

Exactness of the Cuntz-Pimsner construction

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ABSTRACT. In prior work we described how the Cuntz-Pimsner construction may be viewed as a functor. The domain of this functor is a category whose objects are C^* -correspondences and morphisms are isomorphism classes of certain pairs comprised of a C^* -correspondence and an isomorphism. The codomain is the well-studied category whose objects are C^* -algebras and morphisms are isomorphism classes of C^* -correspondences. In this paper we show that certain fundamental results in the theory of Cuntz-Pimsner algebras are direct consequences of the functoriality of the Cuntz-Pimsner construction. In addition, we describe exact sequences in the target and domain categories, and prove that the Cuntz-Pimsner functor is exact.

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1. Introduction

In [Ery22] we introduced a categorical framework for viewing the Cuntz-Pimsner construction as a functor, which thereby allows one to determine relationships between Cuntz-Pimsner algebras from relationships between the defining C^* -correspondences. The domain of this functor is the category $C^*\text{cor}_{\text{pair}}$, introduced in [Ery22], that has C^* -correspondences as objects, and a morphism from ${}_A X_A$ to ${}_B Y_B$ is the isomorphism class of the pair $({}_A M_B, U_M)$, where ${}_A M_B$ is a C^* -correspondence satisfying certain conditions, and

$$U_M : {}_A(X \otimes_A M)_B \rightarrow {}_A(M \otimes_B Y)_B$$

Received August 5, 2025.

2020 *Mathematics Subject Classification.* Primary 46L08; Secondary 18B99.

Key words and phrases. C^* -correspondence, Cuntz-Pimsner algebra, exact sequence, exact functor.

is a C^* -correspondence isomorphism. The codomain category $C^*\text{alg}_{\text{cor}}$, which has sometimes been called the Enchilada Category in the literature, has C^* -algebras as objects and isomorphism classes of C^* -correspondences as morphisms. For any pair $({}_A M_B, U_M)$ one can construct a covariant representation (π, Φ) of ${}_A X_A$ on $\mathcal{K}(M \otimes_B \mathcal{O}_Y)$. Then the universal property of Cuntz-Pimsner algebras assures the existence of an associated homomorphism

$$\sigma_{(\pi, \Phi)} : \mathcal{O}_X \rightarrow \mathcal{K}(M \otimes_B \mathcal{O}_Y),$$

which allows us to view $M \otimes_B \mathcal{O}_Y$ as an \mathcal{O}_X - \mathcal{O}_Y -correspondence. It is shown in [Ery22] that there exists a functor \mathcal{E} from $C^*\text{cor}_{\text{pair}}$ to $C^*\text{alg}_{\text{cor}}$ that maps a C^* -correspondence ${}_A X_A$ to its Cuntz-Pimsner algebra \mathcal{O}_X , and a morphism from ${}_A X_A \rightarrow {}_B Y_B$ is mapped to the isomorphism class of an \mathcal{O}_X - \mathcal{O}_Y -correspondence. The first part of this paper uses this functor to obtain some well-known Cuntz-Pimsner algebra results. Specifically, if (Y, t) is a universal covariant representation of ${}_A X_A$, and I is a positive X -invariant ideal of A , then the following hold:

- (i) \mathcal{O}_{IX} is isomorphic to the smallest hereditary subalgebra of \mathcal{O}_X containing $Y(I)$.
- (ii) \mathcal{O}_{IX} is Morita equivalent to the ideal $\langle Y(I) \rangle$ generated by $Y(I)$ in \mathcal{O}_X .
- (iii) If ${}_A X_A$ is regular and I is an X -invariant ideal, then the C^* -algebras $\mathcal{O}_X / \langle Y(I) \rangle$ and $\mathcal{O}_{X/XI}$ are isomorphic.

Items (i) and (ii) were proven by Katsura in [Kat07, Proposition 9.3 and Proposition 9.5] by using what are called O -pairs. Item (iii) can be shown by combining [MT04, Lemma 6.3] and [FMR03, Theorem 3.1]. Item (iii) can also be deduced by combining [Kat07, Proposition 5.3] and [Kat07, Proposition 8.5]; however, this requires a deep understanding of O -pairs and the properties of C^* -algebras generated by such pairs. The first objective of this paper is to show that the functor established in [Ery22] can be applied to obtain easier and more direct proofs of these three results.

The work that is presented in the second part of this paper was motivated by a question frequently asked by audience members when presenting the results of [3], namely: “Can one define exact sequences in the domain and codomain categories so that the Cuntz-Pimsner functor \mathcal{E} is exact?” In order to answer this question, one needs to know what is meant by an exact sequence in both the domain and codomain categories. One of the difficulties in answering this question is that one cannot identify images in either categories, and it is suspected that images may not exist in either categories. Therefore the usual “kernel-image definition” of exactness may not apply. To circumvent this obstruction in each category, we define $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ to be a short exact sequence when f is a categorical kernel of g and g is a categorical cokernel of f (see Definition 5.8 and Definition 5.10). We characterize this “kernel-cokernel definition” of exactness in each category, showing that kernel-cokernel exactness is a tractable concept in these categories. We prove that with the kernel-cokernel definition of short exact sequence, the Cuntz-Pimsner functor is exact. We end

the paper by showing that as an immediate consequence of exactness one can obtain the results (i),(ii), and (iii) listed above for the case when ${}_A X_A$ is regular.

2. Preliminaries

Let \mathcal{C} be a pointed category (a category with a zero object), and let $f : A \rightarrow B$ be a morphism. A *kernel* of f is a pair (K, k) consists of an object K and a morphism $k : K \rightarrow A$ such that

- $f \circ k = 0$;
- whenever a morphism $h : D \rightarrow A$ satisfies $f \circ h = 0$ there exists a unique morphism $p : D \rightarrow K$ such that $k \circ p = h$.

A *cokernel* of f is a pair (C, c) consisting of an object C and a morphism $c : B \rightarrow C$ such that

- $c \circ f = 0$;
- whenever a morphism $h : B \rightarrow D$ satisfies $h \circ f = 0$ there exists a unique morphism $p : C \rightarrow D$ such that $p \circ c = h$.

We say that f is a *monomorphism* if for all morphisms $g : C \rightarrow A$ and $h : C \rightarrow A$ in \mathcal{C} , we have $f \circ g = f \circ h$ implies $g = h$. We say that f is an *epimorphism* if for all morphisms $g : B \rightarrow C$ and $h : B \rightarrow C$ in \mathcal{C} , we have $g \circ f = h \circ f$ implies $g = h$.

A C^* -correspondence ${}_A X_B$ is a right Hilbert B -module equipped with a left action given by a homomorphism $\varphi_X : A \rightarrow \mathcal{L}(X)$, where $\mathcal{L}(X)$ denotes the C^* -algebra of adjointable operators on X . We denote the kernel of the left action homomorphism φ_X by $\text{Ker } \varphi_X$. For a C^* -correspondence ${}_A X_B$ define

$$A \cdot X = \{\varphi_X(a)x : a \in A, x \in X\}.$$

The correspondence ${}_A X_B$ is called *nondegenerate* if $A \cdot X = X$. In this paper *all our correspondences will be nondegenerate by standing hypothesis*. A C^* -correspondence ${}_A X_B$ is called *injective* if the left action $\varphi_X : A \rightarrow \mathcal{L}(X)$ is injective; it is called *proper* if $\varphi_X(A)$ is contained in the C^* -algebra $\mathcal{K}(X)$ of compact operators on X . A C^* -correspondence ${}_A X_B$ is called *regular* if it is both injective and proper. For a C^* -correspondence ${}_A X_B$ we denote the closed span of B -valued inner products $\langle X, X \rangle_B$ by B_X . One of the C^* -correspondence properties we use frequently in this paper is the following: let I be an ideal of B such that $B_X \subset I$. Then, X can be viewed as an A - I -correspondence [EKQ20, Lemma 3.2].

A C^* -correspondence isomorphism from ${}_A X_B$ to ${}_A Y_B$ is a bijective linear map $\Phi : X \rightarrow Y$ satisfying

- (i) $\Phi(a \cdot x) = a \cdot \Phi(x)$,
- (ii) $\langle x, z \rangle_B = \langle \Phi(x), \Phi(z) \rangle_B$,

for all $a \in A$, and $x, z \in X$. We let $Ad \Phi : \mathcal{L}(X) \rightarrow \mathcal{L}(Y)$ denote the associated C^* -algebra isomorphism defined by $Ad \Phi(T) = \Phi \circ T \circ \Phi^{-1}$.

The *balanced tensor product* $X \otimes_B Y$ of an A - B correspondence X and a B - C correspondence Y is formed as follows: the algebraic tensor product $X \odot Y$ is

a pre-correspondence with the $A - C$ bimodule structure satisfying

$$a(x \otimes y)c = ax \otimes yc \quad \text{for } a \in A, x \in X, y \in Y, c \in C,$$

and the unique C -valued semi-inner product whose values on elementary tensors are given by

$$\langle x \otimes y, u \otimes v \rangle_C = \langle y, \langle x, u \rangle_B \cdot v \rangle_C \quad \text{for } x, u \in X, y, v \in Y.$$

This semi-inner product defines a C -valued inner product on the quotient $X \odot_B Y$ of $X \odot Y$ by the subspace generated by elements of form

$$x \cdot b \otimes y - x \otimes \varphi_Y(b)y \quad (x \in X, y \in Y, b \in B).$$

The completion $X \otimes_B Y$ of $X \odot_B Y$ with respect to the norm coming from the C -valued inner product is an $A - B$ correspondence, where the left action is given by

$$A \rightarrow \mathcal{L}(X \otimes_B Y), \quad a \mapsto \varphi_X(a) \otimes 1_Y,$$

for $a \in A$. We denote the canonical image of $x \otimes y$ in $X \otimes_B Y$ by $x \otimes_B y$.

Proposition 2.1. [EKQ20, Proposition 3.1] *For C^* -correspondences ${}_A X_B$ and ${}_B Y_C$ we have*

$${}_A(X \otimes_B Y)_B \cong {}_A 0_B \iff {}_A(X \otimes_B Y)_B = {}_A 0_B \iff B_X \subset \text{Ker } \varphi_Y.$$

Lemma 2.2 ([FMR03]). *Let X be a C^* -correspondence over A and let ${}_A Y_B$ be an injective C^* -correspondence. Then the map $\iota : T \mapsto T \otimes 1_Y$ gives an isometric homomorphism of $\mathcal{L}(X)$ into $\mathcal{L}(X \otimes_A Y)$. If, in addition, $\varphi_Y(A) \subset \mathcal{K}(Y)$, then ι embeds $\mathcal{K}(X)$ into $\mathcal{K}(X \otimes_A Y)$.*

A Hilbert bimodule ${}_A X_B$ is a C^* -correspondence that is also equipped with an A -valued inner product ${}_A \langle \cdot, \cdot \rangle$, which satisfies

$${}_A \langle a \cdot x, y \rangle = a \cdot {}_A \langle x, y \rangle \quad \text{and} \quad {}_A \langle x, y \rangle^* = {}_A \langle y, x \rangle$$

for all $a \in A, x, y \in X$, as well as the *compatibility property*

$${}_A \langle x, y \rangle \cdot z = x \cdot \langle y, z \rangle_B \quad \text{for } x, y, z \in X.$$

A Hilbert bimodule ${}_A X_B$ is *left-full* if the closed span of ${}_A \langle X, X \rangle$ is all of A .

An *imprimitivity bimodule* ${}_A X_B$ is a Hilbert bimodule that is full on both the left and the right. The *identity correspondence* on A is the Hilbert bimodule ${}_A A_A$ where the bimodule structure is given by multiplication, and the inner products are given by

$${}_A \langle a, b \rangle = ab^*, \quad \langle a, b \rangle_A = a^*b, \quad \text{for } a, b \in A.$$

A *representation* (π, t) of a C^* -correspondence ${}_A X_A$ on a C^* -algebra B consists of a $*$ -homomorphism $\pi : A \rightarrow B$ and a linear map $t : X \rightarrow B$ such that

$$\pi(a)t(x) = t(\varphi_X(a)(x)) \quad \text{and} \quad t(x)^*t(y) = \pi(\langle x, y \rangle_A),$$

for $a \in A$ and $x, y \in X$, where φ_X is the left action homomorphism associated with ${}_A X_A$. For any representation (π, t) of ${}_A X_A$ on B , there is an associated homomorphism $\psi_t : \mathcal{K}(X) \rightarrow B$ satisfying $\psi_t(\theta_{x,x'}) = t(x)t(x')^*$ for $x, x' \in X$.

The representation (π, t) is called *injective* if π is injective, in which case t is an isometry. We denote the C^* -algebra generated by the images of π and t in B by $C^*(\pi, t)$.

Consider a C^* -correspondence ${}_A X_A$. The ideal J_X is defined as

$$\begin{aligned} J_X &= \varphi_X^{-1}(\mathcal{K}(X)) \cap (\text{Ker } \varphi_X)^\perp \\ &= \{a \in A : \varphi_X(a) \in \mathcal{K}(X) \text{ and } ab = 0 \text{ for all } b \in \text{Ker } \varphi_X\}, \end{aligned}$$

and is called the *Katsura ideal*. Notice here that for a regular ${}_A X_A$ we have $J_X = A$.

A representation (π, t) of ${}_A X_A$ is called *covariant* if $\pi(a) = \Psi_t(\varphi_X(a))$, for all $a \in J_X$. The C^* -algebra generated by the universal covariant representation of ${}_A X_A$ is called the *Cuntz-Pimsner algebra* \mathcal{O}_X of ${}_A X_A$.

3. Categories and the covariant representation

In this section we briefly explain the construction of the functor \mathcal{E} defined in [Ery22], and recall the related categories. The range category $C^*\text{alg}_{\text{cor}}$ of \mathcal{E} is sometimes called “the enchilada category” as in [EKQ20]. In this category our objects are C^* -algebras, and a morphism from A to B is the isomorphism class of an A - B -correspondence. The composition $[_B Y_C] \circ [_A X_B]$ is the isomorphism class of the balanced tensor product ${}_A(X \otimes_B Y)_C$; the identity morphism on A is the isomorphism class of the identity correspondence ${}_A A_A$, and the zero morphism $A \rightarrow B$ is $[_A 0_B]$. Note that a morphism $[_A X_B]$ is an isomorphism in $C^*\text{alg}_{\text{cor}}$ if and only if ${}_A X_B$ is an imprimitivity bimodule [EKKR00, Proposition 2.6].

We need the following definition for the domain category.

Definition 3.1. [Ery22, Definition 3.1] For C^* -correspondences ${}_A X_A, {}_B Y_B$, and A - B -correspondences ${}_A M_B, {}_A N_B$, let

$$U_M : X \otimes_A M \rightarrow M \otimes_B Y$$

and

$$U_N : X \otimes_A N \rightarrow N \otimes_B Y$$

be A - B -correspondence isomorphisms. The pairs $({}_A M_B, U_M)$ and $({}_A N_B, U_N)$ are called *isomorphic* if

- there exists an isomorphism $\xi: {}_A M_B \rightarrow {}_A N_B$; and
- the diagram

$$\begin{array}{ccc} X \otimes_A M & \xrightarrow{1 \otimes \xi} & X \otimes_A N \\ U_M \downarrow & & \downarrow U_N \\ M \otimes_B Y & \xrightarrow{\xi \otimes 1_Y} & N \otimes_B Y \end{array}$$

commutes.

We denote the isomorphism class of the pair $({}_A M_B, U_M)$ by $[_A M_B, U_M]$.

Remark 3.2. For a C^* -correspondence ${}_A M_B$, let I and J be ideals of A and B , respectively. We denote the map

$$M \otimes_B J \rightarrow MJ, \quad m \otimes_B j \mapsto m \cdot j$$

by $\xi_{(r,M,J)}$, where $m \in M$, $j \in J$. This map defines an A - B -correspondence isomorphism as well as an A - J -correspondence isomorphism. Similarly, we denote the map

$$I \otimes_A M \rightarrow IM, \quad i \otimes_A m \mapsto i \cdot m$$

by $\xi_{(l,M,I)}$, where $m \in M$, $i \in I$. This map defines an A - B -correspondence isomorphism as well as an I - B -correspondence isomorphism.

Theorem 3.3 ([Ery22, Theorem 3.2]). *There exists a category $C^* \text{cor}_{\text{pair}}$ such that*

- objects are C^* -correspondences ${}_A X_A$;
- morphisms ${}_A X_A \rightarrow {}_B Y_B$ are isomorphism classes $[{}_A M_B, U_M]$ where U_M denotes an A - B -correspondence isomorphism $X \otimes_A M \rightarrow M \otimes_B Y$, and ${}_A M_B$ is a proper correspondence satisfying $J_X \cdot M \subset M \cdot J_Y$;
- the composition $[{}_B N_C, U_N] \circ [{}_A M_B, U_M]$ is given by the isomorphism class

$$[{}_A (M \otimes_B N)_C, U_{M \otimes_B N}]$$

where $U_{M \otimes_B N}$ denotes the isomorphism $(1_M \otimes U_N)(U_M \otimes 1_N)$;

- the identity morphism on ${}_A X_A$ is $[{}_A A_A, U_A]$, where U_A denotes the isomorphism $\xi_{l,X,A}^{-1} \circ \xi_{r,X,A} : X \otimes_A A \rightarrow A \otimes_A X$.

Let $[{}_A M_B, U_M] : {}_A X_A \rightarrow {}_B Y_B$ be a morphism in $C^* \text{cor}_{\text{pair}}$. Denote the universal covariant representation of ${}_B Y_B$ by (Y, t) . Let $V_Y : Y \otimes_B \mathcal{O}_Y \rightarrow \overline{t(Y)\mathcal{O}_Y}$ be the isomorphism determined on elementary tensors by

$$V_Y(y \otimes_B S) = t(y)S$$

for $y \in Y$, $S \in \mathcal{O}_Y$. Define $T : X \rightarrow \mathcal{L}(M, M \otimes_B Y)$ by

$$T(x)(m) = U_M(x \otimes_A m),$$

for $x \in X$, $m \in M$. Next, define a linear map $\Phi : X \rightarrow \mathcal{K}(M \otimes_B \mathcal{O}_Y)$ by

$$\Phi(x) = (1_M \otimes V_Y)(T(x) \otimes 1_Y).$$

and a homomorphism $\pi : A \rightarrow \mathcal{K}(M \otimes_B \mathcal{O}_Y)$ by

$$\pi(a) = \varphi_M \otimes 1_{\mathcal{O}_Y}.$$

The pair (π, Φ) is a covariant representation of ${}_A X_A$ on $\mathcal{K}(M \otimes_B \mathcal{O}_Y)$ [Ery22, Proposition 4.2], and it is called the C -covariant representation of ${}_A X_A$. It is injective when the homomorphism φ_M is. By the universal property of \mathcal{O}_X we obtain a $*$ -homomorphism $\sigma_{(\pi, \Phi)} : \mathcal{O}_X \rightarrow \mathcal{K}(M \otimes_B \mathcal{O}_Y)$, which provides a left action of \mathcal{O}_X on the Hilbert \mathcal{O}_Y -module $M \otimes_B \mathcal{O}_Y$ and allows us to view $M \otimes_B \mathcal{O}_Y$ as a proper \mathcal{O}_X - \mathcal{O}_Y -correspondence. It is important to note that the C -covariant representation (π, Φ) admits a gauge action. Consequently, the homomorphism $\sigma_{(\pi, \Phi)}$ is an isomorphism onto $C^*(\pi, \Phi)$ when ${}_A M_B$ is an injective C^* -correspondence [Ery22, Theorem 4.13].

Theorem 3.4 ([Ery22, Theorem 5.1]). *Let $[{}_A M_B, U_M] : {}_A X_A \rightarrow {}_B Y_B$ be a morphism in $C^* \text{cor}_{\text{pair}}$. Then the assignments ${}_A X_A \mapsto \mathcal{O}_X$ on objects and*

$$[{}_A M_B, U_M] \mapsto [{}_{\mathcal{O}_X} (M \otimes_B \mathcal{O}_Y)_{\mathcal{O}_Y}]$$

on morphisms define a functor \mathcal{E} from $C^ \text{cor}_{\text{pair}}$ to $C^* \text{alg}_{\text{cor}}$.*

The next proposition is crucial for this paper.

Proposition 3.5. *Let $[{}_A M_B, U_M] : {}_A X_A \rightarrow {}_B Y_B$ be a morphism in $C^* \text{cor}_{\text{pair}}$, where ${}_A X_A$ is a regular correspondence, and let $\sigma : \mathcal{O}_X \rightarrow \mathcal{K}(M \otimes_B \mathcal{O}_Y)$ be the associated homomorphism. Denote the universal covariant representation of ${}_A X_A$ by (Y, ι) . Then $\text{Ker } \sigma$ is the ideal $\langle Y(\text{Ker } \varphi_M) \rangle$ generated by $Y(\text{Ker } \varphi_M)$ in \mathcal{O}_X .*

Proof. It suffices to show the equality $\text{Ker } \sigma \cap Y(A) = \langle Y(\text{Ker } \varphi_M) \rangle \cap Y(A)$, since gauge invariant ideals of \mathcal{O}_X are distinguished by their intersection with $Y(A)$ when ${}_A X_A$ is regular [Kat07, Corollary 8.7]. One can easily verify that $\langle Y(\text{Ker } \varphi_M) \rangle \subset \text{Ker } \sigma$. Let $Y(a) \in \text{Ker } \sigma$. Then we have

$$0 = \sigma(Y(a)) = \varphi_M(a) \otimes 1_{\mathcal{O}_Y}.$$

This implies $\varphi_M(a) = 0$ by Lemma 2.2. And thus, $a \in \text{Ker } \varphi_M$, which means $Y(a) \in Y(A) \cap \langle Y(\text{Ker } \varphi_M) \rangle$. \square

4. Invariant Ideals and Structure Theorems

Definition 4.1. Let ${}_A X_A$ be a C^* -correspondence. For an ideal I of A , define an ideal $X^{-1}(I)$ of A by

$$X^{-1}(I) = \{a \in A : \langle x, a \cdot y \rangle_A \in I \text{ for all } x, y \in X\}.$$

An ideal I of A is said to be *positive X -invariant* if $IX \subset XI$, *negative X -invariant* if $J_X \cap X^{-1}(I) \subset I$, and *X -invariant* if I is both positive and negative invariant.

Note that I is a positive X -invariant ideal of A if and only if $\langle X, IX \rangle_A \subset I$. When that's the case, we have $IX = IX \langle IX, IX \rangle_A \subset IXI$. Therefore, we have the equality $IX = IXI$. Consequently, the I - A -correspondence IX can be viewed as a C^* -correspondence over I .

Lemma 4.2. *Let ${}_A X_A$ be a C^* -correspondence and I be a positive X -invariant ideal of A . Denote the I - A -correspondence isomorphism $\xi_{(I, X, I)}^{-1} \circ \xi_{(r, IX, I)} : IX \otimes_I I \rightarrow I \otimes_A X$ by U_I , where $\xi_{(I, X, I)}$ and $\xi_{(r, IX, I)}$ are the I - A -correspondence isomorphisms defined as in Remark 3.2. Then, the isomorphism class $[{}_I I_A, U_I] : {}_I I_X \rightarrow {}_A X_A$ is a morphism in $C^* \text{cor}_{\text{pair}}$.*

Proof. It suffices to show $J_{IX} \cdot I \subset J_X$, which follows immediately from the fact that $J_{IX} = I \cap J_X$ [Kat07, Proposition 9.2]. \square

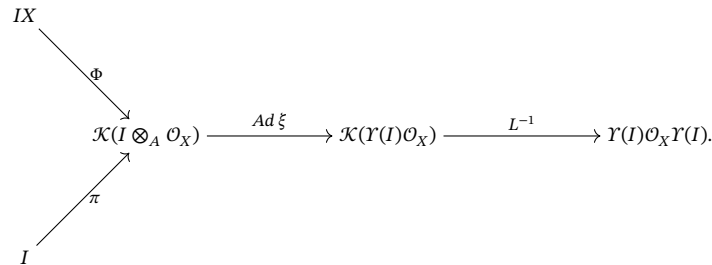
Lemma 4.3. *For C^* -algebras A and B , let $A \subset B$. Then we have the C^* -algebra isomorphism $\mathcal{K}(AB) \cong ABA$, where AB is viewed as a Hilbert B -module.*

Proof. For any $x \in ABA$, consider the operator $T_x : AB \rightarrow AB$ defined by $T_x(y) = xy$, where $y \in AB$. Then each T_x is an element of $\mathcal{K}(AB)$, and the map $L : ABA \rightarrow \mathcal{K}(AB)$ defined by $x \mapsto T_x$ is an injective $*$ -homomorphism. Now take any $\theta_{a_1b_1, a_2b_2} \in \mathcal{K}(AB)$. We have $\theta_{a_1b_1, a_2b_2} = T_{a_1b_1b_2^*a_2^*} = L(a_1b_1b_2^*a_2^*)$. And thus, L is surjective. \square

Theorem 4.4. Let ${}_A X_A$ be a C^* -correspondence and I be a positive X -invariant ideal of A . Let (Y, t) be the universal covariant representation of ${}_A X_A$. Then we have the following:

- (1) $\mathcal{E}([{}_I I_A, U_I]) = [{}_{\mathcal{O}_{IX}}(I \otimes_A \mathcal{O}_X)_{\mathcal{O}_X}]$ is an isomorphism class of a left-full Hilbert bimodule.
- (2) \mathcal{O}_{IX} is isomorphic to the smallest hereditary subalgebra of \mathcal{O}_X containing $Y(I)$ [Kat07, Proposition 9.3].
- (3) \mathcal{O}_{IX} is Morita equivalent to the ideal $\langle Y(I) \rangle$ generated by $Y(I)$ in \mathcal{O}_X [Kat07, Proposition 9.5].

Proof. Let $\xi : I \otimes_A \mathcal{O}_X \rightarrow Y(I)\mathcal{O}_X$ denote the Hilbert \mathcal{O}_X -module isomorphism defined on elementary tensors by $i \otimes_A S \mapsto Y(i)S$. Denote the C -covariant representation of ${}_A X_A$ by (π, Φ) , and let $L : Y(I)\mathcal{O}_X Y(I) \rightarrow \mathcal{K}(Y(I)\mathcal{O}_X)$ be the isomorphism defined as in the proof of Lemma 4.3. Then we have the following diagram



We claim that $\text{Ad } \xi^{-1} \circ L$ is an isomorphism onto $C^*(\pi, \Phi)$. It suffices to show the equalities

$$\text{Ad } \xi[\Phi(ixj)] = L(Y(i)t(x)Y(j)) \text{ and } \text{Ad } \xi[\pi(iaj)] = L(Y(iaj)),$$

for any $i, j \in I, x \in X$, and $a \in A$. Let $V : X \otimes_A \mathcal{O}_X \rightarrow \overline{t(X)\mathcal{O}_X}$ denote the $A - \mathcal{O}_X$ correspondence isomorphism defined on elementary tensors by $x \otimes_A S \mapsto t(x)S$, for any $x \in X, S \in \mathcal{O}_X$. For $k \in I, S \in \mathcal{O}_X$ we have

$$\begin{aligned}
 \xi \Phi(ixj)(k \otimes_A S) &= \xi(1_I \otimes V)U_I(ixj \otimes_I k) \otimes_A S \\
 &= \xi(a \otimes_A t(z)S) \quad (a \in I, z \in X \text{ with } az = ixjk) \\
 &= Y(a)t(z)S \\
 &= t(ixjk)S.
 \end{aligned}$$

On the other hand, we have

$$L(Y(i)t(x)Y(j))\xi(k \otimes_A S) = Y(i)t(x)Y(j)Y(k)S = t(ixjk)S,$$

which proves the first equality. For the second equality, we observe that

$$\xi\pi(iaj)(k \otimes_A S) = \xi(iajk \otimes_A S) = Y(iajk)S = L(Y(iaj))\xi(k \otimes_A S),$$

which proves our claim.

Recall that the left action homomorphism $\sigma : \mathcal{O}_{IX} \rightarrow \mathcal{K}(I \otimes_A \mathcal{O}_X)$ is injective onto $C^*(\pi, \Phi)$. And thus, we may now conclude that the C^* -correspondence ${}_{\mathcal{O}_{IX}}(I \otimes_A \mathcal{O}_X)_{\mathcal{O}_X}$ is a left-full Hilbert bimodule, which implies \mathcal{O}_{IX} and $\langle I \otimes_A \mathcal{O}_X, I \otimes_A \mathcal{O}_X \rangle_{\mathcal{O}_X} = \langle Y(I) \rangle$ are Morita equivalent C^* -algebras. Moreover, by Lemma 4.3 we have $\mathcal{O}_{IX} \cong \mathcal{K}(I \otimes_A \mathcal{O}_X) \cong \mathcal{K}(Y(I)\mathcal{O}_X) \cong Y(I)\mathcal{O}_X Y(I)$, which proves item (2). \square

Remark 4.5. Let ${}_A X_A$ be a C^* -correspondence, and let I be a positive X -invariant ideal of A . Let $p : A \rightarrow A/I$ and $q : X \rightarrow X/XI$ be the natural quotient maps. Then, X/XI can be viewed as a C^* -correspondence over A/I with the module actions and the inner product are given by

$$p(a) \cdot q(x) \cdot q(a') = q(axa') \quad \langle q(x), q(y) \rangle = p(\langle x, y \rangle_A),$$

for $a, a' \in A$, and $x, y \in X$.

Assume ${}_A X_A$ is regular and I is an X -invariant ideal. Then X/XI is a regular correspondence as well: properness of X/XI is straightforward by construction. To see injectivity let $a \in A$, and let $p(a)q(x) = 0$ for all $x \in X$. Then $ax \in XI$ for all $x \in X$, which means $a \in X^{-1}(I)$. Since I is X -invariant and ${}_A X_A$ is regular we have $X^{-1}(I) \subset I$, and thus $p(a) = 0$.

Now, for a regular correspondence ${}_A X_A$ and an X -invariant ideal I , consider the isomorphisms

$$\begin{aligned} i_1 : X \otimes_A A/I &\rightarrow X/XI, & x \otimes_A p(a) &\mapsto q(xa) \\ i_2 : A/I \otimes_{A/I} X/XI &\rightarrow X/XI, & p(a) \otimes_{A/I} q(x) &\mapsto q(ax), \end{aligned}$$

where $x \in X, a \in A$. Then, $[A/I, U_{A/I}] : {}_A X_A \rightarrow {}_{A/I}(X/XI)_{A/I}$ is a morphism in $C^*\text{cor}_{\text{pair}}$, where $U_{A/I} := i_2^{-1} \circ i_1$.

Note that for any $x \in X, a \in A$, we have $U_{A/I}(x \otimes_A p(a)) = p(a') \otimes_{A/I} q(x')$, for some $a' \in A, x' \in X$, satisfying $p(a')q(x') = q(x)p(a)$: $i_1(x \otimes_A p(a)) = q(xa)$. Since $q(xa)$ is an element of the non-degenerate correspondence X/XI , there exists $p(a') \in A/I, q(x') \in X/XI$ such that $p(a')q(x') = q(x)p(a)$.

Theorem 4.6. *Let ${}_A X_A$ be a regular correspondence and let I be an X -invariant ideal. Then, we have the isomorphism $\mathcal{O}_X/\langle Y(I) \rangle \cong \mathcal{O}_{X/XI}$.*

Proof. Let (Y, t) and (\tilde{Y}, \tilde{t}) be universal covariant representations of X and X/XI , respectively. And, let $p : A \rightarrow A/I$ and $q : X \rightarrow X/XI$ be the quotient maps. The map $\xi : A/I \otimes_{A/I} \mathcal{O}_{X/XI} \rightarrow \mathcal{O}_{X/XI}$ defined on elementary tensors by $p(a) \otimes S \mapsto \tilde{Y}(p(a))S$ is a Hilbert- $\mathcal{O}_{X/XI}$ module isomorphism, and extends to a C^* -algebra isomorphism

$$\text{Ad } \xi : \mathcal{K}(A/I \otimes_{A/I} \mathcal{O}_{X/XI}) \rightarrow \mathcal{K}(\mathcal{O}_{X/XI}).$$

Let $L : \mathcal{O}_{X/XI} \rightarrow \mathcal{K}(\mathcal{O}_{X/XI})$ be the C^* -algebra isomorphism defined by $L(S)T = ST$ for $S, T \in \mathcal{O}_{X/XI}$. Denote the C -covariant representation of ${}_A X_A$ on $\mathcal{K}(A/I \otimes_{A/I} \mathcal{O}_{X/XI})$ by (π, Φ) . Then we have the following diagram

$$\begin{array}{ccccc}
 X & & & & \\
 & \searrow \Phi & & & \\
 & & \mathcal{K}(A/I \otimes_{A/I} \mathcal{O}_{X/XI}) & \xrightarrow{\text{Ad } \xi} & \mathcal{K}(\mathcal{O}_{X/XI}) & \xrightarrow{L^{-1}} & \mathcal{O}_{X/XI} \\
 & \nearrow \pi & & & & & \\
 A & & & & & &
 \end{array}$$

We claim $\text{Ad } \xi^{-1} \circ L$ is an isomorphism onto $C^*(\pi, \Phi)$. To prove our claim we first show $\tilde{t}(q(x)) = L^{-1} \text{Ad } \xi(\Phi(x))$, for $x \in X$. Let $a \in A, S \in \mathcal{O}_{X/XI}$. On one hand we have

$$L(\tilde{t}(q(x))\xi(p(a) \otimes_{A/I} S)) = L(\tilde{t}(q(x))\tilde{Y}(p(a))S) = \tilde{t}(q(x))\tilde{Y}(p(a))S = \tilde{t}(q(xa))S.$$

Now, let $V : X/XI \otimes_{A/I} \mathcal{O}_{X/XI} \rightarrow \mathcal{O}_{X/XI}$ denote the isomorphism defined on elementary tensors by $q(z) \otimes_{A/I} T \mapsto \tilde{t}(q(z))T$, where $z \in X, T \in \mathcal{O}_{X/XI}$. Let $q(x)p(a) = p(a')q(x')$ ¹. We have

$$\begin{aligned}
 \xi\Phi(x)(p(a) \otimes_{A/I} S) &= \xi(1_{A/I} \otimes V)(U_{A/I}(x \otimes_A p(a)) \otimes_{A/I} S) \\
 &= \xi(1_{A/I} \otimes V)(p(a') \otimes_{A/I} q(x') \otimes_{A/I} S) \\
 &= \xi[p(a') \otimes_{A/I} \tilde{t}(q(x'))S] \\
 &= \tilde{t}(q(xa))S.
 \end{aligned}$$

It is easy to show $\tilde{Y}(p(a)) = L^{-1} \text{Ad } \xi(\pi(a))$, for any $a \in A$, completing the proof of our claim. We may now conclude that $\sigma : \mathcal{O}_X \rightarrow \mathcal{K}(A/I \otimes_{A/I} \mathcal{O}_{X/XI})$ is surjective. Then by the first isomorphism theorem we have $\mathcal{O}_X / \text{Ker } \sigma \cong \sigma(\mathcal{O}_X)$. By using Proposition 3.5 we obtain $\mathcal{O}_X / \langle Y(I) \rangle \cong \mathcal{K}(A/I \otimes_{A/I} \mathcal{O}_{X/XI}) \cong \mathcal{O}_{X/XI}$. \square

We next give a factorization property in $C^*\text{cor}_{\text{pair}}$, which allows us to generalize the first item of Theorem 4.4. But first we need a Lemma.

Lemma 4.7. *For an A - B -correspondence isomorphism $U_M : X \otimes_A M \rightarrow M \otimes_B Y$ we have the following.*

- (1) *The ideal $B_M = \overline{\langle M, M \rangle_B}$ of B is positive Y -invariant.*
- (2) *$\text{Ker } \varphi_M$ is a positive X -invariant ideal of A . If ${}_A X_A$ and ${}_B Y_B$ are regular correspondences, then $\text{Ker } \varphi_M$ is an X -invariant ideal.*

Proof. For the first item we compute

$$\langle Y, B_M \cdot Y \rangle_B = \langle M \otimes_B Y, M \otimes_B Y \rangle_B = \langle X \otimes_A M, X \otimes_A M \rangle_B = \langle M, A_X \cdot M \rangle_B \subset B_M,$$

¹See the last paragraph of Remark 4.5.

as desired. For the second item denote $\text{Ker } \varphi_M$ by K . We have

$$0 = \langle K \cdot M \otimes_B Y, M \otimes_B Y \rangle_B = \langle K \cdot X \otimes_A M, X \otimes_A M \rangle_B = \langle M, \langle KX, X \rangle_A \cdot M \rangle_B,$$

which implies $\langle KX, X \rangle_A \subset K$, as desired. Now, assume ${}_A X_A$ and ${}_B Y_B$ are regular correspondences. Let $a \in X^{-1}(K)$. Then, $\langle ax, x' \rangle_A \in K$ for any $x, x' \in X$. This means

$$\langle ax \otimes_A m, x' \otimes_A n \rangle_B = \langle m, \langle ax, x' \rangle_A \cdot n \rangle_B = 0,$$

for any $x, x' \in X$ and $m, n \in M$. This implies $a \in \text{Ker } \varphi_{X \otimes_A M} = \text{Ker } \varphi_{M \otimes_B Y}$. Then for any $m, n \in M$ and $y, y' \in Y$, we have

$$0 = \langle a \cdot m \otimes_B y, m' \otimes_B y' \rangle_B = \langle y, \langle am, m' \rangle_B \cdot y' \rangle_B,$$

which implies $\langle am, m' \rangle_B \in \text{Ker } \varphi_Y$. Since ${}_B Y_B$ is a regular correspondence, we conclude that $a \in \text{Ker } \varphi_M$. \square

Let $[{}_A M_B, U_M]: {}_A X_A \rightarrow {}_B Y_B$ be a morphism in $\mathbf{C}^* \text{cor}_{\text{pair}}$. The first item of Lemma 4.7 and Lemma 4.2 together imply that the isomorphism class of the pair $({}_{B_M}(B_M)_B, U_{B_M})$ is a morphism $[{}_{B_M}(B_M)_B, U_{B_M}]: {}_{B_M}(B_M Y)_{B_M} \rightarrow {}_B Y_B$ in $\mathbf{C}^* \text{cor}_{\text{pair}}$.

Proposition 4.8. *For any morphism $[{}_A M_B, U_M]: {}_A X_A \rightarrow {}_B Y_B$ in $\mathbf{C}^* \text{cor}_{\text{pair}}$, there exists a morphism $[{}_A M'_{B_M}, U_{M'}]: {}_A X_A \rightarrow {}_{B_M}(B_M Y)_{B_M}$ such that the equality*

$$[{}_A M_B, U_M] = [{}_{B_M}(B_M)_B, U_{B_M}] \circ [{}_A M'_{B_M}, U_{M'}]$$

holds.

Proof. Let ${}_A M'_{B_M}$ be the Hilbert B -module M viewed as A - B_M -correspondence. Consider the following C^* -correspondence isomorphisms:

$$\begin{aligned} \iota &: (M \otimes_B B_M)_{B_M} \rightarrow {}_A M'_{B_M}, & m \otimes_B b &\mapsto m \cdot b \\ l &: (M' \otimes_{B_M} B_M)_B \rightarrow {}_A M_B, & m \otimes_{B_M} b &\mapsto m \cdot b \\ j &: {}_{B_M}(B_M \otimes_B Y)_B \rightarrow {}_{B_M}(B_M Y)_B, & b \otimes_B y &\mapsto b \cdot y \\ k &: {}_{B_M}(B_M Y \otimes_B B_M)_{B_M} \rightarrow {}_{B_M}(B_M Y)_{B_M}, & \xi \otimes_B b &\mapsto \xi \cdot b \end{aligned}$$

where $b \in B_M, m \in M, y \in Y$, and $\xi \in B_M Y$. Let $U_{M'}$ be the composition of the $A - B_M$ correspondence isomorphisms

$$\begin{array}{ccccc} X \otimes_A M' & \xrightarrow{1_X \otimes \iota^{-1}} & X \otimes_A M \otimes_B B_M & \xrightarrow{U_M \otimes 1_{B_M}} & M \otimes_B Y \otimes_B B_M \\ & & \searrow^{l^{-1} \otimes 1_Y \otimes 1_{B_M}} & & \\ & & & & \swarrow \\ M' \otimes_{B_M} B_M \otimes_B Y \otimes_B B_M & \xrightarrow{1_{M'} \otimes j \otimes 1_{B_M}} & M' \otimes_{B_M} B_M Y \otimes_B B_M & \xrightarrow{1_{M'} \otimes k} & M' \otimes_{B_M} B_M Y. \end{array}$$

To prove $[_{B_M}(B_M)_B, U_{B_M}] \circ [_A M'_{B_M}, U_{M'}] = [_A M_B, U_M]$ we show that the diagram

$$\begin{array}{ccc} X \otimes_A M' \otimes_{B_M} B_M & \xrightarrow{1_X \otimes l} & X \otimes_A M \\ \downarrow (1_{M'} \otimes U_{B_M})(U_{M'} \otimes 1_{B_M}) & & \downarrow U_M \\ M' \otimes_{B_M} B_M \otimes_B Y & \xrightarrow{l \otimes 1_Y} & M \otimes_B Y \end{array}$$

commutes. Take an elementary tensor $x \otimes_A m \in X \otimes_A M'$. By the Cohen-Hewitt factorization theorem there exist $m' \in M, b' \in B_M$ with $m = m' \cdot b'$. Then we have

$$\begin{aligned} & U_{M'}(x \otimes_A m) \\ &= (1_{M'} \otimes k)(1_{M'} \otimes j \otimes 1_{B_M})(l^{-1} \otimes 1_Y \otimes 1_{B_M})(U_M \otimes 1_{B_M})(1_X \otimes l^{-1})(x \otimes_A m) \\ &= (1_M \otimes k)(1_M \otimes j \otimes 1_{B_M})(l^{-1} \otimes 1_Y \otimes 1_{B_M})(U_M \otimes 1_{B_M})(x \otimes_A m' \otimes_B b') \\ &= (1_M \otimes k)(1_M \otimes j \otimes 1_{B_M})(l^{-1} \otimes 1_Y \otimes 1_{B_M}) \lim_{n \rightarrow \infty} \sum_{i=1}^{N_n} m_i^n \otimes_B y_i^n \otimes_B b', \end{aligned}$$

where $\lim_{n \rightarrow \infty} \sum_{i=1}^{N_n} m_i^n \otimes_B y_i^n = U_M(x \otimes_A m')$. Again by Cohen-Hewitt factorization theorem, there exist $\xi_i^n \in M, c_i^n \in B_M$ such that $m_i^n = \xi_i^n \cdot c_i^n$. Then we may continue our computation as

$$\begin{aligned} &= (1_M \otimes k)(1_M \otimes j \otimes 1_{B_M}) \lim_{n \rightarrow \infty} \sum_{i=1}^{N_n} \xi_i^n \otimes_{B_M} c_i^n \otimes_B y_i^n \otimes_B b' \\ &= (1_M \otimes k) \lim_{n \rightarrow \infty} \sum_{i=1}^{N_n} \xi_i^n \otimes_{B_M} c_i^n \cdot y_i^n \otimes_B b' \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^{N_n} \xi_i^n \otimes_{B_M} c_i^n \cdot y_i^n \cdot b'. \end{aligned}$$

Then, for the elementary tensor $x \otimes_A m \otimes_B b$ of $X \otimes_A M \otimes_B B_M$ we have

$$\begin{aligned} & (l \otimes 1_Y)(1_M \otimes U_{B_M})(U_{M'} \otimes 1_{B_M})(x \otimes_A m \otimes_B b) \\ &= (l \otimes 1_Y)(1_M \otimes U_{B_M}) \lim_{n \rightarrow \infty} \sum_{i=1}^{N_n} \xi_i^n \otimes_{B_M} c_i^n \cdot y_i^n \cdot b' \otimes_B b \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^{N_n} m_i^n \otimes_B y_i^n \cdot b' b \\ &= U_M(x \otimes_A m') b' b \\ &= U_M(1_X \otimes l)(x \otimes_A m \otimes_{B_M} b), \end{aligned}$$

as desired. \square

Corollary 4.9. *Let $[{}_A M_B, U_M]: {}_A X_A \rightarrow {}_B Y_B$ be a morphism in $C^* \text{cor}_{\text{pair}}$. If ${}_A M_B$ is a left-full Hilbert bimodule, then so is the associated correspondence ${}_{\mathcal{O}_X}(M \otimes_B \mathcal{O}_Y)_{\mathcal{O}_Y}$.*

Proof. By Proposition 4.8 we have $[{}_{B_M}(B_M)_B, U_{B_M}] \circ [{}_A M'_{B_M}, U_{M'}] = [{}_A M_B, U_M]$, and thus

$$[{}_{\mathcal{O}_X}(M \otimes_B \mathcal{O}_Y)_{\mathcal{O}_Y}] = [{}_{\mathcal{O}_{B_M Y}}(B_M \otimes_B \mathcal{O}_Y)_{\mathcal{O}_Y}] \circ [{}_{\mathcal{O}_X}(M' \otimes_{B_M} \mathcal{O}_{B_M Y})_{\mathcal{O}_{B_M Y}}].$$

Since ${}_A M'_{B_M}$ is an imprimitivity bimodule, $[{}_A M'_{B_M}, U_{M'}]$ is an isomorphism in $C^* \text{cor}_{\text{pair}}$, and thus, $\mathcal{E}([{}_A M'_{B_M}, U_{M'}]) = [{}_{\mathcal{O}_X}(M' \otimes_{B_M} \mathcal{O}_{B_M Y})_{\mathcal{O}_{B_M Y}}]$ is an isomorphism in $C^* \text{alg}_{\text{cor}}$. This means ${}_{\mathcal{O}_X}(M' \otimes_{B_M} \mathcal{O}_{B_M Y})_{\mathcal{O}_{B_M Y}}$ is an imprimitivity bimodule. We also know by Theorem 4.4 that ${}_{\mathcal{O}_{B_M Y}}(B_M \otimes_B \mathcal{O}_Y)_{\mathcal{O}_Y}$ is a left-full Hilbert bimodule. Hence, ${}_{\mathcal{O}_X}(M \otimes_B \mathcal{O}_Y)_{\mathcal{O}_Y}$ is a left-full Hilbert bimodule. \square

5. Exactness

We denote by $C^* \text{cor}_{\text{pair}}^{\text{reg}}$ the subcategory of $C^* \text{cor}_{\text{pair}}$ where all objects are regular C^* -correspondences. Every morphism in $C^* \text{cor}_{\text{pair}}$ has a kernel; however, not every morphism has a cokernel. We show in this section that every kernel in $C^* \text{cor}_{\text{pair}}^{\text{reg}}$ has a cokernel. This observation leads us to study exactness in the subcategory $C^* \text{cor}_{\text{pair}}^{\text{reg}}$ instead of $C^* \text{cor}_{\text{pair}}$.

To study kernels in $C^* \text{cor}_{\text{pair}}^{\text{reg}}$ we need some understanding of monomorphisms in this category. The following lemma is necessary for this purpose.

Lemma 5.1. *Let $\mu : M \otimes_B N \rightarrow M' \otimes_B N$ be an A - C -correspondence isomorphism where M and M' are A - B -correspondences, and ${}_B N_C$ is a left-full Hilbert bimodule. Then, there exists an isomorphism $\iota : {}_A M_B \rightarrow {}_A M'_B$ such that $\iota \otimes 1_N = \mu$.*

Proof. Since ${}_B N_C$ is a left-full Hilbert bimodule, there exists a C - B -correspondence \tilde{N} and a B - B -correspondence isomorphism

$$j : N \otimes_C \tilde{N} \rightarrow B, \quad n_1 \otimes_C \tilde{n}_2 \mapsto {}_B \langle n_1, n_2 \rangle,$$

where $n_1, n_2 \in N$. Define an isomorphism $\iota : {}_A M_B \rightarrow {}_A M'_B$ by

$$\iota = \xi_{(r, M', B)}(1_{M'} \otimes j)(\mu \otimes 1_N)(1_M \otimes j^{-1})(\xi_{(r, M, B)})^{-1},$$

where $\xi_{(r, M, B)}$ and $\xi_{(r, M', B)}$ are the A - B -correspondence isomorphisms defined as in Remark 3.2. It suffices to use elementary tensors to verify the equality $\iota \otimes 1_N = \mu$. Let $m' \in M'$ and $n_1, n_2, n_3 \in N$. Then we have

$$\begin{aligned} & (\xi_{(r, M', B)} \otimes 1_N)(1_{M'} \otimes j \otimes 1_N)(m' \otimes_B n_1 \otimes_C \tilde{n}_2 \otimes_B n_3) \\ &= (\xi_{(r, M', B)} \otimes 1_N)m' \otimes_B B \langle n_1, n_2 \rangle \otimes_B n_3 \\ &= m'_B \langle n_1, n_2 \rangle \otimes_B n_3 \\ &= m' \otimes_B n_1 \langle n_2, n_3 \rangle_C. \end{aligned}$$

This shows that for any $x \in M \otimes_B N$, and $n, n' \in N$ we have

$$(\xi_{(r, M', B)} \otimes 1_N)(1_{M'} \otimes j \otimes 1_N)(\mu \otimes 1_N \otimes 1_N)(x \otimes_C \tilde{n} \otimes_B n') = \mu(x)\langle n, n' \rangle_C$$

and

$$\mu(\xi_{(r, M, B)} \otimes 1_N)(1_M \otimes j \otimes 1_N)(x \otimes_C \tilde{n} \otimes_B n') = \mu(x)\langle n, n' \rangle_C,$$

as desired. \square

Proposition 5.2. *Let $[_B N_C, U_N]: {}_B Y_B \rightarrow {}_C Z_C$ be a morphism in $C^* \text{cor}_{\text{pair}}^{\text{reg}}$. If ${}_B N_C$ is a left-full Hilbert bimodule, then $[_B N_C, U_N]$ is a monomorphism in $C^* \text{cor}_{\text{pair}}^{\text{reg}}$.*

Proof. Let $[_A M_B, U_M], [_A M'_B, U_{M'}]: {}_A X_A \rightarrow {}_B Y_B$ be morphisms in $C^* \text{cor}_{\text{pair}}^{\text{reg}}$ satisfying

$$[_B N_C, U_N] \circ [_A M_B, U_M] = [_B N_C, U_N] \circ [_A M'_B, U_{M'}].$$

Then, there exists an isomorphism $\mu : M \otimes_B N \rightarrow M' \otimes_B N$ with the commutative diagram

$$\begin{array}{ccc} X \otimes_A M \otimes_B N & \xrightarrow{1_X \otimes \mu} & X \otimes_A M' \otimes_B N \\ \downarrow (1_M \otimes U_N)(U_M \otimes 1_N) & & \downarrow (1_{M'} \otimes U_N)(U_{M'} \otimes 1_N) \\ M \otimes_B N \otimes_C Z & \xrightarrow{\mu \otimes 1_Z} & M' \otimes_B N \otimes_C Z. \end{array}$$

Since ${}_B N_C$ is a left-full Hilbert bimodule, by Lemma 5.1, there exists an isomorphism $\iota : {}_A M_B \rightarrow {}_A M'_B$ such that $\mu = \iota \otimes 1_N$. We aim to show that the diagram

$$\begin{array}{ccc} X \otimes_A M & \xrightarrow{1_X \otimes \iota} & X \otimes_A M' \\ U_M \downarrow & & \downarrow U_{M'} \\ M \otimes_B Y & \xrightarrow{\iota \otimes 1_Y} & M' \otimes_B Y \end{array}$$

commutes.

By the first diagram above, we have

$$\begin{aligned} (1_{M'} \otimes U_N)(U_{M'} \otimes 1_N)(1_X \otimes \iota \otimes 1_N) &= (\iota \otimes 1_N \otimes 1_Z)(1_M \otimes U_N)(U_M \otimes 1_N) \\ &= (1_{M'} \otimes U_N)(\iota \otimes 1_Y \otimes 1_N)(U_M \otimes 1_N), \end{aligned}$$

which implies the equality

$$(U_{M'} \otimes 1_N)(1_X \otimes \iota \otimes 1_N) = (\iota \otimes 1_Y \otimes 1_N)(U_M \otimes 1_N).$$

Since ${}_B N_C$ is an injective correspondence, by Lemma 2.2, we have

$$U_{M'}(1_X \otimes \iota) = (\iota \otimes 1_Y)U_M,$$

completing the proof. \square

Remark 5.3. Let $U_M : X \otimes_A M \rightarrow M \otimes_B Y$ be an A - B -correspondence isomorphism. We know by Lemma 4.7 that $\text{Ker } \varphi_M$ is a positive X -invariant ideal of A . And thus we may view KX as a C^* -correspondence over K , where K denotes the ideal $\text{Ker } \varphi_M$. Then, as described in Lemma 4.2, $[_K K_A, U_K]: {}_K KX_K \rightarrow {}_A X_A$

is a morphism in $C^* \text{cor}_{\text{pair}}^{\text{reg}}$ where $U_K(kx \otimes_K k') = k \otimes_A xk'$, for any $k, k' \in K$ and $x \in X$.

We are now ready to determine kernels in $C^* \text{cor}_{\text{pair}}^{\text{reg}}$.

Theorem 5.4. *Let $[{}_A M_B, U_M]: {}_A X_A \rightarrow {}_B Y_B$ be a morphism in $C^* \text{cor}_{\text{pair}}^{\text{reg}}$. Let K denote the kernel of the homomorphism $\varphi_M : A \rightarrow \mathcal{K}(M)$. Then, the object ${}_K(KX)_K$ paired with the morphism $[{}_K K_A, U_K] : {}_K KX_K \rightarrow {}_A X_A$ is a kernel of $[{}_A M_B, U_M]$.*

Proof. We must show the following:

- (1) $[{}_A M_B, U_M] \circ [{}_K K_A, U_K] = [0, 0_{KX,Y}]$; and
- (2) assume that $[{}_C N_A, U_N]: {}_C Z_C \rightarrow {}_A X_A$ is a morphism in $C^* \text{cor}_{\text{pair}}^{\text{reg}}$ satisfying the equality $[{}_A M_B, U_M] \circ [{}_C N_A, U_N] = [0, 0_{Z,Y}]$. Then, there exists a unique morphism $[{}_C T_K, U_T]: {}_C Z_C \rightarrow {}_K KX_K$ such that the equality $[{}_K K_A, U_K] \circ [{}_C T_K, U_T] = [{}_C N_A, U_N]$ holds.

Item (1) is folklore. For (2), notice that since $N \otimes_A M \cong 0$ we have $\langle N, N \rangle_A \subset K$. Thus we may view ${}_C N_A$ as a C - K -correspondence, which we denote by N' . Moreover, we have the isomorphisms

$$\iota : {}_C N \otimes_A K_K \rightarrow {}_C N'_K \quad n \otimes_A k \mapsto n \cdot k$$

and

$$j : {}_C N' \otimes_K K_A \rightarrow {}_C N_A \quad n \otimes_K k \mapsto n \cdot k$$

for $n \in N, k \in K$. Now let $U_{N'}$ be the C - K -correspondence isomorphism

$$\begin{array}{ccccc} {}_C Z \otimes_C N'_K & \xrightarrow{1_Z \otimes \iota^{-1}} & {}_C Z \otimes_C N \otimes_A K_K & \xrightarrow{U_N \otimes 1_K} & {}_C N \otimes_A X \otimes_A K_K \\ & & \searrow^{j^{-1} \otimes 1_X \otimes 1_K} & & \\ {}_C N' \otimes_K K \otimes_A X \otimes_A K_K & \xrightarrow{1_{N'} \otimes \xi_l \otimes 1_K} & {}_C N' \otimes_K KX \otimes_A K_K & \xrightarrow{1_{N'} \otimes \xi_r} & {}_C N' \otimes_K KX_K, \end{array}$$

where ξ_l is the K - A -correspondence isomorphism $\xi_{(l,X,K)} : K \otimes_A X \rightarrow KX$, and ξ_r is the K - K -correspondence isomorphism $\xi_{(r,KX,K)} : KX \otimes_A K \rightarrow KX$, i.e.,

$$U_{N'} := [1_{N'} \otimes \xi_r][1_{N'} \otimes \xi_l \otimes 1_K][j^{-1} \otimes 1_X \otimes 1_K][U_N \otimes 1_K][1_Z \otimes \iota^{-1}].$$

We show that $[{}_C(N' \otimes_K K)_A, (1_{N'} \otimes U_K)(U_{N'} \otimes 1_K)] = [{}_C N_A, U_N]$, i.e., the diagram

$$\begin{array}{ccc} Z \otimes_C N' \otimes_K K & \xrightarrow{1_Z \otimes j} & Z \otimes_C N \\ \downarrow (1_{N'} \otimes U_K)(U_{N'} \otimes 1_K) & & \downarrow U_N \\ N' \otimes_K K \otimes_A X & \xrightarrow{j \otimes 1_X} & N \otimes_A X \end{array}$$

commutes. Consider a tensor $n \otimes_A x \otimes_A k_1 \otimes_K k_2$ of ${}_A(N \otimes_A X \otimes_A K \otimes_K K)_A$. By the Cohen-Hewitt factorization theorem we have $n = n' \cdot k'$ for some $n' \in N$ and $k' \in \langle N, N \rangle_A \subset K$. Then,

$$\begin{aligned} (j \otimes 1_X)(1_{N'} \otimes U_K)(1_{N'} \otimes \xi_r \otimes 1_K)(1_{N'} \otimes \xi_l \otimes 1_K \otimes 1_K)(j^{-1} \otimes 1_X \otimes 1_K \otimes 1_K)(n \otimes_A x \otimes_A k_1 \otimes_K k_2) \\ = (j \otimes 1_X)(1_{N'} \otimes U_K)(1_{N'} \otimes \xi_r \otimes 1_K)(n' \otimes_K k'x \otimes_A k_1 \otimes_K k_2) \end{aligned}$$

$$\begin{aligned}
 &= (j \otimes 1_X)(1_{N'} \otimes U_K)(n' \otimes_K k' x k_1 \otimes_K k_2) \\
 &= (j \otimes 1_X)(n' \otimes_K k' \otimes_A x k_1 k_2) \\
 &= n \otimes_A x k_1 k_2.
 \end{aligned}$$

On the other hand, it is not hard to see that

$$U_N(1_Z \otimes j)(1_Z \otimes \iota \otimes 1_K)(U_N^{-1} \otimes 1_K \otimes 1_K)(n \otimes_A x \otimes_A k_1 \otimes_K k_2) = n \otimes_A x k_1 k_2.$$

Uniqueness of the morphism $[{}_C N'_K, U_{N'}]$ follows from Proposition 5.2, since ${}_K K_A$ is a left-full Hilbert bimodule. \square

We next study cokernels in $\mathbf{C}^* \text{cor}_{\text{pair}}^{\text{reg}}$.

Lemma 5.5. *Let ${}_C X_B$ and ${}_C Y_B$ be two C^* -correspondences, and let ${}_A C_C$ be the C^* -correspondence where the left action is determined by the surjective map $\pi : A \rightarrow C$. If there exists an A - B -correspondence isomorphism $U : C \otimes_C X \rightarrow C \otimes_C Y$, then there exists an isomorphism $V : {}_C X_B \rightarrow {}_C Y_B$ such that $1_C \otimes V = U$.*

Proof. Consider the natural A - B -correspondence isomorphisms

$$\begin{aligned}
 \iota_{C,X} : C \otimes_C X &\rightarrow X & c \otimes_C x &\mapsto c \cdot x \\
 \iota_{C,Y} : C \otimes_C Y &\rightarrow Y & c \otimes_C y &\mapsto c \cdot y
 \end{aligned}$$

[EKQ20, Lemma 3.3] tells us that the map $\iota_{C,Y} \circ U \circ \iota_{C,X}^{-1} : {}_A X_B \rightarrow {}_A Y_B$ preserves the left C -module structure and thus, provides an isomorphism ${}_C X_B \rightarrow {}_C Y_B$. We observe that $1_C \otimes \iota_{C,Y} U \iota_{C,X}^{-1} = U$: let $c, c' \in C, x \in X$. Notice that since $U(c' \otimes_C x) = \lim_{n \rightarrow \infty} \sum_{i=1}^{N_n} c_i^n \otimes_C y_i^n$ for $c_i^n \in C, y_i^n \in Y$, we have

$$\begin{aligned}
 (1_C \otimes \iota_{C,Y})(1_C \otimes U)(c \otimes_C c' \otimes_C x) &= c \otimes_C \lim_{n \rightarrow \infty} \sum_{i=1}^{N_n} c_i^n \cdot y_i^n \\
 &= \lim_{n \rightarrow \infty} \sum_{i=1}^{N_n} c c_i^n \otimes_C y_i^n \\
 &= U(1_C \otimes \iota_{C,X})(c \otimes_C c' \otimes_C x),
 \end{aligned}$$

as desired. \square

Proposition 5.6. *Let ${}_A X_A$ be a regular C^* -correspondence and let I be an X -invariant ideal of A . Then, $[{}_A(A/I)_{A/I}, U_{A/I}] : {}_A X_A \rightarrow {}_{A/I}(X/XI)_{A/I}$ is an epimorphism in $\mathbf{C}^* \text{cor}_{\text{pair}}^{\text{reg}}$.*

Proof. Assume that there exist two morphisms $[{}_{A/I} M_B, U_M], [{}_{A/I} N_B, U_N] : {}_{A/I}(X/XI)_{A/I} \rightarrow {}_B Y_B$ in $\mathbf{C}^* \text{cor}_{\text{pair}}$ such that

$$[{}_{A/I} M_B, U_M] \circ [{}_A(A/I)_{A/I}, U_{A/I}] = [{}_{A/I} N_B, U_N] \circ [{}_A(A/I)_{A/I}, U_{A/I}].$$

Then, there exists an A - B -correspondence isomorphism

$$\mu : A/I \otimes_{A/I} M \rightarrow A/I \otimes_{A/I} N$$

making the diagram

$$\begin{array}{ccc} X \otimes_A A/I \otimes_{A/I} M & \xrightarrow{1_X \otimes \mu} & X \otimes_A A/I \otimes_{A/I} N \\ \downarrow (1_{A/I} \otimes U_M)(U_{A/I} \otimes 1_M) & & \downarrow (1_{A/I} \otimes U_N)(U_{A/I} \otimes 1_N) \\ A/I \otimes_{A/I} M \otimes_B Y & \xrightarrow{\mu \otimes 1_Y} & A/I \otimes_{A/I} N \otimes_B Y \end{array}$$

commute.

Since the C^* -correspondence ${}_A(A/I)_{A/I}$ comes from the surjective homomorphism $A \rightarrow A/I$, by Lemma 5.5, there exists an isomorphism $\xi : {}_{A/I}M_B \rightarrow {}_{A/I}N_B$ such that $\mu = 1_{A/I} \otimes \xi$. Then, by the diagram above, we have

$$\begin{aligned} (1_{A/I} \otimes \xi \otimes 1_Y)(1_{A/I} \otimes U_M)(U_{A/I} \otimes 1_M) &= (1_{A/I} \otimes U_N)(U_{A/I} \otimes 1_N)(1_X \otimes 1_{A/I} \otimes \xi) \\ &= (1_{A/I} \otimes U_N)(1_{A/I} \otimes 1_{X/XI} \otimes \xi)(U_{A/I} \otimes 1_M), \end{aligned}$$

which means $1_{A/I} \otimes (\xi \otimes 1_Y)U_M = 1_{A/I} \otimes U_N(1_{X/XI} \otimes \xi)$. Since $[{}_A(A/I)_{A/I}]$ is an epimorphism in $C^*\text{alg}_{\text{cor}}$, we may now conclude the equality $(\xi \otimes 1_Y)U_M = U_N(1_{X/XI} \otimes \xi)$, which implies $[{}_A M_B, U_M] = [{}_A N_B, U_N]$. \square

Theorem 5.7. *Let $[{}_A M_B, U_M] : {}_A X_A \rightarrow {}_B Y_B$ be a morphism in $C^*\text{cor}_{\text{pair}}^{\text{reg}}$. If B_M is a Y -invariant ideal of B , then a cokernel of $[{}_A M_B, U_M]$ is $[{}_B(B/B_M)_{B/B_M}, U_{B/B_M}] : {}_B Y_B \rightarrow {}_B(B/B_M)(Y/YB_M)_{B/B_M}$.*

Proof. We must show the following:

- (1) $[{}_B(B/B_M)_{B/B_M}, U_{B/B_M}] \circ [{}_A M_B, U_M] = [0, 0_{X,Y/YB_M}]$; and
- (2) assume $[{}_B N_C, U_N] : {}_B Y_B \rightarrow {}_C Z_C$ is a morphism in $C^*\text{cor}_{\text{pair}}^{\text{reg}}$ satisfying the equality $[{}_B N_C, U_N] \circ [{}_A M_B, U_M] = [0, 0_{X,Z}]$. Then, there exists a unique morphism $[{}_B(B/B_M)_{B/B_M}, U_{B/B_M}] : {}_B(B/B_M)(Y/YB_M)_{B/B_M} \rightarrow {}_C Z_C$ satisfying the equality $[{}_B N_C, U_N] = [{}_B(B/B_M)_{B/B_M}, U_{B/B_M}] \circ [{}_A M_B, U_M]$.

The first item is easy to verify. Let $[{}_B N_C, U_N] : {}_B Y_B \rightarrow {}_C Z_C$ be a morphism described as in the second item. Then ${}_B N_C$ can be viewed as a B/B_M - C -correspondence [EKQ20, Lemma 3.3], which we denote by N' . Now, let ξ be the C^* -correspondence isomorphism ${}_B(B/B_M \otimes_{B/B_M} N')_C \rightarrow {}_B N_C$; and consider the B - C -correspondence isomorphism

$$(\xi^{-1} \otimes 1_Z)U_N(1_Y \otimes \xi)(U_{B/B_M}^{-1} \otimes 1_{N'}) : {}_B(B/B_M \otimes_{B/B_M} Y/YB_M \otimes_{B/B_M} N')_C \rightarrow {}_B(B/B_M \otimes_{B/B_M} N')_C \otimes_C Z.$$

By Lemma 5.5 there exists an isomorphism

$$U_{N'} : {}_B(B/B_M)(Y/YB_M \otimes_{B/B_M} N')_C \rightarrow {}_B(B/B_M)(N' \otimes_C Z)_C$$

such that $1_{B/B_M} \otimes U_{N'} = (\xi^{-1} \otimes 1_Z)U_N(1_Y \otimes \xi)(U_{B/B_M}^{-1} \otimes 1_{N'})$. One can now see that the diagram

$$\begin{array}{ccc} Y \otimes_B B/B_M \otimes_{B/B_M} N' & \xrightarrow{1_Y \otimes \xi} & Y \otimes_B N \\ \downarrow (1_{B/B_M} \otimes U_{N'})(U_{B/B_M} \otimes 1_{N'}) & & \downarrow U_N \\ B/B_M \otimes_{B/B_M} N' \otimes_C Z & \xrightarrow{\xi \otimes 1_Z} & N \otimes_C Z \end{array}$$

commutes. The uniqueness of $[_{B/B_M}N'_C, U_{N'}]$ follows from Proposition 5.6. \square

Let $[_AX_B]$ be a morphism in $C^*\text{alg}_{\text{cor}}$. A kernel of $[_AX_B]$ is the pair $(K, [{}_K K_A])$, where K denotes the kernel of $\varphi_X : A \rightarrow \mathcal{L}(X)$ [EKQ20, Theorem 3.11]. A cokernel of $[_AX_B]$ is the pair $(B/B_X, [{}_B(B/B_X)_{B/B_X}])$ [EKQ20, Corollary 3.12].

Definition 5.8. A sequence $0 \rightarrow A \xrightarrow{[_AX_B]} B \xrightarrow{[_BY_C]} C \rightarrow 0$ in $C^*\text{alg}_{\text{cor}}$ is exact if the pair $(A, [_AX_B])$ is a kernel of $[_BY_C]$ and the pair $(B, [_BY_C])$ is a cokernel of $[_AX_B]$.

Proposition 5.9. A sequence $0 \rightarrow A \xrightarrow{[_AX_B]} B \xrightarrow{[_BY_C]} C \rightarrow 0$ in $C^*\text{alg}_{\text{cor}}$ is exact if and only if the following three hold.

- (1) $_AX_B$ is a left-full Hilbert bimodule;
- (2) $B_X = K$, where K denotes the kernel of $\varphi_Y : B \rightarrow \mathcal{L}(Y)$.
- (3) Hilbert C -module Y viewed as a B/K - C -correspondence $_{B/K}Y'_C$ is an imprimitivity bimodule.

Proof. Assume we have (1)-(3). In $C^*\text{alg}_{\text{cor}}$, we know that kernel of $[_BY_C]$ is the pair $(K, [{}_K K_B])$, where K denotes the kernel of $\varphi_Y : B \rightarrow \mathcal{L}(Y)$. On the other hand, item (2) implies that $[_BY_C] \circ [_AX_B] = [_A 0_C]$. Then, by the universal property of kernels there exists a morphism from A to K which $[_AX_B]$ factors through. As shown in [EKQ20, Theorem 3.9] this unique morphism is $[_AX'_K]$ where X is just X' viewed as an A - K -correspondence. Since $_AX'_K$ is an imprimitivity bimodule we have that $[_AX'_K]$ is an isomorphism in $C^*\text{alg}_{\text{cor}}$. It remains to show that $[_BY_C]$ is a cokernel of $[_AX_B]$. We know that a cokernel of $[_AX_B]$ is $[_B B/K_{B/K}]$, and since $[_BY_C] \circ [_AX_B] = [_A 0_C]$, by the universal property of cokernels there exists a unique morphism which $[_BY_C]$ factors through. As shown in [EKQ20, Proposition 3.11], this unique morphism is $[_{B/K}Y'_C]$, which is an isomorphism in $C^*\text{alg}_{\text{cor}}$ by item (3).

For the other direction, assume $[_AX_B]$ is a kernel of $[_BY_C]$ and $[_BY_C]$ is a cokernel of $[_AX_B]$. Since $[_AX_B]$ is a kernel of $[_BY_C]$, the correspondence $_AX_K$ in the kernel factorization $_AX_B \cong _AX \otimes_K K_B$ must be an imprimitivity bimodule, which means $_AX_B$ is a left-full Hilbert bimodule, giving us item (1). Moreover, $_AX_K$ being an imprimitivity bimodule implies that $B_X = K$, which proves item (2). Since $[_BY_C]$ is a cokernel of $[_AX_B]$, the correspondence $_{B/K}Y'_C$ in the cokernel factorization $_B B/K \otimes_{B/K} Y'_C \cong _BY_C$ must be an imprimitivity bimodule, concluding the proof. \square

Definition 5.10. A sequence

$$0 \rightarrow _AX_A \xrightarrow{[_AM_B, U_M]} _BY_B \xrightarrow{[_BN_C, U_N]} _CZ_C \rightarrow 0$$

in $C^*\text{cor}_{\text{pair}}^{\text{reg}}$ is called exact if the pair $(_AX_A, [_AM_B, U_M])$ is a kernel of the morphism $[_BN_C, U_N] : _BY_B \rightarrow _CZ_C$; and the pair $(_BY_B, [_BN_C, U_N])$ is a cokernel of the morphism $[_AM_B, U_M] : _AX_A \rightarrow _BY_B$.

Note that $[{}_A M_B, U_M]$ being a kernel of $[{}_B N_C, U_N]$ in the sequence above implies that $B_M = \text{Ker } \varphi_N$. Then, Lemma 4.7 allows us to conclude that B_M is a Y -invariant ideal of B , and thus, cokernel of $[{}_A M_B, U_M]$ exists.

Corollary 5.11. *A sequence*

$$0 \rightarrow {}_A X_A \xrightarrow{[{}_A M_B, U_M]} {}_B Y_B \xrightarrow{[{}_B N_C, U_N]} {}_C Z_C \rightarrow 0$$

is exact in $\mathbf{C}^* \text{cor}_{\text{pair}}^{\text{reg}}$ if and only if the following holds.

- (1) ${}_A M_B$ is a left-full Hilbert bimodule;
- (2) $B_M = K$, where K denotes the kernel of $\varphi_N : B \rightarrow \mathcal{L}(N)$;
- (3) Hilbert C -module N viewed as a B/K - C -correspondence ${}_{B/K} N'_C$ is an imprimitivity bimodule.

We omit the proof of Corollary 5.11 as it is very similar to the proof of Proposition 5.9. We would like to point out that one must use Theorems 5.4 and 5.7 for this proof.

Theorem 5.12. *The restriction of the functor \mathcal{E} to the category $\mathbf{C}^* \text{cor}_{\text{pair}}^{\text{reg}}$ is exact.*

Proof. Let the sequence

$$0 \rightarrow {}_A X_A \xrightarrow{[{}_A M_B, U_M]} {}_B Y_B \xrightarrow{[{}_B N_C, U_N]} {}_C Z_C \rightarrow 0$$

in $\mathbf{C}^* \text{cor}_{\text{pair}}^{\text{reg}}$ be exact. Denote $\text{Ker } \varphi_N$ by K . We know that ${}_A M_B$ is a left-full Hilbert bimodule, the correspondence ${}_{B/K} N'_C$ is an imprimitivity bimodule, and we have the equality $B_M = K$. The functor \mathcal{E} maps this sequence to

$$0 \rightarrow \mathcal{O}_X \xrightarrow{[\mathcal{O}_X(M \otimes_B \mathcal{O}_Y)_{\mathcal{O}_Y}]} \mathcal{O}_Y \xrightarrow{[\mathcal{O}_Y(N \otimes_C \mathcal{O}_Z)_{\mathcal{O}_Z}]} \mathcal{O}_Z \rightarrow 0.$$

By Corollary 4.9, we have that ${}_{\mathcal{O}_X}(M \otimes_B \mathcal{O}_Y)_{\mathcal{O}_Y}$ is a left-full Hilbert bimodule. Since $[{}_{B/K} N'_C, U_{N'}] : {}_{B/K} Y / Y K_{B/K} \rightarrow {}_C Z_C$ is an isomorphism in $\mathbf{C}^* \text{cor}_{\text{pair}}^{\text{reg}}$, we have that

$$\mathcal{E}([{}_{B/K} N'_C, U_{N'}]) = [{}_{\mathcal{O}_Y / YK}(N' \otimes_C \mathcal{O}_Z)_{\mathcal{O}_Z}]$$

is an isomorphism in $\mathbf{C}^* \text{alg}_{\text{cor}}$, and thus ${}_{\mathcal{O}_Y / YK}(N' \otimes_C \mathcal{O}_Z)_{\mathcal{O}_Z}$ is an imprimitivity bimodule. It remains to prove that $\langle M \otimes_B \mathcal{O}_Y, M \otimes_B \mathcal{O}_Y \rangle_{\mathcal{O}_Y} = \text{Ker } \sigma$, where $\sigma : \mathcal{O}_Y \rightarrow \mathcal{K}(N \otimes_C \mathcal{O}_Z)$ is the left action homomorphism associated to the correspondence ${}_{\mathcal{O}_Y}(N \otimes_C \mathcal{O}_Z)_{\mathcal{O}_Z}$. Let (Y, t) denote the universal covariant representation of ${}_B Y_B$. Then, by Proposition 3.5, we have

$$\langle M \otimes_B \mathcal{O}_Y, M \otimes_B \mathcal{O}_Y \rangle_{\mathcal{O}_Y} = \langle \mathcal{O}_Y, K \cdot \mathcal{O}_Y \rangle = \langle Y(K) \rangle = \text{Ker } \sigma,$$

as desired. \square

Example 5.13. By using Theorem 5.12 we can easily see Theorem 4.6 and Theorem 4.4 for the case when ${}_A X_A$ is a regular correspondence: let I be an X -invariant ideal of A . Then, the sequence

$$0 \rightarrow {}_I X_I \xrightarrow{[{}_I I_A, U_I]} {}_A X_A \xrightarrow{[{}_A A / I_{A/I}, U_{A/I}]} {}_{A/I} X / X I_{A/I} \rightarrow 0$$

is exact in $C^*\text{cor}_{\text{pair}}^{\text{reg}}$. And thus, the sequence

$$0 \rightarrow \mathcal{O}_{IX} \xrightarrow{[I \otimes_A \mathcal{O}_X]} \mathcal{O}_X \xrightarrow{[A/I \otimes_{A/I} \mathcal{O}_{X/XI}]} \mathcal{O}_{X/XI} \rightarrow 0$$

is exact in $C^*\text{alg}_{\text{cor}}$. This implies by Proposition 5.9 that ${}_{\mathcal{O}_{IX}}(I \otimes_A \mathcal{O}_X)_{\mathcal{O}_X}$ is a left-full Hilbert bimodule, which means ${}_{\mathcal{O}_{IX}}(I \otimes_A \mathcal{O}_X)_{\langle Y(I) \rangle}$ is an imprimitivity bimodule, where (Y, t) is the universal covariant representation of ${}_A X_A$. Consequently, we have $\mathcal{O}_{IX} \cong \mathcal{K}(I \otimes_A \mathcal{O}_X) \cong Y(I)\mathcal{O}_X Y(I)$. On the other hand, again by Proposition 5.9, we know that ${}_{\mathcal{O}_X/\langle Y(I) \rangle}(A/I \otimes_{A/I} \mathcal{O}_{X/XI})_{\mathcal{O}_{X/XI}}$ is an imprimitivity bimodule. This allows us to conclude the isomorphism $\mathcal{O}_X/\langle Y(I) \rangle \cong \mathcal{K}(A/I \otimes_{A/I} \mathcal{O}_{X/XI}) \cong \mathcal{O}_{X/XI}$.

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This paper is available via <http://nyjm.albany.edu/j/2026/32-31.html>.