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Representing rational integers by generalized quadratic forms over quadratic fields

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ABSTRACT. We investigate generalized quadratic forms with values in the set of rational integers over quadratic fields. We characterize the real quadratic fields which admit a positive definite binary generalized form of this type representing every positive integer. We also show that there are only finitely many such fields where a ternary generalized form with these properties exists.

CONTENTS

1. Introduction

A positive definite quadratic form over $\mathbb{Z}$ is called *universal* if it represents every positive integer n. The history of universal quadratic forms begins with Lagrange's proof of the Four Square Theorem in 1770: *Every positive integer n is of the form* $x^2+y^2+z^2+w^2$. Diagonal quaternary universal forms were classified

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by Ramanujan [23], who stated that there exist 55 such forms. Dickson [9] corrected Ramanujan's statement by pointing out that the form $x^2+2y^2+5z^2+5u^2$ does not represent 15. Thus there exist exactly 54 diagonal universal forms. Dickson [10] also proved various results on representation by non-diagonal quaternary forms.

In 1993, John H. Conway and W. A. Schneeberger proved the celebrated 15-Theorem. A quadratic form having an integer matrix is called *classical*.

Theorem 1.1 (15-Theorem, Conway-Schneeberger)**.** *A classical quadratic form Q over $\mathbb{Z}$ is universal if and only if it represents the integers*

$$1, 2, 3, 5, 6, 7, 10, 14, \textit{and } 15.$$

The proof was simplified by Bhargava [1]. As a corollary, one obtains all universal classical quaternary forms – there are exactly 207 of them. Bhargava and Hanke [2] extended the result to non-classical forms by proving the 290-Theorem: *If a positive definite quadratic form Q over $\mathbb{Z}$ represents every positive integer $n \leq 290$, then Q is universal.* In fact, for Q to be universal, it is enough that it represents every integer from a list of 29 numbers, the largest number on the list being 290. A corollary is that there are exactly 6436 universal quaternary forms [2, Theorem 4].

It is natural to extend the notion of universality from $\mathbb{Z}$ to the ring of integers $\mathcal{O}_K$ in a number field. Assume that the number field K is *totally real*. A positive integral quadratic form over K is called *universal* if it represents all totally positive integers in K. A version of the local-global principal over number fields proved by Hsia, Kitaoka, and Kneser [12] implies that a universal form over K always exists. Maaß [21] showed that the sum of three squares $x^2+y^2+z^2$ is universal over $\mathbb{Q}(\sqrt{5})$. By a result of Siegel [26], if a sum of squares is universal over K, then $K = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{5})$. In 1993, Chan, Kim, and Raghavan [7] proved that a ternary universal quadratic form exists over a real quadratic field $K = \mathbb{Q}(\sqrt{D})$ if and only if D equals 2, 3, or 5. Moreover, they identified all ternary universal forms over these fields up to equivalence. For forms of higher ranks, B. M. Kim [17] proved that there exist only finitely many D's such that $\mathbb{Q}(\sqrt{D})$ admits a universal form in 7 variables (see also [20] for an explicit estimate on the size of D). On the other hand, a 8-ary universal form exists over each $\mathbb{Q}(\sqrt{n^2-1})$ whenever n^2-1 is squarefree [18]. Blomer and Kala [3] showed: *For any positive integer r, there are infinitely many quadratic fields $\mathbb{Q}(\sqrt{D})$ that do* not *have a universal lattice of rank $\leq r$.* Lower bounds for the rank of a universal form depending on D were obtained by Kala and Tinková [15].

One of the principal open problems in this area is Kitaoka's conjecture: *There are only finitely many totally real number fields K having a ternary universal form.* The conjecture was proved by Kala and Yatsyna [16] for fields of an arbitrary but fixed degree d.

For a more comprehensive account of the theory of universal quadratic forms, especially over number fields, we refer the reader to the survey article [14].

In the same vein, one can study Hermitian forms over imaginary quadratic fields. We call a positive definite Hermitian form over an imaginary quadratic field $\mathbb{Z}$-*universal* if it represents every $n \in \mathbb{Z}_{\geq 1}$. Earnest and Khosravani [11] proved that universal binary Hermitian forms exist over $K = \mathbb{Q}(\sqrt{m})$ for only finitely many m, and identified all such forms in the case when K has class number 1. B. M. Kim, J. Y. Kim, and P.-S. Park [19] proved an analogy of the 15-Theorem for Hermitian quadratic forms: *If a positive definite Hermitian form H represents* 1, 2, 3, 5, 6, 7, 10, 13, 14, *and* 15, *then H is universal.* They also completely determined the minimal rank of a positive Hermitian form over all imaginary quadratic fields.

Our paper concerns generalized quadratic forms introduced recently by Browning, Pierce, and Schindler [5, Definition 1.1]. They can be defined for an arbitrary Galois extension K of $\mathbb{Q}$ but we will consider them only over quadratic fields. A quadratic field $K = \mathbb{Q}(\sqrt{D})$ has two $\mathbb{Q}$-automorphisms: the identity and the conjugation map

$$\begin{aligned} \tau : K &\hookrightarrow K \\ a + b\sqrt{D} &\mapsto a - b\sqrt{D}. \end{aligned}$$

Informally speaking, a generalized quadratic form may contain also the conjugates of the variables. For example,

$$G(z, w) = z^2 - z\tau(z) + \tau(z)^2 + w^2 + w\tau(w) + \tau(w)^2 \tag{1}$$

is a generalized quadratic form (and we will show later that it represents every $a \in \mathbb{Z}_{\geq 1}$ over $\mathbb{Q}(\sqrt{2})$). Thus, the concept generalizes both quadratic and Hermitian forms.

A generalized quadratic form is a special case of a generalized polynomial (see our Definition 2.1). The values of a generalized polynomial g in n variables with coefficients in K also lie in K. However, we restrict our attention to generalized polynomials with values in $\mathbb{Q}$ and find a minimal set of generators for the ring of such polynomials in Theorem 2.4.

If $G(z_1, z_2, \dots, z_n)$ is a generalized form in n variables and

$$G(\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{Z}, \quad \forall \alpha_1, \alpha_2, \dots, \alpha_n \in \mathcal{O}_K,$$

then we say that G is $\mathbb{Z}$-*valued*. The properties of being positive definite, integral, and classical also readily extend to this setting. A positive definite integral $\mathbb{Z}$-valued generalized form which represents every $a \in \mathbb{Z}_{\geq 1}$ will be called $\mathbb{Z}$-*universal*. Our main theorem is the following.

Theorem 1.2. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z}_{\geq 2}$ is squarefree.*

(1) Assume that $D \equiv 2, 3 \pmod 4$. A binary $\mathbb{Z}$-universal generalized quadratic form exists over K if and only if $D \in \{2, 3, 6, 7, 10\}$.

(2) Assume that $D \equiv 1 \pmod 4$. A classical *binary $\mathbb{Z}$-universal generalized quadratic form exists over K if and only if $D = 5$.*

We further extend this result to ternary generalized forms by proving the following.

Theorem 1.3. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z}_{\geq 2}$ is squarefree. Assume that one of the following two conditions is satisfied:*

(1) $D \equiv 2, 3 \pmod 4$ *and there exists a ternary $\mathbb{Z}$-universal generalized quadratic form over K,*
(2) $D \equiv 1 \pmod 4$ *and there exists a* classical *ternary $\mathbb{Z}$-universal generalized quadratic form over K.*

Then $D \leq 110$.

The proofs heavily rely on the 15-Theorem. A natural question is what happens when we remove the assumption that the form is classical in the case $D \equiv 1 \pmod 4$. By an application of the 290-Theorem, we can find the answer.

Theorem 1.4. *There are only finitely many real quadratic fields which admit a binary or ternary $\mathbb{Z}$-universal generalized quadratic form.*

We will prove these as Theorems 4.10 to 4.12 in Section 4. In contrast with the situation for binary and ternary forms, it is not very difficult to construct a generalized form in four variables which is $\mathbb{Z}$-universal over every real quadratic field (and we do this later in Proposition 4.13). This provides a fairly complete picture of the minimal rank required for a generalized form to be $\mathbb{Z}$-universal.

The preceding discussion relates only to forms which are positive definite. As regards indefinite forms, we prove the following in Theorem 5.1: If $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z}_{\geq 2}$ is squarefree, then the binary generalized form

$$G(z, w) = z\tau(z) + w\tau(w)$$

represents every $a \in \mathbb{Z}$.

If we start with an integral $\mathbb{Z}$-valued generalized form and express the variables in the integral basis, then we end up with an integral quadratic form over $\mathbb{Z}$ in twice as many variables (this will be proved formally later). For example, if we let $z = x_1 + y_1\sqrt{2}$ and $w = x_2 + y_2\sqrt{2}$ in (1), then we obtain the quadratic form

$$Q(x_1, y_1, x_2, y_2) = x_1^2 + 6y_1^2 + 3x_2^2 + 2y_2^2.$$

(It can be checked that Q is universal by the 15-Theorem.) We call Q the quadratic form *associated* to G. A natural question is which universal quaternary quadratic forms arise in this way. In other words, we ask which quaternary quadratic forms are associated to some $\mathbb{Z}$-universal binary generalized form. We find all such forms (up to equivalence) for $D \in \{2, 3, 6, 7, 10\}$. We also find all quadratic forms associated to *classical* $\mathbb{Z}$-universal binary generalized forms (again, up to equivalence) for $D = 5$.

The rest of the paper is organized as follows. In Section 2, we find a minimal set of generators of the ring of $\mathbb{Q}$-valued generalized polynomials. The theory of generalized quadratic forms is developed in Section 3. We prove Theorems 1.2

to 1.4 at the end of Section 4 and Theorem 5.1 in Section 5. In the Appendix, which can be found at arXiv:2403.07171v2 (we will further cite it as [8]), we give the proofs of Lemmas 4.5-4.9 in Section A4 and a more detailed approach to the proof of Theorem 5.1 in Section A5. In Section A6 we explicitly show all quaternary forms associated to $\mathbb{Z}$-universal binary generalized forms for $D \in \{2, 3, 6, 7, 10\}$ (and to classical ones for $D = 5$).

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2. Generalized polynomials with rational values

Throughout the article let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{0, 1\}$ is squarefree. If we let

$$\omega_D = \begin{cases} \sqrt{D}, & \text{if } D \equiv 2, 3 \pmod 4, \\ \frac{1+\sqrt{D}}{2}, & \text{if } D \equiv 1 \pmod 4, \end{cases}$$

then $(1, \omega_D)$ is a basis of $\mathcal{O}_K$. For $\alpha \in K$, we let $\mathrm{Tr}(\alpha) = \alpha + \tau(\alpha)$ and $\mathrm{N}(\alpha) = \alpha\tau(\alpha)$ be the *trace* and *norm* of α, respectively.

We begin by defining generalized polynomials over a quadratic field.

Definition 2.1. Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{-1, 0\}$ is squarefree. A *generalized polynomial* in n variables over K is an expression of the form

$$g(z_1, z_2, \dots, z_n) = \sum_{\substack{i_1, j_1, \dots, i_n, j_n \geq 0 \\ i_1 + j_1 + \dots + i_n + j_n \leq k}} \alpha_{i_1, j_1, \dots i_n, j_n} z_1^{i_1} \tau(z_1)^{j_1} \cdots z_n^{i_n} \tau(z_n)^{j_n} \qquad (2)$$

where $k \in \mathbb{Z}_{\geq 0}$, $\alpha_{i_1, j_1, \dots, i_n, j_n} \in K$ for every $i_1, j_1, \dots, i_n, j_n \in \{0, 1, \dots, k\}$ and $\tau : K \hookrightarrow \mathbb{C}$ where $\tau(a + b\sqrt{D}) = a - b\sqrt{D}$ is the conjugation map.

We will often use the multiindex notation, where $\mathbf{i} = (i_1, i_2, \dots, i_n) \in \mathbb{Z}^n_{\geq 0}$ is a multiindex, $|\mathbf{i}| = \sum_{r=1}^n i_r$ is the sum of its components, and for $\mathbf{x} = (x_1, x_2, \dots, x_n)$ we let $\mathbf{x}^{\mathbf{i}} = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$. The lexicographic order on $\mathbb{Z}^n_{\geq 0}$ is defined as follows: Let $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^n_{\geq 0}$, $\mathbf{i} = (i_1, i_2, \dots, i_n)$ and $\mathbf{j} = (j_1, j_2, \dots, j_n)$. Then $\mathbf{i} < \mathbf{j}$ if and only if there exists $s \in \{1, 2, \dots, n\}$ such that $i_r = j_r$ for $1 \leq r < s$ and $i_s < j_s$.

Hence if $\mathbf{z} = (z_1, z_2, \dots, z_n)$, then we let $\tau(\mathbf{z}) = (\tau(z_1), \tau(z_2), \dots, \tau(z_n))$ and we can write the generalized polynomial g as

$$g(\mathbf{z}) = \sum_{|\mathbf{i}| + |\mathbf{j}| \leq k} \alpha_{\mathbf{ij}} \mathbf{z}^{\mathbf{i}} \tau(\mathbf{z})^{\mathbf{j}}.$$

If g is a generalized polynomial of the form (2), then we define the *degree* of g as

$$\deg(g) = \max\{|\mathbf{i}| + |\mathbf{j}| : \alpha_{\mathbf{ij}} \neq 0\}.$$

We see that g is a generalized polynomial in n variables if and only if there exists a polynomial $\tilde{g}$ in $2n$ variables such that

$$g(z_1, z_2, \dots, z_n) = \tilde{g}(z_1, \tau(z_1), \dots, z_n, \tau(z_n)).$$

We are particularly interested in generalized polynomials g with values in $\mathbb{Q}$. They are characterized by the property $g(\mathbf{a}) = \tau(g(\mathbf{a}))$ for every $\mathbf{a} \in K^n$. We prove below that the coefficients of the two conjugate terms $\mathbf{z}^{\mathbf{i}}\tau(\mathbf{z})^{\mathbf{j}}$ and $\tau(\mathbf{z})^{\mathbf{j}}\mathbf{z}^{\mathbf{i}}$ in such a polynomial must be conjugate, i.e., $\alpha_{\mathbf{ji}} = \tau(\alpha_{\mathbf{ij}})$.

Lemma 2.2. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{0, 1\}$ is squarefree and let g be a generalized polynomial in n variables over K. If $g(\mathbf{a}) = 0$ for every $\mathbf{a} \in K^n$, then g is the zero polynomial.*

Proof. Let $\tilde{g}$ be a polynomial in $2n$ variables such that

$$g(z_1, z_2, \dots, z_n) = \tilde{g}(z_1, \tau(z_1), \dots, z_n, \tau(z_n)).$$

First, suppose that K is real. We know that $\sigma(K) = \{(\alpha, \tau(\alpha)) : \alpha \in K\}$ is dense in $V = \mathbb{R}^2$. Consequently, $\sigma(K)^n$ is dense in $\mathbb{R}^{2n}$. Because the polynomial $\tilde{g}$ vanishes on $\sigma(K)^n$, it vanishes on the whole set $\mathbb{R}^{2n}$, and hence $\tilde{g}$ is the zero polynomial.

Secondly, suppose that K is imaginary. In this case, $\sigma(K) = K$ is dense in $V = \mathbb{C}$. Thus K^n is dense in $\mathbb{C}^n$ and the polynomial $\tilde{g}$ is zero on the set

$$\{(z_1, \overline{z_1}, \dots, z_n, \overline{z_n}) : z_1, \dots, z_n \in K\}.$$

Let h be another polynomial in $2n$ variables defined by

$$h(x_1, y_1, \dots, x_n, y_n) = \tilde{g}(x_1 + iy_1, x_1 - iy_1, \dots, x_n + iy_n, x_n - iy_n).$$

Since h vanishes on $\mathbb{R}^{2n}$ and

$$g(z_1, z_2, \dots, z_n) = h\left(\frac{z_1 + \overline{z_1}}{2}, \frac{z_1 - \overline{z_1}}{2i}, \dots, \frac{z_n + \overline{z_n}}{2}, \frac{z_n - \overline{z_n}}{2i}\right),$$

g is the zero polynomial. □

Proposition 2.3. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{0, 1\}$ is squarefree and let g be a generalized polynomial in n variables over K given by (2). We have $g(\mathbf{a}) \in \mathbb{Q}$ for every $\mathbf{a} \in K^n$ if and only if $\alpha_{\mathbf{ji}} = \tau(\alpha_{\mathbf{ij}})$ for every $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^n_{\geq 0}$.*

Proof. Consider the two generalized polynomials

$$g(\mathbf{z}) = \sum_{|\mathbf{i}|+|\mathbf{j}| \leq k} \alpha_{\mathbf{ij}} \mathbf{z}^{\mathbf{i}} \tau(\mathbf{z})^{\mathbf{j}} = \sum_{|\mathbf{i}|+|\mathbf{j}| \leq k} \alpha_{\mathbf{ji}} \mathbf{z}^{\mathbf{j}} \tau(\mathbf{z})^{\mathbf{i}}$$

and

$$\tau(g)(\mathbf{z}) = \sum_{|\mathbf{i}|+|\mathbf{j}| \leq k} \tau(\alpha_{\mathbf{ij}}) \tau(\mathbf{z})^{\mathbf{i}} \mathbf{z}^{\mathbf{j}}.$$

The condition $g(\mathbf{a}) \in \mathbb{Q}$ for every $\mathbf{a} \in K$ is equivalent to the condition $g(\mathbf{a}) = \tau(g)(\mathbf{a})$ for every $\mathbf{a} \in K$. By Lemma 2.2, this is equivalent to $g - \tau(g)$ being the zero polynomial, which is in turn equivalent to $\alpha_{\mathbf{ji}} = \tau(\alpha_{\mathbf{ij}})$ for every $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^n_{\geq 0}$. □

Let K be a quadratic field and let Γ^n_K be the ring of $\mathbb{Q}$-valued generalized polynomials in n variables, i.e.,

$$\Gamma^n_K = \{g(z_1, \dots, z_n) \in K[z_1, \tau(z_1), \dots, z_n, \tau(z_n)] : \; g(\mathbf{a}) \in \mathbb{Q} \text{ for every } \mathbf{a} \in K^n\}.$$

For $d \in \mathbb{Z}_{\geq 0}$, let

$$\Gamma^{n,d}_K = \left\{g \in \Gamma^n_K : \; \deg(g) \leq d\right\}.$$

Clearly, $\Gamma^{n,d}_K$ is a vector space over $\mathbb{Q}$.

The main theorem of this section is the following one.

Theorem 2.4. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{0, 1\}$ is squarefree. If $n \in \mathbb{Z}_{\geq 1}$, then*

$$\Gamma^n_K = \mathbb{Q}\left[z_1 + \tau(z_1), \sqrt{D}(z_1 - \tau(z_1)), \dots, z_n + \tau(z_n), \sqrt{D}(z_n - \tau(z_n))\right].$$

The first step is to prove the theorem for $n = 1$. In this case, we denote the single variable by z instead of z_1, thus

$$\Gamma^1_K = \{g(z) \in K[z, \tau(z)] : \; g(\alpha) \in \mathbb{Q} \text{ for every } \alpha \in K\}.$$

Lemma 2.5. *If $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{0, 1\}$ is squarefree, then $\Gamma^1_K = \mathbb{Q}\left[z + \tau(z), \sqrt{D}(z - \tau(z))\right]$.*

Proof. Let $S_1 = \mathbb{Q}\left[z + \tau(z), \sqrt{D}(z - \tau(z))\right]$. We have $S_1 \subset \Gamma^1_K$ because $z + \tau(z)$ and $\sqrt{D}(z - \tau(z))$ are $\mathbb{Q}$-valued, and we want to prove $\Gamma^1_K \subset S_1$.

We claim that $z\tau(z) \in S_1$. Since

$$D(z - \tau(z))^2 = \left(\sqrt{D}(z - \tau(z))\right)^2 \in S_1,$$

we have also $(z - \tau(z))^2 \in S_1$, and hence

$$z\tau(z) = \frac{1}{2}\left((z + \tau(z))^2 - (z - \tau(z))^2\right) \in S_1.$$

This proves the claim.

Let $g \in \Gamma^1_K$. We prove $g \in S_1$ by induction on $d = \deg(g)$. If $d = 0$, then $g(z) = a \in \mathbb{Q}$ is a constant polynomial. Thus we assume that $d \geq 1$ and that the $\mathbb{Q}$-valued generalized polynomials of degree $< d$ are contained in S_1. We write g as

$$g(z) = \sum_{\substack{i,j \geq 0 \\ i+j \leq d}} \alpha_{i,j} z^i \tau(z)^j = \sum_{\substack{i,j \geq 1 \\ i+j \leq d}} \alpha_{i,j} z^i \tau(z)^j + \sum_{i=0}^{d} \alpha_{i,0} z^i + \sum_{j=0}^{d} \alpha_{0,j} \tau(z)^j.$$

By Proposition 2.3, $\alpha_{j,i} = \tau(\alpha_{i,j})$. Let

$$g_1(z) = \sum_{\substack{i,j\geq 1\\ i+j\leq d}} \alpha_{i,j} z^i \tau(z)^j,$$

$$g_2(z) = \sum_{i=0}^{d} \alpha_{i,0} z^i + \tau(\alpha_{i,0})\tau(z)^i,$$

so that $g(z) = g_1(z) + g_2(z)$. We have $g_1(z) = z\tau(z) \cdot h_1(z)$ where

$$h_1(z) = \sum_{\substack{i,j\geq 1\\ i+j\leq d}} \alpha_{i,j} z^{i-1} \tau(z)^{j-1}$$

is a $\mathbb{Q}$-valued generalized polynomial of degree $< d$. By the inductive hypothesis, $h_1 \in S_1$, and because we also know that $z\tau(z) \in S_1$, we get $g_1 \in S_1$.

Next, we have $g_2(z) = \alpha_{d,0} z^d + \tau(\alpha_{d,0})\tau(z)^d + h_2(z)$ where

$$h_2(z) = \sum_{i=0}^{d-1} \alpha_{i,0} z^i + \tau(\alpha_{i,0})\tau(z)^i$$

is a $\mathbb{Q}$-valued generalized polynomial of degree $< d$. By the inductive hypothesis, $h_2 \in S_2$. We show $\alpha_{d,0} z^d + \tau(\alpha_{d,0})\tau(z)^d \in S_1$. If we let $\alpha_{d,0} = a + b\sqrt{D}$ for $a, b \in \mathbb{Q}$, then

$$\alpha_{d,0} z^d + \tau(\alpha_{d,0})\tau(z)^d = a(z^d + \tau(z)^d) + b\sqrt{D}(z^d - \tau(z)^d).$$

We have $z^d + \tau(z)^d = (z + \tau(z))^d - z\tau(z) \cdot r_1(z)$ where

$$r_1(z) = \sum_{i=1}^{d-1} \binom{d}{i} z^{i-1} \tau(z)^{d-i-1}$$

is a $\mathbb{Q}$-valued generalized polynomial of degree $< d$, hence $r_1 \in S_1$. Because $(z + \tau(z))^d$ and $z\tau(z)$ also belong to S_1, we get $z^d + \tau(z)^d \in S_1$.

We also have $\sqrt{D}(z^d - \tau(z)^d) = \sqrt{D}(z - \tau(z)) \cdot r_2(z)$ where

$$r_2(z) = \sum_{i=0}^{d-1} z^{d-1-i} \tau(z)^i$$

is a $\mathbb{Q}$-valued generalized polynomial of degree $< d$, hence $r_2 \in S_1$. Because $\sqrt{D}(z-\tau(z)) \in S_1$, we get $\sqrt{D}(z^d-\tau(z)^d) \in S_1$. This proves $\alpha_{d,0}z^d+\tau(\alpha_{d,0})\tau(z)^d \in S_1$, thus $g_2 \in S_1$.

Finally, because $g(z) = g_1(z) + g_2(z)$, we get $g \in S_1$. □

Next, we find a basis of the vector space $\Gamma_K^{n,d}$ over $\mathbb{Q}$. Let $n \in \mathbb{Z}_{\geq 1}$ and $d \in \mathbb{Z}_{\geq 0}$. We define the polynomials $g_{\mathbf{ij}}$ for $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^n_{\geq 0}$, $|\mathbf{i}| + |\mathbf{j}| \leq d$ as follows: If

$\mathbf{i} < \mathbf{j}$, then

$$g_{\mathbf{ij}}(\mathbf{z}) = \mathbf{z}^{\mathbf{i}}\tau(\mathbf{z})^{\mathbf{j}} + \tau(\mathbf{z})^{\mathbf{i}}\mathbf{z}^{\mathbf{j}},$$
$$g_{\mathbf{ji}}(\mathbf{z}) = \sqrt{D}\left(\mathbf{z}^{\mathbf{i}}\tau(\mathbf{z})^{\mathbf{j}} - \tau(\mathbf{z})^{\mathbf{i}}\mathbf{z}^{\mathbf{j}}\right)$$

and if $\mathbf{i} = \mathbf{j}$, then

$$g_{\mathbf{ii}} = \mathbf{z}^{\mathbf{i}}\tau(\mathbf{z})^{\mathbf{i}}.$$

Lemma 2.6. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{0,1\}$ is squarefree, $n \in \mathbb{Z}_{\geq 1}$, and $d \in \mathbb{Z}_{\geq 0}$. The elements $g_{\mathbf{ij}}$ for $\mathbf{i},\mathbf{j} \in \mathbb{Z}^n_{\geq 0}$, $|\mathbf{i}| + |\mathbf{j}| \leq d$ form a basis of the vector space $\Gamma^{n,d}_K$ over $\mathbb{Q}$.*

Proof. Let

$$g(\mathbf{z}) = \sum_{|\mathbf{i}|+|\mathbf{j}|\leq d} \alpha_{\mathbf{ij}}\mathbf{z}^{\mathbf{i}}\tau(\mathbf{z})^{\mathbf{j}} \in \Gamma^{n,d}_K.$$

By Proposition 2.3, $\alpha_{\mathbf{ji}} = \tau(\alpha_{\mathbf{ij}})$ for every $\mathbf{i},\mathbf{j} \in \mathbb{Z}^n_{\geq 0}$, hence

$$g(\mathbf{z}) = \sum_{\substack{\mathbf{i}<\mathbf{j}\\ |\mathbf{i}|+|\mathbf{j}|\leq d}} \alpha_{\mathbf{ij}}\mathbf{z}^{\mathbf{i}}\tau(\mathbf{z})^{\mathbf{j}} + \tau(\alpha_{\mathbf{ij}})\tau(\mathbf{z})^{\mathbf{i}}\mathbf{z}^{\mathbf{j}} + \sum_{2|\mathbf{i}|\leq d} \alpha_{\mathbf{ii}}\mathbf{z}^{\mathbf{i}}\tau(\mathbf{z})^{\mathbf{i}}.$$

Let $\alpha_{\mathbf{ij}} = a_{\mathbf{ij}} + b_{\mathbf{ij}}\sqrt{D}$ where $a_{\mathbf{ij}}, b_{\mathbf{ij}} \in \mathbb{Q}$. In particular, $\alpha_{\mathbf{ii}} = \tau(\alpha_{\mathbf{ii}})$ implies $\alpha_{\mathbf{ii}} = a_{\mathbf{ii}}$. We get

$$g(\mathbf{z}) = \sum_{\substack{\mathbf{i}<\mathbf{j}\\ |\mathbf{i}|+|\mathbf{j}|\leq d}} a_{\mathbf{ij}}\left(\mathbf{z}^{\mathbf{i}}\tau(\mathbf{z})^{\mathbf{j}} + \tau(\mathbf{z})^{\mathbf{i}}\mathbf{z}^{\mathbf{j}}\right) + b_{\mathbf{ij}}\sqrt{D}\left(\mathbf{z}^{\mathbf{i}}\tau(\mathbf{z})^{\mathbf{j}} - \tau(\mathbf{z})^{\mathbf{i}}\mathbf{z}^{\mathbf{j}}\right)$$
$$+ \sum_{2|\mathbf{i}|\leq d} a_{\mathbf{ii}}\mathbf{z}^{\mathbf{i}}\tau(\mathbf{z})^{\mathbf{i}},$$

hence g is a $\mathbb{Q}$-linear combination of the elements $g_{\mathbf{ij}}$.

Let

$$B = \{g_{\mathbf{ij}} : \mathbf{i},\mathbf{j} \in \mathbb{Z}^n_{\geq 0}, |\mathbf{i}| + |\mathbf{j}| \leq d\}.$$

If $\mathbf{i} \neq \mathbf{j}$, then the only generalized polynomials in B containing the term $\mathbf{z}^{\mathbf{i}}\tau(\mathbf{z})^{\mathbf{j}}$ are $g_{\mathbf{ij}}$ and $g_{\mathbf{ji}}$. Since they are linearly independent, neither can be expressed as a linear combination of the remaining elements in B. Similarly, if $\mathbf{i} = \mathbf{j}$, then $\mathbf{z}^{\mathbf{i}}\tau(\mathbf{z})^{\mathbf{i}}$ is contained only in $g_{\mathbf{ii}}$, and thus $g_{\mathbf{ii}}$ cannot be expressed as a linear combination of the other elements in B. This proves that the $g_{\mathbf{ij}}$ are linearly independent. □

Proof of Theorem 2.4. Let

$$S_n = \mathbb{Q}\left[z_1 + \tau(z_1), \sqrt{D}(z_1 - \tau(z_1)), \dots, z_n + \tau(z_n), \sqrt{D}(z_n - \tau(z_n))\right].$$

We have $S_n \subset \Gamma^n_K$ because the generators of S_n are $\mathbb{Q}$-valued, and we want to prove $\Gamma^n_K \subset S_n$.

We proceed by induction on n. For $n = 1$, this is Lemma 2.5. Suppose that $n \geq 2$ and the statement is true for polynomials with $< n$ variables.

Let $g \in \Gamma_K^n$. We show $g \in S_n$ by induction on $d = \deg(g)$. If $d = 0$, then $g(\mathbf{z}) = a \in \mathbb{Q}$ is a constant polynomial. Thus we assume that $d \geq 1$ and that the $\mathbb{Q}$-valued generalized polynomials of degree $< d$ in n variables are contained in S_n. The polynomial g is a $\mathbb{Q}$-linear combination of the elements in the basis from Lemma 2.6. Thus it is enough to show that the basis elements of degree d belong to S_n.

We show that if $\mathbf{i}, \mathbf{j} \in \mathbb{Z}_{\geq 0}^n$, $|\mathbf{i}| + |\mathbf{j}| = d$, and $\mathbf{i} < \mathbf{j}$, then

$$g_{\mathbf{ij}}(\mathbf{z}) = \mathbf{z}^{\mathbf{i}}\tau(\mathbf{z})^{\mathbf{j}} + \tau(\mathbf{z})^{\mathbf{i}}\mathbf{z}^{\mathbf{j}} \in S_n,$$
$$g_{\mathbf{ji}}(\mathbf{z}) = \sqrt{D}\left(\mathbf{z}^{\mathbf{i}}\tau(\mathbf{z})^{\mathbf{j}} - \tau(\mathbf{z})^{\mathbf{i}}\mathbf{z}^{\mathbf{j}}\right) \in S_n.$$

If $i_n = 0$ and $j_n = 0$, then the variable z_n does not appear in $g_{\mathbf{ij}}$ and $g_{\mathbf{ji}}$, hence $g_{\mathbf{ij}}, g_{\mathbf{ji}} \in S_n$ by the inductive hypothesis.

If $i_n \geq 1$, then we let

$$h(\mathbf{z}) = z_1^{i_1}\tau(z_1)^{j_1} z_2^{i_2}\tau(z_2)^{j_2} \cdots z_n^{i_n-1}\tau(z_n)^{j_n},$$
$$g_{\mathbf{ij}}^1(\mathbf{z}) = (z_n + \tau(z_n)) \cdot (h(\mathbf{z}) + \tau(h(\mathbf{z}))),$$
$$g_{\mathbf{ij}}^2(\mathbf{z}) = (z_n - \tau(z_n)) \cdot (h(\mathbf{z}) - \tau(h(\mathbf{z}))),$$
$$g_{\mathbf{ji}}^1(\mathbf{z}) = \sqrt{D}(z_n - \tau(z_n)) \cdot (h(\mathbf{z}) + \tau(h(\mathbf{z}))),$$
$$g_{\mathbf{ji}}^2(\mathbf{z}) = (z_n + \tau(z_n)) \cdot \sqrt{D}(h(\mathbf{z}) - \tau(h(\mathbf{z}))).$$

so that

$$g_{\mathbf{ij}}(\mathbf{z}) = z_n h(\mathbf{z}) + \tau(z_n)\tau(h(\mathbf{z})) = \frac{1}{2}\left(g_{\mathbf{ij}}^1 + g_{\mathbf{ij}}^2\right),$$
$$g_{\mathbf{ji}}(\mathbf{z}) = \sqrt{D}\left(z_n h(\mathbf{z}) - \tau(z_n)\tau(h(\mathbf{z}))\right) = \frac{1}{2}\left(g_{\mathbf{ji}}^1 + g_{\mathbf{ji}}^2\right).$$

Because $h(\mathbf{z})$ has degree $d-1$, we can apply induction on $h(\mathbf{z}) + \tau(h(\mathbf{z}))$ and $\sqrt{D}(h(\mathbf{z}) - \tau(h(\mathbf{z})))$. Since

$$z_n + \tau(z_n),\ \sqrt{D}(z_n - \tau(z_n)),\ h(\mathbf{z}) + \tau(h(\mathbf{z})),\ \sqrt{D}(h(\mathbf{z}) - \tau(h(\mathbf{z}))) \in S_n,$$

we immediately obtain $g_{\mathbf{ij}}^1, g_{\mathbf{ji}}^1, g_{\mathbf{ji}}^2 \in S_n$. We also get

$$g_{\mathbf{ij}}^2 = \frac{1}{D} \cdot \sqrt{D}(z_n - \tau(z_n)) \cdot \sqrt{D}(h(\mathbf{z}) - \tau(h(\mathbf{z}))) \in S_n,$$

and $g_{\mathbf{ij}}, g_{\mathbf{ji}} \in S_n$ follows.

If $i_n = 0$ and $j_n \geq 1$, then the proof is analogous to the case $i_n \geq 1$.

Next, we show that if $\mathbf{i} \in \mathbb{Z}_{\geq 0}^n$ and $2|\mathbf{i}| = d$, then $g_{\mathbf{ii}}(\mathbf{z}) = \mathbf{z}^{\mathbf{i}}\tau(\mathbf{z})^{\mathbf{i}} \in S_n$.

If $i_n = 0$, then $\mathbf{g}_{ii}$ is a polynomial in $n-1$ variables, hence $\mathbf{g}_{ii} \in S_n$ by the inductive hypothesis.

If $i_n \geq 1$, then we let

$$h(\mathbf{z}) = z_1^{i_1}\tau(z_1)^{i_1} z_2^{i_2}\tau(z_2)^{i_2} \cdots z_n^{i_n-1}\tau(z_n)^{i_n-1}.$$

We have $g_{\mathbf{ii}}(\mathbf{z}) = z_n\tau(z_n) \cdot h(\mathbf{z})$. Since the theorem holds for generalized polynomials in one variable, we get

$$z_n\tau(z_n) \in \mathbb{Q}\left[z_n + \tau(z_n), \sqrt{D}\,(z_n - \tau(z_n))\right] \subset S_n.$$

The generalized polynomial h has degree $d-2$, hence $h \in S_n$ by the inductive hypothesis, and $g_{\mathbf{ii}} \in S_n$ follows. □

3. Generalized quadratic forms

Generalized quadratic forms are defined in [5, Definition 1.1] over a Galois extension $K/\mathbb{Q}$ of degree d (which is assumed to be totally real). We need the definition only over quadratic fields.

Definition 3.1. Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{0, 1\}$ is squarefree. A *generalized quadratic form* in n variables over K is a generalized polynomial

$$G(z_1, z_2, \dots, z_n) = \sum_{1\le i\le j\le n} \alpha_{ij} z_i z_j + \sum_{1\le i,j\le n} \beta_{ij} z_i \tau(z_j) + \sum_{1\le i\le j\le n} \gamma_{ij} \tau(z_i)\tau(z_j) \quad (3)$$

where $\alpha_{ij}, \gamma_{ij} \in K$ for $1 \le i \le j \le n$ and $\beta_{ij} \in K$ for $1 \le i, j \le n$.

A generalized quadratic form G given by (3) will be called *integral* if $\alpha_{ij}, \gamma_{ij} \in \mathcal{O}_K$ for $1 \le i \le j \le n$ and $\beta_{ij} \in \mathcal{O}_K$ for $1 \le i, j \le n$. An integral generalized quadratic form will be called *classical* if $\alpha_{ij}, \gamma_{ij} \in 2\mathcal{O}_K$ for $i < j$ and $\beta_{ij} \in 2\mathcal{O}_K$ for $1 \le i, j \le n$.

Let us recall that a quadratic form Q in n variables over K is a homogeneous polynomial of degree 2 over K. We can also obtain it as a generalized quadratic form with $\beta_{ij}, \gamma_{ij} = 0$ for all $1 \le i, j \le n$.

Then if G is a generalized quadratic form in n variables over K, then we can define a *quadratic form Q associated to G* in $2n$ variables by

$$Q(x_1, y_1, \dots, x_n, y_n) = G(x_1 + y_1\omega_D, \dots, x_n + y_n\omega_D). \quad (4)$$

In other words, Q is obtained from G by expressing the variables $z_1, z_2, \dots, z_n$ in the integral basis $(1, \omega_D)$. It has coefficients in K and the variables $x_1, y_1, \dots, x_n, y_n$ take values in $\mathbb{Q}$.

If G is a generalized form in n variables over K given by (3), then we define the *matrix of G* as the $2n \times 2n$ matrix

$$M_G = \begin{pmatrix} A & B \\ B^\top & C \end{pmatrix}$$

where

$$A = \begin{pmatrix} \alpha_{11} & \frac{\alpha_{12}}{2} & \dots & \frac{\alpha_{1n}}{2} \\ \frac{\alpha_{12}}{2} & \alpha_{22} & \dots & \frac{\alpha_{2n}}{2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\alpha_{1n}}{2} & \frac{\alpha_{2n}}{2} & \dots & \alpha_{nn} \end{pmatrix}, \; B = \begin{pmatrix} \frac{\beta_{11}}{2} & \frac{\beta_{12}}{2} & \dots & \frac{\beta_{1n}}{2} \\ \frac{\beta_{21}}{2} & \frac{\beta_{22}}{2} & \dots & \frac{\beta_{2n}}{2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\beta_{n1}}{2} & \frac{\beta_{n2}}{2} & \dots & \frac{\beta_{nn}}{2} \end{pmatrix},$$

$$C = \begin{pmatrix} \gamma_{11} & \frac{\gamma_{12}}{2} & \dots & \frac{\gamma_{1n}}{2} \\ \frac{\gamma_{12}}{2} & \gamma_{22} & \dots & \frac{\gamma_{2n}}{2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\gamma_{1n}}{2} & \frac{\gamma_{2n}}{2} & \dots & \gamma_{nn} \end{pmatrix}.$$

Thus M_G is the matrix such that

$$G(\mathbf{z}) = (\mathbf{z}, \tau(\mathbf{z}))M_G \begin{pmatrix} \mathbf{z} \\ \tau(\mathbf{z}) \end{pmatrix}$$

for $\mathbf{z} \in K^n$. The *rank* of G is the rank of the matrix M_G over K.

The $2n \times 2n$ matrix

$$T = \begin{pmatrix} 1 & \omega_D & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \omega_D & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 1 & \omega_D \\ 1 & \tau(\omega_D) & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \tau(\omega_D) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 1 & \tau(\omega_D) \end{pmatrix}$$

determines the change of variables

$$(x_1, y_1, \dots, x_n, y_n) \mapsto (z_1, \dots, z_n, \tau(z_1), \dots, \tau(z_n)).$$

Thus, if the quadratic form Q is associated to the quadratic form G defined by (4), and M_Q, M_G are their respective matrices, then

$$M_Q = T^\top M_G T.$$

If G is a generalized quadratic form over $\mathbb{Q}$ then we say that G *represents* $b \in \mathbb{Z}$ over $\mathbb{Z}$ if there exists $\mathbf{a} \in \mathbb{Z}^n$ such that $G(\mathbf{a}) = b$. In particular, according to our definition, every quadratic form represents 0. If there exists $\mathbf{a} \in \mathbb{Z}^n \setminus \{(0, 0, \dots, 0)\}$ such that $G(\mathbf{a}) = 0$, then we say that G represents 0 *non-trivially*. The form G is called *positive definite* if M_G is a positive definite matrix and G is called *universal* if it is a positive definite integral form such that it represents every $b \in \mathbb{Z}_{\geq 1}$.

Lemma 3.2. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{0, 1\}$ is squarefree, G be a generalized form in n variables over K given by (3), and M_G be the matrix of G. If Q is the quadratic form in $2n$ variables associated to G and M_Q is the matrix of Q, then*

$$\det(M_Q) = D^n \cdot \begin{cases} \det(2M_G), & \text{if } D \equiv 2, 3 \pmod 4, \\ \det(M_G), & \text{if } D \equiv 1 \pmod 4. \end{cases}$$

Proof. Since $M_G = T^\top M_Q T$, we get

$$\det(M_Q) = (\det T)^2 \det(M_G).$$

Let us compute the determinant of T. If π is the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 2n-1 & 2n \\ 1 & n+1 & 2 & n+2 & \dots & n & 2n \end{pmatrix}$$

(the out-shuffle), then $\operatorname{sgn}(\pi) = (-1)^{\frac{n(n-1)}{2}}$. Applying this permutation to the rows of T, we obtain a block-diagonal matrix where each block equals

$$M = \begin{pmatrix} 1 & \omega_D \\ 1 & \tau(\omega_D) \end{pmatrix}.$$

If $D \equiv 2, 3 \pmod 4$, then

$$\det M = \det \begin{pmatrix} 1 & \sqrt{D} \\ 1 & -\sqrt{D} \end{pmatrix} = -2\sqrt{D},$$

while if $D \equiv 1 \pmod 4$, then

$$\det M = \det \begin{pmatrix} 1 & \frac{1+\sqrt{D}}{2} \\ 1 & \frac{1-\sqrt{D}}{2} \end{pmatrix} = -\sqrt{D}.$$

Thus

$$\det T = (-1)^{\frac{n(n-1)}{2}} (\det M)^n = \begin{cases} (-1)^{\frac{n(n+1)}{2}} D^{\frac{n}{2}} 2^n, & \text{if } D \equiv 2, 3 \pmod 4, \\ (-1)^{\frac{n(n+1)}{2}} D^{\frac{n}{2}}, & \text{if } D \equiv 1 \pmod 4. \end{cases}$$

In the case $D \equiv 2, 3 \pmod 4$, we get

$$\det(M_Q) = D^n 2^{2n} \det(M_G) = D^n \det(2M_G),$$

and in the case $D \equiv 1 \pmod 4$, we get

$$\det(M_Q) = D^n \det(M_G). \qquad \square$$

Corollary 3.3. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{0, 1\}$ is squarefree, G be a generalized form in n variables over K given by (3), Q be the quadratic form in $2n$ variables associated to G, and M_Q be the matrix of Q. If one of the following conditions is satisfied:*

(1) $D \equiv 2, 3 \pmod 4$ and G is integral,
(2) $D \equiv 1 \pmod 4$ and G is classical,

then $D^n \mid \det(M_Q)$.

Proof. First, assume that $D \equiv 2, 3 \pmod 4$ and G is integral. Since the elements of M_G are in $\frac{1}{2}\mathcal{O}_K$, we must have $\det(2M_G) \in \mathcal{O}_K$. Thus $D^n \mid \det(M_Q)$ by Lemma 3.2.

Secondly, assume that $D \equiv 1 \pmod 4$ and G is classical. The elements of M_G are in $\mathcal{O}_K$, hence $\det(M_G) \in \mathcal{O}_K$, and $D^n \mid \det(M_Q)$ by Lemma 3.2. $\square$

Lemma 3.4. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{0, 1\}$ is squarefree and let G be a generalized form in n variables over K given by (3). Assume that $G(\mathbf{a}) \in \mathbb{Q}$ for every $\mathbf{a} \in K^n$. Let Q be the quadratic form in $2n$ variables associated to G. If $1 \le i \le j \le n$, then we let a_{ij}, b_{ij}, b_{ji}, and c_{ij} be the coefficients of the terms $x_i x_j$, $x_i y_j$, $x_j y_i$, and $y_i y_j$ in Q, respectively, so that the matrix M_Q of the quadratic form Q is*

$$M_Q = \begin{pmatrix} a_{11} & \frac{b_{11}}{2} & \frac{a_{12}}{2} & \frac{b_{12}}{2} & \cdots & \frac{a_{1n}}{2} & \frac{b_{1n}}{2} \\ \frac{b_{11}}{2} & c_{11} & \frac{b_{21}}{2} & \frac{c_{12}}{2} & \cdots & \frac{b_{n1}}{2} & \frac{c_{1n}}{2} \\ \frac{a_{12}}{2} & \frac{b_{21}}{2} & a_{22} & \frac{b_{22}}{2} & & \frac{a_{2n}}{2} & \frac{b_{2n}}{2} \\ \frac{b_{12}}{2} & \frac{c_{12}}{2} & \frac{b_{22}}{2} & c_{22} & \cdots & \frac{b_{n2}}{2} & \frac{c_{2n}}{2} \\ \vdots & & \vdots & & \ddots & \vdots & \\ \frac{a_{1n}}{2} & \frac{b_{n1}}{2} & \frac{a_{2n}}{2} & \frac{b_{n2}}{2} & & a_{nn} & \frac{b_{nn}}{2} \\ \frac{b_{1n}}{2} & \frac{c_{1n}}{2} & \frac{b_{2n}}{2} & \frac{c_{2n}}{2} & \cdots & \frac{b_{nn}}{2} & c_{nn} \end{pmatrix}.$$

If $1 \le i < j \le n$, then

$$\begin{aligned} a_{ij} &= \mathrm{Tr}(\alpha_{ij}) + \mathrm{Tr}(\beta_{ij}), \\ b_{ij} &= \mathrm{Tr}(\alpha_{ij}\omega_D) + \mathrm{Tr}(\beta_{ij}\tau(\omega_D)), \\ b_{ji} &= \mathrm{Tr}(\alpha_{ij}\omega_D) + \mathrm{Tr}(\beta_{ij}\omega_D), \\ c_{ij} &= \mathrm{Tr}(\alpha_{ij}\omega_D^2) + \mathrm{Tr}(\beta_{ij})\,\mathrm{N}(\omega_D). \end{aligned}$$

Moreover, if $1 \le i \le n$, then

$$\begin{aligned} a_{ii} &= \mathrm{Tr}(\alpha_{ii}) + \beta_{ii}, \\ b_{ii} &= \mathrm{Tr}(2\alpha_{ii}\omega_D) + \beta_{ii}\,\mathrm{Tr}(\omega_D), \\ c_{ii} &= \mathrm{Tr}(\alpha_{ii}\omega_D^2) + \beta_{ii}\,\mathrm{N}(\omega_D). \end{aligned}$$

Proof. The quadratic form Q is given by

$$\begin{aligned} Q(x_1, y_1, \dots, x_n, y_n) = &\sum_{1 \le i \le j \le n} \alpha_{ij}(x_i + y_i\omega)(x_j + y_j\omega) \\ &+ \sum_{1 \le i, j \le n} \beta_{ij}(x_i + y_i\omega)(x_j + y_j\tau(\omega)) \\ &+ \sum_{1 \le i \le j \le n} \gamma_{ij}(x_i + y_i\tau(\omega))(x_j + y_j\tau(\omega)). \end{aligned}$$

By Proposition 2.3, $G(\mathbf{a}) \in \mathbb{Q}$ for every $\mathbf{a} \in K^n$ if and only if the coefficients of conjugate terms are conjugate, i.e., $\gamma_{ij} = \tau(\alpha_{ij})$ for $1 \le i \le j \le n$ and $\beta_{ji} = \tau(\beta_{ij})$ for $1 \le i, j \le n$. If $i < j$, then

$$\begin{aligned} a_{ij} &= \alpha_{ij} + \beta_{ij} + \beta_{ji} + \gamma_{ij} = \mathrm{Tr}(\alpha_{ij}) + \mathrm{Tr}(\beta_{ij}), \\ b_{ij} &= \alpha_{ij}\omega + \beta_{ij}\tau(\omega) + \beta_{ji}\omega + \gamma_{ij}\tau(\omega) = \mathrm{Tr}(\alpha_{ij}\omega) + \mathrm{Tr}(\beta_{ij}\tau(\omega)), \\ b_{ji} &= \alpha_{ij}\omega + \beta_{ij}\omega + \beta_{ji}\tau(\omega) + \gamma_{ij}\tau(\omega) = \mathrm{Tr}(\alpha_{ij}\omega) + \mathrm{Tr}(\beta_{ij}\omega), \\ c_{ij} &= \alpha_{ij}\omega^2 + (\beta_{ij} + \beta_{ji})\,\mathrm{N}(\omega) + \gamma_{ij}\tau(\omega)^2 = \mathrm{Tr}(\alpha_{ij}\omega^2) + \mathrm{Tr}(\beta_{ij})\,\mathrm{N}(\omega). \end{aligned}$$

Moreover,

$$\begin{aligned}
a_{ii} &= \alpha_{ii} + \beta_{ii} + \gamma_{ii} = \operatorname{Tr}(\alpha_{ii}) + \beta_{ii},\\
b_{ii} &= 2\alpha_{ii}\omega + \beta_{ii}\operatorname{Tr}(\omega) + 2\gamma_{ii}\tau(\omega) = \operatorname{Tr}(2\alpha_{ii}\omega) + \beta_{ii}\operatorname{Tr}(\omega),\\
c_{ii} &= \alpha_{ii}\omega^2 + \beta_{ii}\operatorname{N}(\omega) + \gamma_{ii}\tau(\omega)^2 = \operatorname{Tr}(\alpha_{ii}\omega^2) + \beta_{ii}\operatorname{N}(\omega). \qquad \square
\end{aligned}$$

Lemma 3.5. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{0,1\}$ is squarefree and let G be a generalized form in n variables over K given by (3). Assume that G is $\mathbb{Z}$-valued. Let Q be the quadratic form in $2n$ variables associated to G. If G is integral, then Q is integral. Moreover, if one of the following conditions holds:*

(1) $D \equiv 2,3 \pmod 4$ and G is integral,
(2) $D \equiv 1 \pmod 4$ and G is classical,

then Q is classical.

Proof. As before, we let a_{ij}, b_{ij}, b_{ji}, and c_{ij} be the coefficients of the terms x_ix_j, x_iy_j, x_jy_i, and y_iy_j in Q, respectively, for $1 \le i \le j \le n$. If G is integral, then $\alpha_{ij}, \beta_{ij} \in \mathcal{O}_K$ for $i \le j$. The elements of $\mathcal{O}_K$ have trace in $\mathbb{Z}$, hence $a_{ij}, b_{ij}, b_{ji}, c_{ij} \in \mathbb{Z}$ by Lemma 3.4 and Q is integral.

If G is classical, then $\alpha_{ij}, \beta_{ij} \in 2\mathcal{O}_K$ for $i < j$ and $\beta_{ii} \in 2\mathbb{Z}$. It follows from Lemma 3.4 that $a_{ij}, b_{ij}, b_{ji}, c_{ij} \in 2\mathbb{Z}$ for $i < j$ and $b_{ii} \in 2\mathbb{Z}$, hence Q is classical.

If $D \equiv 2,3 \pmod 4$, then $\operatorname{Tr}(\alpha)$ is even for every $\alpha \in \mathcal{O}_K$. Thus it is sufficient to assume that G is integral to conclude that Q is classical. $\square$

Lemma 3.6. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{0,1\}$ is squarefree and let G be a generalized form in n variables given by (3). Assume that $G(\mathbf{a}) \in \mathbb{Q}$ for every $\mathbf{a} \in K^n$. If $1 \le i \le j \le n$, then we let $\alpha_{ij} = r_{ij} + s_{ij}\omega_D$ and $\beta_{ij} = t_{ij} + u_{ij}\omega_D$ where $r_{ij}, s_{ij}, t_{ij}, u_{ij} \in \mathbb{Q}$. Let Q be the quadratic form in $2n$ variables associated to G. If $1 \le i \le j \le n$, then we let a_{ij}, b_{ij}, b_{ji}, and c_{ij} be the coefficients of the terms x_ix_j, x_iy_j, x_jy_i, and y_iy_j in Q, respectively.*

If $D \equiv 2,3 \pmod 4$ and $1 \le i < j \le n$, then

$$\begin{aligned}
a_{ij} &= 2r_{ij} + 2t_{ij},\\
b_{ij} &= 2Ds_{ij} - 2Du_{ij},\\
b_{ji} &= 2Ds_{ij} + 2Du_{ij},\\
c_{ij} &= 2Dr_{ij} - 2Dt_{ij}.
\end{aligned}$$

Moreover, if $1 \le i \le n$, then

$$\begin{aligned}
a_{ii} &= 2r_{ii} + t_{ii},\\
b_{ii} &= 4Ds_{ii},\\
c_{ii} &= 2Dr_{ii} - Dt_{ii}.
\end{aligned}$$

If $D \equiv 1 \pmod 4$ *and* $1 \le i < j \le n$*, then*

$$\begin{aligned} a_{ij} &= 2r_{ij} + s_{ij} + 2t_{ij} + u_{ij}, \\ b_{ij} &= r_{ij} + \frac{1+D}{2}s_{ij} + t_{ij} + \frac{1-D}{2}u_{ij}, \\ b_{ji} &= r_{ij} + \frac{1+D}{2}s_{ij} + t_{ij} + \frac{1+D}{2}u_{ij}, \\ c_{ij} &= \frac{1+D}{2}r_{ij} + \frac{1+3D}{4}s_{ij} + \frac{1-D}{2}t_{ij} + \frac{1-D}{4}u_{ij}. \end{aligned}$$

Moreover, if $1 \le i \le n$*, then*

$$\begin{aligned} a_{ii} &= 2r_{ii} + s_{ii} + t_{ii}, \\ b_{ii} &= 2r_{ii} + (1+D)s_{ii} + t_{ii}, \\ c_{ii} &= \frac{1+D}{2}r_{ii} + \frac{1+3D}{4}s_{ii} + \frac{1-D}{4}t_{ii}. \end{aligned}$$

Proof. We express the coefficients of Q using Lemma 3.4. If $1 \le i < j \le n$, then

$$\begin{aligned} a_{ij} &= \mathrm{Tr}(\alpha_{ij}) + \mathrm{Tr}(\beta_{ij}) = \mathrm{Tr}(r_{ij} + s_{ij}\omega_D) + \mathrm{Tr}(t_{ij} + u_{ij}\omega_D) \\ &= 2r_{ij} + s_{ij}\,\mathrm{Tr}(\omega_D) + 2t_{ij} + u_{ij}\,\mathrm{Tr}(\omega_D), \\ b_{ij} &= \mathrm{Tr}(\alpha_{ij}\omega_D) + \mathrm{Tr}(\beta_{ij}\tau(\omega_D)) = \mathrm{Tr}(r_{ij}\omega_D + s_{ij}\omega_D^2) + \mathrm{Tr}(t_{ij}\tau(\omega_D) + u_{ij}\,\mathrm{N}(\omega_D)) \\ &= r_{ij}\,\mathrm{Tr}(\omega_D) + s_{ij}\,\mathrm{Tr}(\omega_D^2) + t_{ij}\,\mathrm{Tr}(\omega_D) + 2u_{ij}\,\mathrm{N}(\omega_D), \\ b_{ji} &= \mathrm{Tr}(\alpha_{ij}\omega_D) + \mathrm{Tr}(\beta_{ij}\omega_D) = \mathrm{Tr}(r_{ij}\omega_D + s_{ij}\omega_D^2) + \mathrm{Tr}(t_{ij}\omega_D + u_{ij}\omega_D^2) \\ &= r_{ij}\,\mathrm{Tr}(\omega_D) + s_{ij}\,\mathrm{Tr}(\omega_D^2) + t_{ij}\,\mathrm{Tr}(\omega_D) + u_{ij}\,\mathrm{Tr}(\omega_D^2), \\ c_{ij} &= \mathrm{Tr}(\alpha_{ij}\omega_D^2) + \mathrm{Tr}(\beta_{ij})\,\mathrm{N}(\omega_D) = \mathrm{Tr}(r_{ij}\omega_D^2 + s_{ij}\omega_D^3) + \mathrm{Tr}(t_{ij} + u_{ij}\omega_D)\,\mathrm{N}(\omega_D) \\ &= r_{ij}\,\mathrm{Tr}(\omega_D^2) + s_{ij}\,\mathrm{Tr}(\omega_D^3) + 2t_{ij}\,\mathrm{N}(\omega_D) + u_{ij}\,\mathrm{Tr}(\omega_D)\,\mathrm{N}(\omega_D). \end{aligned}$$

Moreover, if $1 \le i \le n$, then

$$\begin{aligned} a_{ii} &= \mathrm{Tr}(\alpha_{ii}) + \beta_{ii} = \mathrm{Tr}(r_{ii} + s_{ii}\omega_D) + t_{ii} = 2r_{ii} + s_{ii}\,\mathrm{Tr}(\omega_D) + t_{ii}, \\ b_{ii} &= \mathrm{Tr}(2\alpha_{ii}\omega_D) + \beta_{ii}\,\mathrm{Tr}(\omega_D) = \mathrm{Tr}(2r_{ii}\omega_D + 2s_{ii}\omega_D^2) + t_{ii}\,\mathrm{Tr}(\omega_D) \\ &= 2r_{ii}\,\mathrm{Tr}(\omega_D) + 2s_{ii}\,\mathrm{Tr}(\omega_D^2) + t_{ii}\,\mathrm{Tr}(\omega_D), \\ c_{ii} &= \mathrm{Tr}(\alpha_{ii}\omega_D^2) + \beta_{ii}\,\mathrm{N}(\omega_D) = \mathrm{Tr}(r_{ii}\omega_D^2 + s_{ii}\omega_D^3) + t_{ii}\,\mathrm{N}(\omega_D) \\ &= r_{ii}\,\mathrm{Tr}(\omega_D^2) + s_{ii}\,\mathrm{Tr}(\omega_D^3) + t_{ii}\,\mathrm{N}(\omega_D). \end{aligned}$$

If $D \equiv 2, 3 \pmod 4$, then $\omega_D = \sqrt{D}$, hence $\mathrm{Tr}(\omega_D) = 0$, $\mathrm{Tr}(\omega_D^2) = \mathrm{Tr}(D) = 2D$, $\mathrm{Tr}(\omega_D^3) = \mathrm{Tr}(D\sqrt{D}) = 0$, and $\mathrm{N}(\omega_D) = -D$.

If $D \equiv 1 \pmod 4$, then $\omega_D = \frac{1+\sqrt{D}}{2}$, so $\mathrm{Tr}(\omega_D) = 1$, $\mathrm{Tr}(\omega_D^2) = \mathrm{Tr}\left(\frac{1+D}{4} + \frac{\sqrt{D}}{2}\right) = \frac{1+D}{2}$, $\mathrm{Tr}(\omega_D^3) = \mathrm{Tr}\left(\frac{1+3D}{8} + \frac{3+D}{8}\sqrt{D}\right) = \frac{1+3D}{4}$, and $\mathrm{N}(\omega_D) = \frac{1-D}{4}$.

After substituting these expressions into the formulas above, we get the lemma. □

If p is a prime and M a matrix with entries in $\mathbb{Z}$, then we let $M \bmod p$ denote the matrix with entries in $\mathbb{Z}/p\mathbb{Z}$ obtained by reducing the elements of M modulo p.

Proposition 3.7. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{0,1\}$ is squarefree, $p \mid D$ be a prime number, and G an integral generalized quadratic form in n variables over K. Assume that G is $\mathbb{Z}$-valued. If Q is the quadratic form associated to G and M_Q is the matrix of Q, then* $\operatorname{rank}(M_Q \bmod p) \leq n$.

Proof. We introduce the following notation: if M is a $2n \times 2n$ matrix and $1 \leq k \leq 2n$, let $[M]_k$ denote the k-th row of M.

If $1 \leq i \leq j \leq n$, then we let $\alpha_{ij} = r_{ij} + s_{ij}\omega_D$ and $\beta_{ij} = t_{ij} + u_{ij}\omega_D$ where $r_{ij}, s_{ij}, t_{ij}, u_{ij} \in \mathbb{Q}$.

First assume that $D \equiv 2, 3 \pmod 4$. If $1 \leq k \leq n$, then the $2k$-th row of M_Q equals

$$[M_Q]_{2k} = \begin{pmatrix} \frac{b_{1k}}{2} & \frac{c_{1k}}{2} & \dots & \frac{b_{kk}}{2} & c_{kk} & \dots & \frac{b_{nk}}{2} & \frac{c_{kn}}{2} \end{pmatrix}.$$

By Lemma 3.6, if $1 \leq i < j \leq n$, then

$$\frac{b_{ij}}{2} = D(s_{ij} - u_{ij}),\ \frac{b_{ji}}{2} = D(s_{ij} + u_{ij}),\ \frac{c_{ij}}{2} = D(r_{ij} - t_{ij}),$$

and if $1 \leq i \leq n$, then

$$\frac{b_{ii}}{2} = 2Ds_{ii},\ c_{ii} = D(2r_{ii} - t_{ii}).$$

Since we assume that G is integral, $r_{ij}, s_{ij}, t_{ij}, u_{ij} \in \mathbb{Z}$. Thus D divides every element of $[M_Q]_{2k}$. It follows that $M_Q \bmod p$ contains n zero rows, hence $\operatorname{rank}(M_Q \bmod p) \leq n$.

Secondly, assume that $D \equiv 1 \pmod 4$. Let N_Q be the matrix obtained from M_Q by subtracting two times the $2k$-th row of M_Q from the $(2k-1)$-th row of M_Q for every $1 \leq k \leq n$. The $(2k-1)$-th row of N_Q equals $[M_Q]2k-1-2[M_Q]2k =$
$\begin{pmatrix} \frac{a_{1k}}{2} - b_{1k} & \frac{b_{k1}}{2} - c_{1k} & \dots & a_{kk} - b_{kk} & \frac{b_{kk}}{2} - 2c_{kk} & \dots & \frac{a_{kn}}{2} - b_{nk} & \frac{b_{kn}}{2} - c_{kn} \end{pmatrix}$

By Lemma 3.6, if $1 \leq i < j \leq n$, then

$$\begin{aligned}
\frac{a_{ij}}{2} - b_{ij} &= \frac{1}{2}\left(2r_{ij} + s_{ij} + 2t_{ij} + u_{ij}\right) - \left(r_{ij} + \frac{1+D}{2}s_{ij} + t_{ij} + \frac{1-D}{2}u_{ij}\right) \\
&= D \cdot \frac{u_{ij} + s_{ij}}{2}, \\
\frac{b_{ji}}{2} - c_{ij} &= \frac{1}{2}\left(r_{ij} + \frac{1+D}{2}s_{ij} + t_{ij} + \frac{1+D}{2}u_{ij}\right) \\
&\quad - \left(\frac{1+D}{2}r_{ij} + \frac{1+3D}{4}s_{ij} + \frac{1-D}{2}t_{ij} + \frac{1-D}{4}u_{ij}\right) \\
&= D \cdot \frac{t_{ij} + u_{ij} - r_{ij} - s_{ij}}{2},
\end{aligned}$$

and if $1 \le i \le n$, then

$$\begin{aligned} a_{ii} - b_{ii} &= 2r_{ii} + s_{ii} + t_{ii} - (2r_{ii} + (1+D)s_{ii} + t_{ii}) = -Ds_{ii}, \\ \frac{b_{ii}}{2} - 2c_{ii} &= \frac{1}{2}(2r_{ii} + (1+D)s_{ii} + t_{ii}) - 2\left(\frac{1+D}{2}r_{ii} + \frac{1+3D}{4}s_{ii} + \frac{1-D}{4}t_{ii}\right) \\ &= -Dr_{ii} - Ds_{ii} + \frac{D}{2}t_{ii}. \end{aligned}$$

We have $r_{ij}, s_{ij}, t_{ij}, u_{ij} \in \frac{1}{2}\mathbb{Z}$ because G is integral. Since we assume $D \equiv 1 \pmod 4$, p is an odd prime and 2 is invertible modulo p. We see that every element of $[N_Q]_{2k-1}$ vanishes modulo p, hence $N_Q \bmod p$ contains n zero rows and $\operatorname{rank}(M_Q \bmod p) = \operatorname{rank}(N_Q \bmod p) \le n$. □

4. Binary and ternary generalized quadratic forms

Let us formally state a definition from the introduction.

Definition 4.1. Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z}_{\ge 2}$ is squarefree and let G be an integral generalized form in n variables over K. Assume that G is $\mathbb{Z}$-valued. If for every $a \in \mathbb{Z}_{\ge 1}$, there exist $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathcal{O}_K$ such that

$$G(\alpha_1, \alpha_2, \dots, \alpha_n) = a,$$

then we say that G is *$\mathbb{Z}$-universal*.

The purpose of this section is to investigate binary and ternary $\mathbb{Z}$-universal forms. We begin by proving one implication in statements (i) and (ii) of Theorem 1.2.

Proposition 4.2. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z}_{\ge 2}$ is squarefree.*

(1) If $D \equiv 2, 3 \pmod 4$ and there exists a positive definite binary generalized form over K which is $\mathbb{Z}$-universal, then $D \in \{2, 3, 6, 7, 10\}$.
(2) If $D \equiv 1 \pmod 4$ and there exists a classical *positive definite binary generalized form over K which is $\mathbb{Z}$-universal, then $D = 5$.*

Proof. Suppose that G is a positive definite binary generalized form over K which is $\mathbb{Z}$-universal. If $D \equiv 1 \pmod 4$, we further assume that G is classical. If Q is the associated quaternary quadratic form defined by (4), then Q is positive definite and universal.

By Lemma 3.5, Q is also classical. Bhargava obtained, as a corollary of his proof of the 15-Theorem, a complete list of classical quaternary positive definite universal quadratic forms up to equivalence [1, Table 5]. Every form on this list has determinant ≤ 112. By Corollary 3.3, $D^2 \mid \det(M_Q)$, thus $D^2 \le 112$. This yields the proof. □

Lemma 4.3. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{0,1\}$ is squarefree and let G be a binary generalized form over K given by the matrix*

$$M_G = \begin{pmatrix} a & \frac{b}{2} & \frac{d}{2} & \frac{e}{2} \\ \frac{b}{2} & c & \frac{\tau(e)}{2} & \frac{f}{2} \\ \frac{d}{2} & \frac{\tau(e)}{2} & \tau(a) & \frac{\tau(b)}{2} \\ \frac{e}{2} & \frac{f}{2} & \frac{\tau(b)}{2} & \tau(c) \end{pmatrix} \tag{5}$$

where $a, b, c, e \in K$ and $d, f \in \mathbb{Q}$. Let $a = a_1 + a_2\omega_D$, $b = b_1 + b_2\omega_D$, $c = c_1 + c_2\omega_D$, and $e = e_1 + e_2\omega_D$. Let Q be the quaternary quadratic form associated to G and M_Q be the matrix of Q.

If $D \equiv 2, 3 \pmod 4$, then

$$M_Q = \begin{pmatrix} 2a_1 + d & 2Da_2 & b_1 + e_1 & D(b_2 - e_2) \\ 2Da_2 & D(2a_1 - d) & D(b_2 + e_2) & D(b_1 - e_1) \\ b_1 + e_1 & D(b_2 + e_2) & 2c_1 + f & 2Dc_2 \\ D(b_2 - e_2) & D(b_1 - e_1) & 2Dc_2 & D(2c_1 - f) \end{pmatrix}.$$

If $D \equiv 1 \pmod 4$, then

$$M_Q = \begin{pmatrix} M_{Q,1} & M_{Q,2} \\ M_{Q,2}^{\top} & M_{Q,3} \end{pmatrix}$$

where

$$M_{Q,1} = \begin{pmatrix} 2a_1 + a_2 + d & a_1 + \frac{1+D}{2}a_2 + \frac{d}{2} \\ a_1 + \frac{1+D}{2}a_2 + \frac{d}{2} & \frac{1+D}{2}a_1 + \frac{1+3D}{4}a_2 + \frac{1-D}{4}d \end{pmatrix},$$

$$M_{Q,2} = \begin{pmatrix} b_1 + \frac{b_2}{2} + e_1 + \frac{e_2}{2} & \frac{b_1}{2} + \frac{1+D}{4}b_2 + \frac{e_1}{2} + \frac{1-D}{4}e_2 \\ \frac{b_1}{2} + \frac{1+D}{4}b_2 + \frac{e_1}{2} + \frac{1+D}{4}e_2 & \frac{1+D}{4}b_1 + \frac{1+3D}{8}b_2 + \frac{1-D}{4}e_1 + \frac{1-D}{8}e_2 \end{pmatrix},$$

$$M_{Q,3} = \begin{pmatrix} 2c_1 + c_2 + f & c_1 + \frac{1+D}{2}c_2 + \frac{f}{2} \\ c_1 + \frac{1+D}{2}c_2 + \frac{f}{2} & \frac{1+D}{2}c_1 + \frac{1+3D}{4}c_2 + \frac{1-D}{4}f \end{pmatrix}.$$

Proof. The matrix of G is

$$M_G = \begin{pmatrix} \alpha_{11} & \frac{\alpha_{12}}{2} & \frac{\beta_{11}}{2} & \frac{\beta_{12}}{2} \\ \frac{\alpha_{12}}{2} & \alpha_{22} & \frac{\tau(\beta_{12})}{2} & \frac{\beta_{22}}{2} \\ \frac{\beta_{11}}{2} & \frac{\tau(\beta_{12})}{2} & \tau(\alpha_{11}) & \frac{\tau(\alpha_{12})}{2} \\ \frac{\beta_{12}}{2} & \frac{\beta_{22}}{2} & \frac{\tau(\alpha_{12})}{2} & \tau(\alpha_{22}) \end{pmatrix}$$

where $\alpha_{11} = a$, $\alpha_{12} = b$, $\alpha_{22} = c$, $\beta_{11} = d$, $\beta_{12} = e$, and $\beta_{22} = f$. In the notation of Lemma 3.6, $\alpha_{ij} = r_{ij} + s_{ij}\omega_D$ and $\beta_{ij} = t_{ij} + u_{ij}\omega_D$, hence $r_{11} = a_1$, $s_{11} = a_2$, $r_{12} = b_1$, $s_{12} = b_2$, $r_{22} = c_1$, $s_{22} = c_2$, $t_{11} = d$, $t_{12} = e_1$, $u_{12} = e_2$, and $t_{22} = f$.

The matrix of Q is

$$M_Q = \begin{pmatrix} a_{11} & \frac{b_{11}}{2} & \frac{a_{12}}{2} & \frac{b_{12}}{2} \\ \frac{b_{11}}{2} & c_{11} & \frac{b_{21}}{2} & \frac{c_{12}}{2} \\ \frac{a_{12}}{2} & \frac{b_{21}}{2} & a_{22} & \frac{b_{22}}{2} \\ \frac{b_{12}}{2} & \frac{c_{12}}{2} & \frac{b_{22}}{2} & c_{22} \end{pmatrix}.$$

If $D \equiv 2, 3 \pmod 4$, then by Lemma 3.6,

$$\begin{aligned}
a_{11} &= 2r_{11} + t_{11} = 2a_1 + d, \\
b_{11} &= 4Ds_{11} = 4Da_2, \\
c_{11} &= 2Dr_{11} - Dt_{11} = D(2a_1 - d), \\
a_{12} &= 2r_{12} + 2t_{12} = 2(b_1 + e_1), \\
b_{12} &= 2Ds_{12} - 2Du_{12} = 2D(b_2 - e_2), \\
b_{21} &= 2Ds_{12} + 2Du_{12} = 2D(b_2 + e_2), \\
c_{12} &= 2Dr_{12} - 2Dt_{12} = 2D(b_1 - e_1), \\
a_{22} &= 2r_{22} + t_{22} = 2c_1 + f, \\
b_{22} &= 4Ds_{22} = 4Dc_2, \\
c_{22} &= 2Dr_{22} - Dt_{22} = D(2c_1 - f).
\end{aligned}$$

If $D \equiv 1 \pmod 4$, then by Lemma 3.6,

$$\begin{aligned}
a_{11} &= 2r_{11} + s_{11} + t_{11} = 2a_1 + a_2 + d, \\
b_{11} &= 2r_{11} + (1+D)s_{11} + t_{11} = 2a_1 + (1+D)a_2 + d, \\
c_{11} &= \frac{1+D}{2}r_{11} + \frac{1+3D}{4}s_{11} + \frac{1-D}{4}t_{11} = \frac{1+D}{2}a_1 + \frac{1+3D}{4}a_2 + \frac{1-D}{4}d, \\
a_{12} &= 2r_{12} + s_{12} + 2t_{12} + u_{12} = 2b_1 + b_2 + 2e_1 + e_2, \\
b_{12} &= r_{12} + \frac{1+D}{2}s_{12} + t_{12} + \frac{1-D}{2}u_{12} = b_1 + \frac{1+D}{2}b_2 + e_1 + \frac{1-D}{2}e_2, \\
b_{21} &= r_{12} + \frac{1+D}{2}s_{12} + t_{12} + \frac{1+D}{2}u_{12} = b_1 + \frac{1+D}{2}b_2 + e_1 + \frac{1+D}{2}e_2, \\
c_{12} &= \frac{1+D}{2}r_{12} + \frac{1+3D}{4}s_{12} + \frac{1-D}{2}t_{12} + \frac{1-D}{4}u_{12} \\
&= \frac{1+D}{2}b_1 + \frac{1+3D}{4}b_2 + \frac{1-D}{2}e_1 + \frac{1-D}{4}e_2, \\
a_{22} &= 2r_{22} + s_{22} + t_{22} = 2c_1 + c_2 + f, \\
b_{22} &= 2r_{22} + (1+D)s_{22} + t_{22} = 2c_1 + (1+D)c_2 + f, \\
c_{22} &= \frac{1+D}{2}r_{22} + \frac{1+3D}{4}s_{22} + \frac{1-D}{4}t_{22} = \frac{1+D}{2}c_1 + \frac{1+3D}{4}c_2 + \frac{1-D}{4}f. \quad \square
\end{aligned}$$

In Lemmas 4.4 to 4.9, we find an example of a $\mathbb{Z}$-universal generalized form over $\mathbb{Q}(\sqrt{D})$ for $D \in \{2, 3, 6, 7, 10\}$ and of a classical $\mathbb{Z}$-universal generalized form over $\mathbb{Q}(\sqrt{5})$. We prove only Lemma 4.4, the rest is stated and proved in the Appendix in Section A4 [8].

Lemma 4.4. *Let* $K = \mathbb{Q}(\sqrt{2})$ *and*

$$G_2(z,w) = z^2 - \sqrt{2}zw + w^2 - z\tau(z) - w\tau(w) + \tau(z)^2 + \sqrt{2}\tau(z)\tau(w) + \tau(w)^2.$$

The generalized form G_2 *is* $\mathbb{Z}$*-universal over* K.

Proof. The coefficients of G_2 are $a = 1$, $b = -\sqrt{2}$, $c = 1$, $d = -1$, $e = 0$, $f = -1$. Let Q_2 be the quaternary quadratic form $Q_2(x_1, y_1, x_2, y_2) = G_2(x_1 + y_1\sqrt{2}, x_2 + y_2\sqrt{2})$ and let M_{Q_2} be the matrix of Q_2. By Lemma 4.3 with $D = 2$,

$$M_{Q_2} = \begin{pmatrix} 2a_1 + d & 4a_2 & b_1 + e_1 & 2(b_2 - e_2) \\ 4a_2 & 2(2a_1 - d) & 2(b_2 + e_2) & 2(b_1 - e_1) \\ b_1 + e_1 & 2(b_2 + e_2) & 2c_1 + f & 4c_2 \\ 2(b_2 - e_2) & 2(b_1 - e_1) & 4c_2 & 2(2c_1 - f) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 6 & -2 & 0 \\ 0 & -2 & 1 & 0 \\ -2 & 0 & 0 & 6 \end{pmatrix}.$$

We get

$$M_{Q_2} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & -2 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Since the last matrix is positive definite, G_2 is also positive definite. Moreover, the diagonal quadratic form $x^2 + 2y^2 + z^2 + 2w^2$ is universal by the 15-Theorem, hence G_2 is $\mathbb{Z}$-universal. □

Lemma 4.5. *Let* $K = \mathbb{Q}(\sqrt{3})$ *and*

$$\begin{aligned} G_3(z,w) = {} & 2z^2 - 2\sqrt{3}zw + 2w^2 \\ & - 3z\tau(z) - \sqrt{3}z\tau(w) + \sqrt{3}w\tau(z) \\ & + 2\tau(z)^2 + 2\sqrt{3}\tau(z)\tau(w) + 2\tau(w)^2. \end{aligned}$$

The generalized form G_3 *is* $\mathbb{Z}$*-universal over* K.

Lemma 4.6. *Let* $K = \mathbb{Q}(\sqrt{5})$ *and*

$$G_5(z,w) = (3 - \omega_5)z^2 + w^2 - 4z\tau(z) + (3 - \tau(\omega_5))\tau(z)^2 + \tau(w)^2$$

where $\omega_5 = \frac{1+\sqrt{5}}{2}$. *The generalized form* G_5 *is classical and* $\mathbb{Z}$*-universal over* K.

Lemma 4.7. *Let $K = \mathbb{Q}(\sqrt{6})$ and G_6 be the generalized form with the matrix*

$$M_{G_6} = \begin{pmatrix} 3 & \sqrt{6} & -\frac{5}{2} & \frac{\sqrt{6}}{2} \\ \sqrt{6} & 3 & -\frac{\sqrt{6}}{2} & -\frac{1}{2} \\ -\frac{5}{2} & -\frac{\sqrt{6}}{2} & 3 & -\sqrt{6} \\ \frac{\sqrt{6}}{2} & -\frac{1}{2} & -\sqrt{6} & 3 \end{pmatrix}.$$

The generalized form G_6 is $\mathbb{Z}$-universal over K.

Lemma 4.8. *Let $K = \mathbb{Q}(\sqrt{7})$ and G_7 be the generalized quadratic form with the matrix*

$$M_{G_7} = \begin{pmatrix} 3 & \frac{1+2\sqrt{7}}{2} & -\frac{5}{2} & \frac{-1+2\sqrt{7}}{2} \\ \frac{1+2\sqrt{7}}{2} & 4 & \frac{-1-2\sqrt{7}}{2} & \frac{3}{2} \\ -\frac{5}{2} & \frac{-1-2\sqrt{7}}{2} & 3 & \frac{1-2\sqrt{7}}{2} \\ \frac{-1+2\sqrt{7}}{2} & \frac{3}{2} & \frac{1-2\sqrt{7}}{2} & 4 \end{pmatrix}.$$

The generalized form G_7 is $\mathbb{Z}$-universal over K.

Lemma 4.9. *Let $K = \mathbb{Q}(\sqrt{10})$. If G_{10} is the generalized quadratic form with the matrix*

$$M_{G_{10}} = \begin{pmatrix} 3 & \frac{\sqrt{10}}{2} & -\frac{5}{2} & \frac{\sqrt{10}}{2} \\ \frac{\sqrt{10}}{2} & 5 & -\frac{\sqrt{10}}{2} & \frac{7}{2} \\ -\frac{5}{2} & -\frac{\sqrt{10}}{2} & 3 & \frac{-\sqrt{10}}{2} \\ \frac{\sqrt{10}}{2} & \frac{7}{2} & \frac{\sqrt{10}}{2} & 5 \end{pmatrix},$$

then G_{10} is $\mathbb{Z}$-universal over K.

Now we are ready to prove Theorem 1.2, which we restate here for the convenience of the reader.

Theorem 4.10. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z}_{\geq 2}$ is squarefree.*

(1) Assume that $D \equiv 2, 3 \pmod 4$. A binary $\mathbb{Z}$-universal generalized quadratic form exists over K if and only if $D \in \{2, 3, 6, 7, 10\}$.

(2) Assume that $D \equiv 1 \pmod 4$. A classical *binary $\mathbb{Z}$-universal generalized quadratic form exists over K if and only if $D = 5$.*

Proof. First we prove (i). We assume that $D \equiv 2, 3 \pmod 4$. If a binary $\mathbb{Z}$-universal generalized form exists over $K = \mathbb{Q}(\sqrt{D})$, then $D \in \{2, 3, 6, 7, 10\}$ by Proposition 4.2 (i). Conversely, in Lemmas 4.4, 4.5 and 4.7 to 4.9 we found an example of a generalized form with the required properties over each of these fields.

Next, we prove (ii). We assume that $D \equiv 1 \pmod 4$. If a classical binary $\mathbb{Z}$-universal generalized form exists over $K = \mathbb{Q}(\sqrt{D})$, then $D = 5$ by Proposition 4.2 (ii). We found an example of such a form over $\mathbb{Q}(\sqrt{5})$ in Lemma 4.6. □

Theorem 1.3 is an immediate consequence of the following slightly refined version.

Theorem 4.11. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z}_{\geq 2}$ is squarefree. Assume that one of the following two conditions is satisfied:*

(1) $D \equiv 2, 3 \pmod 4$ and there exists a ternary $\mathbb{Z}$-universal generalized quadratic form over K,

(2) $D \equiv 1 \pmod 4$ and there exists a classical *ternary $\mathbb{Z}$-universal generalized quadratic form over K.*

Then $D \in \{1, \dots, 62\} \cup \{66, 70, 74, 77, 78, 82, 85, 86, 87, 93, 94, 95, 102, 106, 110\}$.

Proof. Let G be a ternary generalized quadratic form over K which is $\mathbb{Z}$-universal. If $D \equiv 1 \pmod 4$, then we further assume that G is classical. Let Q be the quadratic form in 6 variables associated to G and let M_Q be the matrix of Q. By Lemma 3.5, Q is classical.

A classical universal quadratic form has a quaternary subform (not necessarily universal) obtained by the method of escalations [1, p. 31]. Let R be such a subform of Q and M_R the matrix of R. If $p \mid D$ is a prime, then $\operatorname{rank}(M_Q \bmod p) \leq 3$ by Proposition 3.7. The rank of R over $\mathbb{Z}/p\mathbb{Z}$ is less than or equal to the rank of Q over $\mathbb{Z}/p\mathbb{Z}$, hence $\operatorname{rank}(M_R \bmod p) \leq 3$. Since M_R is a 4×4 matrix, $\det(M_R) \equiv 0 \pmod p$.

This proves $D \mid \det(M_R)$. Every quaternary form obtained by the method of escalations is listed in [1, Table 3] and in particular, has determinant ≤ 112. It remains to determine which squarefree $D \leq 112$ divide the determinant of some form on the list. Every squarefree $D \in \{1, \dots, 62\}$ satisfies this condition. Additionally, the squarefree determinants greater than 62 on the list are 66, 70, 74, 77, 78, 82, 85, 86, 87, 93, 94, 95, 102, 106, 110. □

Using the 290-Theorem, we can remove the “classical" assumption in the case $D \equiv 1 \pmod 4$.

Theorem 4.12. *There are only finitely many real quadratic fields which admit a binary or ternary $\mathbb{Z}$-universal generalized quadratic form.*

Proof. We show that there exists a positive constant C such that $D \leq C$ for every such field $K = \mathbb{Q}(\sqrt{D})$. If $D \equiv 2, 3 \pmod 4$, then the statement follows from Theorems 4.10 and 4.11. Hence, we can assume that $D \equiv 1 \pmod 4$. Let G be a $\mathbb{Z}$-universal generalized form over K with matrix M_G and Q be the quadratic form associated to G with matrix M_Q.

First, assume that G is binary. If M_G is the matrix of G and M_Q is the matrix of Q, then $\det(M_Q) = D^2 \det(M_G)$ by Lemma 3.2. The matrix M_G is a 4×4 matrix with entries in $\frac{1}{2}\mathcal{O}_K$, hence $16 \det(M_G) \in \mathcal{O}_K \cap \mathbb{Q} = \mathbb{Z}$, and we get $D^2 \mid 16 \det(M_Q)$.

A corollary of the 290-Theorem is that there are exactly 6436 universal quaternary forms [2, Theorem 4]. If we let m be the maximum of their determinants, then $D^2 \leq 16m$.

Secondly, assume that G is ternary. A universal quadratic form contains a quaternary subform equivalent to one of 6560 basic escalators [2, p. 4]. Let R be such a subform of Q and M_R the matrix of R. If $p \mid D$ is a prime, then $\operatorname{rank}(M_Q \bmod p) \leq 3$ by Proposition 3.7, hence $\operatorname{rank}(M_R \bmod p) \leq 3$ as well. Since M_R is a 4×4 matrix, $\det(M_R) \equiv 0 \pmod p$, and since the entries of M_R are in $\frac{1}{2}\mathbb{Z}$, we have $16\det(M_R) \in \mathbb{Z}$.

This proves $D \mid 16\det(M_R)$. If we let m be the maximum of the determinants of the basic escalators, then $D \leq 16m$. □

In contrast to Theorem 4.12, it is not difficult to construct a $\mathbb{Z}$-universal generalized form in 4 variables over every real quadratic field.

Proposition 4.13. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z}_{\geq 2}$ is squarefree. The generalized quadratic form*

$$G(z_1, z_2, z_3, z_4) = z_1^2 - z_1\tau(z_1) + \tau(z_1)^2 + z_2^2 - z_2\tau(z_2) + \tau(z_2)^2 + z_3^2 - z_3\tau(z_3) + \tau(z_3)^2 + z_4^2 - z_4\tau(z_4) + \tau(z_4)^2$$

is $\mathbb{Z}$-universal over K. Thus, the minimal rank of a $\mathbb{Z}$-universal generalized form over K is at most 4.

Proof. Clearly, G is positive definite. It is enough to show that G represents every $a \in \mathbb{Z}_{\geq 1}$ over $\mathbb{Z}$. If $z_i \in \mathbb{Z}$ for $i = 1, \dots, 4$, then $\tau(z_i) = z_i$, hence

$$G(z_1, z_2, z_3, z_4) = z_1^2 + z_2^2 + z_3^2 + z_4^2.$$

This quadratic form is universal by Lagrange's Four Square Theorem. □

We end this section with a proposition that is used in Section A6 in [8].

Proposition 4.14. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{0, 1\}$ is squarefree, $D \equiv 2, 3 \pmod 4$. Let G be an integral binary generalized form over K. Assume that G is $\mathbb{Z}$-valued. If Q is the quaternary quadratic form associated to G and M_Q is the matrix of Q, then*

$$\frac{\det(M_Q)}{D^2} \equiv 0, 1 \pmod 4.$$

Proof. By Lemma 3.2,

$$\det(M_Q) = D^2 \cdot \det(2M_G).$$

The matrix $2M_G$ is

$$2M_G = \begin{pmatrix} 2a & b & d & e \\ b & 2c & \tau(e) & f \\ d & \tau(e) & 2\tau(a) & \tau(b) \\ e & f & \tau(b) & 2\tau(c) \end{pmatrix}$$

where $a, b, c, e \in \mathcal{O}_K$ and $d, f \in \mathbb{Z}$ because G is integral. Expanding $\det(2M_G)$ along the first column, we get

$$\det(2M_G) = 2a \cdot \det\begin{pmatrix} 2c & \tau(e) & f \\ \tau(e) & 2\tau(a) & \tau(b) \\ f & \tau(b) & 2\tau(c) \end{pmatrix} - b \cdot \det\begin{pmatrix} b & d & e \\ \tau(e) & 2\tau(a) & \tau(b) \\ f & \tau(b) & 2\tau(c) \end{pmatrix} + \\ + d \cdot \det\begin{pmatrix} b & d & e \\ 2c & \tau(e) & f \\ f & \tau(b) & 2\tau(c) \end{pmatrix} - e \cdot \det\begin{pmatrix} b & d & e \\ 2c & \tau(e) & f \\ \tau(e) & 2\tau(a) & \tau(b) \end{pmatrix},$$

hence

$$\begin{aligned}\det(2M_G) \equiv\, & 2a\Big(\tau(e)\tau(b)f + f\tau(e)\tau(b)\Big) - b\Big(d\tau(b)f \\ & + e\tau(e)\tau(b) - 2f\tau(a)e - \tau(b)^2 b - 2\tau(c)\tau(e)d\Big) \\ & + d\Big(2b\tau(e)\tau(c) + df^2 + 2ec\tau(b) - f\tau(e)e - \tau(b)fb\Big) \\ & - e\Big(b\tau(e)\tau(b) + df\tau(e) - \tau(e)^2 e - 2\tau(a)fb - 2\tau(b)cd\Big) \\ \equiv & - \mathrm{N}(b)df - \mathrm{N}(b)\,\mathrm{N}(e) + 2\tau(a)bef + \mathrm{N}(b)^2 + 2b\tau(c)d\tau(e) \\ & + 2b\tau(c)d\tau(e) + d^2f^2 + 2\tau(b)cde - d\,\mathrm{N}(e)f - \mathrm{N}(b)df \\ & - \mathrm{N}(b)\,\mathrm{N}(e) - d\,\mathrm{N}(e)f + \mathrm{N}(e)^2 + 2\tau(a)bef + 2\tau(b)cde \\ \equiv & - 2\,\mathrm{N}(b)df - 2\,\mathrm{N}(b)\,\mathrm{N}(e) + \mathrm{N}(b)^2 + d^2f^2 - 2d\,\mathrm{N}(e)f + \mathrm{N}(e)^2 \\ \equiv & \big(\mathrm{N}(b) + \mathrm{N}(e) + df\big)^2 \pmod 4.\end{aligned}$$

Thus,

$$\frac{\det(M_Q)}{D^2} = \det(2M_G) \equiv 0, 1 \pmod 4. \qquad \square$$

5. Indefinite generalized quadratic forms

We end with a short section providing a shortened proof of the universality of $G(z, w) = z\tau(z) + w\tau(w)$ over real quadratic fields, where the form is indefinite. For a more detailed approach, please consult Section A5 of the Appendix [8].

Theorem 5.1. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z}_{\geq 2}$ is squarefree. The binary generalized form*

$$G(z, w) = z\tau(z) + w\tau(w)$$

represents every $a \in \mathbb{Z}$.

Proof. It is sufficient to show that G represents every $a \in \mathbb{Z} \setminus \{0\}$ over $\mathbb{Z}[\sqrt{D}]$ because $\mathbb{Z}[\sqrt{D}] \subset \mathcal{O}_K$.

Let $z = x_1 + y_1\sqrt{D}$ and $w = x_2 + y_2\sqrt{D}$ where $x_1, y_1, x_2, y_2 \in \mathbb{Z}$. Let

$$Q(x_1, y_1, x_2, y_2) = G(x_1 + y_1\sqrt{D}, x_2 + y_2\sqrt{D}).$$

We have

$$\begin{aligned} Q(x_1, y_1, x_2, y_2) &= (x_1 + y_1\sqrt{D})(x_1 - y_1\sqrt{D}) \\ &\quad + (x_2 + y_2\sqrt{D})(x_2 - y_2\sqrt{D}) \\ &= x_1^2 + x_2^2 - Dy_1^2 - Dy_2^2. \end{aligned}$$

By the local-global principle for representations over $\mathbb{Z}$ by indefinite forms (see, e.g., [6, Chap. 9, Thm. 1.5]), it suffices to verify that each $a \in \mathbb{Z}$ is represented over $\mathbb{Z}_p$ for all primes p. This is indeed straightforward to verify; for details see Section A5 of the Appendix. □

It is natural to be interested in the ways a quadratic form represents 0 non-trivially. We show that the quadratic form $Q(x, y, z, w) = x^2 + y^2 - Dz^2 - Dw^2$ represents 0 non-trivially if and only if D is a sum of two squares. The proof of this can be found in the Appendix, see Proposition A5.6. in [8].

References

[1] Bhargava, Manjul. On the Conway-Schneeberger fifteen theorem. *Contemp. Math.* **272** (2000), 27–37. MR1803359, Zbl 0987.11027, doi: 10.1090/conm/272/04395. 692, 708, 713

[2] Bhargava, Manjul; Hanke, Jonathan. Universal quadratic forms and the 290-Theorem. *Preprint*. `https://math.stanford.edu/~vakil/files/290-Theorem-preprint.pdf`. 692, 714

[3] Blomer, Valentin; Kala, Vítězslav. Number fields without n-ary universal quadratic forms. *Math. Proc. Cambridge Philos. Soc.* **159** (2015), no. 2, 239–252. MR3395370, Zbl 1371.11084, doi: 10.1017/S030500411500033X. 692

[4] Blomer, Valentin; Kala, Vítězslav. On the rank of universal quadratic forms over real quadratic fields. *Doc. Math.* **23** (2018), 15–34. MR3846065, Zbl 1396.11061, doi: 10.4171/DM/611.

[5] Browning, Tim; Pierce, Lillian B.; Schindler, Damaris. Generalised quadratic forms over totally real number fields. *J. Inst. Math. Jussieu* **23** (2024), no. 6, 2859–2912. MR4825130, Zbl 1567.11120, doi: 10.1017/S1474748024000161. 693, 701

[6] Cassels, John W. S. Rational quadratic forms. London Mathematical Society Monographs. *Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York*, 1978. xvi+413 pp. ISBN:0-12-163260-1. MR522835, Zbl 0496.10008. 716

[7] Chan, Wai-kiu; Kim, Myung-Hwan; Raghavan, Srinivasacharya. Ternary universal integral quadratic forms over real quadratic fields. *Japan. J. Math. (N.S.)* **22** (1996), no. 2, 263–273. MR1432376, Zbl 0868.11020, doi: 10.4099/math1924.22.263. 692

[8] Chwiedziuk, Ondřej; Doležálek, Matěj; Pěchoučková, Emma; Pezlar, Zdeněk; Prakash, Om; Romeo, Giuliano; Růžičková, Anna; Zindulka, Mikuláš. Representing rational integers by generalized quadratic forms over quadratic fields. arXiv:https://arxiv.org/abs/2403.07171v2. 695, 711, 714, 715, 716

[9] Dickson, Leonard E. Integers represented by positive ternary quadratic forms. *Bull. Amer. Math. Soc.*, **33** (1927), no. 1, 63–70. MR1561323, Zbl 53.0133.04, doi: 10.1090/S0002-9904-1927-04312-9 692

[10] Dickson, Leonard E. Quaternary Quadratic Forms Representing all Integers. *Amer. J. Math.* **49** (1927), no. 1, 39–56. MR1432376, Zbl 0868.11020, doi: 10.4099/math1924.22.263. 692

[11] Earnest, A. G.; Khosravani, Azar. Universal binary Hermitian forms. *Math. Comp.* **66** (1997), no. 219, 1161–1168. MR1422787, Zbl 0877.11028, doi: 10.1090/S0025-5718-97-00860-0. 693

[12] Hsia, John S. and Kitaoka, Yoshiyuki and Kneser, Martin. Representations of positive definite quadratic forms. *J. Reine Angew. Math.* **301** (1978), 132–141. MR560499, Zbl 0374.10013, doi: 10.1515/crll.1978.301.132. 692

[13] Kala, Vítězslav. Universal quadratic forms and elements of small norm in real quadratic fields. *Bull. Aust. Math. Soc.* **94** (2016), no. 1, 7–14. MR3539315, Zbl 1345.11025, doi: 10.1017/S0004972715001495.

[14] Kala, Vítězslav. Universal quadratic forms and indecomposables in number fields: a survey. *Commun. Math.* **31** (2023), no. 2, 81–114. MR4621253, Zbl 1558.11030, doi: 10.46298/cm.10896. 692

[15] Kala, Vítězslav; Tinková, Magdaléna. Universal quadratic forms, small norms, and traces in families of number fields. *Int. Math. Res. Not. IMRN* (2023), no. 9, 7541–7577. MR4584708, Zbl 1531.11037, doi: 10.1093/imrn/rnac073. 692

[16] Kala, Vítězslav; Yatsyna, Pavlo. Lifting problem for universal quadratic forms. *Adv. Math.* **377** (2021), Paper No. 107497, 24. MR4182914, Zbl 1462.11028, doi: 10.1016/j.aim.2020.107497. 692

[17] Kim, Byeong M. Finiteness of real quadratic fields which admit positive integral diagonal septanary universal forms. *Manuscripta Math.* **99** (1999), no. 2, 181–184. MR1697212, Zbl 0961.11016, doi: 10.1007/s002290050168. 692

[18] Kim, Byeong M. Universal octonary diagonal forms over some real quadratic fields. *Comment. Math. Helv.* **75** (2000), no. 3, 410–414. MR1793795, Zbl 1120.11301, doi: 10.1007/s000140050133. 692

[19] Kim, Byeong M.; Kim, Ji Y.; Park, Poo-Sung. The fifteen theorem for universal Hermitian lattices over imaginary quadratic fields. *Math. Comp.* **79** (2010), no. 270, 1123–1144. MR2600559, Zbl 1216.11046, doi: 10.1090/S0025-5718-09-02287-X. 693

[20] Kim, Byeong M.; Kim, Myung-Hwan; Park, Dayoon. Real quadratic fields admitting universal lattices of rank 7. *J. Number Theory* **233** (2022), 456–466. MR4356860, Zbl 1487.11037, doi: 10.1016/j.jnt.2021.06.018. 692

[21] Maass, Hans. Über die Darstellung total positiver Zahlen des Körpers $R(5)$ als Summe von drei Quadraten. *Abh. Math. Sem. Hansischen Univ.* **14** (1941), 185–191. MR5505, Zbl 67.0103.02, doi: 10.1007/BF02940744. 692

[22] Milne, James S. Algebraic Number Theory (v3.08). (2022), 166 pages. `www.jmilne.org/math/`.

[23] Ramanujan, Srinivasa. On the expression of a number in the form $ax^2 + by^2 + cz^2 + du^2$. *Proc. Cambridge Philos. Soc.* **19** (1917), 11–21. MR2280863, Zbl 46.0240.01. 692

[24] `https://doc.sagemath.org/html/en/reference/quadratic_forms/sage/quadratic_forms/genera/normal_form.html`.

[25] Serre, Jean-Pierre. A course in arithmetic. Graduate Texts in Mathematics *Springer-Verlag, New York-Heidelberg* **No. 7** (1973). viii+115. MR344216, Zbl 0432.10001.

[26] Siegel, Carl L. Sums of mth powers of algebraic integers. *Ann. of Math. (2)* **46** (1945), 313–339. MR12630, Zbl 0063.07010, doi: 10.2307/1969026. 692

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