

Representing rational integers by generalized quadratic forms over quadratic fields

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ABSTRACT. We investigate generalized quadratic forms with values in the set of rational integers over quadratic fields. We characterize the real quadratic fields which admit a positive definite binary generalized form of this type representing every positive integer. We also show that there are only finitely many such fields where a ternary generalized form with these properties exists.

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1. Introduction

A positive definite quadratic form over \mathbb{Z} is called *universal* if it represents every positive integer n . The history of universal quadratic forms begins with Lagrange's proof of the Four Square Theorem in 1770: *Every positive integer n is of the form $x^2 + y^2 + z^2 + w^2$* . Diagonal quaternary universal forms were classified

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by Ramanujan [23], who stated that there exist 55 such forms. Dickson [9] corrected Ramanujan's statement by pointing out that the form $x^2 + 2y^2 + 5z^2 + 5u^2$ does not represent 15. Thus there exist exactly 54 diagonal universal forms. Dickson [10] also proved various results on representation by non-diagonal quaternary forms.

In 1993, John H. Conway and W. A. Schneeberger proved the celebrated 15-Theorem. A quadratic form having an integer matrix is called *classical*.

Theorem 1.1 (15-Theorem, Conway-Schneeberger). *A classical quadratic form Q over \mathbb{Z} is universal if and only if it represents the integers*

1, 2, 3, 5, 6, 7, 10, 14, and 15.

The proof was simplified by Bhargava [1]. As a corollary, one obtains all universal classical quaternary forms – there are exactly 207 of them. Bhargava and Hanke [2] extended the result to non-classical forms by proving the 290-Theorem: *If a positive definite quadratic form Q over \mathbb{Z} represents every positive integer $n \leq 290$, then Q is universal.* In fact, for Q to be universal, it is enough that it represents every integer from a list of 29 numbers, the largest number on the list being 290. A corollary is that there are exactly 6436 universal quaternary forms [2, Theorem 4].

It is natural to extend the notion of universality from \mathbb{Z} to the ring of integers \mathcal{O}_K in a number field. Assume that the number field K is *totally real*. A positive integral quadratic form over K is called *universal* if it represents all totally positive integers in K . A version of the local-global principle over number fields proved by Hsia, Kitaoka, and Kneser [12] implies that a universal form over K always exists. Maaß [21] showed that the sum of three squares $x^2 + y^2 + z^2$ is universal over $\mathbb{Q}(\sqrt{5})$. By a result of Siegel [26], if a sum of squares is universal over K , then $K = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{5})$. In 1993, Chan, Kim, and Raghavan [7] proved that a ternary universal quadratic form exists over a real quadratic field $K = \mathbb{Q}(\sqrt{D})$ if and only if D equals 2, 3, or 5. Moreover, they identified all ternary universal forms over these fields up to equivalence. For forms of higher ranks, B. M. Kim [17] proved that there exist only finitely many D 's such that $\mathbb{Q}(\sqrt{D})$ admits a universal form in 7 variables (see also [20] for an explicit estimate on the size of D). On the other hand, a 8-ary universal form exists over each $\mathbb{Q}(\sqrt{n^2 - 1})$ whenever $n^2 - 1$ is squarefree [18]. Blomer and Kala [3] showed: *For any positive integer r , there are infinitely many quadratic fields $\mathbb{Q}(\sqrt{D})$ that do not have a universal lattice of rank $\leq r$.* Lower bounds for the rank of a universal form depending on D were obtained by Kala and Tinková [15].

One of the principal open problems in this area is Kitaoka's conjecture: *There are only finitely many totally real number fields K having a ternary universal form.* The conjecture was proved by Kala and Yatsyna [16] for fields of an arbitrary but fixed degree d .

For a more comprehensive account of the theory of universal quadratic forms, especially over number fields, we refer the reader to the survey article [14].

In the same vein, one can study Hermitian forms over imaginary quadratic fields. We call a positive definite Hermitian form over an imaginary quadratic field \mathbb{Z} -universal if it represents every $n \in \mathbb{Z}_{\geq 1}$. Earnest and Khosravani [11] proved that universal binary Hermitian forms exist over $K = \mathbb{Q}(\sqrt{m})$ for only finitely many m , and identified all such forms in the case when K has class number 1. B. M. Kim, J. Y. Kim, and P.-S. Park [19] proved an analogy of the 15-Theorem for Hermitian quadratic forms: *If a positive definite Hermitian form H represents 1, 2, 3, 5, 6, 7, 10, 13, 14, and 15, then H is universal.* They also completely determined the minimal rank of a positive Hermitian form over all imaginary quadratic fields.

Our paper concerns generalized quadratic forms introduced recently by Browning, Pierce, and Schindler [5, Definition 1.1]. They can be defined for an arbitrary Galois extension K of \mathbb{Q} but we will consider them only over quadratic fields. A quadratic field $K = \mathbb{Q}(\sqrt{D})$ has two \mathbb{Q} -automorphisms: the identity and the conjugation map

$$\begin{aligned} \tau : K &\hookrightarrow K \\ a + b\sqrt{D} &\mapsto a - b\sqrt{D}. \end{aligned}$$

Informally speaking, a generalized quadratic form may contain also the conjugates of the variables. For example,

$$G(z, w) = z^2 - z\tau(z) + \tau(z)^2 + w^2 + w\tau(w) + \tau(w)^2 \tag{1}$$

is a generalized quadratic form (and we will show later that it represents every $a \in \mathbb{Z}_{\geq 1}$ over $\mathbb{Q}(\sqrt{2})$). Thus, the concept generalizes both quadratic and Hermitian forms.

A generalized quadratic form is a special case of a generalized polynomial (see our Definition 2.1). The values of a generalized polynomial g in n variables with coefficients in K also lie in K . However, we restrict our attention to generalized polynomials with values in \mathbb{Q} and find a minimal set of generators for the ring of such polynomials in Theorem 2.4.

If $G(z_1, z_2, \dots, z_n)$ is a generalized form in n variables and

$$G(\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{Z}, \quad \forall \alpha_1, \alpha_2, \dots, \alpha_n \in \mathcal{O}_K,$$

then we say that G is \mathbb{Z} -valued. The properties of being positive definite, integral, and classical also readily extend to this setting. A positive definite integral \mathbb{Z} -valued generalized form which represents every $a \in \mathbb{Z}_{\geq 1}$ will be called \mathbb{Z} -universal. Our main theorem is the following.

Theorem 1.2. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z}_{\geq 2}$ is squarefree.*

- (1) *Assume that $D \equiv 2, 3 \pmod{4}$. A binary \mathbb{Z} -universal generalized quadratic form exists over K if and only if $D \in \{2, 3, 6, 7, 10\}$.*
- (2) *Assume that $D \equiv 1 \pmod{4}$. A classical binary \mathbb{Z} -universal generalized quadratic form exists over K if and only if $D = 5$.*

We further extend this result to ternary generalized forms by proving the following.

Theorem 1.3. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z}_{\geq 2}$ is squarefree. Assume that one of the following two conditions is satisfied:*

- (1) $D \equiv 2, 3 \pmod{4}$ and there exists a ternary \mathbb{Z} -universal generalized quadratic form over K ,
- (2) $D \equiv 1 \pmod{4}$ and there exists a classical ternary \mathbb{Z} -universal generalized quadratic form over K .

Then $D \leq 110$.

The proofs heavily rely on the 15-Theorem. A natural question is what happens when we remove the assumption that the form is classical in the case $D \equiv 1 \pmod{4}$. By an application of the 290-Theorem, we can find the answer.

Theorem 1.4. *There are only finitely many real quadratic fields which admit a binary or ternary \mathbb{Z} -universal generalized quadratic form.*

We will prove these as Theorems 4.10 to 4.12 in Section 4. In contrast with the situation for binary and ternary forms, it is not very difficult to construct a generalized form in four variables which is \mathbb{Z} -universal over every real quadratic field (and we do this later in Proposition 4.13). This provides a fairly complete picture of the minimal rank required for a generalized form to be \mathbb{Z} -universal.

The preceding discussion relates only to forms which are positive definite. As regards indefinite forms, we prove the following in Theorem 5.1: If $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z}_{\geq 2}$ is squarefree, then the binary generalized form

$$G(z, w) = z\tau(z) + w\tau(w)$$

represents every $a \in \mathbb{Z}$.

If we start with an integral \mathbb{Z} -valued generalized form and express the variables in the integral basis, then we end up with an integral quadratic form over \mathbb{Z} in twice as many variables (this will be proved formally later). For example, if we let $z = x_1 + y_1\sqrt{2}$ and $w = x_2 + y_2\sqrt{2}$ in (1), then we obtain the quadratic form

$$Q(x_1, y_1, x_2, y_2) = x_1^2 + 6y_1^2 + 3x_2^2 + 2y_2^2.$$

(It can be checked that Q is universal by the 15-Theorem.) We call Q the quadratic form *associated* to G . A natural question is which universal quaternary quadratic forms arise in this way. In other words, we ask which quaternary quadratic forms are associated to some \mathbb{Z} -universal binary generalized form. We find all such forms (up to equivalence) for $D \in \{2, 3, 6, 7, 10\}$. We also find all quadratic forms associated to *classical* \mathbb{Z} -universal binary generalized forms (again, up to equivalence) for $D = 5$.

The rest of the paper is organized as follows. In Section 2, we find a minimal set of generators of the ring of \mathbb{Q} -valued generalized polynomials. The theory of generalized quadratic forms is developed in Section 3. We prove Theorems 1.2

to 1.4 at the end of Section 4 and Theorem 5.1 in Section 5. In the Appendix, which can be found at arXiv:2403.07171v2 (we will further cite it as [8]), we give the proofs of Lemmas 4.5-4.9 in Section A4 and a more detailed approach to the proof of Theorem 5.1 in Section A5. In Section A6 we explicitly show all quaternary forms associated to \mathbb{Z} -universal binary generalized forms for $D \in \{2, 3, 6, 7, 10\}$ (and to classical ones for $D = 5$).

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2. Generalized polynomials with rational values

Throughout the article let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{0, 1\}$ is squarefree. If we let

$$\omega_D = \begin{cases} \sqrt{D}, & \text{if } D \equiv 2, 3 \pmod{4}, \\ \frac{1+\sqrt{D}}{2}, & \text{if } D \equiv 1 \pmod{4}, \end{cases}$$

then $(1, \omega_D)$ is a basis of \mathcal{O}_K . For $\alpha \in K$, we let $\text{Tr}(\alpha) = \alpha + \tau(\alpha)$ and $N(\alpha) = \alpha\tau(\alpha)$ be the *trace* and *norm* of α , respectively.

We begin by defining generalized polynomials over a quadratic field.

Definition 2.1. Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{-1, 0\}$ is squarefree. A *generalized polynomial* in n variables over K is an expression of the form

$$g(z_1, z_2, \dots, z_n) = \sum_{\substack{i_1, j_1, \dots, i_n, j_n \geq 0 \\ i_1 + j_1 + \dots + i_n + j_n \leq k}} \alpha_{i_1, j_1, \dots, i_n, j_n} z_1^{i_1} \tau(z_1)^{j_1} \dots z_n^{i_n} \tau(z_n)^{j_n} \quad (2)$$

where $k \in \mathbb{Z}_{\geq 0}$, $\alpha_{i_1, j_1, \dots, i_n, j_n} \in K$ for every $i_1, j_1, \dots, i_n, j_n \in \{0, 1, \dots, k\}$ and $\tau : K \hookrightarrow \mathbb{C}$ where $\tau(a + b\sqrt{D}) = a - b\sqrt{D}$ is the conjugation map.

We will often use the multiindex notation, where $\mathbf{i} = (i_1, i_2, \dots, i_n) \in \mathbb{Z}_{\geq 0}^n$ is a multiindex, $|\mathbf{i}| = \sum_{r=1}^n i_r$ is the sum of its components, and for $\mathbf{x} = (x_1, x_2, \dots, x_n)$ we let $\mathbf{x}^{\mathbf{i}} = x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$. The lexicographic order on $\mathbb{Z}_{\geq 0}^n$ is defined as follows: Let $\mathbf{i}, \mathbf{j} \in \mathbb{Z}_{\geq 0}^n$, $\mathbf{i} = (i_1, i_2, \dots, i_n)$ and $\mathbf{j} = (j_1, j_2, \dots, j_n)$. Then $\mathbf{i} < \mathbf{j}$ if and only if there exists $s \in \{1, 2, \dots, n\}$ such that $i_r = j_r$ for $1 \leq r < s$ and $i_s < j_s$.

Hence if $\mathbf{z} = (z_1, z_2, \dots, z_n)$, then we let $\tau(\mathbf{z}) = (\tau(z_1), \tau(z_2), \dots, \tau(z_n))$ and we can write the generalized polynomial g as

$$g(\mathbf{z}) = \sum_{|\mathbf{i}|+|\mathbf{j}|\leq k} \alpha_{\mathbf{i}\mathbf{j}} \mathbf{z}^{\mathbf{i}} \tau(\mathbf{z})^{\mathbf{j}}.$$

If g is a generalized polynomial of the form (2), then we define the *degree* of g as

$$\deg(g) = \max\{|\mathbf{i}| + |\mathbf{j}| : \alpha_{\mathbf{ij}} \neq 0\}.$$

We see that g is a generalized polynomial in n variables if and only if there exists a polynomial \tilde{g} in $2n$ variables such that

$$g(z_1, z_2, \dots, z_n) = \tilde{g}(z_1, \tau(z_1), \dots, z_n, \tau(z_n)).$$

We are particularly interested in generalized polynomials g with values in \mathbb{Q} . They are characterized by the property $g(\mathbf{a}) = \tau(g(\mathbf{a}))$ for every $\mathbf{a} \in K^n$. We prove below that the coefficients of the two conjugate terms $\mathbf{z}^{\mathbf{i}}\tau(\mathbf{z})^{\mathbf{j}}$ and $\tau(\mathbf{z})^{\mathbf{j}}\mathbf{z}^{\mathbf{i}}$ in such a polynomial must be conjugate, i.e., $\alpha_{\mathbf{ji}} = \tau(\alpha_{\mathbf{ij}})$.

Lemma 2.2. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{0, 1\}$ is squarefree and let g be a generalized polynomial in n variables over K . If $g(\mathbf{a}) = 0$ for every $\mathbf{a} \in K^n$, then g is the zero polynomial.*

Proof. Let \tilde{g} be a polynomial in $2n$ variables such that

$$g(z_1, z_2, \dots, z_n) = \tilde{g}(z_1, \tau(z_1), \dots, z_n, \tau(z_n)).$$

First, suppose that K is real. We know that $\sigma(K) = \{(\alpha, \tau(\alpha)) : \alpha \in K\}$ is dense in $V = \mathbb{R}^2$. Consequently, $\sigma(K)^n$ is dense in \mathbb{R}^{2n} . Because the polynomial \tilde{g} vanishes on $\sigma(K)^n$, it vanishes on the whole set \mathbb{R}^{2n} , and hence \tilde{g} is the zero polynomial.

Secondly, suppose that K is imaginary. In this case, $\sigma(K) = K$ is dense in $V = \mathbb{C}$. Thus K^n is dense in \mathbb{C}^n and the polynomial \tilde{g} is zero on the set

$$\{(z_1, \bar{z}_1, \dots, z_n, \bar{z}_n) : z_1, \dots, z_n \in K\}.$$

Let h be another polynomial in $2n$ variables defined by

$$h(x_1, y_1, \dots, x_n, y_n) = \tilde{g}(x_1 + iy_1, x_1 - iy_1, \dots, x_n + iy_n, x_n - iy_n).$$

Since h vanishes on \mathbb{R}^{2n} and

$$g(z_1, z_2, \dots, z_n) = h\left(\frac{z_1 + \bar{z}_1}{2}, \frac{z_1 - \bar{z}_1}{2i}, \dots, \frac{z_n + \bar{z}_n}{2}, \frac{z_n - \bar{z}_n}{2i}\right),$$

g is the zero polynomial. □

Proposition 2.3. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{0, 1\}$ is squarefree and let g be a generalized polynomial in n variables over K given by (2). We have $g(\mathbf{a}) \in \mathbb{Q}$ for every $\mathbf{a} \in K^n$ if and only if $\alpha_{\mathbf{ji}} = \tau(\alpha_{\mathbf{ij}})$ for every $\mathbf{i}, \mathbf{j} \in \mathbb{Z}_{\geq 0}^n$.*

Proof. Consider the two generalized polynomials

$$g(\mathbf{z}) = \sum_{|\mathbf{i}|+|\mathbf{j}| \leq k} \alpha_{\mathbf{ij}} \mathbf{z}^{\mathbf{i}} \tau(\mathbf{z})^{\mathbf{j}} = \sum_{|\mathbf{i}|+|\mathbf{j}| \leq k} \alpha_{\mathbf{ji}} \mathbf{z}^{\mathbf{j}} \tau(\mathbf{z})^{\mathbf{i}}$$

and

$$\tau(g)(\mathbf{z}) = \sum_{|\mathbf{i}|+|\mathbf{j}| \leq k} \tau(\alpha_{\mathbf{ij}}) \tau(\mathbf{z})^{\mathbf{i}} \mathbf{z}^{\mathbf{j}}.$$

The condition $g(\mathbf{a}) \in \mathbb{Q}$ for every $\mathbf{a} \in K$ is equivalent to the condition $g(\mathbf{a}) = \tau(g)(\mathbf{a})$ for every $\mathbf{a} \in K$. By Lemma 2.2, this is equivalent to $g - \tau(g)$ being the zero polynomial, which is in turn equivalent to $\alpha_{ji} = \tau(\alpha_{ij})$ for every $\mathbf{i}, \mathbf{j} \in \mathbb{Z}_{\geq 0}^n$. \square

Let K be a quadratic field and let Γ_K^n be the ring of \mathbb{Q} -valued generalized polynomials in n variables, i.e.,

$$\Gamma_K^n = \{g(z_1, \dots, z_n) \in K[z_1, \tau(z_1), \dots, z_n, \tau(z_n)] : g(\mathbf{a}) \in \mathbb{Q} \text{ for every } \mathbf{a} \in K^n\}.$$

For $d \in \mathbb{Z}_{\geq 0}$, let

$$\Gamma_K^{n,d} = \{g \in \Gamma_K^n : \deg(g) \leq d\}.$$

Clearly, $\Gamma_K^{n,d}$ is a vector space over \mathbb{Q} .

The main theorem of this section is the following one.

Theorem 2.4. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{0, 1\}$ is squarefree. If $n \in \mathbb{Z}_{\geq 1}$, then*

$$\Gamma_K^n = \mathbb{Q} \left[z_1 + \tau(z_1), \sqrt{D}(z_1 - \tau(z_1)), \dots, z_n + \tau(z_n), \sqrt{D}(z_n - \tau(z_n)) \right].$$

The first step is to prove the theorem for $n = 1$. In this case, we denote the single variable by z instead of z_1 , thus

$$\Gamma_K^1 = \{g(z) \in K[z, \tau(z)] : g(\alpha) \in \mathbb{Q} \text{ for every } \alpha \in K\}.$$

Lemma 2.5. *If $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{0, 1\}$ is squarefree, then $\Gamma_K^1 = \mathbb{Q} \left[z + \tau(z), \sqrt{D}(z - \tau(z)) \right]$.*

Proof. Let $S_1 = \mathbb{Q} \left[z + \tau(z), \sqrt{D}(z - \tau(z)) \right]$. We have $S_1 \subset \Gamma_K^1$ because $z + \tau(z)$ and $\sqrt{D}(z - \tau(z))$ are \mathbb{Q} -valued, and we want to prove $\Gamma_K^1 \subset S_1$.

We claim that $z\tau(z) \in S_1$. Since

$$D(z - \tau(z))^2 = \left(\sqrt{D}(z - \tau(z)) \right)^2 \in S_1,$$

we have also $(z - \tau(z))^2 \in S_1$, and hence

$$z\tau(z) = \frac{1}{2} \left((z + \tau(z))^2 - (z - \tau(z))^2 \right) \in S_1.$$

This proves the claim.

Let $g \in \Gamma_K^1$. We prove $g \in S_1$ by induction on $d = \deg(g)$. If $d = 0$, then $g(z) = a \in \mathbb{Q}$ is a constant polynomial. Thus we assume that $d \geq 1$ and that the \mathbb{Q} -valued generalized polynomials of degree $< d$ are contained in S_1 . We write g as

$$g(z) = \sum_{\substack{i,j \geq 0 \\ i+j \leq d}} \alpha_{i,j} z^i \tau(z)^j = \sum_{\substack{i,j \geq 1 \\ i+j \leq d}} \alpha_{i,j} z^i \tau(z)^j + \sum_{i=0}^d \alpha_{i,0} z^i + \sum_{j=0}^d \alpha_{0,j} \tau(z)^j.$$

By Proposition 2.3, $\alpha_{j,i} = \tau(\alpha_{i,j})$. Let

$$g_1(z) = \sum_{\substack{i,j \geq 1 \\ i+j \leq d}} \alpha_{i,j} z^i \tau(z)^j,$$

$$g_2(z) = \sum_{i=0}^d \alpha_{i,0} z^i + \tau(\alpha_{i,0}) \tau(z)^i,$$

so that $g(z) = g_1(z) + g_2(z)$. We have $g_1(z) = z\tau(z) \cdot h_1(z)$ where

$$h_1(z) = \sum_{\substack{i,j \geq 1 \\ i+j \leq d}} \alpha_{i,j} z^{i-1} \tau(z)^{j-1}$$

is a \mathbb{Q} -valued generalized polynomial of degree $< d$. By the inductive hypothesis, $h_1 \in S_1$, and because we also know that $z\tau(z) \in S_1$, we get $g_1 \in S_1$.

Next, we have $g_2(z) = \alpha_{d,0} z^d + \tau(\alpha_{d,0}) \tau(z)^d + h_2(z)$ where

$$h_2(z) = \sum_{i=0}^{d-1} \alpha_{i,0} z^i + \tau(\alpha_{i,0}) \tau(z)^i$$

is a \mathbb{Q} -valued generalized polynomial of degree $< d$. By the inductive hypothesis, $h_2 \in S_2$. We show $\alpha_{d,0} z^d + \tau(\alpha_{d,0}) \tau(z)^d \in S_1$. If we let $\alpha_{d,0} = a + b\sqrt{D}$ for $a, b \in \mathbb{Q}$, then

$$\alpha_{d,0} z^d + \tau(\alpha_{d,0}) \tau(z)^d = a(z^d + \tau(z)^d) + b\sqrt{D}(z^d - \tau(z)^d).$$

We have $z^d + \tau(z)^d = (z + \tau(z))^d - z\tau(z) \cdot r_1(z)$ where

$$r_1(z) = \sum_{i=1}^{d-1} \binom{d}{i} z^{i-1} \tau(z)^{d-i-1}$$

is a \mathbb{Q} -valued generalized polynomial of degree $< d$, hence $r_1 \in S_1$. Because $(z + \tau(z))^d$ and $z\tau(z)$ also belong to S_1 , we get $z^d + \tau(z)^d \in S_1$.

We also have $\sqrt{D}(z^d - \tau(z)^d) = \sqrt{D}(z - \tau(z)) \cdot r_2(z)$ where

$$r_2(z) = \sum_{i=0}^{d-1} z^{d-1-i} \tau(z)^i$$

is a \mathbb{Q} -valued generalized polynomial of degree $< d$, hence $r_2 \in S_1$. Because $\sqrt{D}(z - \tau(z)) \in S_1$, we get $\sqrt{D}(z^d - \tau(z)^d) \in S_1$. This proves $\alpha_{d,0} z^d + \tau(\alpha_{d,0}) \tau(z)^d \in S_1$, thus $g_2 \in S_1$.

Finally, because $g(z) = g_1(z) + g_2(z)$, we get $g \in S_1$. \square

Next, we find a basis of the vector space $\Gamma_K^{n,d}$ over \mathbb{Q} . Let $n \in \mathbb{Z}_{\geq 1}$ and $d \in \mathbb{Z}_{\geq 0}$. We define the polynomials $g_{\mathbf{i}, \mathbf{j}}$ for $\mathbf{i}, \mathbf{j} \in \mathbb{Z}_{\geq 0}^n$, $|\mathbf{i}| + |\mathbf{j}| \leq d$ as follows: If

$\mathbf{i} < \mathbf{j}$, then

$$\begin{aligned} g_{ij}(\mathbf{z}) &= \mathbf{z}^i \tau(\mathbf{z})^j + \tau(\mathbf{z})^i \mathbf{z}^j, \\ g_{ji}(\mathbf{z}) &= \sqrt{D} (\mathbf{z}^i \tau(\mathbf{z})^j - \tau(\mathbf{z})^i \mathbf{z}^j) \end{aligned}$$

and if $\mathbf{i} = \mathbf{j}$, then

$$g_{ii} = \mathbf{z}^i \tau(\mathbf{z})^i.$$

Lemma 2.6. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{0, 1\}$ is squarefree, $n \in \mathbb{Z}_{\geq 1}$, and $d \in \mathbb{Z}_{\geq 0}$. The elements g_{ij} for $\mathbf{i}, \mathbf{j} \in \mathbb{Z}_{\geq 0}^n$, $|\mathbf{i}| + |\mathbf{j}| \leq d$ form a basis of the vector space $\Gamma_K^{n,d}$ over \mathbb{Q} .*

Proof. Let

$$g(\mathbf{z}) = \sum_{|\mathbf{i}|+|\mathbf{j}| \leq d} \alpha_{ij} \mathbf{z}^i \tau(\mathbf{z})^j \in \Gamma_K^{n,d}.$$

By Proposition 2.3, $\alpha_{ji} = \tau(\alpha_{ij})$ for every $\mathbf{i}, \mathbf{j} \in \mathbb{Z}_{\geq 0}^n$, hence

$$g(\mathbf{z}) = \sum_{\substack{\mathbf{i} < \mathbf{j} \\ |\mathbf{i}|+|\mathbf{j}| \leq d}} \alpha_{ij} \mathbf{z}^i \tau(\mathbf{z})^j + \tau(\alpha_{ij}) \tau(\mathbf{z})^i \mathbf{z}^j + \sum_{2|\mathbf{i}| \leq d} \alpha_{ii} \mathbf{z}^i \tau(\mathbf{z})^i.$$

Let $\alpha_{ij} = a_{ij} + b_{ij} \sqrt{D}$ where $a_{ij}, b_{ij} \in \mathbb{Q}$. In particular, $\alpha_{ii} = \tau(\alpha_{ii})$ implies $\alpha_{ii} = a_{ii}$. We get

$$\begin{aligned} g(\mathbf{z}) &= \sum_{\substack{\mathbf{i} < \mathbf{j} \\ |\mathbf{i}|+|\mathbf{j}| \leq d}} a_{ij} (\mathbf{z}^i \tau(\mathbf{z})^j + \tau(\mathbf{z})^i \mathbf{z}^j) + b_{ij} \sqrt{D} (\mathbf{z}^i \tau(\mathbf{z})^j - \tau(\mathbf{z})^i \mathbf{z}^j) \\ &\quad + \sum_{2|\mathbf{i}| \leq d} a_{ii} \mathbf{z}^i \tau(\mathbf{z})^i, \end{aligned}$$

hence g is a \mathbb{Q} -linear combination of the elements g_{ij} .

Let

$$B = \{g_{ij} : \mathbf{i}, \mathbf{j} \in \mathbb{Z}_{\geq 0}^n, |\mathbf{i}| + |\mathbf{j}| \leq d\}.$$

If $\mathbf{i} \neq \mathbf{j}$, then the only generalized polynomials in B containing the term $\mathbf{z}^i \tau(\mathbf{z})^j$ are g_{ij} and g_{ji} . Since they are linearly independent, neither can be expressed as a linear combination of the remaining elements in B . Similarly, if $\mathbf{i} = \mathbf{j}$, then $\mathbf{z}^i \tau(\mathbf{z})^i$ is contained only in g_{ii} , and thus g_{ii} cannot be expressed as a linear combination of the other elements in B . This proves that the g_{ij} are linearly independent. \square

Proof of Theorem 2.4. Let

$$S_n = \mathbb{Q} \left[z_1 + \tau(z_1), \sqrt{D}(z_1 - \tau(z_1)), \dots, z_n + \tau(z_n), \sqrt{D}(z_n - \tau(z_n)) \right].$$

We have $S_n \subset \Gamma_K^n$ because the generators of S_n are \mathbb{Q} -valued, and we want to prove $\Gamma_K^n \subset S_n$.

We proceed by induction on n . For $n = 1$, this is Lemma 2.5. Suppose that $n \geq 2$ and the statement is true for polynomials with $< n$ variables.

Let $g \in \Gamma_K^n$. We show $g \in S_n$ by induction on $d = \deg(g)$. If $d = 0$, then $g(\mathbf{z}) = a \in \mathbb{Q}$ is a constant polynomial. Thus we assume that $d \geq 1$ and that the \mathbb{Q} -valued generalized polynomials of degree $< d$ in n variables are contained in S_n . The polynomial g is a \mathbb{Q} -linear combination of the elements in the basis from Lemma 2.6. Thus it is enough to show that the basis elements of degree d belong to S_n .

We show that if $\mathbf{i}, \mathbf{j} \in \mathbb{Z}_{\geq 0}^n$, $|\mathbf{i}| + |\mathbf{j}| = d$, and $\mathbf{i} < \mathbf{j}$, then

$$\begin{aligned} \mathbf{g}_{\mathbf{ij}}(\mathbf{z}) &= \mathbf{z}^{\mathbf{i}}\tau(\mathbf{z})^{\mathbf{j}} + \tau(\mathbf{z})^{\mathbf{i}}\mathbf{z}^{\mathbf{j}} \in S_n, \\ \mathbf{g}_{\mathbf{ji}}(\mathbf{z}) &= \sqrt{D}(\mathbf{z}^{\mathbf{i}}\tau(\mathbf{z})^{\mathbf{j}} - \tau(\mathbf{z})^{\mathbf{i}}\mathbf{z}^{\mathbf{j}}) \in S_n. \end{aligned}$$

If $i_n = 0$ and $j_n = 0$, then the variable z_n does not appear in $\mathbf{g}_{\mathbf{ij}}$ and $\mathbf{g}_{\mathbf{ji}}$, hence $\mathbf{g}_{\mathbf{ij}}, \mathbf{g}_{\mathbf{ji}} \in S_n$ by the inductive hypothesis.

If $i_n \geq 1$, then we let

$$\begin{aligned} h(\mathbf{z}) &= z_1^{i_1}\tau(z_1)^{j_1}z_2^{i_2}\tau(z_2)^{j_2} \cdots z_n^{i_n-1}\tau(z_n)^{j_n}, \\ \mathbf{g}_{\mathbf{ij}}^1(\mathbf{z}) &= (z_n + \tau(z_n)) \cdot (h(\mathbf{z}) + \tau(h(\mathbf{z}))), \\ \mathbf{g}_{\mathbf{ij}}^2(\mathbf{z}) &= (z_n - \tau(z_n)) \cdot (h(\mathbf{z}) - \tau(h(\mathbf{z}))), \\ \mathbf{g}_{\mathbf{ji}}^1(\mathbf{z}) &= \sqrt{D}(z_n - \tau(z_n)) \cdot (h(\mathbf{z}) + \tau(h(\mathbf{z}))), \\ \mathbf{g}_{\mathbf{ji}}^2(\mathbf{z}) &= (z_n + \tau(z_n)) \cdot \sqrt{D}(h(\mathbf{z}) - \tau(h(\mathbf{z}))). \end{aligned}$$

so that

$$\begin{aligned} \mathbf{g}_{\mathbf{ij}}(\mathbf{z}) &= z_n h(\mathbf{z}) + \tau(z_n)\tau(h(\mathbf{z})) = \frac{1}{2}(\mathbf{g}_{\mathbf{ij}}^1 + \mathbf{g}_{\mathbf{ij}}^2), \\ \mathbf{g}_{\mathbf{ji}}(\mathbf{z}) &= \sqrt{D}(z_n h(\mathbf{z}) - \tau(z_n)\tau(h(\mathbf{z}))) = \frac{1}{2}(\mathbf{g}_{\mathbf{ji}}^1 + \mathbf{g}_{\mathbf{ji}}^2). \end{aligned}$$

Because $h(\mathbf{z})$ has degree $d - 1$, we can apply induction on $h(\mathbf{z}) + \tau(h(\mathbf{z}))$ and $\sqrt{D}(h(\mathbf{z}) - \tau(h(\mathbf{z})))$. Since

$$z_n + \tau(z_n), \sqrt{D}(z_n - \tau(z_n)), h(\mathbf{z}) + \tau(h(\mathbf{z})), \sqrt{D}(h(\mathbf{z}) - \tau(h(\mathbf{z}))) \in S_n,$$

we immediately obtain $\mathbf{g}_{\mathbf{ij}}^1, \mathbf{g}_{\mathbf{ji}}^1, \mathbf{g}_{\mathbf{ij}}^2 \in S_n$. We also get

$$\mathbf{g}_{\mathbf{ij}}^2 = \frac{1}{D} \cdot \sqrt{D}(z_n - \tau(z_n)) \cdot \sqrt{D}(h(\mathbf{z}) - \tau(h(\mathbf{z}))) \in S_n,$$

and $\mathbf{g}_{\mathbf{ij}}, \mathbf{g}_{\mathbf{ji}} \in S_n$ follows.

If $i_n = 0$ and $j_n \geq 1$, then the proof is analogous to the case $i_n \geq 1$.

Next, we show that if $\mathbf{i} \in \mathbb{Z}_{\geq 0}^n$ and $2|\mathbf{i}| = d$, then $\mathbf{g}_{\mathbf{ii}}(\mathbf{z}) = \mathbf{z}^{\mathbf{i}}\tau(\mathbf{z})^{\mathbf{i}} \in S_n$.

If $i_n = 0$, then $\mathbf{g}_{\mathbf{ii}}$ is a polynomial in $n - 1$ variables, hence $\mathbf{g}_{\mathbf{ii}} \in S_n$ by the inductive hypothesis.

If $i_n \geq 1$, then we let

$$h(\mathbf{z}) = z_1^{i_1}\tau(z_1)^{i_1}z_2^{i_2}\tau(z_2)^{i_2} \cdots z_n^{i_n-1}\tau(z_n)^{i_n-1}.$$

We have $g_{ii}(z) = z_n \tau(z_n) \cdot h(z)$. Since the theorem holds for generalized polynomials in one variable, we get

$$z_n \tau(z_n) \in \mathbb{Q} \left[z_n + \tau(z_n), \sqrt{D} (z_n - \tau(z_n)) \right] \subset S_n.$$

The generalized polynomial h has degree $d - 2$, hence $h \in S_n$ by the inductive hypothesis, and $g_{ii} \in S_n$ follows. \square

3. Generalized quadratic forms

Generalized quadratic forms are defined in [5, Definition 1.1] over a Galois extension K/\mathbb{Q} of degree d (which is assumed to be totally real). We need the definition only over quadratic fields.

Definition 3.1. Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{0, 1\}$ is squarefree. A *generalized quadratic form* in n variables over K is a generalized polynomial

$$G(z_1, z_2, \dots, z_n) = \sum_{1 \leq i < j \leq n} \alpha_{ij} z_i z_j + \sum_{1 \leq i, j \leq n} \beta_{ij} z_i \tau(z_j) + \sum_{1 \leq i \leq j \leq n} \gamma_{ij} \tau(z_i) \tau(z_j) \quad (3)$$

where $\alpha_{ij}, \gamma_{ij} \in K$ for $1 \leq i < j \leq n$ and $\beta_{ij} \in K$ for $1 \leq i, j \leq n$.

A generalized quadratic form G given by (3) will be called *integral* if $\alpha_{ij}, \gamma_{ij} \in \mathcal{O}_K$ for $1 \leq i < j \leq n$ and $\beta_{ij} \in \mathcal{O}_K$ for $1 \leq i, j \leq n$. An integral generalized quadratic form will be called *classical* if $\alpha_{ij}, \gamma_{ij} \in 2\mathcal{O}_K$ for $i < j$ and $\beta_{ij} \in 2\mathcal{O}_K$ for $1 \leq i, j \leq n$.

Let us recall that a quadratic form Q in n variables over K is a homogeneous polynomial of degree 2 over K . We can also obtain it as a generalized quadratic form with $\beta_{ij}, \gamma_{ij} = 0$ for all $1 \leq i, j \leq n$.

Then if G is a generalized quadratic form in n variables over K , then we can define a *quadratic form* Q associated to G in $2n$ variables by

$$Q(x_1, y_1, \dots, x_n, y_n) = G(x_1 + y_1 \omega_D, \dots, x_n + y_n \omega_D). \quad (4)$$

In other words, Q is obtained from G by expressing the variables z_1, z_2, \dots, z_n in the integral basis $(1, \omega_D)$. It has coefficients in K and the variables $x_1, y_1, \dots, x_n, y_n$ take values in \mathbb{Q} .

If G is a generalized form in n variables over K given by (3), then we define the *matrix of* G as the $2n \times 2n$ matrix

$$M_G = \begin{pmatrix} A & B \\ B^\top & C \end{pmatrix}$$

where

$$A = \begin{pmatrix} \alpha_{11} & \frac{\alpha_{12}}{2} & \dots & \frac{\alpha_{1n}}{2} \\ \frac{\alpha_{12}}{2} & \alpha_{22} & \dots & \frac{\alpha_{2n}}{2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\alpha_{1n}}{2} & \frac{\alpha_{2n}}{2} & \dots & \alpha_{nn} \end{pmatrix}, \quad B = \begin{pmatrix} \frac{\beta_{11}}{2} & \frac{\beta_{12}}{2} & \dots & \frac{\beta_{1n}}{2} \\ \frac{\beta_{21}}{2} & \frac{\beta_{22}}{2} & \dots & \frac{\beta_{2n}}{2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\beta_{n1}}{2} & \frac{\beta_{n2}}{2} & \dots & \frac{\beta_{nn}}{2} \end{pmatrix},$$

$$C = \begin{pmatrix} \gamma_{11} & \frac{\gamma_{12}}{2} & \cdots & \frac{\gamma_{1n}}{2} \\ \frac{\gamma_{12}}{2} & \gamma_{22} & \cdots & \frac{\gamma_{2n}}{2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\gamma_{1n}}{2} & \frac{\gamma_{2n}}{2} & \cdots & \gamma_{nn} \end{pmatrix}.$$

Thus M_G is the matrix such that

$$G(\mathbf{z}) = (\mathbf{z}, \tau(\mathbf{z}))M_G \begin{pmatrix} \mathbf{z} \\ \tau(\mathbf{z}) \end{pmatrix}$$

for $\mathbf{z} \in K^n$. The *rank* of G is the rank of the matrix M_G over K .

The $2n \times 2n$ matrix

$$T = \begin{pmatrix} 1 & \omega_D & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \omega_D & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 1 & \omega_D \\ 1 & \tau(\omega_D) & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \tau(\omega_D) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 1 & \tau(\omega_D) \end{pmatrix}$$

determines the change of variables

$$(x_1, y_1, \dots, x_n, y_n) \mapsto (z_1, \dots, z_n, \tau(z_1), \dots, \tau(z_n)).$$

Thus, if the quadratic form Q is associated to the quadratic form G defined by (4), and M_Q, M_G are their respective matrices, then

$$M_Q = T^\top M_G T.$$

If G is a generalized quadratic form over \mathbb{Q} then we say that G represents $b \in \mathbb{Z}$ over \mathbb{Z} if there exists $\mathbf{a} \in \mathbb{Z}^n$ such that $G(\mathbf{a}) = b$. In particular, according to our definition, every quadratic form represents 0. If there exists $\mathbf{a} \in \mathbb{Z}^n \setminus \{(0, 0, \dots, 0)\}$ such that $G(\mathbf{a}) = 0$, then we say that G represents 0 *non-trivially*. The form G is called *positive definite* if M_G is a positive definite matrix and G is called *universal* if it is a positive definite integral form such that it represents every $b \in \mathbb{Z}_{\geq 1}$.

Lemma 3.2. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{0, 1\}$ is squarefree, G be a generalized form in n variables over K given by (3), and M_G be the matrix of G . If Q is the quadratic form in $2n$ variables associated to G and M_Q is the matrix of Q , then*

$$\det(M_Q) = D^n \cdot \begin{cases} \det(2M_G), & \text{if } D \equiv 2, 3 \pmod{4}, \\ \det(M_G), & \text{if } D \equiv 1 \pmod{4}. \end{cases}$$

Proof. Since $M_G = T^\top M_Q T$, we get

$$\det(M_Q) = (\det T)^2 \det(M_G).$$

Let us compute the determinant of T . If π is the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 2n-1 & 2n \\ 1 & n+1 & 2 & n+2 & \dots & n & 2n \end{pmatrix}$$

(the out-shuffle), then $\text{sgn}(\pi) = (-1)^{\frac{n(n-1)}{2}}$. Applying this permutation to the rows of T , we obtain a block-diagonal matrix where each block equals

$$M = \begin{pmatrix} 1 & \omega_D \\ 1 & \tau(\omega_D) \end{pmatrix}.$$

If $D \equiv 2, 3 \pmod{4}$, then

$$\det M = \det \begin{pmatrix} 1 & \sqrt{D} \\ 1 & -\sqrt{D} \end{pmatrix} = -2\sqrt{D},$$

while if $D \equiv 1 \pmod{4}$, then

$$\det M = \det \begin{pmatrix} 1 & \frac{1+\sqrt{D}}{2} \\ 1 & \frac{1-\sqrt{D}}{2} \end{pmatrix} = -\sqrt{D}.$$

Thus

$$\det T = (-1)^{\frac{n(n-1)}{2}} (\det M)^n = \begin{cases} (-1)^{\frac{n(n+1)}{2}} D^{\frac{n}{2}} 2^n, & \text{if } D \equiv 2, 3 \pmod{4}, \\ (-1)^{\frac{n(n+1)}{2}} D^{\frac{n}{2}}, & \text{if } D \equiv 1 \pmod{4}. \end{cases}$$

In the case $D \equiv 2, 3 \pmod{4}$, we get

$$\det(M_Q) = D^n 2^{2n} \det(M_G) = D^n \det(2M_G),$$

and in the case $D \equiv 1 \pmod{4}$, we get

$$\det(M_Q) = D^n \det(M_G). \quad \square$$

Corollary 3.3. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{0, 1\}$ is squarefree, G be a generalized form in n variables over K given by (3), Q be the quadratic form in $2n$ variables associated to G , and M_Q be the matrix of Q . If one of the following conditions is satisfied:*

- (1) $D \equiv 2, 3 \pmod{4}$ and G is integral,
- (2) $D \equiv 1 \pmod{4}$ and G is classical,

then $D^n \mid \det(M_Q)$.

Proof. First, assume that $D \equiv 2, 3 \pmod{4}$ and G is integral. Since the elements of M_G are in $\frac{1}{2}\mathcal{O}_K$, we must have $\det(2M_G) \in \mathcal{O}_K$. Thus $D^n \mid \det(M_Q)$ by Lemma 3.2.

Secondly, assume that $D \equiv 1 \pmod{4}$ and G is classical. The elements of M_G are in \mathcal{O}_K , hence $\det(M_G) \in \mathcal{O}_K$, and $D^n \mid \det(M_Q)$ by Lemma 3.2. \square

Lemma 3.4. Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{0, 1\}$ is squarefree and let G be a generalized form in n variables over K given by (3). Assume that $G(\mathbf{a}) \in \mathbb{Q}$ for every $\mathbf{a} \in K^n$. Let Q be the quadratic form in $2n$ variables associated to G . If $1 \leq i \leq j \leq n$, then we let a_{ij} , b_{ij} , b_{ji} , and c_{ij} be the coefficients of the terms $x_i x_j$, $x_i y_j$, $x_j y_i$, and $y_i y_j$ in Q , respectively, so that the matrix M_Q of the quadratic form Q is

$$M_Q = \begin{pmatrix} a_{11} & \frac{b_{11}}{2} & \frac{a_{12}}{2} & \frac{b_{12}}{2} & \cdots & \frac{a_{1n}}{2} & \frac{b_{1n}}{2} \\ \frac{b_{11}}{2} & c_{11} & \frac{b_{21}}{2} & \frac{c_{12}}{2} & \cdots & \frac{b_{n1}}{2} & \frac{c_{1n}}{2} \\ \frac{a_{12}}{2} & \frac{b_{21}}{2} & a_{22} & \frac{b_{22}}{2} & \cdots & \frac{a_{2n}}{2} & \frac{b_{2n}}{2} \\ \frac{b_{12}}{2} & \frac{c_{12}}{2} & \frac{b_{22}}{2} & c_{22} & \cdots & \frac{b_{n2}}{2} & \frac{c_{2n}}{2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{a_{1n}}{2} & \frac{b_{n1}}{2} & \frac{a_{2n}}{2} & \frac{b_{n2}}{2} & \cdots & a_{nn} & \frac{b_{nn}}{2} \\ \frac{b_{1n}}{2} & \frac{c_{1n}}{2} & \frac{b_{2n}}{2} & \frac{c_{2n}}{2} & \cdots & \frac{b_{nn}}{2} & c_{nn} \end{pmatrix}.$$

If $1 \leq i < j \leq n$, then

$$\begin{aligned} a_{ij} &= \text{Tr}(\alpha_{ij}) + \text{Tr}(\beta_{ij}), \\ b_{ij} &= \text{Tr}(\alpha_{ij}\omega_D) + \text{Tr}(\beta_{ij}\tau(\omega_D)), \\ b_{ji} &= \text{Tr}(\alpha_{ij}\omega_D) + \text{Tr}(\beta_{ij}\omega_D), \\ c_{ij} &= \text{Tr}(\alpha_{ij}\omega_D^2) + \text{Tr}(\beta_{ij})N(\omega_D). \end{aligned}$$

Moreover, if $1 \leq i \leq n$, then

$$\begin{aligned} a_{ii} &= \text{Tr}(\alpha_{ii}) + \beta_{ii}, \\ b_{ii} &= \text{Tr}(2\alpha_{ii}\omega_D) + \beta_{ii} \text{Tr}(\omega_D), \\ c_{ii} &= \text{Tr}(\alpha_{ii}\omega_D^2) + \beta_{ii} N(\omega_D). \end{aligned}$$

Proof. The quadratic form Q is given by

$$\begin{aligned} Q(x_1, y_1, \dots, x_n, y_n) &= \sum_{1 \leq i \leq j \leq n} \alpha_{ij}(x_i + y_i\omega)(x_j + y_j\omega) \\ &+ \sum_{1 \leq i, j \leq n} \beta_{ij}(x_i + y_i\omega)(x_j + y_j\tau(\omega)) \\ &+ \sum_{1 \leq i \leq j \leq n} \gamma_{ij}(x_i + y_i\tau(\omega))(x_j + y_j\tau(\omega)). \end{aligned}$$

By Proposition 2.3, $G(\mathbf{a}) \in \mathbb{Q}$ for every $\mathbf{a} \in K^n$ if and only if the coefficients of conjugate terms are conjugate, i.e., $\gamma_{ij} = \tau(\alpha_{ij})$ for $1 \leq i \leq j \leq n$ and $\beta_{ji} = \tau(\beta_{ij})$ for $1 \leq i, j \leq n$. If $i < j$, then

$$\begin{aligned} a_{ij} &= \alpha_{ij} + \beta_{ij} + \beta_{ji} + \gamma_{ij} = \text{Tr}(\alpha_{ij}) + \text{Tr}(\beta_{ij}), \\ b_{ij} &= \alpha_{ij}\omega + \beta_{ij}\tau(\omega) + \beta_{ji}\omega + \gamma_{ij}\tau(\omega) = \text{Tr}(\alpha_{ij}\omega) + \text{Tr}(\beta_{ij}\tau(\omega)), \\ b_{ji} &= \alpha_{ij}\omega + \beta_{ij}\omega + \beta_{ji}\tau(\omega) + \gamma_{ij}\tau(\omega) = \text{Tr}(\alpha_{ij}\omega) + \text{Tr}(\beta_{ij}\omega), \\ c_{ij} &= \alpha_{ij}\omega^2 + (\beta_{ij} + \beta_{ji})N(\omega) + \gamma_{ij}\tau(\omega)^2 = \text{Tr}(\alpha_{ij}\omega^2) + \text{Tr}(\beta_{ij})N(\omega). \end{aligned}$$

Moreover,

$$\begin{aligned} a_{ii} &= \alpha_{ii} + \beta_{ii} + \gamma_{ii} = \text{Tr}(\alpha_{ii}) + \beta_{ii}, \\ b_{ii} &= 2\alpha_{ii}\omega + \beta_{ii} \text{Tr}(\omega) + 2\gamma_{ii}\tau(\omega) = \text{Tr}(2\alpha_{ii}\omega) + \beta_{ii} \text{Tr}(\omega), \\ c_{ii} &= \alpha_{ii}\omega^2 + \beta_{ii} N(\omega) + \gamma_{ii}\tau(\omega)^2 = \text{Tr}(\alpha_{ii}\omega^2) + \beta_{ii} N(\omega). \end{aligned} \quad \square$$

Lemma 3.5. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{0, 1\}$ is squarefree and let G be a generalized form in n variables over K given by (3). Assume that G is \mathbb{Z} -valued. Let Q be the quadratic form in $2n$ variables associated to G . If G is integral, then Q is integral. Moreover, if one of the following conditions holds:*

- (1) $D \equiv 2, 3 \pmod{4}$ and G is integral,
- (2) $D \equiv 1 \pmod{4}$ and G is classical,

then Q is classical.

Proof. As before, we let a_{ij}, b_{ij}, b_{ji} , and c_{ij} be the coefficients of the terms $x_i x_j, x_i y_j, x_j y_i$, and $y_i y_j$ in Q , respectively, for $1 \leq i \leq j \leq n$. If G is integral, then $\alpha_{ij}, \beta_{ij} \in \mathcal{O}_K$ for $i \leq j$. The elements of \mathcal{O}_K have trace in \mathbb{Z} , hence $a_{ij}, b_{ij}, b_{ji}, c_{ij} \in \mathbb{Z}$ by Lemma 3.4 and Q is integral.

If G is classical, then $\alpha_{ij}, \beta_{ij} \in 2\mathcal{O}_K$ for $i < j$ and $\beta_{ii} \in 2\mathbb{Z}$. It follows from Lemma 3.4 that $a_{ij}, b_{ij}, b_{ji}, c_{ij} \in 2\mathbb{Z}$ for $i < j$ and $b_{ii} \in 2\mathbb{Z}$, hence Q is classical.

If $D \equiv 2, 3 \pmod{4}$, then $\text{Tr}(\alpha)$ is even for every $\alpha \in \mathcal{O}_K$. Thus it is sufficient to assume that G is integral to conclude that Q is classical. □

Lemma 3.6. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{0, 1\}$ is squarefree and let G be a generalized form in n variables given by (3). Assume that $G(\mathbf{a}) \in \mathbb{Q}$ for every $\mathbf{a} \in K^n$. If $1 \leq i \leq j \leq n$, then we let $\alpha_{ij} = r_{ij} + s_{ij}\omega_D$ and $\beta_{ij} = t_{ij} + u_{ij}\omega_D$ where $r_{ij}, s_{ij}, t_{ij}, u_{ij} \in \mathbb{Q}$. Let Q be the quadratic form in $2n$ variables associated to G . If $1 \leq i \leq j \leq n$, then we let a_{ij}, b_{ij}, b_{ji} , and c_{ij} be the coefficients of the terms $x_i x_j, x_i y_j, x_j y_i$, and $y_i y_j$ in Q , respectively.*

If $D \equiv 2, 3 \pmod{4}$ and $1 \leq i < j \leq n$, then

$$\begin{aligned} a_{ij} &= 2r_{ij} + 2t_{ij}, \\ b_{ij} &= 2Ds_{ij} - 2Du_{ij}, \\ b_{ji} &= 2Ds_{ij} + 2Du_{ij}, \\ c_{ij} &= 2Dr_{ij} - 2Dt_{ij}. \end{aligned}$$

Moreover, if $1 \leq i \leq n$, then

$$\begin{aligned} a_{ii} &= 2r_{ii} + t_{ii}, \\ b_{ii} &= 4Ds_{ii}, \\ c_{ii} &= 2Dr_{ii} - Dt_{ii}. \end{aligned}$$

If $D \equiv 1 \pmod{4}$ and $1 \leq i < j \leq n$, then

$$\begin{aligned} a_{ij} &= 2r_{ij} + s_{ij} + 2t_{ij} + u_{ij}, \\ b_{ij} &= r_{ij} + \frac{1+D}{2}s_{ij} + t_{ij} + \frac{1-D}{2}u_{ij}, \\ b_{ji} &= r_{ij} + \frac{1+D}{2}s_{ij} + t_{ij} + \frac{1+D}{2}u_{ij}, \\ c_{ij} &= \frac{1+D}{2}r_{ij} + \frac{1+3D}{4}s_{ij} + \frac{1-D}{2}t_{ij} + \frac{1-D}{4}u_{ij}. \end{aligned}$$

Moreover, if $1 \leq i \leq n$, then

$$\begin{aligned} a_{ii} &= 2r_{ii} + s_{ii} + t_{ii}, \\ b_{ii} &= 2r_{ii} + (1+D)s_{ii} + t_{ii}, \\ c_{ii} &= \frac{1+D}{2}r_{ii} + \frac{1+3D}{4}s_{ii} + \frac{1-D}{4}t_{ii}. \end{aligned}$$

Proof. We express the coefficients of Q using Lemma 3.4. If $1 \leq i < j \leq n$, then

$$\begin{aligned} a_{ij} &= \text{Tr}(\alpha_{ij}) + \text{Tr}(\beta_{ij}) = \text{Tr}(r_{ij} + s_{ij}\omega_D) + \text{Tr}(t_{ij} + u_{ij}\omega_D) \\ &= 2r_{ij} + s_{ij} \text{Tr}(\omega_D) + 2t_{ij} + u_{ij} \text{Tr}(\omega_D), \\ b_{ij} &= \text{Tr}(\alpha_{ij}\omega_D) + \text{Tr}(\beta_{ij}\tau(\omega_D)) = \text{Tr}(r_{ij}\omega_D + s_{ij}\omega_D^2) + \text{Tr}(t_{ij}\tau(\omega_D) + u_{ij}N(\omega_D)) \\ &= r_{ij} \text{Tr}(\omega_D) + s_{ij} \text{Tr}(\omega_D^2) + t_{ij} \text{Tr}(\omega_D) + 2u_{ij}N(\omega_D), \\ b_{ji} &= \text{Tr}(\alpha_{ij}\omega_D) + \text{Tr}(\beta_{ij}\omega_D) = \text{Tr}(r_{ij}\omega_D + s_{ij}\omega_D^2) + \text{Tr}(t_{ij}\omega_D + u_{ij}\omega_D^2) \\ &= r_{ij} \text{Tr}(\omega_D) + s_{ij} \text{Tr}(\omega_D^2) + t_{ij} \text{Tr}(\omega_D) + u_{ij} \text{Tr}(\omega_D^2), \\ c_{ij} &= \text{Tr}(\alpha_{ij}\omega_D^2) + \text{Tr}(\beta_{ij})N(\omega_D) = \text{Tr}(r_{ij}\omega_D^2 + s_{ij}\omega_D^3) + \text{Tr}(t_{ij} + u_{ij}\omega_D)N(\omega_D) \\ &= r_{ij} \text{Tr}(\omega_D^2) + s_{ij} \text{Tr}(\omega_D^3) + 2t_{ij}N(\omega_D) + u_{ij} \text{Tr}(\omega_D)N(\omega_D). \end{aligned}$$

Moreover, if $1 \leq i \leq n$, then

$$\begin{aligned} a_{ii} &= \text{Tr}(\alpha_{ii}) + \beta_{ii} = \text{Tr}(r_{ii} + s_{ii}\omega_D) + t_{ii} = 2r_{ii} + s_{ii} \text{Tr}(\omega_D) + t_{ii}, \\ b_{ii} &= \text{Tr}(2\alpha_{ii}\omega_D) + \beta_{ii} \text{Tr}(\omega_D) = \text{Tr}(2r_{ii}\omega_D + 2s_{ii}\omega_D^2) + t_{ii} \text{Tr}(\omega_D) \\ &= 2r_{ii} \text{Tr}(\omega_D) + 2s_{ii} \text{Tr}(\omega_D^2) + t_{ii} \text{Tr}(\omega_D), \\ c_{ii} &= \text{Tr}(\alpha_{ii}\omega_D^2) + \beta_{ii}N(\omega_D) = \text{Tr}(r_{ii}\omega_D^2 + s_{ii}\omega_D^3) + t_{ii}N(\omega_D) \\ &= r_{ii} \text{Tr}(\omega_D^2) + s_{ii} \text{Tr}(\omega_D^3) + t_{ii}N(\omega_D). \end{aligned}$$

If $D \equiv 2, 3 \pmod{4}$, then $\omega_D = \sqrt{D}$, hence $\text{Tr}(\omega_D) = 0$, $\text{Tr}(\omega_D^2) = \text{Tr}(D) = 2D$, $\text{Tr}(\omega_D^3) = \text{Tr}(D\sqrt{D}) = 0$, and $N(\omega_D) = -D$.

If $D \equiv 1 \pmod{4}$, then $\omega_D = \frac{1+\sqrt{D}}{2}$, so $\text{Tr}(\omega_D) = 1$, $\text{Tr}(\omega_D^2) = \text{Tr}\left(\frac{1+D}{4} + \frac{\sqrt{D}}{2}\right) = \frac{1+D}{2}$, $\text{Tr}(\omega_D^3) = \text{Tr}\left(\frac{1+3D}{8} + \frac{3+D}{8}\sqrt{D}\right) = \frac{1+3D}{4}$, and $N(\omega_D) = \frac{1-D}{4}$.

After substituting these expressions into the formulas above, we get the lemma. \square

If p is a prime and M a matrix with entries in \mathbb{Z} , then we let $M \bmod p$ denote the matrix with entries in $\mathbb{Z}/p\mathbb{Z}$ obtained by reducing the elements of M modulo p .

Proposition 3.7. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{0, 1\}$ is squarefree, $p \mid D$ be a prime number, and G an integral generalized quadratic form in n variables over K . Assume that G is \mathbb{Z} -valued. If Q is the quadratic form associated to G and M_Q is the matrix of Q , then $\text{rank}(M_Q \bmod p) \leq n$.*

Proof. We introduce the following notation: if M is a $2n \times 2n$ matrix and $1 \leq k \leq 2n$, let $[M]_k$ denote the k -th row of M .

If $1 \leq i \leq j \leq n$, then we let $\alpha_{ij} = r_{ij} + s_{ij}\omega_D$ and $\beta_{ij} = t_{ij} + u_{ij}\omega_D$ where $r_{ij}, s_{ij}, t_{ij}, u_{ij} \in \mathbb{Q}$.

First assume that $D \equiv 2, 3 \pmod{4}$. If $1 \leq k \leq n$, then the $2k$ -th row of M_Q equals

$$[M_Q]_{2k} = \left(\frac{b_{1k}}{2} \quad \frac{c_{1k}}{2} \quad \dots \quad \frac{b_{kk}}{2} \quad c_{kk} \quad \dots \quad \frac{b_{nk}}{2} \quad \frac{c_{kn}}{2} \right).$$

By Lemma 3.6, if $1 \leq i < j \leq n$, then

$$\frac{b_{ij}}{2} = D(s_{ij} - u_{ij}), \quad \frac{b_{ji}}{2} = D(s_{ij} + u_{ij}), \quad \frac{c_{ij}}{2} = D(r_{ij} - t_{ij}),$$

and if $1 \leq i \leq n$, then

$$\frac{b_{ii}}{2} = 2Ds_{ii}, \quad c_{ii} = D(2r_{ii} - t_{ii}).$$

Since we assume that G is integral, $r_{ij}, s_{ij}, t_{ij}, u_{ij} \in \mathbb{Z}$. Thus D divides every element of $[M_Q]_{2k}$. It follows that $M_Q \bmod p$ contains n zero rows, hence $\text{rank}(M_Q \bmod p) \leq n$.

Secondly, assume that $D \equiv 1 \pmod{4}$. Let N_Q be the matrix obtained from M_Q by subtracting two times the $2k$ -th row of M_Q from the $(2k-1)$ -th row of M_Q for every $1 \leq k \leq n$. The $(2k-1)$ -th row of N_Q equals $[M_Q]_{2k-1} - 2[M_Q]_{2k} =$

$$\left(\frac{a_{1k}}{2} - b_{1k} \quad \frac{b_{k1}}{2} - c_{1k} \quad \dots \quad a_{kk} - b_{kk} \quad \frac{b_{kk}}{2} - 2c_{kk} \quad \dots \quad \frac{a_{kn}}{2} - b_{nk} \quad \frac{b_{kn}}{2} - c_{kn} \right)$$

By Lemma 3.6, if $1 \leq i < j \leq n$, then

$$\begin{aligned} \frac{a_{ij}}{2} - b_{ij} &= \frac{1}{2} (2r_{ij} + s_{ij} + 2t_{ij} + u_{ij}) - \left(r_{ij} + \frac{1+D}{2}s_{ij} + t_{ij} + \frac{1-D}{2}u_{ij} \right) \\ &= D \cdot \frac{u_{ij} + s_{ij}}{2}, \end{aligned}$$

$$\begin{aligned} \frac{b_{ji}}{2} - c_{ij} &= \frac{1}{2} \left(r_{ij} + \frac{1+D}{2}s_{ij} + t_{ij} + \frac{1+D}{2}u_{ij} \right) \\ &\quad - \left(\frac{1+D}{2}r_{ij} + \frac{1+3D}{4}s_{ij} + \frac{1-D}{2}t_{ij} + \frac{1-D}{4}u_{ij} \right) \\ &= D \cdot \frac{t_{ij} + u_{ij} - r_{ij} - s_{ij}}{2}, \end{aligned}$$

and if $1 \leq i \leq n$, then

$$\begin{aligned} a_{ii} - b_{ii} &= 2r_{ii} + s_{ii} + t_{ii} - (2r_{ii} + (1 + D)s_{ii} + t_{ii}) = -Ds_{ii}, \\ \frac{b_{ii}}{2} - 2c_{ii} &= \frac{1}{2}(2r_{ii} + (1 + D)s_{ii} + t_{ii}) - 2\left(\frac{1 + D}{2}r_{ii} + \frac{1 + 3D}{4}s_{ii} + \frac{1 - D}{4}t_{ii}\right) \\ &= -Dr_{ii} - Ds_{ii} + \frac{D}{2}t_{ii}. \end{aligned}$$

We have $r_{ij}, s_{ij}, t_{ij}, u_{ij} \in \frac{1}{2}\mathbb{Z}$ because G is integral. Since we assume $D \equiv 1 \pmod{4}$, p is an odd prime and 2 is invertible modulo p . We see that every element of $[N_Q]_{2k-1}$ vanishes modulo p , hence $N_Q \pmod{p}$ contains n zero rows and $\text{rank}(M_Q \pmod{p}) = \text{rank}(N_Q \pmod{p}) \leq n$. \square

4. Binary and ternary generalized quadratic forms

Let us formally state a definition from the introduction.

Definition 4.1. Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z}_{\geq 2}$ is squarefree and let G be an integral generalized form in n variables over K . Assume that G is \mathbb{Z} -valued. If for every $a \in \mathbb{Z}_{\geq 1}$, there exist $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathcal{O}_K$ such that

$$G(\alpha_1, \alpha_2, \dots, \alpha_n) = a,$$

then we say that G is \mathbb{Z} -universal.

The purpose of this section is to investigate binary and ternary \mathbb{Z} -universal forms. We begin by proving one implication in statements (i) and (ii) of Theorem 1.2.

Proposition 4.2. Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z}_{\geq 2}$ is squarefree.

- (1) If $D \equiv 2, 3 \pmod{4}$ and there exists a positive definite binary generalized form over K which is \mathbb{Z} -universal, then $D \in \{2, 3, 6, 7, 10\}$.
- (2) If $D \equiv 1 \pmod{4}$ and there exists a classical positive definite binary generalized form over K which is \mathbb{Z} -universal, then $D = 5$.

Proof. Suppose that G is a positive definite binary generalized form over K which is \mathbb{Z} -universal. If $D \equiv 1 \pmod{4}$, we further assume that G is classical. If Q is the associated quaternary quadratic form defined by (4), then Q is positive definite and universal.

By Lemma 3.5, Q is also classical. Bhargava obtained, as a corollary of his proof of the 15-Theorem, a complete list of classical quaternary positive definite universal quadratic forms up to equivalence [1, Table 5]. Every form on this list has determinant ≤ 112 . By Corollary 3.3, $D^2 \mid \det(M_Q)$, thus $D^2 \leq 112$. This yields the proof. \square

Lemma 4.3. Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{0, 1\}$ is squarefree and let G be a binary generalized form over K given by the matrix

$$M_G = \begin{pmatrix} a & \frac{b}{2} & \frac{d}{2} & \frac{e}{2} \\ \frac{b}{2} & c & \frac{\tau(e)}{2} & \frac{f}{2} \\ \frac{d}{2} & \frac{\tau(e)}{2} & \tau(a) & \frac{\tau(b)}{2} \\ \frac{e}{2} & \frac{f}{2} & \frac{\tau(b)}{2} & \tau(c) \end{pmatrix} \tag{5}$$

where $a, b, c, e \in K$ and $d, f \in \mathbb{Q}$. Let $a = a_1 + a_2\omega_D$, $b = b_1 + b_2\omega_D$, $c = c_1 + c_2\omega_D$, and $e = e_1 + e_2\omega_D$. Let Q be the quaternary quadratic form associated to G and M_Q be the matrix of Q .

If $D \equiv 2, 3 \pmod{4}$, then

$$M_Q = \begin{pmatrix} 2a_1 + d & 2Da_2 & b_1 + e_1 & D(b_2 - e_2) \\ 2Da_2 & D(2a_1 - d) & D(b_2 + e_2) & D(b_1 - e_1) \\ b_1 + e_1 & D(b_2 + e_2) & 2c_1 + f & 2Dc_2 \\ D(b_2 - e_2) & D(b_1 - e_1) & 2Dc_2 & D(2c_1 - f) \end{pmatrix}.$$

If $D \equiv 1 \pmod{4}$, then

$$M_Q = \begin{pmatrix} M_{Q,1} & M_{Q,2} \\ M_{Q,2}^\dagger & M_{Q,3} \end{pmatrix}$$

where

$$M_{Q,1} = \begin{pmatrix} 2a_1 + a_2 + d & a_1 + \frac{1+D}{2}a_2 + \frac{d}{2} \\ a_1 + \frac{1+D}{2}a_2 + \frac{d}{2} & \frac{1+D}{2}a_1 + \frac{1+3D}{4}a_2 + \frac{1-D}{4}d \end{pmatrix},$$

$$M_{Q,2} = \begin{pmatrix} b_1 + \frac{b_2}{2} + e_1 + \frac{e_2}{2} & \frac{b_1}{2} + \frac{1+D}{4}b_2 + \frac{e_1}{2} + \frac{1-D}{4}e_2 \\ \frac{b_1}{2} + \frac{1+D}{4}b_2 + \frac{e_1}{2} + \frac{1+D}{4}e_2 & \frac{1+D}{4}b_1 + \frac{1+3D}{8}b_2 + \frac{1-D}{4}e_1 + \frac{1-D}{8}e_2 \end{pmatrix},$$

$$M_{Q,3} = \begin{pmatrix} 2c_1 + c_2 + f & c_1 + \frac{1+D}{2}c_2 + \frac{f}{2} \\ c_1 + \frac{1+D}{2}c_2 + \frac{f}{2} & \frac{1+D}{2}c_1 + \frac{1+3D}{4}c_2 + \frac{1-D}{4}f \end{pmatrix}.$$

Proof. The matrix of G is

$$M_G = \begin{pmatrix} \alpha_{11} & \frac{\alpha_{12}}{2} & \frac{\beta_{11}}{2} & \frac{\beta_{12}}{2} \\ \frac{\alpha_{12}}{2} & \alpha_{22} & \frac{\tau(\beta_{12})}{2} & \frac{\beta_{22}}{2} \\ \frac{\beta_{11}}{2} & \frac{\tau(\beta_{12})}{2} & \tau(\alpha_{11}) & \frac{\tau(\alpha_{12})}{2} \\ \frac{\beta_{12}}{2} & \frac{\beta_{22}}{2} & \frac{\tau(\alpha_{12})}{2} & \tau(\alpha_{22}) \end{pmatrix}$$

where $\alpha_{11} = a$, $\alpha_{12} = b$, $\alpha_{22} = c$, $\beta_{11} = d$, $\beta_{12} = e$, and $\beta_{22} = f$. In the notation of Lemma 3.6, $\alpha_{ij} = r_{ij} + s_{ij}\omega_D$ and $\beta_{ij} = t_{ij} + u_{ij}\omega_D$, hence $r_{11} = a_1$, $s_{11} = a_2$, $r_{12} = b_1$, $s_{12} = b_2$, $r_{22} = c_1$, $s_{22} = c_2$, $t_{11} = d$, $t_{12} = e_1$, $u_{12} = e_2$, and $t_{22} = f$.

The matrix of Q is

$$M_Q = \begin{pmatrix} a_{11} & \frac{b_{11}}{2} & \frac{a_{12}}{2} & \frac{b_{12}}{2} \\ \frac{b_{11}}{2} & c_{11} & \frac{b_{21}}{2} & \frac{c_{12}}{2} \\ \frac{a_{12}}{2} & \frac{b_{21}}{2} & a_{22} & \frac{b_{22}}{2} \\ \frac{b_{12}}{2} & \frac{c_{12}}{2} & \frac{b_{22}}{2} & c_{22} \end{pmatrix}.$$

If $D \equiv 2, 3 \pmod{4}$, then by Lemma 3.6,

$$\begin{aligned} a_{11} &= 2r_{11} + t_{11} = 2a_1 + d, \\ b_{11} &= 4Ds_{11} = 4Da_2, \\ c_{11} &= 2Dr_{11} - Dt_{11} = D(2a_1 - d), \\ a_{12} &= 2r_{12} + 2t_{12} = 2(b_1 + e_1), \\ b_{12} &= 2Ds_{12} - 2Du_{12} = 2D(b_2 - e_2), \\ b_{21} &= 2Ds_{12} + 2Du_{12} = 2D(b_2 + e_2), \\ c_{12} &= 2Dr_{12} - 2Dt_{12} = 2D(b_1 - e_1), \\ a_{22} &= 2r_{22} + t_{22} = 2c_1 + f, \\ b_{22} &= 4Ds_{22} = 4Dc_2, \\ c_{22} &= 2Dr_{22} - Dt_{22} = D(2c_1 - f). \end{aligned}$$

If $D \equiv 1 \pmod{4}$, then by Lemma 3.6,

$$\begin{aligned} a_{11} &= 2r_{11} + s_{11} + t_{11} = 2a_1 + a_2 + d, \\ b_{11} &= 2r_{11} + (1+D)s_{11} + t_{11} = 2a_1 + (1+D)a_2 + d, \\ c_{11} &= \frac{1+D}{2}r_{11} + \frac{1+3D}{4}s_{11} + \frac{1-D}{4}t_{11} = \frac{1+D}{2}a_1 + \frac{1+3D}{4}a_2 + \frac{1-D}{4}d, \\ a_{12} &= 2r_{12} + s_{12} + 2t_{12} + u_{12} = 2b_1 + b_2 + 2e_1 + e_2, \\ b_{12} &= r_{12} + \frac{1+D}{2}s_{12} + t_{12} + \frac{1-D}{2}u_{12} = b_1 + \frac{1+D}{2}b_2 + e_1 + \frac{1-D}{2}e_2, \\ b_{21} &= r_{12} + \frac{1+D}{2}s_{12} + t_{12} + \frac{1+D}{2}u_{12} = b_1 + \frac{1+D}{2}b_2 + e_1 + \frac{1+D}{2}e_2, \\ c_{12} &= \frac{1+D}{2}r_{12} + \frac{1+3D}{4}s_{12} + \frac{1-D}{2}t_{12} + \frac{1-D}{4}u_{12} \\ &= \frac{1+D}{2}b_1 + \frac{1+3D}{4}b_2 + \frac{1-D}{2}e_1 + \frac{1-D}{4}e_2, \\ a_{22} &= 2r_{22} + s_{22} + t_{22} = 2c_1 + c_2 + f, \\ b_{22} &= 2r_{22} + (1+D)s_{22} + t_{22} = 2c_1 + (1+D)c_2 + f, \\ c_{22} &= \frac{1+D}{2}r_{22} + \frac{1+3D}{4}s_{22} + \frac{1-D}{4}t_{22} = \frac{1+D}{2}c_1 + \frac{1+3D}{4}c_2 + \frac{1-D}{4}f. \quad \square \end{aligned}$$

In Lemmas 4.4 to 4.9, we find an example of a \mathbb{Z} -universal generalized form over $\mathbb{Q}(\sqrt{D})$ for $D \in \{2, 3, 6, 7, 10\}$ and of a classical \mathbb{Z} -universal generalized

form over $\mathbb{Q}(\sqrt{5})$. We prove only Lemma 4.4, the rest is stated and proved in the Appendix in Section A4 [8].

Lemma 4.4. *Let $K = \mathbb{Q}(\sqrt{2})$ and*

$$G_2(z, w) = z^2 - \sqrt{2}zw + w^2 - z\tau(z) - w\tau(w) + \tau(z)^2 + \sqrt{2}\tau(z)\tau(w) + \tau(w)^2.$$

The generalized form G_2 is \mathbb{Z} -universal over K .

Proof. The coefficients of G_2 are $a = 1, b = -\sqrt{2}, c = 1, d = -1, e = 0, f = -1$. Let Q_2 be the quaternary quadratic form $Q_2(x_1, y_1, x_2, y_2) = G_2(x_1 + y_1\sqrt{2}, x_2 + y_2\sqrt{2})$ and let M_{Q_2} be the matrix of Q_2 . By Lemma 4.3 with $D = 2$,

$$M_{Q_2} = \begin{pmatrix} 2a_1 + d & 4a_2 & b_1 + e_1 & 2(b_2 - e_2) \\ 4a_2 & 2(2a_1 - d) & 2(b_2 + e_2) & 2(b_1 - e_1) \\ b_1 + e_1 & 2(b_2 + e_2) & 2c_1 + f & 4c_2 \\ 2(b_2 - e_2) & 2(b_1 - e_1) & 4c_2 & 2(2c_1 - f) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 6 & -2 & 0 \\ 0 & -2 & 1 & 0 \\ -2 & 0 & 0 & 6 \end{pmatrix}.$$

We get

$$M_{Q_2} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & -2 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Since the last matrix is positive definite, G_2 is also positive definite. Moreover, the diagonal quadratic form $x^2 + 2y^2 + z^2 + 2w^2$ is universal by the 15-Theorem, hence G_2 is \mathbb{Z} -universal. \square

Lemma 4.5. *Let $K = \mathbb{Q}(\sqrt{3})$ and*

$$G_3(z, w) = 2z^2 - 2\sqrt{3}zw + 2w^2 - 3z\tau(z) - \sqrt{3}z\tau(w) + \sqrt{3}w\tau(z) + 2\tau(z)^2 + 2\sqrt{3}\tau(z)\tau(w) + 2\tau(w)^2.$$

The generalized form G_3 is \mathbb{Z} -universal over K .

Lemma 4.6. *Let $K = \mathbb{Q}(\sqrt{5})$ and*

$$G_5(z, w) = (3 - \omega_5)z^2 + w^2 - 4z\tau(z) + (3 - \tau(\omega_5))\tau(z)^2 + \tau(w)^2$$

where $\omega_5 = \frac{1+\sqrt{5}}{2}$. *The generalized form G_5 is classical and \mathbb{Z} -universal over K .*

Lemma 4.7. Let $K = \mathbb{Q}(\sqrt{6})$ and G_6 be the generalized form with the matrix

$$M_{G_6} = \begin{pmatrix} 3 & \sqrt{6} & -\frac{5}{2} & \frac{\sqrt{6}}{2} \\ \sqrt{6} & 3 & -\frac{\sqrt{6}}{2} & -\frac{1}{2} \\ -\frac{5}{2} & -\frac{\sqrt{6}}{2} & 3 & -\sqrt{6} \\ \frac{\sqrt{6}}{2} & -\frac{1}{2} & -\sqrt{6} & 3 \end{pmatrix}.$$

The generalized form G_6 is \mathbb{Z} -universal over K .

Lemma 4.8. Let $K = \mathbb{Q}(\sqrt{7})$ and G_7 be the generalized quadratic form with the matrix

$$M_{G_7} = \begin{pmatrix} 3 & \frac{1+2\sqrt{7}}{2} & -\frac{5}{2} & \frac{-1+2\sqrt{7}}{2} \\ \frac{1+2\sqrt{7}}{2} & 4 & \frac{-1-2\sqrt{7}}{2} & \frac{3}{2} \\ -\frac{5}{2} & \frac{-1-2\sqrt{7}}{2} & 3 & \frac{1-2\sqrt{7}}{2} \\ \frac{-1+2\sqrt{7}}{2} & \frac{3}{2} & \frac{1-2\sqrt{7}}{2} & 4 \end{pmatrix}.$$

The generalized form G_7 is \mathbb{Z} -universal over K .

Lemma 4.9. Let $K = \mathbb{Q}(\sqrt{10})$. If G_{10} is the generalized quadratic form with the matrix

$$M_{G_{10}} = \begin{pmatrix} 3 & \frac{\sqrt{10}}{2} & -\frac{5}{2} & \frac{\sqrt{10}}{2} \\ \frac{\sqrt{10}}{2} & 5 & -\frac{\sqrt{10}}{2} & \frac{7}{2} \\ -\frac{5}{2} & -\frac{\sqrt{10}}{2} & 3 & -\frac{\sqrt{10}}{2} \\ \frac{\sqrt{10}}{2} & \frac{7}{2} & \frac{\sqrt{10}}{2} & 5 \end{pmatrix},$$

then G_{10} is \mathbb{Z} -universal over K .

Now we are ready to prove Theorem 1.2, which we restate here for the convenience of the reader.

Theorem 4.10. Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z}_{\geq 2}$ is squarefree.

- (1) Assume that $D \equiv 2, 3 \pmod{4}$. A binary \mathbb{Z} -universal generalized quadratic form exists over K if and only if $D \in \{2, 3, 6, 7, 10\}$.
- (2) Assume that $D \equiv 1 \pmod{4}$. A classical binary \mathbb{Z} -universal generalized quadratic form exists over K if and only if $D = 5$.

Proof. First we prove (i). We assume that $D \equiv 2, 3 \pmod{4}$. If a binary \mathbb{Z} -universal generalized form exists over $K = \mathbb{Q}(\sqrt{D})$, then $D \in \{2, 3, 6, 7, 10\}$ by Proposition 4.2 (i). Conversely, in Lemmas 4.4, 4.5 and 4.7 to 4.9 we found an example of a generalized form with the required properties over each of these fields.

Next, we prove (ii). We assume that $D \equiv 1 \pmod{4}$. If a classical binary \mathbb{Z} -universal generalized form exists over $K = \mathbb{Q}(\sqrt{D})$, then $D = 5$ by Proposition 4.2 (ii). We found an example of such a form over $\mathbb{Q}(\sqrt{5})$ in Lemma 4.6. \square

Theorem 1.3 is an immediate consequence of the following slightly refined version.

Theorem 4.11. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z}_{\geq 2}$ is squarefree. Assume that one of the following two conditions is satisfied:*

- (1) $D \equiv 2, 3 \pmod{4}$ and there exists a ternary \mathbb{Z} -universal generalized quadratic form over K ,
- (2) $D \equiv 1 \pmod{4}$ and there exists a classical ternary \mathbb{Z} -universal generalized quadratic form over K .

Then $D \in \{1, \dots, 62\} \cup \{66, 70, 74, 77, 78, 82, 85, 86, 87, 93, 94, 95, 102, 106, 110\}$.

Proof. Let G be a ternary generalized quadratic form over K which is \mathbb{Z} -universal. If $D \equiv 1 \pmod{4}$, then we further assume that G is classical. Let Q be the quadratic form in 6 variables associated to G and let M_Q be the matrix of Q . By Lemma 3.5, Q is classical.

A classical universal quadratic form has a quaternary subform (not necessarily universal) obtained by the method of escalations [1, p. 31]. Let R be such a subform of Q and M_R the matrix of R . If $p \mid D$ is a prime, then $\text{rank}(M_Q \pmod{p}) \leq 3$ by Proposition 3.7. The rank of R over $\mathbb{Z}/p\mathbb{Z}$ is less than or equal to the rank of Q over $\mathbb{Z}/p\mathbb{Z}$, hence $\text{rank}(M_R \pmod{p}) \leq 3$. Since M_R is a 4×4 matrix, $\det(M_R) \equiv 0 \pmod{p}$.

This proves $D \mid \det(M_R)$. Every quaternary form obtained by the method of escalations is listed in [1, Table 3] and in particular, has determinant ≤ 112 . It remains to determine which squarefree $D \leq 112$ divide the determinant of some form on the list. Every squarefree $D \in \{1, \dots, 62\}$ satisfies this condition. Additionally, the squarefree determinants greater than 62 on the list are 66, 70, 74, 77, 78, 82, 85, 86, 87, 93, 94, 95, 102, 106, 110. \square

Using the 290-Theorem, we can remove the ‘‘classical’’ assumption in the case $D \equiv 1 \pmod{4}$.

Theorem 4.12. *There are only finitely many real quadratic fields which admit a binary or ternary \mathbb{Z} -universal generalized quadratic form.*

Proof. We show that there exists a positive constant C such that $D \leq C$ for every such field $K = \mathbb{Q}(\sqrt{D})$. If $D \equiv 2, 3 \pmod{4}$, then the statement follows from Theorems 4.10 and 4.11. Hence, we can assume that $D \equiv 1 \pmod{4}$. Let G be a \mathbb{Z} -universal generalized form over K with matrix M_G and Q be the quadratic form associated to G with matrix M_Q .

First, assume that G is binary. If M_G is the matrix of G and M_Q is the matrix of Q , then $\det(M_Q) = D^2 \det(M_G)$ by Lemma 3.2. The matrix M_G is a 4×4 matrix with entries in $\frac{1}{2}\mathcal{O}_K$, hence $16 \det(M_G) \in \mathcal{O}_K \cap \mathbb{Q} = \mathbb{Z}$, and we get $D^2 \mid 16 \det(M_Q)$.

A corollary of the 290-Theorem is that there are exactly 6436 universal quaternary forms [2, Theorem 4]. If we let m be the maximum of their determinants, then $D^2 \leq 16m$.

Secondly, assume that G is ternary. A universal quadratic form contains a quaternary subform equivalent to one of 6560 basic escalators [2, p. 4]. Let R be such a subform of Q and M_R the matrix of R . If $p \mid D$ is a prime, then $\text{rank}(M_Q \bmod p) \leq 3$ by Proposition 3.7, hence $\text{rank}(M_R \bmod p) \leq 3$ as well. Since M_R is a 4×4 matrix, $\det(M_R) \equiv 0 \pmod{p}$, and since the entries of M_R are in $\frac{1}{2}\mathbb{Z}$, we have $16 \det(M_R) \in \mathbb{Z}$.

This proves $D \mid 16 \det(M_R)$. If we let m be the maximum of the determinants of the basic escalators, then $D \leq 16m$. \square

In contrast to Theorem 4.12, it is not difficult to construct a \mathbb{Z} -universal generalized form in 4 variables over every real quadratic field.

Proposition 4.13. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z}_{\geq 2}$ is squarefree. The generalized quadratic form*

$$G(z_1, z_2, z_3, z_4) = z_1^2 - z_1\tau(z_1) + \tau(z_1)^2 + z_2^2 - z_2\tau(z_2) + \tau(z_2)^2 \\ + z_3^2 - z_3\tau(z_3) + \tau(z_3)^2 + z_4^2 - z_4\tau(z_4) + \tau(z_4)^2$$

is \mathbb{Z} -universal over K . Thus, the minimal rank of a \mathbb{Z} -universal generalized form over K is at most 4.

Proof. Clearly, G is positive definite. It is enough to show that G represents every $a \in \mathbb{Z}_{\geq 1}$ over \mathbb{Z} . If $z_i \in \mathbb{Z}$ for $i = 1, \dots, 4$, then $\tau(z_i) = z_i$, hence

$$G(z_1, z_2, z_3, z_4) = z_1^2 + z_2^2 + z_3^2 + z_4^2.$$

This quadratic form is universal by Lagrange's Four Square Theorem. \square

We end this section with a proposition that is used in Section A6 in [8].

Proposition 4.14. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z} \setminus \{0, 1\}$ is squarefree, $D \equiv 2, 3 \pmod{4}$. Let G be an integral binary generalized form over K . Assume that G is \mathbb{Z} -valued. If Q is the quaternary quadratic form associated to G and M_Q is the matrix of Q , then*

$$\frac{\det(M_Q)}{D^2} \equiv 0, 1 \pmod{4}.$$

Proof. By Lemma 3.2,

$$\det(M_Q) = D^2 \cdot \det(2M_G).$$

The matrix $2M_G$ is

$$2M_G = \begin{pmatrix} 2a & b & d & e \\ b & 2c & \tau(e) & f \\ d & \tau(e) & 2\tau(a) & \tau(b) \\ e & f & \tau(b) & 2\tau(c) \end{pmatrix}$$

where $a, b, c, e \in \mathcal{O}_K$ and $d, f \in \mathbb{Z}$ because G is integral. Expanding $\det(2M_G)$ along the first column, we get

$$\begin{aligned} \det(2M_G) = & 2a \cdot \det \begin{pmatrix} 2c & \tau(e) & f \\ \tau(e) & 2\tau(a) & \tau(b) \\ f & \tau(b) & 2\tau(c) \end{pmatrix} - b \cdot \det \begin{pmatrix} b & d & e \\ \tau(e) & 2\tau(a) & \tau(b) \\ f & \tau(b) & 2\tau(c) \end{pmatrix} + \\ & + d \cdot \det \begin{pmatrix} b & d & e \\ 2c & \tau(e) & f \\ f & \tau(b) & 2\tau(c) \end{pmatrix} - e \cdot \det \begin{pmatrix} b & d & e \\ 2c & \tau(e) & f \\ \tau(e) & 2\tau(a) & \tau(b) \end{pmatrix}, \end{aligned}$$

hence

$$\begin{aligned} \det(2M_G) \equiv & 2a(\tau(e)\tau(b)f + f\tau(e)\tau(b)) - b(d\tau(b)f \\ & + e\tau(e)\tau(b) - 2f\tau(a)e - \tau(b)^2b - 2\tau(c)\tau(e)d) \\ & + d(2b\tau(e)\tau(c) + df^2 + 2e\tau(c)b - f\tau(e)e - \tau(b)fb) \\ & - e(b\tau(e)\tau(b) + df\tau(e) - \tau(e)^2e - 2\tau(a)fb - 2\tau(b)cd) \\ \equiv & -N(b)df - N(b)N(e) + 2\tau(a)bef + N(b)^2 + 2b\tau(c)d\tau(e) \\ & + 2b\tau(c)d\tau(e) + d^2f^2 + 2\tau(b)cde - dN(e)f - N(b)df \\ & - N(b)N(e) - dN(e)f + N(e)^2 + 2\tau(a)bef + 2\tau(b)cde \\ \equiv & -2N(b)df - 2N(b)N(e) + N(b)^2 + d^2f^2 - 2dN(e)f + N(e)^2 \\ \equiv & (N(b) + N(e) + df)^2 \pmod{4}. \end{aligned}$$

Thus,

$$\frac{\det(M_Q)}{D^2} = \det(2M_G) \equiv 0, 1 \pmod{4}. \quad \square$$

5. Indefinite generalized quadratic forms

We end with a short section providing a shortened proof of the universality of $G(z, w) = z\tau(z) + w\tau(w)$ over real quadratic fields, where the form is indefinite. For a more detailed approach, please consult Section A5 of the Appendix [8].

Theorem 5.1. *Let $K = \mathbb{Q}(\sqrt{D})$ where $D \in \mathbb{Z}_{\geq 2}$ is squarefree. The binary generalized form*

$$G(z, w) = z\tau(z) + w\tau(w)$$

represents every $a \in \mathbb{Z}$.

Proof. It is sufficient to show that G represents every $a \in \mathbb{Z} \setminus \{0\}$ over $\mathbb{Z}[\sqrt{D}]$ because $\mathbb{Z}[\sqrt{D}] \subset \mathcal{O}_K$.

Let $z = x_1 + y_1\sqrt{D}$ and $w = x_2 + y_2\sqrt{D}$ where $x_1, y_1, x_2, y_2 \in \mathbb{Z}$. Let

$$Q(x_1, y_1, x_2, y_2) = G(x_1 + y_1\sqrt{D}, x_2 + y_2\sqrt{D}).$$

We have

$$\begin{aligned} Q(x_1, y_1, x_2, y_2) &= (x_1 + y_1\sqrt{D})(x_1 - y_1\sqrt{D}) \\ &\quad + (x_2 + y_2\sqrt{D})(x_2 - y_2\sqrt{D}) \\ &= x_1^2 + x_2^2 - Dy_1^2 - Dy_2^2. \end{aligned}$$

By the local-global principle for representations over \mathbb{Z} by indefinite forms (see, e.g., [6, Chap. 9, Thm. 1.5]), it suffices to verify that each $a \in \mathbb{Z}$ is represented over \mathbb{Z}_p for all primes p . This is indeed straightforward to verify; for details see Section A5 of the Appendix. \square

It is natural to be interested in the ways a quadratic form represents 0 non-trivially. We show that the quadratic form $Q(x, y, z, w) = x^2 + y^2 - Dz^2 - Dw^2$ represents 0 non-trivially if and only if D is a sum of two squares. The proof of this can be found in the Appendix, see Proposition A5.6. in [8].

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