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On S-J-Noetherian rings

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ABSTRACT. Let R be a commutative ring with identity, $S \subseteq R$ be a multiplicative set and J be an ideal of R. In this paper, we introduce the concept of S-J-Noetherian rings, which generalizes both J-Noetherian rings and S-Noetherian rings. We study several properties and characterizations of this new class of rings. For instance, we prove Cohen's-type theorem for S-J-Noetherian rings. Among other results, we establish the existence of S-primary decomposition in S-J-Noetherian rings as a generalization of classical Lasker-Noether theorem.

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1. Introduction

Throughout the paper, let R be a commutative ring with identity, $S \subseteq R$ be a multiplicative set, and J be a fixed ideal of R. For an ideal I of R, we denote $\overline{S} = \{s + I \mid s \in S\}$ which is a multiplicative closed subset of R/I. The Noetherian property of rings plays a crucial role in areas such as commutative algebra and algebraic geometry. Given the significance of Noetherian rings, numerous authors attempted to generalize the concept of Noetherian rings (see [2], [3], [7], [8], [9], [12], and [13]). As one of its crucial generalizations, Anderson and Dumitrescu [3] introduced the concept of S-Noetherian rings. An ideal I of R is S-finite if there exists an element $s \in S$ and a finitely generated ideal F of R such that $sI \subseteq F \subseteq I$. A ring R is called S-Noetherian if every ideal of R is S-finite. Recently, Alhazmy et al. [2] introduced the concept of J-Noetherian rings as a generalization of Noetherian ring. An ideal I of R is called a J-ideal if $I \nsubseteq J$ and R is said to be J-Noetherian if every J-ideal is finitely generated. A

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particular interesting case occurs when $J = Nil(R)$, the ideal consisting of all nilpotent elements of R. In this situation, a J-Noetherian ring is referred to as a Nonnil-Noetherian ring, which was first introduced and studied by Badawi in [5]. Furthermore, when $J = J(R)$, the Jacobson radical of R, the J-Noetherian ring is termed a non-J-Noetherian ring. This class of rings was first introduced by Dabbabi [7] et al. in 2024, where they characterized various properties of non-J-Noetherian rings.

The primary objective of this paper is to introduce and study the notion of S-J-Noetherian rings. We present an example of an S-J-Noetherian ring which is not an S-Noetherian ring (see Example 2.4). We generalize various properties and characterizations of both J-Noetherian and S-Noetherian rings to this new class of rings. For instance, we establish Cohen-type theorem for S-J-Noetherian rings and prove that the polynomial ring $R[X]$ is S-J-Noetherian if and only if it is S-Noetherian. Also, we show that the quotient of an S-J-Noetherian ring is an $\overline{S}$-Noetherian ring (see Proposition 2.11). Moreover, we provide necessary and sufficient conditions for an S-J-Noetherian ring to belong to the class of S-Noetherian rings (see Theorem 2.7 and 2.18). In [14, Theorem 2.10], among the other result, Singh et al. generalized the classical Lasker-Noether theorem for S-Noetherian modules. We end the paper by extending the classical Lasker-Noether theorem for the class of S-J-Noetherian rings (see Theorem 2.22).

2. Main Results

We begin by introducing the concept of S-J-Noetherian rings.

Definition 2.1. *Let R be a ring, $S \subseteq R$ be a multiplicative set, and J an ideal of R. An ideal I of R is said to be a J-ideal if $I \nsubseteq J$. We say that R is an S-J-Noetherian ring if each J-ideal of R is S-finite.*

It is evident that every J-Noetherian ring is an S-J-Noetherian ring when $S = \{1\}$. However, the following example illustrates that the converse is not true in general.

Example 2.2. *Consider the ring $R = \mathcal{F}[X_1, X_2, \dots]$, where $\mathcal{F}$ is a field, and let $J = (0)$. Define the ideal $I = (X_1, \dots, X_n, \dots)$. Clearly, I is a J-ideal but is not finitely generated. Hence, R is not a J-Noetherian ring. Now, let $S = R \setminus \{0\}$ be the multiplicative closed subset of R. Let K be a nonzero proper ideal of R. Evidently, K is J-ideal and $K \cap S \neq \emptyset$. Therefore, by* [3, Proposition 2(a)]*, K is S-finite. Hence, R is S-J-Noetherian.*

Cohen's theorem is the classical result which states that a ring is Noetherian if all its prime ideals are finitely generated. Now, we extend this result for S-J-Noetherian rings.

Theorem 2.3. *A ring R is S-J-Noetherian if and only if its prime J-ideals (disjoint from S) are S-finite.*

Proof. If R is S-J-Noetherian, then it is obvious that all prime J-ideals of R are S-finite. Now, suppose that all prime J-ideals (disjoint from S) of R are S-finite and assume that R is not S-J-Noetherian. Therefore, the set $\mathcal{F}$ of all J-ideals that are non-S-finite is a non-empty set which is ordered by the inclusion. By Zorn's lemma, choose P maximal in $\mathcal{F}$. This implies P is not a S-finite and so $P \cap S = \emptyset$. We show that P is a prime ideal of R. This makes P a J-prime ideal (disjoint from S) that is S-finite, which is a contradiction to the fact that $P \in \mathcal{F}$. Suppose there exists $a, b \in R \backslash P$ such that $ab \in P$. If $P + aR \subseteq J$, $P \subseteq J$, a contradiction, as P is J-ideal. Therefore, $P + aR$ is J-ideal. Since $P \subsetneq P + aR$, it follows that $P + aR$ is S-finite since P is a maximal element of $\mathcal{F}$. Then, there exist $s \in S, \alpha_1, \dots, \alpha_n \in P$ and $x_1, \dots x_n \in R$ such that $s(P + aR) \subseteq (\alpha_1 + ax_1, \dots, \alpha_n + ax_n) \subseteq P + aR$. Consider the ideal $Q = (P : a) = \{x \in R \mid ax \in P\}$. Evidently, Q is J-ideal and $P \subsetneq Q$ as $b \in Q \setminus P$. By the maximality of P, Q is an S-finite ideal. Then there exist $t \in S$ and $\beta_1, \dots, \beta_k \in Q$ such that $tQ \subseteq (\beta_1, \dots, \beta_k) \subseteq Q$. Let $x \in P$. Then $sx \in s(P + aR) \subseteq (\alpha_1 + ax_1, \dots, \alpha_n + ax_n)$, and so there exist $u_1, \dots, u_n \in R$ such that $sx = u_1(\alpha_1 + ax_1) + \cdots + u_n(\alpha_n + ax_n) = u_1\alpha_1 + \cdots + u_n a_n + a(u_1x_1 + \cdots + u_nx_n)$.So $a(u_1x_1 + \cdots + u_nx_n) = sx - (u_1\alpha_1 + \cdots + u_n\alpha_n) \in P$. Then $u_1x_1 + \cdots + u_nx_n \in (P : a) = Q$. Therefore, we can find $w_1, \dots, w_k \in R$ such that $t(u_1x_1 + \dots + u_nx_n) = w_1\beta_1 + \cdots + w_k\beta_k$, which states that

$$\begin{aligned} stx &= t(u_1\alpha_1 + \cdots + u_n a_n) + at(u_1x_1 + \cdots + u_nx_n) \\ &= t(u_1\alpha_1 + \cdots + u_n a_n) + a(w_1\beta_1 + \cdots + w_k\beta_k). \end{aligned}$$

Hence, we obtain $uP \subseteq (\alpha_1, \dots, \alpha_n, a\beta_1, \dots, a\beta_k) \subseteq P$, where $u = st \in S$, which means that P is S-finite. This contradicts to the choice of P. Thus, R is S-J-Noetherian. □

Every S-Noetherian ring is clearly an S-J-Noetherian ring. However, an S-J-Noetherian ring need not be an S-Noetherian ring. For this, consider the following example.

Example 2.4. *Consider a ring $R_1 = \mathcal{F}[X_1, \dots, X_n, \dots]$, where $\mathcal{F}$ is a field, and $I = (X_i^2; i \in \mathbb{N})$ be an ideal of R_1. Let $R = R_1/I$. Consider the prime ideal $P = (X_i; i \in \mathbb{N})$ of R_1. Note that any prime ideal of the ring R contains P/I. Then the unique minimal prime ideal of R is P/I. Take $J = P/I$ and $S = R \setminus (P/I)$ is a multiplicative subset of R. Any J-prime ideal P' of R contains properly P/I, and then $P' \cap S \neq \emptyset$. By* [3, Proposition 2(a)]*, P' is S-finite. By Theorem* 2.3*, R is an S-J-Noetherian ring. Next, our aim is to show that R is not a S-Noetherian ring. Suppose the ideal P/I is S-finite. There exist $\bar{s} \in S$ and $i_1, \dots, i_n \in \mathbb{N}$ such that $\bar{s}(P/I) \subseteq (\overline{X_{i_1}}, \dots, \overline{X_{i_n}}) \subseteq (P/I)$. The polynomial $\bar{s}$ of R uses a finite number of variables $X_{j_1}, \dots, X_{j_m}$ and its constant term $d \neq 0$. Let $k \in \mathbb{N} \backslash \{i_1, \dots, i_n, j_1, \dots, j_m\}$. Then, $\bar{s}\overline{X_k} = \bar{f}_1\overline{X_{i_1}} + \cdots + \bar{f}_n\overline{X_{i_n}}$, where $\bar{f}_1, \dots, \bar{f}_n \in R$. Thus, $sX_k - f_1X_{i_1} - \cdots - f_nX_{i_n} \in I$. This implies that $X_{i_1} = \cdots = X_{i_n} = X_{j_1} = \cdots = X_{j_m} = 0$, we obtain $dX_k \in \left(X_i^2 \mid i \in \mathbb{N} \setminus \{i_1, \dots, i_n, j_1, \dots, j_m\}\right)$. This is a contradiction.*

Examples 2.2 and 2.4 demonstrate that the concept of S-J-Noetherian rings is a proper generalization of both the J-Noetherian rings and S-Noetherian rings.

Recall [6], let E be a family of ideals of a ring R. An element $I \in E$ is said to be an S-*maximal element* of E if there exists an $s \in S$ such that for each $J \in E$, if $I \subseteq J$, then $sJ \subseteq I$. Also, a chain of ideals $(I_i)_{i\in\wedge}$ of R is called S-*stationary* if there exist $k \in \wedge$ and $s \in S$ such that $sI_i \subseteq I_k$ for all $i \in \wedge$, where $\wedge$ is an arbitrary indexing set. A family $\mathcal{F}$ of ideals of R is said to be S-saturated if it satisfies the following property: for every ideal I of R, if there exist $s \in S$ and $J \in \mathcal{F}$ such that $sI \subseteq J$, then $I \in \mathcal{F}$.

Theorem 2.5. *Let J be a proper ideal of R. Then the following statements are equivalent.*

(1) *R is an S-J-Noetherian.*
(2) *Every ascending chain of J-ideals of R is S-stationary.*
(3) *Every nonempty S-saturated set of J-ideals of R has a maximal element.*
(4) *Every nonempty family of J-ideals has an S-maximal element with respect to inclusion.*

Proof.
(1) ⇒ (2). Let $(I_n)_{n\in\wedge}$ be an increasing sequence of J-ideals of R. Define the ideal $I = \bigcup_{n\in\wedge} I_n$. If $I \subseteq J$, then $I_n \subseteq J$, which is not possible since each I_n is a J-ideal. Thus, I is a J-ideal of R. Also, I is S-finite since R is S-J-Noetherian. Consequently, there exist a finitely generated ideal $F \subseteq R$ and $s \in S$ such that $sI \subseteq F \subseteq I$. Since F is finitely generated, there is a $k \in \wedge$ satisfying $F \subseteq I_k$. Then we have $sI \subseteq F \subseteq I_k$, from which it follows that $sI_n \subseteq I_k$ for each $n \in \wedge$.

(2) ⇒ (3). Let $\mathcal{D}$ be an S-saturated set of J-ideals of R. Given any chain $\{I_n\}_{n\in\wedge} \subseteq \mathcal{D}$, we claim that $I = \bigcup_{n\in\wedge} I_n$ belongs to $\mathcal{D}$, which will establish that I as an upper bound for the chain. Indeed, by (2), there exist $k \in \wedge$ and $s \in S$ such that $sI_n \subseteq I_k$ for every $n \in \wedge$. Consequently, we obtain $sI = s\left(\bigcup_{n\in\wedge} I_n\right) \subseteq I_k$. Since $\mathcal{D}$ is S-saturated, it follows that $I \in \mathcal{D}$, as required. Applying Zorn's lemma, we conclude that $\mathcal{D}$ has a maximal element.

(3) ⇒ (4). Let $\mathcal{D}$ be a nonempty set of J-ideals of R. Consider the family $\mathcal{D}^S$ of all J-ideals $L \subseteq R$ such that there exist some $s \in S$ and $L_0 \in \mathcal{D}$ with $sL \subseteq L_0$. Clearly, $\mathcal{D} \subseteq \mathcal{D}^S$, so $\mathcal{D}^S \neq \emptyset$. It is straightforward to see that $\mathcal{D}^S$ is S-saturated. Thus, by (3) $\mathcal{D}^S$ has a maximal element $K \in \mathcal{D}^S$. Fix $s \in S$ and $Q \in \mathcal{D}$ such that $sK \subseteq Q$. Now, we claim that Q is an S-maximal element of $\mathcal{D}$; specifically, given $L \in \mathcal{D}$ with $Q \subseteq L$, we will show that $sL \subseteq Q$. Note that $K + L$ satisfies $s(K+L) = sK + sL \subseteq Q + L \subseteq L$, so that $K + L \in \mathcal{D}^S$. Also, if $(K+L) \subseteq J$, then $K \subseteq J$, which is not possible since K is a J-ideal of R. Thus, $K + L$ is a J-ideal of R. Therefore, maximality of K implies $K = K + L$, so that $L \subseteq K$. But then $sL \subseteq sK \subseteq Q$, as desired.

(4) ⇒ (1). Let I be a J-ideal of R, which we will prove to be S-finite. Let $\mathcal{D}$ be the family of finitely generated J-ideals of R such that $J \subseteq I$. Choose $x \in I \setminus J$. Then $L = (x) \subseteq I$, and $L \nsubseteq J$. This implies that $L \in \mathcal{D}$, and so $\mathcal{D}$ is nonempty. Then $\mathcal{D}$ has an S-maximal element $K \in \mathcal{D}$. Fixing $x \in I$, take a finitely generated ideal of the form $Q = K + xR$. Since $K \subseteq I$ and $x \in I$, so $Q \subseteq I$. Consequently, $Q \in \mathcal{D}$ such that $K \subseteq Q$. This implies that there exists $s \in S$ such that $sQ \subseteq K$; in particular, $sx \in K$. This verifies $sI \subseteq K \subseteq I$, so that I is S-finite. It follows that R is S-J-Noetherian. □

Let $f : R \to R'$ be a homomorphism and S a multiplicative closed subset of R. Then it is easy to see that $f(S)$ is a multiplicative closed subset of R' if $0 \notin f(S)$ and $1 \in f(S)$.

Proposition 2.6. *Let $f : R \to R'$ be an epimorphism and J be an ideal of R'. If R is an S-$f^{-1}(J)$-Noetherian ring with $0 \notin f(S)$, then R' is a $f(S)$-J-Noetherian ring, where $f(S)$ is a multiplicative closed subset of R' containing* 1.

Proof. Suppose $\{I_i\}_{i\in\wedge}$ is any increasing chain of J-ideals of R'. Then $I_i \nsubseteq J$ for each $i \in \wedge$. Suppose on the contrary that for each i there exist $\alpha_i \in I_i \setminus J$ such that $f^{-1}(\alpha_i) \subseteq f^{-1}(J)$. Then $\alpha_i \in f(f^{-1}(\alpha_i)) \subseteq f(f^{-1}(J)) = J$, for f is an epimorphism. This is a contradiction, as $\alpha_i \notin J$. Thus, $f^{-1}(I_i) \nsubseteq f^{-1}(J)$ for each $i \in \wedge$ and hence $f^{-1}(I_i)$ is $f^{-1}(J)$ ideal of R. Then we have an increasing chain $\{f^{-1}(I_i)\}_{i\in\wedge}$ of $f^{-1}(J)$-ideal of R. Since R is an S-$f^{-1}(J)$-Noetherian, there exist $k \in \wedge$ and $s \in S$ such that $sf^{-1}(I_i) \subseteq f^{-1}(I_k)$ for all $i \in \wedge$. Applying f to both sides, we obtain $f(sf^{-1}(I_i)) = f(s)f(f^{-1}(I_i)) \subseteq f(f^{-1}(I_k))$ for all $i \in \wedge$. Since f is an epimorphism, it follows that $f(s)I_i \subseteq I_k$ for all $i \in \wedge$. Hence, by Theorem 2.5, R' is a $f(S)$-J-Noetherian ring. □

Theorem 2.7. *Let S be a multiplicative subset of a ring R. The following statements are equivalent:*

(1) *R is S-Noetherian.*
(2) *R is S-J-Noetherian and J is an S-finite ideal of R.*

Proof. (1) ⇒ (2). This implication is obvious. (2) ⇒ (1). Let P be a prime ideal of R. If $P \subseteq J$, then P is S-finite by the assumption. Suppose that P contains properly in J. Then P is a J-ideal of R disjoint with S. Since R is S-J-Noetherian, then P is S-finite disjoint from S. So, by [3, Corollary 5], R is S-Noetherian. □

Let R be a ring and S be a multiplicative subset of R. Recall [3], let S be an anti-Archimedean subset of R if $\bigcap_{n\geq 1} s^n R \cap S \neq \emptyset$ for all $s \in S$.

Corollary 2.8. *Let $S \subseteq R$ be an anti-Archimedean multiplicative set and J is S-finite. If R is S-J-Noetherian, then the polynomial ring $R[X_1, \ldots, X_n]$ is also S-J-Noetherian.*

Proof. By Theorem 2.7, R is S-Noetherian ring. Then, by [3, Proposition 9], $R[X_1, \ldots, X_n]$ is S-Noetherian. This implies $R[X_1, \ldots, X_n]$ is S-J-Noetherian. □

Recall [4], let M be an R-module. Then, the idealization of R-module M, $R(+)M = \{(r, m) \mid r \in R, m \in M\}$ is a commutative ring with componentwise addition and multiplication defined by $(\alpha_1, m_1)(\alpha_2, m_2) = (\alpha_1\alpha_2, \alpha_1 m_2 + \alpha_2 m_1)$ for all $\alpha_1, \alpha_2 \in R$ and $m_1, m_2 \in M$. It is straightforward to verify that $S(+)M = \{(s, m) \mid s \in S, m \in M\}$ forms a multiplicative set in $R(+)M$. The following example shows that the polynomial ring over an S-J-Noetherian ring need not be S-J-Noetherian.

Example 2.9. *Let V be an infinite dimensional vector space over a field K. Then $R = K(+)V$ is an S-J-Noetherian ring for every multiplicative subset S of R. Moreover, if $0 \notin S$, then $R[X]$ is not an S-J-Noetherian ring. In particular, if $J = Nil(R)$, then the proof follows from* [10, Example 2.4].

We next show that the polynomial ring $R[X]$ is S-J-Noetherian if and only if it is S-Noetherian.

Corollary 2.10. *Let R be a ring, $S \subseteq R$ be a multiplicative set and J be an ideal of R. Then $R[X]$ is an S-$J[X]$-Noetherian ring if and only if $R[X]$ is an S-Noetherian ring.*

Proof. Suppose $R[X]$ is an S-$J[X]$-Noetherian ring. Then we show that $R[X]$ is an S-Noetherian ring. To prove this, by Theorem 2.7, it is sufficient to show that $J[X]$ is S-finite. Define the ideal $Q = J[X] + XR[X]$ of $R[X]$. Note that Q is a $J[X]$-ideal since $Q \nsubseteq J[X]$. Therefore, Q is S-finite. So there exist $s \in S$ and $f_1, \dots, f_n \in R[X]$ such that $s(J[X] + XR[X]) \subseteq f_1R[X] + \cdots + f_nR[X] \subseteq J[X] + XR[X]$. As a result, we get $sJ \subseteq f_1(0)R + \cdots + f_n(0)R \subseteq J$. This implies that $sJ[X] \subseteq f_1(0)R[X] + \cdots + f_n(0)R[X] \subseteq J[X]$. Thus, $J[X]$ is an S-finite ideal of $R[X]$. The converse is trivially true. □

Proposition 2.11. *Let R be an S-J-Noetherian ring. Then R/J is an $\overline{S}$-Noetherian ring.*

Proof. A nonzero prime ideal (disjoint from $\overline{S}$) of R/J is of the form P/J with $P \in \text{Spec}(R)$ and $J \subsetneq P$. Evidently, P is a J-ideal with $P \cap S = \emptyset$ since P/J is nonzero and $P/J \cap \overline{S} = \emptyset$. By the hypothesis, P is S-finite. Then there exist $s \in S$ and $p_1, \dots, p_n \in P$ such that $sP \subseteq (p_1, \dots, p_n) \subseteq P$. Let $x \in P$. Then we can find $a_1, \dots, a_n \in R$ such that $sx = a_1p_1 + \cdots + a_np_n$. It follows that $(s + J)(x + J) = (a_1 + J)(p_1 + J) + \dots + (a_n + J)(p_n + J)$, where $s + J \in \overline{S}$ and $a_1 + J, \dots, a_n + J \in R/J$. This implies that $(s+J)(P/J) \subseteq (p_1+J, \dots, p_n+J) \subseteq P/J$, i.e., P/J is $\overline{S}$-finite. By [3, Corollary 5], R/J is $\overline{S}$-Noetherian. □

Corollary 2.12. *Let $S \subseteq R$ be an anti-Archimedean multiplicative set. If R is S-J-Noetherian, then polynomial ring $(R/J)[X_1, \dots, X_n]$ is $\overline{S}$-$J[X_1, \dots, X_n]$-Noetherian.*

Proof. By Proposition 2.11, R/J is $\overline{S}$-Noetherian. Then, by [3, Proposition 9], $(R/J)[X_1, \dots, X_n]$ is also $\overline{S}$-Noetherian. This implies that $(R/J)[X_1, \dots, X_n]$ is $\overline{S}$-$J[X_1, \dots, X_n]$-Noetherian. □

Definition 2.13. [5] *An ideal I of a ring R is called divided if $I \subset xR$ for every $x \in R \backslash I$.*

Theorem 2.14. *Let R be an S-J-Noetherian ring, and I be a J-ideal of R disjoint from S. If J is divided ideal, then there exist $s \in S$ and S-prime ideals $P_1, \ldots, P_n$ of R such that $s(P_1 \cdots P_n) \subseteq I$.*

Proof. Since $I \nsubseteq J$ and J is divided, then $J \subset (x) \subseteq I$ for some $x \in I \setminus J$. Thus, I/J is an ideal of the $\overline{S}$-Noetherian ring R/J. Since $I \cap S = \emptyset$, then $(I/J) \cap \overline{S} = \emptyset$. For this, if $(I/J) \cap \overline{S} \neq \emptyset$, then $s + J = i + J$ for some $s \in S$ and $i \in I$. Consequently, $s - i \in J \subset I$, and so $s \in I$, a contradiction as $I \cap S = \emptyset$. Thus, I/J is disjoint from $\overline{S}$. It follows that there exist $\bar{s} \in \overline{S}$ and $\overline{S}$-prime ideals $Q_1, \ldots, Q_n$ of R/J containing I/J such that $\bar{s}(Q_1 \ldots Q_n) \subseteq I/J$, by [1, Theorem 5]. Clearly, $Q_i \cap \overline{S} = \emptyset$ for each $i = 1, \ldots, n$ since each Q_i is $\overline{S}$-prime. Then, by [1, Proposition 3], for each $1 \leq i \leq n$, there exists an S-prime ideal P_i of R containing J such that $Q_i = P_i/J$. Therefore, $\bar{s}((P_1 \cdots P_n)/J) \subseteq I/J$ since $P_1/J \cdots P_n/J = (P_1 \cdots P_n)/J$. For every $a \in P_1 \cdots P_n$, $(s+J)(a+J) = b+J$ for some $b \in I$. Conseuently, $sa - b \in J \subset I$, and so $sa \in I + Rb \subseteq I + J = I$. Thus, $s(P_1 \cdots P_n) \subseteq I$. □

Proposition 2.15. *Let $R \subseteq R'$ be an extension of rings such that $IR' \cap R = I$ for each ideal I of R, and let $S \subseteq R$ be a multiplicative set. If R' is an S-J-Noetherian ring, then R is S-J-Noetherian.*

Proof. Let I be a J-ideal of R and $I \subseteq IR'$. If $IR' \subseteq J$, then $I \subseteq J$, which is not possible since $I \nsubseteq J$. Thus, IR' is a J-ideal of R'. Since the ring R' is S-J-Noetherian, there exist $s \in S$ and $i_1, \ldots, i_n \in I$ such that $sIR' \subseteq (i_1, \ldots, i_n)R' \subseteq IR'$. By hypothesis, $sI = sIR' \cap R \subseteq (i_1, \ldots, i_n)R' \cap R \subseteq IR' \cap R = I$. Then I is an S-finite ideal of R, as desired. □

Proposition 2.16. *Let R be an S-J-Noetherian ring and I be a J-ideal of R disjoint from S. Then there exist $t \in S$ and $m \in \mathbb{N}$, such that $t(rad(I))^m \subseteq I$.*

Proof. Let I be a J-ideal of R. Then $rad(I)$ is also a J-ideal of R, and hence $rad(I)$ is S-finite. Consequently, there exist $s \in S$ and $x_1, \ldots, x_n \in rad(I)$ such that $s(rad(I)) \subseteq K \subseteq rad(I)$, where $K = (x_1, \ldots, x_n)$. Let $m_i \in \mathbb{N}$ be such that $x_i^{m_i} \in I$ for any $1 \leq i \leq n$. Then choose sufficiently large $m \in \mathbb{N}$ such that $K^m \subseteq I$. Therefore, $t(rad(I))^m \subseteq I$, where $t = s^m \in S$. □

Lemma 2.17. *Let R be an S-J-Noetherian and I be an J-ideal of R. Then, R/I is an $\overline{S}$-Noetherian ring.*

Proof. Let $\{I_i/I\}_{i \in \wedge}$ be an ascending chain of non-zero ideals of R/I. As a result, $\{I_i\}_{i \in \wedge}$ is an ascending chain of J-ideal of R and hence, by Theorem 2.5, there exist $s \in S$ and $k \in \wedge$ such that $sI_i \subseteq I_k$ for every $i \in \wedge$. Therefore, $(s + I)(I_i/I) \subseteq I_k/I$ for every $i \in \wedge$ and hence $(I_i/I)_{n \in \wedge}$ is $\overline{S}$-stationary. By [6, Theorem 2.3], R/I is $\overline{S}$-Noetherian. □

Recall that a ring R is said to be decomposable if R admits a non-trivial idempotent. Let $Idem(R)$ denote the set of idempotent elements of R.

Theorem 2.18. *Let R be a decomposable ring and J be an ideal of R with $eJ \neq (e)$ for each $e \in Idem(R) \setminus \{0,1\}$. Then, R is S-J-Noetherian if and only if R is S-Noetherian.*

Proof. It is sufficient to prove that if R is S-J-Noetherian, then R is S-Noetherian. To prove this, first we prove that $R/(e)$ is $\overline{S}$-Noetherian for each $e \in Idem(R) \setminus \{0,1\}$. Consider $e \in Idem(R) \setminus \{0,1\}$. Let L be an ideal of R which contains (e). Then $e \notin J$ since $eJ \neq (e)$, and so $L \nsubseteq J$. Thus, L is a J-ideal, and so by Lemma 2.17, R/L is $\overline{S}$-Noetherian. This implies that $R/(e)$ is $\overline{S}$-Noetherian since $(e) \subseteq L$. Now, let K be an ideal of R such that $K \subseteq (e)$ for each $e \in Idem(R) \setminus \{0,1\}$. We claim that K is S-finite. Clearly, $eK = K$. If $K = (0)$, then K is S-finite. So we may assume that $K \neq 0$. If $K \subseteq (1-e)$, then $eK \subseteq (e-e^2) = (0)$, i.e., $eK = K = 0$, a contradiction as $K \neq 0$. Therefore, $K \nsubseteq (1-e)$. Since $1-e \in Idem(R) \setminus \{0,1\}$, $R/(1-e)$ is a $\overline{S}$-Noetherian ring. Set $I = (1-e)$ for simplicity. Then, $L = (K+I)/I$ is an $\overline{S}$-finite ideal of R/I. Then there exist $\alpha_1 + I, \ldots, \alpha_n + I \in R/I$, where $\alpha_1, \ldots, \alpha_n \in K$ and $s' = s + I \in \overline{S}$ such that $s'L \subseteq (\alpha_1 + I, \ldots, \alpha_n + I) \subseteq L$. Let $\beta \in K + I$. Then $\beta + I \in L$, and so $s\beta + I \in s'L \subseteq (\alpha_1 + I, \ldots, \alpha_n + I)$. This implies that $s\beta + I = (u_1+I)(\alpha_1+I) + \cdots + (u_n+I)(\alpha_n+I)$ for some $u_1 + I, \ldots, u_n + I \in R/I$. Consequently, $s\beta - (u_1\alpha_1 + \cdots + u_n\alpha_n) \in I$, $s\beta - (u_1\alpha_1 + \cdots + u_n\alpha_n) \in F$, where $F = (\alpha_1, \ldots, \alpha_n, 1-e)$ since $I \subseteq F$. Thus, $s\beta \in F$, and hence $s(K + (1-e)) \subseteq F \subseteq K + (1-e)$. Therefore, $K + (1-e)$ is S-finite. Consequently, $K = Ke = (K+(1-e))e$ is an S-finite ideal of R, as claimed. Now, let T be an ideal of R. Since $eT \subseteq (e)$ and $(1-e)T \subseteq K+(1-e)T \subseteq K+(1-e)$ for each $e \in Idem(R) \setminus \{0,1\}$, eT and $(1-e)T$ are S-finite. It follows that $T = eT + (1-e)T$ is S-finite, and hence R is S-Noetherian ring. □

Definition 2.19. [14] *An ideal Q (disjoint from S) of the ring R is called S-irreducible if $s(I \cap K) \subseteq Q \subseteq I \cap K$ for some $s \in S$ and some ideals I, K of R, then there exists $s' \in S$ such that either $ss'I \subseteq Q$ or $ss'K \subseteq Q$.*

It is clear from the definition that every irreducible ideal is an S-irreducible ideal. However, the following example shows that an S-irreducibile ideal need not be irreducible.

Example 2.20. *Let $R = \mathbb{Z}$, $S = \mathbb{Z} \setminus 3\mathbb{Z}$ and $I = 6\mathbb{Z}$. Since $I = 2\mathbb{Z} \cap 3\mathbb{Z}$, therefore I is not an irreducible ideal of R. Now, take $s = 2 \in S$. Then, $2(3\mathbb{Z}) = 6\mathbb{Z} \subseteq I$. Thus, I is an S-irreducible ideal of R.*

Recall [11, Definition 2.1], a proper ideal Q of a ring R disjoint from S is said to be S-primary if there exists an $s \in S$ such that for all $a, b \in R$, if $ab \in Q$, then either $sa \in Q$ or $sb \in rad(Q)$. Following from [14], let I be an ideal of R such that $I \cap S = \emptyset$. Then, I admits S-primary decomposition if I can be written as a finite intersection of S-primary ideals of R.

Now, we extend S-primary decomposition theorem for S-J-Noetherian rings. We start with the following lemma.

Lemma 2.21. *Let R be an S-J-Noetherian ring. Then, every S-irreducible J-ideal of R is S-primary.*

Proof. Suppose Q is an S-irreducible J-ideal of R. Let $a, b \in R$ be such that $ab \in Q$ and $sb \notin Q$ for all $s \in S$. Our aim is to show that there exists $t \in S$ such that $ta \in rad(Q)$. Consider $A_n = \{x \in R \mid a^n x \in Q\}$ for $n \in \mathbb{N}$. Since Q is a J ideal, there exists $\alpha \in Q \setminus J$. Then, $a^n\alpha \in Q$ for each $n \in \mathbb{N}$. This implies that $\alpha \in A_n$ but $\alpha \notin J$ for each $n \in \mathbb{N}$. Consequently, each A_n is a J-ideal of R and $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ is an increasing chain of ideals of R. Since R is a S-J-Noetherian, by Theorem 2.5, this chain is S-stationary, i.e., there exist $k \in \mathbb{N}$ and $s \in S$ such that $sA_n \subseteq A_k$ for all $n \geq k$. Consider the two ideals $I = (a^k) + Q$ and $K = (b) + Q$ of R. Then, $Q \subseteq I \cap K$. For the reverse containment, let $y \in I \cap K$. Write $y = a^k z + q$ for some $z \in R$ and $q \in Q$. Since $ab \in Q$, $aK \subseteq Q$; whence $ay \in Q$. Now, $a^{k+1}z = a(a^k z) = a(y - q) \in Q$. This implies that $z \in A_{k+1}$, and so $sz \in sA_{k+1} \subseteq A_k$. Consequently, $a^k sz \in Q$ which implies that $a^k sz + sq = sy \in Q$. Thus, we have $s(I \cap K) \subseteq Q \subseteq I \cap K$. This implies that there exists $s' \in S$ such that either $ss'I \subseteq Q$ or $ss'K \subseteq Q$ since Q is S-irreducible. If $ss'K \subseteq Q$, then $ss'b \in Q$ which is not possible. Therefore, $ss'I \subseteq Q$ which implies that $ss'a^k \in Q$. Put $t = ss' \in S$. Then $(ta)^k \in Q$, and hence $ta \in rad(Q)$, as desired. □

Theorem 2.22. *Let R be an S-J-Noetherian ring. Then, every proper J-ideal of R disjoint with S can be written as a finite intersection of S-primary ideals.*

Proof. Let E be the collection of J-ideals of R which are disjoint with S and can not be written as a finite intersection of S-primary ideals. We wish to show $E = \emptyset$. On the contrary suppose $E \neq \emptyset$. Since R is an S-J-Noetherian ring, by Theorem 2.5, there exists an S-maximal element in E, say I. Evidently, I is not an S-primary ideal. Thus, by Lemma 2.21, I is not an S-irreducible ideal, and so I is not an irreducible ideal. This implies that $I = K \cap L$ for some ideals K and L of R with $I \neq K$ and $I \neq L$. As I is not S-irreducible, and so $sK \nsubseteq I$ and $sL \nsubseteq I$ for all $s \in S$. Now, we claim that $K, L \notin E$. For this, if K (respectively, L) belongs to E, then since I is an S-maximal element of E and $I \subset K$ (respectively, $I \subset L$), there exists s' (respectively, s'') from S such that $s'K \subseteq I$ (respectively, $s''L \subseteq I$). This is not possible, as I is not S-irreducible. Therefore, $K, L \notin E$. Also, if $K \cap S \neq \emptyset$ $(respectively, L \cap S \neq \emptyset)$, then there exist $s_1 \in K \cap S$ $(respectively, s_2 \in L \cap S)$. This implies that $s's_1 \in s'K \subseteq I$ $(respectively, s''s_2 \in s''L \subseteq I)$, which is a contradiction because I disjoint with S. Thus, K and L are also disjoint with S. This implies that K and L can be written as a finite intersection of S-primary ideals. Consequently, I can also be written as a finite intersection of S-primary ideals since $I = K \cap L$, a contradiction as $I \in E$. Thus, $E = \emptyset$, i.e., every proper J-ideal of R disjoint with S can be written as a finite intersection of S-primary ideals. □

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