in G$. Suppose that the homology class $p_1(x)$ in $H_1(T; \mathbb{Z})$ is $m[\alpha] + n[\beta]$ and that the homology class $p_2(g)$ in $H_1(\hat{T}; \mathbb{Z})$ is*

$$\sum_{\gamma \in \mathcal{B}} k_\gamma \gamma.$$

Then the homology class of $p_2(xgx^{-1})$ in $H_1(\hat{T}; \mathbb{Z})$ is

$$\sum_{\gamma \in \mathcal{B}} k_\gamma s_{(m,n)}(\gamma)$$

where $s_{(m,n)}(\gamma)$ is the shift of γ by (m, n), as in Definition 4.13.

Proof. First note that the set $\{y\alpha\beta\alpha^{-1}\beta^{-1}y^{-1} \mid y \in G\}$ generates G_2. Therefore we can write

$$g = \prod_i y_i(\alpha\beta\alpha^{-1}\beta^{-1})^{\varepsilon_i} y_i^{-1}$$

for some finite collection of $y_i \in G$ and $\varepsilon_i \in \{\pm 1\}$.

Considering Figure 4.6a, we see that a lift of $\alpha\beta\alpha^{-1}\beta^{-1}$ to $\hat{T}$ is a loop that goes around a boundary component of $\hat{T}$. Thus $p_2(\alpha\beta\alpha^{-1}\beta^{-1})$ is an element of $\mathcal{B}$. Lifting a conjugate $y\alpha\beta\alpha^{-1}\beta^{-1}y^{-1}$ gives a loop around some other boundary component of $\hat{T}$. Thus, for any $y \in G$, we have that the element $\gamma = p_2(y\alpha\beta\alpha^{-1}\beta^{-1}y^{-1})$ is also a member of $\mathcal{B}$. Using this, we write

$$p_2(g) = \sum_i p_2(y_i(\alpha\beta\alpha^{-1}\beta^{-1})^{\varepsilon_i} y_i^{-1}) = \sum_{\gamma \in \mathcal{B}} k_\gamma \gamma.$$

We can calculate the value of k_γ from the y_i and ϵ_i as follows. For each pair (y_i, ϵ_i), lift y_i to $\hat{T}$ with basepoint on γ_0. The endpoint is on some boundary component, γ_i, say. Then k_γ is the sum of $\{\epsilon_i \mid \gamma_i = \gamma\}$.

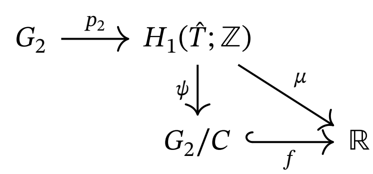
Conjugating by x, we can write

$$xgx^{-1} = \prod_i xy_i(\alpha\beta\alpha^{-1}\beta^{-1})^{\epsilon_i} y_i^{-1}x^{-1}$$

Recall that the homology class $p_1(x)$ is $m[\alpha] + n[\beta] \in H_1(T;\mathbb{Z})$. It follows that if $p_2(y\alpha\beta\alpha^{-1}\beta^{-1}y^{-1}) = \gamma$ then $p_2(xy\alpha\beta\alpha^{-1}\beta^{-1}y^{-1}x^{-1})$ is the shift $s_{(m,n)}(\gamma)$. Thus

$$p_2(xgx^{-1}) = \sum_i p_2(xy_i(\alpha\beta\alpha^{-1}\beta^{-1})^{\epsilon_i} y_i^{-1}x^{-1}) = \sum_{\gamma\in\mathcal{B}} k_\gamma s_{(m,n)}(\gamma). \qquad \square$$

4.16. Proof of Theorem 1.4. We will prove Theorem 1.4 through a series of lemmas. First note that since C is convex, $<_2$ induces a bi-ordering of G_2/C. By Theorem 2.15, there is an injective homomorphism $f : G_2/C \to \mathbb{R}$ respecting the bi-orders. Also, since G_2/C is abelian the quotient map factors through quotient maps $p_2 : G_2 \to H_1(\hat{T};\mathbb{Z}) \cong G_2/[G_2, G_2]$ and $\psi : H_1(\hat{T};\mathbb{Z}) \to G_2/C$. Define $\mu : H_1(\hat{T};\mathbb{Z}) \to \mathbb{R}$ to be $f \circ \psi$ as in the following diagram.



Let P be the positive cone associated to $<_2$. Let $Q^+ = \mu^{-1}(\mathbb{R}^+)$ and $Q^- = \mu^{-1}(\mathbb{R}^-)$. Since $\mu \circ p_2$ and $<_2$ induce the bi-ordering of the image of μ, we have

$$Q^+ = p_2(P - C),$$

$$Q^- = p_2(P^{-1} - C),$$

and

$$\ker \mu = p_2(C).$$

Lemma 4.17. *Suppose that $\phi \in \mathrm{Aut}(G_2)$ preserves C and the bi-ordering $(<_2, P)$. Let $\phi_+ \in \mathrm{Aut}(H_1(\hat{T};\mathbb{Z}))$ be the automorphism induced by ϕ. Then the sets Q^+, Q^-, and $\ker \mu$ are invariant under ϕ_+.*

Proof. By assumption ϕ preserves P and C. Therefore, by the definition of an induced map, ϕ_+ preserves $p_2(P - C)$, $p_2(P^{-1} - C)$, and $p_2(C)$. $\square$

In other words, for all $v \in H_1(\hat{T};\mathbb{Z})$, the sign of $\mu(v)$ is the same as the sign of $\mu(\phi_+(v))$. This fact is also a consequence of the following classical lemma, if we take ρ to be the automorphism on the image of μ defined by $\rho(\mu(v)) = \mu(\phi_+(v))$.

Lemma 4.18 (Hion [Hion54]). *Suppose that G is a subgroup of $(\mathbb{R}, +)$ and ρ is an order-preserving automorphism of $\mathbb{R}$. Then there is some $\lambda \in \mathbb{R}^+$ so that $\rho(x) = \lambda x$ for all $x \in \mathbb{R}$.*

Lemma 4.19. *Let $v, w \in H_1(\hat{T};\mathbb{Z})$ with $\mu(v) \neq 0$ and $\mu(w) \neq 0$. Suppose that $\phi \in \mathsf{Aut}(G_2)$ preserves C and the bi-ordering $<_2$. Let $\phi_+ \in \mathsf{Aut}(H_1(\hat{T};\mathbb{Z}))$ be the automorphism induced by ϕ. Then*

$$\frac{\mu(v)}{\mu(w)} = \frac{\mu(\phi_+(v))}{\mu(\phi_+(w))}.$$

Proof. By Lemma 4.18, we have that $\mu(\phi_+(v)) = \lambda\mu(v)$ for all $v \in H_1(\hat{T};\mathbb{Z})$. The result follows by rearranging the following.

$$\frac{\mu(\phi_+(v))}{\mu(v)} = \lambda = \frac{\mu(\phi_+(w))}{\mu(w)}$$ □

The map μ is determined by its value on the basis $\mathcal{B}$. For each pair of integers m, n, define $\gamma_{m,n}$ to be $q^{-1}(m,n)$, where q is as given in Definition 4.10. Thus

$$\gamma_{0,0} = \gamma_0 \qquad \text{and} \qquad \gamma_{m,n} = s_{(m,n)}(\gamma_{0,0})$$

Lemma 4.20. *For all $m, n \in \mathbb{Z}$,*

$$\mu(\gamma_{m,n}) = \mu(\gamma_{0,0})a^m b^n$$

for some $a, b \in \mathbb{R}^+$.

Proof. For each pair of integers m, n,

$$\gamma_{m,n} = p_2(\alpha^m\beta^n[\alpha,\beta]\beta^{-n}\alpha^{-m}]).$$

Since conjugation preserves every positive cone of the free group G, Lemma 4.17 implies that either $\mu(\mathcal{B}) \subset \mathbb{R}^+$, $\mu(\mathcal{B}) \subset \mathbb{R}^-$, or $\mu(\mathcal{B}) = \{0\}$. Since μ is not a trivial map, $\mu(\gamma_{m,n}) \neq 0$ for all integer pairs m and n.

Let a and b be defined as follows.

$$a = \frac{\mu(\gamma_{1,0})}{\mu(\gamma_{0,0})}$$

and

$$b = \frac{\mu(\gamma_{0,1})}{\mu(\gamma_{0,0})}$$

By Lemma 4.19,

$$a = \frac{\mu(\gamma_{m+1,n})}{\mu(\gamma_{m,n})}$$

and

$$b = \frac{\mu(\gamma_{m,n+1})}{\mu(\gamma_{m,n})}$$

for all m and n. It follows that $\mu(\gamma_{m,n}) = \mu(\gamma_{0,0})a^m b^n$. □

The following lemma is proved using *basic commutators*, see [CMZ17, Chapter 3] for the details.

Lemma 4.21. $G_2/G_3 \cong \mathbb{Z}$.

Proof. By Theorem 3.1 of Clement-Majewics-Zyman [CMZ17], any element of G_2/G_3 is generated modulo G_3 by the basic commutators of weight two. In our case there is only one commutator of weight two, namely $[\beta, \alpha]$. The result follows. □

Lemma 4.22. *If μ is constant on $\mathcal{B}$, then $C = G_3$.*

Proof. Suppose that μ is constant on $\mathcal{B}$. First we show that $G_3 \subset C$. Since f is injective, $C = \ker(\psi \circ p_2) = \ker(\mu \circ p_2)$. It is sufficient to show that for arbitrary elements $g \in G_2$ and $x \in G$, the element $gxg^{-1}x^{-1} \in G_3$ is in $\ker(\mu \circ p_2)$. To show this, suppose that the homology class $p_1(x)$ in $H_1(T; \mathbb{Z})$ is $m[\alpha] + n[\beta]$ and that the homology class $p_2(g)$ in $H_1(\hat{T}; \mathbb{Z})$ is

$$\sum_{\gamma \in \mathcal{B}} k_\gamma \gamma.$$

By Lemma 4.15,

$$p_2(gxg^{-1}x^{-1}) = \sum_{\gamma \in \mathcal{B}} k_\gamma \gamma - \sum_{\gamma \in \mathcal{B}} k_\gamma s_{(m,n)}(\gamma) = \sum_{\gamma \in \mathcal{B}} k_\gamma (\gamma - s_{(m,n)}(\gamma)).$$

Since μ is constant on $\mathcal{B}$, we have that

$$\mu(p_2(gxg^{-1}x^{-1})) = \sum_{\gamma \in \mathcal{B}} k_\gamma \big[\mu(\gamma) - \mu(s_{(m,n)}(\gamma))\big] = 0.$$

Thus, $G_3 \subset C$.

By the third isomorphism theorem,

$$\frac{G_2/G_3}{C/G_3} \cong \frac{G_2}{C}.$$

By Lemma 4.21, we have that G_2/C is a non-trivial bi-orderable quotient of $\mathbb{Z}$. Since $\mathbb{Z}$ is the only bi-orderable cyclic group, C/G_3 must be trivial. Therefore, $C = G_3$. □

Recall that from Definition 4.5, we lift the monodromy h to a map $\hat{h}$ on the cover $\hat{T}$. Then $\hat{h}_+$ is the induced map on homology.

Lemma 4.23. *For all $v \in H_1(\hat{T}; \mathbb{Z})$,*

$$\mu(\hat{h}_+(v)) = \mu(v).$$

Proof. Let $v \in H_1(\hat{T}; \mathbb{Z})$. The map $\hat{h}_+$ is induced by h_*, and by assumption, h_* preserves P. By Lemma 4.17, $\mu(v) = 0$ if and only if $\mu(\hat{h}_+(v)) = 0$.

When $\mu(v) \neq 0$,

$$\frac{\mu(v)}{\mu(\gamma_{0,0})} = \frac{\mu(\hat{h}_+(v))}{\mu(\hat{h}_+(\gamma_{0,0}))} \tag{4.24}$$

by Lemma 4.19. Since $\gamma_{0,0}$ is fixed by $\hat{h}_+$, we have $\mu(\gamma_{0,0}) = \mu(\hat{h}_+(\gamma_{0,0}))$. Therefore, it follows from (4.24) that $\mu(\hat{h}_+(v)) = \mu(v)$. □

Lemma 4.25. *Suppose for $i = 1, 2, 3$ that $\gamma_{m_i,n_i} \in \mathcal{B}$ with*

$$\mu(\gamma_{m_1,n_1}) = \mu(\gamma_{m_2,n_2}) = \mu(\gamma_{m_3,n_3}).$$

If the points $\{(m_i, n_i) \mid i = 1, 2, 3\}$ are not collinear in $\mathbb{Z}^2$ then μ is constant on $\mathcal{B}$.

Proof. By Lemma 4.20, there are numbers $a, b \in \mathbb{R}^+$ such that

$$\mu(\gamma_{m,n}) = \mu(\gamma_{0,0})a^m b^n$$

for all integer pairs m and n. Then

$$\log \circ \mu(\gamma_{m,n}) = \log \circ \mu(\gamma_{0,0}) + m \log(a) + n \log(b)$$

Thus $\log \circ \mu$ is a linear function of m and n. Therefore if it is constant at three non-collinear points then it is constant. □

Proof of Theorem 1.4. Suppose that M hyperbolic so h is a pseudo-Anosov map. Consider the action $\hat{h}_+$ on $\mathcal{B}$. By Lemma 4.12(2), the action of $\hat{h}_+$ is determined by the induced map h_+. Since h is pseudo-Anosov, its eigenvectors have irrational slope. In particular, h_+ doesn't preserve any proper non-trivial subgroups of $H_1(T; \mathbb{Z}) = \mathbb{Z}^2$. Thus, none of the orbits of $\mathcal{B}$ under $\hat{h}_+$ contain all collinear points unless the orbit is a single fixed point.

Consider $\gamma_{m,n} \in \mathcal{B} - \{\gamma_{0,0}\}$. Since $\gamma_{m,n}$ is not a fixed point of $\hat{h}_+$, the orbit of $\gamma_{m,n}$ contains three distinct elements γ, γ', and γ'' which are not collinear. By Lemma 4.23,

$$\mu(\gamma) = \mu(\gamma') = \mu(\gamma'').$$

Therefore, by Lemma 4.25 the function μ is constant on $\mathcal{B}$, and by Lemma 4.22 we have that $C = G_3$. □

5. A non-standard bi-order

To prove Theorem 1.5, we construct non-standard bi-orders on untwisted hyperbolic punctured torus bundles following Construction 2.20.

5.1. The first quotient.

The map $h : T \to T$ induces a map $h_+ \in \mathrm{SL}(2, \mathbb{Z})$ on $H_1(T; \mathbb{Z}) \cong \mathbb{Z}^2$ with eigenvalues $(\lambda, 1/\lambda)$ and corresponding eigenbasis $(e_\lambda, e_{1/\lambda})$. The bundle is untwisted, so the eigenvalues are positive. Using this eigenbasis, we can write h_+ as the matrix

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}.$$

It follows that h_+ preserves four positive cones Q_1 of $\mathbb{Z}^2$. Each of these is given by taking one of $\{e_\lambda, e_{1/\lambda}, -e_\lambda, -e_{1/\lambda}\}$ for the vector v in Example 2.18(2). See Figure 5.2.

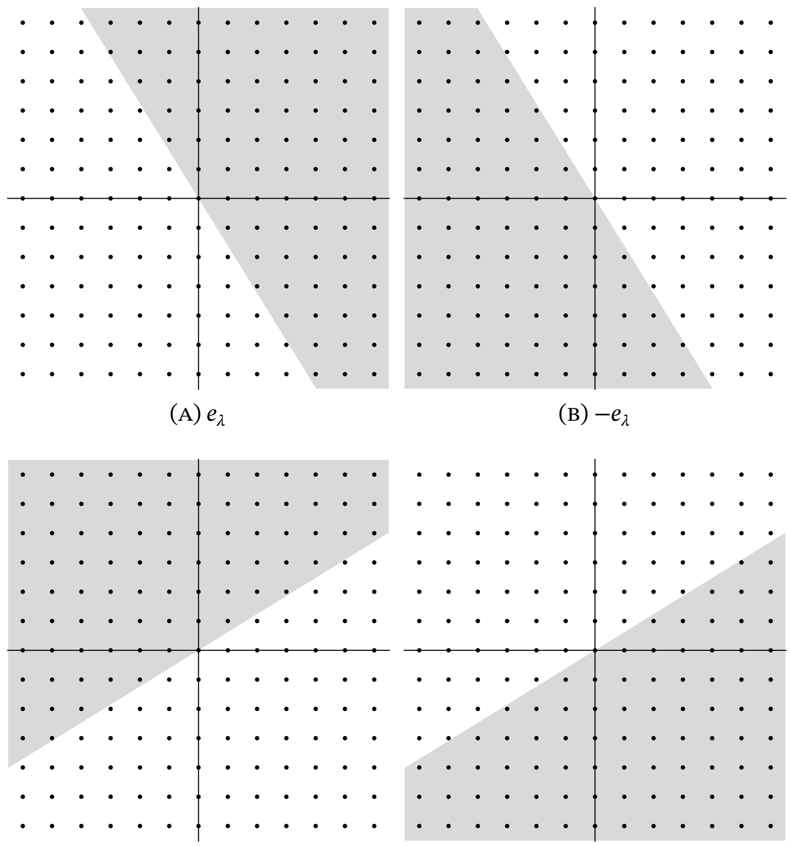


FIGURE 5.2. The four positive cones (shaded) in the example of the figure eight knot complement.

5.3. The second quotient. Recall the definition of the map $\hat{h}$ and basis $\mathcal{B}$ from Definitions 4.5 and 4.7. The goal is to define a positive cone of $H_1(\hat{T};\mathbb{Z})$ invariant under $\hat{h}_+$. In short, the plan is to find an ordering on $\mathcal{B}$ that is invariant under $\hat{h}_+$, and then define a bi-ordering of $H_1(\hat{T};\mathbb{Z})$ in a lexicographic fashion relative to the ordering of $\mathcal{B}$ as in Example 2.19.

Definition 5.4. Let e be one of $\{e_\lambda, e_{1/\lambda}, -e_\lambda, -e_{1/\lambda}\}$. Using Example 2.18(2) again, we generate an order $\prec_e$ on $\mathbb{Z}^2$ (possibly different from the choice made in Section 5.1). This order induces an ordering $<_e$ on $\mathcal{B}$ via q. That is, $\gamma_1 <_e \gamma_2$ if and only if $q(\gamma_1) \prec_e q(\gamma_2)$. ♢

Lemma 5.5. *The order $<_e$ on the basis $\mathcal{B}$ is invariant under shifting by any vector $(m,n) \in \mathbb{Z}^2$.*

Proof. Addition preserves the order $\prec_e$ on $\mathbb{Z}^2$. The result then follows from Definition 4.13. □

Lemma 5.6. *The order $<_e$ on $\mathcal{B}$ is invariant under $\hat{h}_+$.*

Proof. Let γ_1 and γ_2 be elements of $\mathcal{B}$. Suppose that $\gamma_1 <_e \gamma_2$. Therefore $q(\gamma_1) \prec_e q(\gamma_2)$. Since $\prec_e$ is invariant under the action of h_+ we have that

$$h_+ \circ q(\gamma_1) \prec_e h_+ \circ q(\gamma_2).$$

By Lemma 4.12(2), we have

$$q \circ \hat{h}_+(\gamma_1) \prec_e q \circ \hat{h}_+(\gamma_2),$$

and so

$$\hat{h}_+(\gamma_1) <_e \hat{h}_+(\gamma_2). \qquad \square$$

We next define a positive cone Q_2 on $H_1(\hat{T}; \mathbb{Z})$ as follows.

Definition 5.7. Given a non-zero element $v \in H_1(\hat{T}; \mathbb{Z})$, we define $\max(v)$ to be the maximum (under $<_e$) basis element $\gamma \in \mathcal{B}$ such that the coefficient of γ in v is nonzero. Let $k(v)$ be this coefficient. We then define

$$Q_2 = \{v \in H_1(\hat{T}; \mathbb{Z}) \mid k(v) > 0\}. \qquad \lozenge$$

Lemma 5.8. *The set Q_2 is a positive cone. Moreover Q_2 is invariant under the action of $\hat{h}_+$.*

Proof. We check the three properties in Definition 2.2. First suppose $v, w \in Q_2$. If $\max(v) = \max(w)$ then $\max(v+w) = \max(v) = \max(w)$ and $k(v+w) = k(v) + k(w) > 0$. If not, then breaking symmetry, assume that $\max(v) <_e \max(w)$. In this case, $\max(v + w) = \max(w)$ and $k(v + w) = k(w) > 0$. This proves (1). Now suppose that $v \in H_1(\hat{T}; \mathbb{Z}) - \{0\}$. Note that $\max(-v) = \max(v)$ and $k(-v) = -k(v)$. Therefore $v \in Q_2$ if and only if $-v \in Q_2^{-1}$. By definition, $0 \notin Q_2 \cup Q_2^{-1}$, so we have (2). Since $H_1(\hat{T}; \mathbb{Z})$ is abelian, we have (3).

Finally we show that Q_2 is invariant under $\hat{h}_+$. Let $v \in Q_2$. Thus $k(v) > 0$. For some set of coefficients $\{k_\gamma\}_{\gamma \in \mathcal{B}}$,

$$v = \sum_{\gamma \in \mathcal{B}} k_\gamma \gamma.$$

Thus,

$$\hat{h}_+(v) = \sum_{\gamma \in \mathcal{B}} k_\gamma \hat{h}_+(\gamma).$$

By Lemma 5.6, the ordering of elements in $\mathcal{B}$ is invariant under $\hat{h}_+$. Therefore, $\max(v) = \max(\hat{h}_+(v))$. Thus $k(\hat{h}_+(v)) = k(v) > 0$. Therefore, we have that $\hat{h}_+(v) \in Q_2$. $\square$

By the definition of $\hat{T}$, the fundamental group $\pi_1(\hat{T})$ is isomorphic to G_2. Abelianizing, we see that $G_2/[G_2, G_2]$ is naturally isomorphic to $H_1(\hat{T}; \mathbb{Z})$. Under this isomorphism, Q_2 defines a positive cone of $G_2/[G_2, G_2]$.

5.9. The remainder. We are now ready to build a bi-ordering of the free group G that is invariant under the action of h. We require one more piece of data: we choose an arbitrary positive cone R of $\pi_1(M)$ (recall that in the statement of Theorem 1.5 we assume that $\pi_1(M)$ is bi-orderable). We refer to the intersection $R \cap [G_2, G_2]$ as Q_3.

5.10. Putting it all together. We now apply Construction 2.20 to build a bi-ordering of the free group G, invariant under the action of h, as follows. Recall that $p_1 : G \to \mathbb{Z}^2$ is the abelianization map and that $p_2 : G_2 \to G_2/[G_2, G_2]$ is the quotient map.

Definition 5.11. Define P to be the set of all $g \in G$ such that either

(1) $p_1(g) \in Q_1$, or
(2) $g \in G_2$ and $p_2(g) \in Q_2$, or
(3) $g \in Q_3$. ◊

Lemma 5.12. *P is a positive cone of G.*

Proof. The goal is to show that P is a positive cone by repeatedly applying Lemma 2.6. By definition, Q_3 is a positive cone of $[G_2, G_2]$ and is invariant under conjugation. We first apply Lemma 2.6 with p_2 as the surjective homomorphism, mapping from G_2 to $G_2/[G_2, G_2]$. We obtain that $P \cap G_2$ is a positive cone on G_2.

Next, we want to apply Lemma 2.6 with p_1 as the surjective homomorphism, mapping from G to $G/G_2 \cong H_1(T; \mathbb{Z}) \cong \mathbb{Z}^2$. In order to do so, we need to show that $P \cap G_2$ is invariant under conjugation by the elements of G.

Suppose that $g \in G_2 \cap P$ and that $x \in G$. There are two cases. If $g \in Q_3$ then again by definition $xgx^{-1} \in Q_3 \subset G_2 \cap P$. Otherwise we have that $g \in G_2$ and $p_2(g) \in Q_2$. In this case, suppose that the homology class $p_1(x)$ is $m[\alpha]+n[\beta] \in H_1(T, \mathbb{Z})$, and the homology class $p_2(g)$ in $H_1(\hat{T}; \mathbb{Z})$ is

$$\sum_{\gamma \in \mathcal{B}} k_\gamma \gamma.$$

By Lemma 4.15, the homology class of $p_2(xgx^{-1})$ in $H_1(\hat{T}; \mathbb{Z})$ is

$$\sum_{\gamma \in \mathcal{B}} k_\gamma s_{(m,n)}(\gamma).$$

Lemma 5.5 implies that the maximal element of $\mathcal{B}$ appearing in the sum for $x^{-1}gx$ is the shift of the maximal element appearing in the sum for g. The coefficients for these elements are equal, and therefore have the same sign. By Definition 5.7, we have that $p_2(g) \in Q_2$ if and only if $p_2(x^{-1}gx) \in Q_2$. Thus we have obtained the hypotheses of Lemma 2.6 and we are done. □

Lemma 5.13. *The positive cone P is invariant under the map induced by the monodromy h_*.*

Proof. Suppose that $g \in P$. There are three possibilities, as given in Definition 5.11. First suppose that $p_1(g) \in Q_1$. Since $p_1 \circ h_*(g) = h_+ \circ p_1(g)$ and by definition, h_+ preserves Q_1, we have that $p_1(h_*(g)) \in Q_1$.

Next suppose that $g \in G_2$ and $p_2(g) \in Q_2$. Since G_2 is characteristic, we have that $h_*(g) \in G_2$. The positive cone Q_2 is invariant under $\hat{h}_*$ by Lemma 5.8, so it follows that $\hat{h}_* \circ p_2(g) \in Q_2$. Therefore $p_2 \circ h_*(g) = \hat{h}_* \circ p_2(g) \in Q_2$.

Finally, suppose that $g \in Q_3$. Recall that $Q_3 = R \cap [G_2, G_2]$, where R is our arbitrarily chosen positive cone of $\pi_1(M)$. In particular, R is invariant under conjugation, as is $[G_2, G_2]$, since it is characteristic. Thus the intersection Q_3 is also invariant under conjugation. The automorphism h_* acts by conjugation in $\pi_1(M)$ by τ (see Figure 3.3). Thus, $h_*(g) \in Q_3$. □

Lemma 5.14. *$G_3 = [G_2, G]$ in the lower central series for G is not convex with respect to P.*

Proof. We will apply Proposition 2.9, finding elements $y \in P$ and $z \in P^{-1}$ in the same nontrivial coset of G_3.

Let $\gamma_0 = \alpha\beta\alpha^{-1}\beta^{-1}$, and let $\gamma_1 = \beta\alpha^{-1}\beta^{-1}\alpha \in G_2 = [G, G]$. Since $\gamma_0 = [\alpha, \gamma_1]\gamma_1$, we have that γ_0 and γ_1 are in the same coset $S = \gamma_0 G_3 = \gamma_1 G_3$. (Recall that G_3 is normal, so left and right cosets are the same.) Consider the elements $y = \gamma_0^2\gamma_1^{-1}$ and $z = \gamma_0^{-1}\gamma_1^2$. Since γ_0 and γ_1 are in S, we have that $\gamma_0\gamma_1^{-1}$ and $\gamma_0^{-1}\gamma_1$ are in G_3. Thus y and z are also in S.

By the definition of $\hat{T}$, we have that $G_2 \cong \pi_1(\hat{T}, \hat{x})$. We define the *winding number* of an element $f \in G_2$ by

$$w(f) = \sum_{\gamma \in \mathcal{B}} k_\gamma$$

where

$$p_2(f) = \sum_{\gamma \in \mathcal{B}} k_\gamma \gamma.$$

The group G_3 is generated by terms $[g, f]$ where $g \in G$ and $f \in G_2$. By Lemma 4.15, we have that $w(gfg^{-1}) = w(f)$, so $w([g, f]) = 0$. Therefore G_3 is contained in $\ker(w)$. Since $w(\gamma_0) = w(\gamma_1) = 1$, we have that $S \neq G_3$.

We have that $p_2(y) = 2p_2(\gamma_0) - p_2(\gamma_1)$ and $p_2(z) = -p_2(\gamma_0) + 2p_2(\gamma_1)$. Since $p_2(\gamma_0)$ and $p_2(\gamma_1)$ are in $\mathcal{B}$ and the signs of the coefficients of $p_2(y)$ and $p_2(z)$ are opposite, exactly one of the homology classes is in Q_2 and the other is in Q_2^{-1}. Therefore, one of y and z is in P and the other is in P^{-1}. □

Proof of Theorem 1.5. Lemma 5.12 tells us that P is a positive cone. From Lemma 5.14 and Proposition 2.27 we deduce that P is not the positive cone of a standard bi-order of G. □

Corollary 5.15. *Let P be the positive cone given in Definition 5.11. Then G_2 has no maximal convex subgroup with respect to P.*

Proof. This follows from Lemma 5.14 and Theorem 1.4. It can also be seen directly by exhibiting an increasing sequence of convex subgroups. Namely, for each $\gamma \in \mathcal{B}$, let

$$C_\gamma = \{g \in G_2 \mid \max(p_2(g)) \leq_e \gamma\} \cup \{\mathrm{id}_G\}$$

where max is the function given in Definition 5.7. Each C_γ is a convex subgroup, $\gamma <_e \delta$ implies that $C_\gamma \subsetneq C_\delta \subsetneq G_2$, and $\bigcup_{\gamma \in \mathcal{B}} C_\gamma = G_2$. □

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(Jonathan Johnson) DEPARTMENT OF MATHEMATICS AND STATISTICS, SAM HOUSTON STATE UNIVERSITY, LEE DRAIN BUILDING, ROOM 420, 1900 AVENUE I, HUNTSVILLE, TEXAS 77340, USA
jcj055@shsu.edu

(Henry Segerman) DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY, 401 MATHEMATICAL SCIENCES, STILLWATER, OK 74078, USA
henry.segerman@okstate.edu

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