New York Journal of Mathematics

New York J. Math. **32** (2026) 676–690.

Positive scalar curvature and crystallographic fundamental groups

Noé Bárcenas and **Mario Velásquez**

ABSTRACT. We examine positive and negative results for the Gromov-Lawson-Rosenberg Conjecture within the class of crystallographic groups. We give necessary conditions within the class of split extensions of free abelian by cyclic groups to satisfy the unstable Gromov-Lawson-Rosenberg Conjecture. We also give necessary conditions within the same class of groups which are counterexamples for the conjecture.

CONTENTS

1. Introduction

The (unstable) Gromov-Lawson-Rosenberg Conjecture for a discrete group Γ predicts that a connected closed spin n-dimensional manifold M^n where $n \geq 5$, with fundamental group Γ and classifying map for the fundamental group $f : M \to B\Gamma$, the vanishing of the group homomorphism

$$\alpha(M) = A \circ p_{B\Gamma} D(f_M) : \Omega_n^{\text{spin}}(B\Gamma) \longrightarrow KO_n(C_*^*(\Gamma))$$

given as the composition

$$\Omega_n^{\text{spin}}(B\Gamma) \xrightarrow{D(f_M)} ko_n(B\Gamma) \xrightarrow{p_{B\Gamma}} KO_n(B\Gamma) \xrightarrow{A} KO_n(C_r^*(\Gamma)),$$

decides about the existence of a metric of positive scalar curvature on M. Some explanations are due. D is the map which sends a spin bordism class f_M :

Received August 1, 2025.

2020 *Mathematics Subject Classification.* 46L80, 53C27.

Key words and phrases. Gromov-Lawson-Rosenberg Conjecture, Group homology, (connective) topological K-theory, extensions of $\mathbb{Z}^n$ by $\mathbb{Z}/m$.

The first named author aknowledges support of DGAPA-PAPIIT Grant IN101423. Both authors thank the exchange program CIC-UNAM..

ISSN 1076-9803/2026

$M \to B\Gamma$ to the image of the ko-fundamental class $f_*([M]) \in ko_n(B\Gamma)$. The map $p_{B\Gamma} : ko_n(B\Gamma) \to KO_n(B\Gamma)$ is the natural transformation of periodicity, and A denotes the real assembly map $A : KO_n(B\Gamma) \to KO_n(C_r^*(\Gamma))$.

There exist counterexamples to this conjecture. T. Schick in [Sch98] showed that for the group $\Gamma = \mathbb{Z}^4 \times \mathbb{Z}/3$, there exists a five dimensional manifold M with fundamental group $\Gamma = \pi_1(M) = \mathbb{Z}^4 \times \mathbb{Z}/3$ for which

$$\alpha(M) = 0 \in KO_5(C_r^*(\Gamma)),$$

but M admits no metric of positive scalar curvature.

The result initiated a series of subsequent articles stating group cohomological conditions which produce counterexamples for the (unstable) Gromov-Lawson-Rosenberg Conjecture, including [DP03], and specially [DSS03], where the techniques are used to construct a torsionfree example, which is even a fundamental group of a compact manifold admitting a CAT(0)-cubical complex structure.

We would like to mention [DP03], [DL13] and [Hug21] as some sources for positive results on the stable Gromov-Lawson-Rosenberg-Conjecture. The positive results therein concern groups satisfying the Baum Connes Isomorphism conjecture, which satisfy condition 1.1, and some other conditions about the maximal finite subgroups.

In this article, we will consider for a group homomorphism $\rho : \mathbb{Z}/m \to Gl_n(\mathbb{Z})$, split extensions of the type

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma = \mathbb{Z}^n \rtimes_\rho \mathbb{Z}/m \longrightarrow \mathbb{Z}/m \longrightarrow 1.$$

The integral cohomology of such groups Γ has been computed in a series of articles under several additional sets of hypothesis including:

- The action is compatible in the sense of [AGPP08], which allows for a specific resolution of the trivial $\mathbb{Z}[\Gamma]$-module $\mathbb{Z}$, deduce the collapse of the Lyndon-Hochschild-Serre spectral sequence computing the integral cohomology of Γ without extension problems at the E_2-term.
- The action of Γ on $\mathbb{R}^n - \{0\}$ is free outside of the origin. [LL12].
- The action of Γ on $\mathbb{R}^n - \{0\}$ is free outside of the origin, and m is a prime number [DL13].
- The natural number m is square-free, without further assumption on the action [SV24a].

With the exception of [SV24a] and [AGPP08], these conditions are used because they imply the following maximality properties within the family of finite subgroups of Γ.

Condition 1.1. [Conditions (M) and (NM)]

- Each finite subgroup H of Γ is a subgroup of a unique maximal finite subgroup M. We denote by $\mathcal{M}$ to the collection of conjugacy classes of maximal finite subgroups.
- The normalizer in Γ of a maximal finite subgroup M is M itself.

Davis and Lück in [DL13] used the above conditions to derive computations of complex, real, and real connective K-homology of both the classifying space $B\Gamma$, and the classifying space for proper actions $\underline{E}\Gamma$.

Extending these results, the second named author and Sánchez performed computations of both the complex KU- homology of the classifying spaces $B(\mathbb{Z}^n \rtimes \mathbb{Z}/m)$, denoted by $KU_*(B(\mathbb{Z}^n \rtimes \mathbb{Z}/m))$, and the equivariant KU-homology groups of the classifying spaces for proper actions, denoted by $KU_*^{\mathbb{Z}^n \rtimes \mathbb{Z}/m}(\underline{E}(\mathbb{Z}^n \rtimes \mathbb{Z}/m))$, where m is square-free and without further assumption about the action of $\mathbb{Z}/m$ on $\mathbb{Z}^n$.

In this work we will make structural statements about the algebraic structure of real connective ko-homology groups of $B\Gamma$, denoted by $ko_*(B\Gamma)$, which will be the base for positive and negative results for the (unstable) Gromov-Lawson-Rosenberg conjecture for high dimensional smooth spin manifolds with fundamental group Γ.

The hypothesis that we will impose on the group Γ is the following

Condition 1.2. [Condition for positive results] Let m be and odd natural number and assume that $\rho : \mathbb{Z}/m \to Gl_n(\mathbb{Z})$ is a group homomorphism such that the group action of $\mathbb{Z}/m$ on $\mathbb{R}^n - \{0\}$ is free.

The following is our main positive result on the Gromov-Lawson-Rosenberg conjecture.

Theorem 1.3. *Let M^n be a connected, closed, n-dimensional smooth spin manifold, where $n \geq 5$, and with fundamental group isomorphic to Γ, where Γ satisfies condition 1.2. Assume that $\alpha(M) = 0$. Then M admits a metric of positive scalar curvature.*

Theorem 1.3 will be proved for n even as Theorem 3.1, and n odd as Theorem 4.1. The main structural statements for their proof, namely Lemmas 2.2 and 2.6 for the even case, and Lemma 4.2 for the odd dimensional case, are of different nature and therefore stated and proved separately.

Within the class of crystallographic groups

$$1 \to \mathbb{Z}^n \to \Gamma = \mathbb{Z}^n \rtimes_\rho \mathbb{Z}/m \to \mathbb{Z}/m \to 1,$$

there exist groups for which the unstable Gromov-Lawson-Rosenberg conjecture is known to be true, namely

- The number m is prime and the action is free outside of the origin, according to [DL13].
- The groups adressed in section 2.

On the other hand side, the group $\mathbb{Z}^4 \times \mathbb{Z}/3$ belongs to the family of split extensions

$$1 \to \mathbb{Z}^n \to \Gamma = \mathbb{Z}^n \rtimes_\rho \mathbb{Z}/m \to \mathbb{Z}/m \to 1,$$

for a representation $\rho : \mathbb{Z}/3 \to Gl_4(\mathbb{Z})$ whose action on $\mathbb{R}^4$ is trivial.

We address the following question:

Is there a condition on the representation ρ allowing us to construct an ample family of counterexamples inside the family of semidirect products of $\mathbb{Z}^n$ by $\mathbb{Z}/m$?

To answer this question, we use some results in [SV24a] concerning the decomposition of a finite index submodule of $\mathbb{Z}^n$ as

$$M_1 \oplus M_2 \oplus M_3$$

such that, where when we restrict the action to $\mathbb{Z}/p$ (where p is fixed a prime number such that $p \mid m$) and localizing at p, we have that M_1 is isomorphic to r-copies of the irreducible representation $\mathbb{Z}_{(p)}$ (trivial representation), M_2 is isomorphic to s-copies of the irreducible representation $\mathbb{Z}_{(p)}[\mathbb{Z}/p]$, and M_3 is isomorphic to t-copies of the irreducible representation I (augmentation ideal).

By a very simple examination of the group cohomology of $\mathbb{Z}^n \rtimes \mathbb{Z}/m$, we obtain a condition on (r, s, t) for which the method introduced by Schick in [Sch98] applies. The following is our main negative result, which will be proved as Theorem 5.1.

Theorem. *Suppose m is square-free. Let $\mathbb{Z}^n$ be a $\mathbb{Z}/m$-module, and suppose that there exists a prime $p \mid m$ such that if we consider the (r, s, t) decomposition of $\mathbb{Z}^n$ viewed as a $\mathbb{Z}/p$-module, and $r \geq 4$, then $\mathbb{Z}^n \rtimes \mathbb{Z}/m$ is a counter-example for the unstable Gromov-Lawson-Rosenberg conjecture.*

1.1. Aknowledgements. The first named author aknowledges support of DGAPA-PAPIIT Grant IN101423. Both authors thank the exchange program CIC-UNAM.

2. Connective *ko*-homology of $B\Gamma$.

Recall that Γ fits in an extension

$$1 \to \mathbb{Z}^n \to \Gamma \xrightarrow{\pi} \mathbb{Z}/m \to 1, \tag{1}$$

Assume that condition 1.2 holds. Then, as a consequence of [LW12], if $\mathcal{N}$ denotes the family of conjugacy classes of maximal finite subgroups of Γ, as in condition 1.1, we obtain the following result.

Theorem 2.1. *There is a commutative diagram with upper exact row*

$$\begin{array}{ccccc}
\bigoplus_{(N)\in\mathcal{N}} \widetilde{ko}_r(BN) & \longrightarrow & ko_r(B\Gamma) & \xrightarrow{\beta} & ko_r(\underline{B}\Gamma) \\
 & & \Big\downarrow{\scriptstyle A\circ p_{B\Gamma}} & & \Big\downarrow{\scriptstyle p_{\underline{B}\Gamma}} \\
 & & KO_r(C_r^*(\Gamma;\mathbb{R})) & \longrightarrow & KO_r(\underline{B}\Gamma)
\end{array}$$

where the bottom map is the composite of the inverse of the Baum–Connes map and the map $KO_m^\Gamma(\underline{E}\Gamma) \to KO_m(\underline{B}\Gamma)$ is the induction map with respect to the homomorphism $\Gamma \to 1$.

Proof. It is a consequence of the cellular pushout relating $B\Gamma$ and $\underline{B}\Gamma$ when the group Γ satisfies conditions M and NM. □

Now suppose that $f_M : M \to B\Gamma$ is a classifying map of M and

$$\alpha(M) = A \circ p_{B\Gamma}(D[f_M]) = 0.$$

First, by the commutativity of the above diagram, we have $p_{\underline{B}\Gamma} \circ \beta(D[f_M]) = 0$. Now we will analyze $\ker(p_{\underline{B}\Gamma})$.

Lemma 2.2. $\ker(p_{\underline{B}\Gamma})$ *only contains torsion of order dividing* m.

Proof. Let p be a prime dividing m such that $m = p^s m'$ with $(p, m') = 1$. By Lemma 2.3 the quotient map $\underline{B}\Gamma \to \underline{B}(\Gamma/\mathbb{Z}/p^s)$ induces an isomorphism

$$ko_*(\underline{B}\Gamma)_{\mathbb{Z}/p^s} \otimes \mathbb{Z}[1/p] \to ko_*(\underline{B}(\Gamma/\mathbb{Z}/p^s)) \otimes \mathbb{Z}[1/p].$$

We have a commutative diagram

$$\begin{array}{ccc} ko_*(\underline{B}\Gamma)_{\mathbb{Z}/p^s} \otimes \mathbb{Z}[1/p] & \xrightarrow{\cong} & ko_*(\underline{B}(\Gamma/\mathbb{Z}/p^s)) \otimes \mathbb{Z}[1/p] \\ \downarrow p_{\underline{B}\Gamma} & & \downarrow p_{\underline{B}(\Gamma/\mathbb{Z}/p^s)} \\ KO_*(\underline{B}\Gamma)_{\mathbb{Z}/p^s} \otimes \mathbb{Z}[1/p] & \xrightarrow{\cong} & KO_*(\underline{B}(\Gamma/\mathbb{Z}/p^s)) \otimes \mathbb{Z}[1/p] \end{array}$$

Then it is enough to prove that

$$ko_*(\underline{B}(\Gamma/\mathbb{Z}/p^s))_{\mathbb{Z}/p^s} \otimes \mathbb{Z}[1/p] \to KO_*(\underline{B}(\Gamma/\mathbb{Z}/p^s))_{\mathbb{Z}/p^s} \otimes \mathbb{Z}[1/p]$$

only contains m/p^s-torsion.

The commutativity of the previous diagram follows from the fact that periodicity is a transformation of homology theories, and that the composition of a homology theory with the formation of coinvariants is a homology theory.

Proceeding inductively over all primes dividing m it is enough to prove that

$$ko_*(B\mathbb{Z}^n)_{\mathbb{Z}/m} \otimes \mathbb{Z}[1/m] \xrightarrow{p_{B\mathbb{Z}^n}} KO_*(B\mathbb{Z}^n)_{\mathbb{Z}/m} \otimes \mathbb{Z}[1/m]$$

is injective. But it is a consequence of the injectivity of the maps on coefficients from connective real K-theory and periodic real K-theory as in noted by Davis and Lück in the proof of Thm. 0.7 in [DL13]. Then we find that $\ker(p_{\underline{B}\Gamma})$ only contains m-torsion. □

Now we need to recall a couple of results that we will need in the following.

Lemma 2.3. *Let p be a prime number and let G be a p-group. For any G-CW complex X with quotient map $\pi : X \to G \setminus X$ and any homology theory $\mathcal{H}_*(-)$, the induced map $\pi_r : (\mathcal{H}_r(X) \otimes \mathbb{Z}[1/p])_G \to \mathcal{H}_r(G \setminus X) \otimes \mathbb{Z}[1/p]$ is an isomorphism for all $r \in \mathbb{Z}$.*

Proof. The argument is similar to the one given for Prop. A.4. in [DL13]. Given a G-CW-complex X, we get a natural transformation

$$j_* : (\mathcal{H}_m(X) \otimes \mathbb{Z}[1/p])_G \to \mathcal{H}_m(G \setminus X) \otimes \mathbb{Z}[1/p].$$

Both sides are G-homology theories and moreover j_* is an isomorphism when $X = G/H$ we have that j_* is an isomorphism for every X. □

We aso need the following result of Lück-Weiermann.

Theorem 2.4 (Corollary 2.8 in [LW12])**.** *Let $\mathcal{F} \subseteq \mathcal{G}$ be families of subgroups of a group Γ such that every element in $\mathcal{G} - \mathcal{F}$ is contained in a unique maximal element in $\mathcal{G} - \mathcal{F}$. Let $\mathcal{M}$ be a complete system of representatives of the conjugacy classes of subgroups in $\mathcal{G} - \mathcal{F}$ which are maximal in $\mathcal{G} - \mathcal{F}$. Let $\mathcal{SUB}(M)$ be the family of subgroups of M. Then, there is a cellular Γ-pushout*

$$\begin{array}{ccc} \coprod_{M\in\mathcal{M}} B_{\mathcal{F}\cap N_\Gamma M}(N_\Gamma M) & \xrightarrow{\ i\ } & B_{\mathcal{F}}(\Gamma) \\ \downarrow \lambda & & \downarrow \\ \coprod_{M\in\mathcal{M}} B_{\mathcal{SUB}(M)\cup(\mathcal{F}\cap N_\Gamma M)}(N_\Gamma M) & \longrightarrow & X \end{array} \tag{2}$$

such that X is a model for $B_{\mathcal{G}}(\Gamma)$.

For a group Γ given by (1) let us denote by $\Gamma_p = \pi^{-1}(\mathbb{Z}/p^s)$, where p^s is the highest power of p which divides m. For any subgroup $G \subseteq \mathbb{Z}/m$ we denote by G_p the subgroup of Γ defined as $G_p = G/(\mathbb{Z}/p^s \cap G)$.

For a homology theory $\mathcal{H}_*(-)$ denote by $\mathcal{H}_*(-)_{(p)}$ the localization of $\mathcal{H}_*$ at p, that is, $\mathcal{H}_*(-)\otimes\mathbb{Z}_{(p)}$, where $\mathbb{Z}_{(p)}$ is the ring of integers localized at the prime ideal (p).

Lemma 2.5. *For any (reduced or unreduced) homology theory $\mathcal{H}_*(-)$, $p \mid m$, and G_p as above, we have that*

(i) $\mathcal{H}_*(B\Gamma)_{(p)} \cong \left(\mathcal{H}_*(B(\Gamma_p))_{(p)}\right)_{G_p}$

(ii) $\mathcal{H}_*(\underline{B}\Gamma)_{(p)} \cong \left(\mathcal{H}_*(\underline{B}(\Gamma_p))_{(p)}\right)_{G_p}$.

Proof. (i) The extension induces a fibration of classifying spaces.

$$B\Gamma_p \to B\Gamma \to BG_p$$

Consider the Leray-Serre spectral sequence associated to this fibration, for the homology theory $\mathcal{H}_*(-)_{(p)}$. By [DK01, Thm. 9.6] this spectral sequence converges to $\mathcal{H}_*(B\Gamma)_{(p)}$ with second page:

$$E^2_{\alpha,\beta} = H_\alpha\left(G_p; \mathcal{H}_\beta(B\Gamma_p)_{(p)}\right) = \begin{cases} \left(\mathcal{H}_\beta(B\Gamma_p)_{(p)}\right)_{G_p} & \alpha = 0 \\ 0 & \alpha \neq 0. \end{cases}$$

This implies that the sequence collapses without extension problems, so we have proved the first statement.

(ii) The idea of the proof is similar to that of Theorem 3.6 in [SV24b]. Let $\mathcal{F}$ be a family of subgroups of Γ, we say that Γ satifies the $(p^s, \mathcal{F})$-*condition* if for every homology theory $\mathcal{H}_*(-)$, the induction

$$\left(\mathcal{H}_*(B_{\mathcal{F}\cap\Gamma_p}\Gamma_p)_{(p)}\right)_{G_p} \xrightarrow{Ind_{\Gamma_p}^{\Gamma}} \mathcal{H}_*(B_{\mathcal{F}}\Gamma)_{(p)},$$

is an isomorphism. Here, $B_{\mathcal{F}}\Gamma$ denotes the orbit space of the classifying space of Γ with respect to the family $\mathcal{F}$.

Suppose that m is the product of powers of distinct primes $p_1^{s_1}\cdots p_r^{s_r}$. Let

$$\mathcal{F}_i = \{H \subset \Gamma \mid H \text{ is finite and } |H| \text{ divides } p_1^{s_1}\cdots p_i^{s_i}\}.$$

Now we will show that by induction on i that Γ satisfies the $(p^s, \mathcal{F}_i)$-condition, for every i.

For $i = 0$, it is the statement (i).

Now we will construct a model for $B_{\mathcal{SUB}(M)\cup(\mathcal{F}_i\cap N_\Gamma M)}(N_\Gamma M)$.

We have a pushout

$$\begin{array}{ccc} B_{\mathcal{SUB}(M)\cap(\mathcal{F}_i\cap N_\Gamma M)}(N_\Gamma M) & \xrightarrow{\;i\;} & B_{\mathcal{F}_i\cap N_\Gamma M}(N_\Gamma M) \\ \downarrow{\scriptstyle\lambda} & & \downarrow \\ B_{\mathcal{SUB}(M)}(N_\Gamma M) & \longrightarrow & B_{\mathcal{SUB}(M)\cup(\mathcal{F}_i\cap N_\Gamma M)}(N_\Gamma M) \end{array} \tag{3}$$

In both cases, the families $\mathcal{SUB}(M)$ and $\mathcal{SUB}(M)\cap\mathcal{F}_i\cap N_\Gamma M$ have a maximal element, M and $M_{i+1} = M/(\mathbb{Z}/p_{i+1}^{s_{i+1}})$ respectively, then $E(W_\Gamma M)$ with the action induced by the quotient map is a model for $E_{\mathcal{SUB}(M)}N_\Gamma M$, then

$$B_{\mathcal{SUB}(M)}N_\Gamma M = BW_\Gamma M$$

and $E(N_\Gamma M/(M_{i+1}))$ with the action induced by the quotient map is a model for
$E_{\mathcal{SUB}(M)\cap\mathcal{F}_i\cap N_\Gamma M}N_\Gamma M$, and hence

$$B_{\mathcal{SUB}(M)\cap\mathcal{F}_i\cap N_\Gamma M}N_\Gamma M = B(N_\Gamma M/M_{i+1}).$$

Notice that both groups $W_\Gamma M$ and $N_\Gamma M/M_{i+1}$ are finite subgroups of $\mathbb{Z}/m$. We have

$$\begin{aligned}\mathcal{H}_*(B_{\mathcal{SUB}(M)}N_\Gamma M)_{(p)} &\cong \mathcal{H}_*(BW_\Gamma M)_{(p)} \\ &\cong \left(\mathcal{H}_*(B(W_\Gamma M\cap\Gamma_p/M))_{(p)}\right)_{G_p} \text{ by (i)} \\ &\cong \left(\mathcal{H}_*(B_{\mathcal{SUB}(M)}(N_\Gamma M\cap\Gamma_p))_{(p)}\right)_{G_p}.\end{aligned}$$

and in a similar way

$$\mathcal{H}_*(B_{\mathcal{SUB}(M)\cap\mathcal{F}_i\cap N_\Gamma M}N_\Gamma M)_{(p)} \cong \left(\mathcal{H}_*(B_{\mathcal{SUB}(M)\cap\mathcal{F}_i\cap N_\Gamma M}(N_\Gamma M\cap\Gamma_p))_{(p)}\right)_{G_p}.$$

In others words, $N_\Gamma M$ satisfies $(p^s, \mathcal{SUB}(M))$-condition and $(p^s, \mathcal{SUB}(M)\cap\mathcal{F}_i\cap N_\Gamma M)$-condition. By the five lemma applied to the morphism of Mayer-Vietoris sequences given by restrict the pushout (3) to Γ_p we get that $N_\Gamma M$ satisfies the $(p^s, \mathcal{SUB}(M)\cup(\mathcal{F}_i\cap N_\Gamma M))$-condition.

Finally again by the five lemma applied to the morphism of Mayer-Vietoris sequences given by restriction to Γ_p of the pushout (2) associated to families $\mathcal{F}_i \subseteq \mathcal{F}_{i+1}$ we get Γ satisfies the $(p^s, \mathcal{F}_i)$-condition for every i and for $i = r$ we obtain (ii).

□

The following result was proved as Lemma 4.4 in [LL12].

Lemma 2.6. *In the exact sequence*

$$0 \longrightarrow H^{2r}(\underline{B}(\mathbb{Z}^n \rtimes \mathbb{Z}/p^s)) \xrightarrow{\bar{f}_*} H^{2r}(\mathbb{Z}^n \rtimes \mathbb{Z}/p^s) \xrightarrow{\varphi^{2r}} \bigoplus_{P \in \mathcal{M}} H^{2r}(BP),$$

- *The homomorphism φ^{2r} has torsionfree kernel.*
- *The abelian group $H^{2r}(\underline{B}(\mathbb{Z}^n \rtimes \mathbb{Z}/p^s))$ is finitely generated and torsion-free.*

Applying Lemma 2.5 and the Universal Coefficient Theorem, we obtain the following result.

Corollary 2.7. *The abelian groups*

$$H_{2r+1}(\underline{B}\Gamma, \mathbb{Z})$$

are finitely generated and torsion-free.

3. Proof of the positive result in the even dimensional case.

In this section we will prove the even version of Theorem 1.3.

Theorem 3.1. *Let M^n be an n-dimensional smooth spin manifold, where $n \geq 5$ is even, and with fundamental group isomorphic to Γ, where Γ satisfies condition 1.2. Denote by $f_M : M \to B\Gamma$ the classifying map for the fundamental group. Assume that $\alpha(M) = 0$. Then M admits a metric of positive scalar curvature.*

The proof of this theorem requires some work.

Lemma 3.2. *For $*$ even, $ko_*(B\Gamma)$ does not contain m-torsion*

Proof. By Lemma 2.5 it is enough to prove that for any odd prime p dividing m, we have that $ko_*(B(\mathbb{Z}^n \rtimes \mathbb{Z}/p^s))_{G_p}$ does not contain p^s-torsion.

Consider the Atiyah-Hirzebruch- Leray-Serre spectral sequence associated to the extension

$$0 \to \mathbb{Z}^n \to \mathbb{Z}^n \rtimes \mathbb{Z}/p^s \to \mathbb{Z}/p^s \to 0,$$

coverging to $ko_*(B(\mathbb{Z}^n \rtimes \mathbb{Z}/p^s))$. The second page is given by

$$E^2_{p,q} = H_i(\mathbb{Z}/p^s; ko_j(B\mathbb{Z}^n)).$$

Let us first compute $ko_j(B\mathbb{Z}^n))$. By Lemma 5.3 in [DL13] we have isomorphisms of abelian groups

$$ko_j(B\mathbb{Z}^n) \otimes \mathbb{Z}[1/2] \cong \bigoplus_{l=0}^{n} H_l(\mathbb{Z}^n) \otimes ko_j(*) \otimes \mathbb{Z}[1/2] \tag{4}$$

$$ko_j(B\mathbb{Z}^n) \otimes \mathbb{Z}_{(2)} \cong \bigoplus_{l=0}^{n} H_l(\mathbb{Z}^n) \otimes ko_j(*)_{(2)}. \tag{5}$$

Now we will prove that both are actually isomorphisms of $\mathbb{Z}[\mathbb{Z}/p^s]$-modules. Note that $ko_j(*) \otimes \mathbb{Z}[1/2]$ is torsion free, then the Chern character shows that the map (4) is an isomorphism of $\mathbb{Z}[\mathbb{Z}/p^s]$-modules. The isomorphism (5) implies that the Atiyah-Hirzebruch spectral sequence with second page

$$E^2_{i,j} = H_i(\mathbb{Z}^n; ko_j(*)_{(2)})$$

converging to $ko_{i+j}(B\mathbb{Z}^n)_{(2)}$ collapses. On the other hand, this spectral sequence is natural with respect to $\mathbb{Z}/p^s$-module structure of $\mathbb{Z}^n$. We have a filtration of $\mathbb{Z}_{(2)}[\mathbb{Z}/p^s]$-modules

$$ko_r(B\mathbb{Z}^n)_{(2)} = F_{0,r} \supseteq \dots \supseteq F_{r,0} \supseteq F_{r+1,-1} = 0$$

and exact sequences of $\mathbb{Z}_{(2)}[\mathbb{Z}/p^s]$-modules

$$0 \to F_{i+1,r-i-1} \to F_{i,r-i} \to H_i(\mathbb{Z}^n) \otimes ko_{r-i}(*)_{(2)} \to 0$$

It is enough to prove that the above exact sequence splits as $\mathbb{Z}_{(2)}[\mathbb{Z}/p^s]$-modules.

Suppose first that $r - i \equiv 3, 5, 6, 7 \pmod 8$, then $ko_{r-i}(*) = 0$, and so the exact sequence splits trivially.

If $r-i \equiv 0, 4 \pmod 8$, then $ko_{r-i}(*) \cong \mathbb{Z}$. On the other hand, $H_i(\mathbb{Z}^n) \otimes ko_{r-i}(*)_{(2)}$ is a finite generated $\mathbb{Z}_{(2)}[\mathbb{Z}/p^s]$-module that is torsion free as $\mathbb{Z}_{(2)}$-module, as the action is free outside the origin, the norm element x of $\mathbb{Z}_{(2)}[\mathbb{Z}/p^s]$ acts by zero, then $H_i(\mathbb{Z}^n) \otimes ko_{m-i}(*)_{(2)}$ can be considered as a $\mathbb{Z}_{(2)}[\mathbb{Z}/p^s]/\langle x \rangle$-module. But $\mathbb{Z}_{(2)}[\mathbb{Z}/p^s]/x$ is a Dedekind domain (being isomorphic to the p^s-cyclotomic ring), then $H_i(\mathbb{Z}^n) \otimes ko_{m-i}(*)_{(2)}$ is $\mathbb{Z}_{(2)}[\mathbb{Z}/p^s]/x$-projective, and hence it is $\mathbb{Z}_{(2)}[\mathbb{Z}/p^s]$-projective, and hence the sequence splits.

Finally, if $r - i \equiv 1, 2 \pmod 8$. As the Atiyah-Hirzebruch spectral sequence collapses, we have an splitting of abelian groups

$$s : H_i(\mathbb{Z}^n) \otimes ko_{r-i}(*)_{(2)} \to F_{i,r-i}$$

such that $\pi \circ s = id$. Define

$$\widetilde{s} : H_i(\mathbb{Z}^n) \otimes ko_{r-i}(*)_{(2)} \to F_{i,r-i}$$
$$y \mapsto \sum_{g \in \mathbb{Z}/p^s} g \cdot s(g^{-1}x).$$

$\widetilde{s}$ is a homomorphism of $\mathbb{Z}_{(2)}[\mathbb{Z}/p^s]$-modules and as p is odd $\pi \circ \widetilde{s}$ is multiplication by p^s, but it is the identity because $ko_{m-i}(*)$ is isomorphic to $\mathbb{Z}/2$.

Then the maps (4) and (5) are isomorphisms of $\mathbb{Z}[\mathbb{Z}/p^s]$-modules. Using that we have

$$\widehat{H}^{i+1}(\mathbb{Z}/p^s; ko_j(B\mathbb{Z}^n)) \cong \bigoplus_l \widehat{H}^{i+1}(\mathbb{Z}/p^s; H_{j-4l}(\mathbb{Z}^n)).$$

Then using the universal coefficient theorem and Theorem 3.2 in [LL12] we obtain

$$\widehat{H}^{i+1}(\mathbb{Z}/p^s; ko_j(B\mathbb{Z}^n)) = 0 \text{ if } i + j \text{ is even .}$$

In particular this implies that the canonical map

$$E^2_{0,2r} = ko_{2r}(B\mathbb{Z}^n)_{\mathbb{Z}/p^s} \to ko_{2r}(B\mathbb{Z}^n)^{\mathbb{Z}/p^s}$$

is injective, because $\widehat{H}^{-1}(\mathbb{Z}/p^s; ko_{2r}(B\mathbb{Z}^n)) = 0$. Then $ko_{2r}(B\mathbb{Z}^n)_{\mathbb{Z}/p^s}$ does not contain p^s-torsion. On the other hand, $E^2_{i,j}$ is zero if $i + j$ is even and positive, then

$$ko_{2r}(B\mathbb{Z}^n)_{\mathbb{Z}/p^s} = E^2_{0,2r} = E^\infty_{0,2r} \cong ko_{2r}(B(\mathbb{Z}^n \rtimes \mathbb{Z}/p^s)),$$

then $ko_{2r}(B(\mathbb{Z}^n \rtimes \mathbb{Z}/p^s))$ does not contain p^s-torsion and we conclude that $ko_*(B\Gamma)$ does not contains m-torsion if $*$ is even. □

Proof. (Theorem 3.1) Notice that from the Atiyah-Hirzebruch spectral sequence we obtain, for even indices, $\widetilde{ko}_*(B\mathbb{Z}/p^s) = 0$. Then from Theorem 2.1 we conclude that for even degrees β is injective and from Lemma 2.2 we get $D[f_M] = 0$, by Prop. 12.1 in [DL13] we get that M admits a metric with positive scalar curvature.

□

4. Proof of the positive result in the odd dimensional case.

In this section we will prove the odd version of Theorem 1.3

Theorem 4.1. *Let M^n be an n-dimensional smooth spin manifold, where $n \geq 5$ is odd, and with fundamental group isomorphic to Γ, satisfying 1.2. Denote by $f_M : M \to B\Gamma$ the classifying map for the fundamental group. Assume that $\alpha(M) = 0$. Then M admits a metric of positive scalar curvature.*

As in the previous section the proof of this theorem requires some lemmas.

Lemma 4.2. *Let r be a natural number. Then, the ko-homology group*

$$ko_{2r+1}(\underline{B}\Gamma)$$

does not contain p-torsion for $p \neq 2$.

Proof. By the Hasse principle, it suffices to prove the localization at p is torsion free. Recall the Atiyah-Hirzebruch spectral sequence converging to the ko-homology groups localized at p, $ko_*(\underline{B}\Gamma)_{(p)}$ with E^2-term

$$E^2_{i,j} = H_i(\underline{B}\Gamma, ko_j(*)_{(p)}).$$

The relevant elements on the E^2-term of the Atiyah-Hirzebruch spectral sequence for the computation of $ko_{2r+1}(\underline{B}\Gamma)$ are those for which $i + j$ is odd. Let us distinguish the following cases:

- Assume i is odd, and notice that because of 2.7, the homology groups $H_i(\underline{B}\Gamma, \mathbb{Z})$ are torsionfree abelian. Since $i + j$ is odd, it follows that j is even. Let us analyze first the case where $j \equiv 0$ or $\equiv 4$ modulo 8. In both cases, $ko_j(*)$ is free abelian of rank one, and $ko_j(*)_{(p)}$ is a free $\mathbb{Z}_{(p)}$-module of rank one. For $j \equiv 2$ modulo 8, the group $ko_j(*)_{(p)}$ is zero, and $ko_j(*) = 0$ for $j \equiv 6$ modulo 8.
- Assume that i is even. Then j is odd. If $j \equiv 3,\ 5,$ or 7 modulo 8, then $ko_j(*) = 0$. If $j \equiv 1$ modulo 8, then $ko_j(*)_{(p)} = 0$.

In either case, we see that $E^2_{i,j}$ is either zero or a free $\mathbb{Z}_{(p)}$-module of finite rank. Moreover, because of the rational triviality of the differentials of the Atiyah-Hirzebruch spectral sequence for $ko_{2r+1}(\underline{B}\Gamma)_{(p)}$, the spectral sequence collapses without differentials and extension problems, converging to free $\mathbb{Z}_{(p)}$-modules. □

The following two results concern computations of the spin bordism groups of the classifying space, and the real K-theory of the real group C^*-algebra of the finite group $\mathbb{Z}/p^s$.

Lemma 4.3. *For any m odd, the map*

$$\widetilde{D} : \widetilde{\Omega}^{Spin}_*(B\mathbb{Z}/m) \to \widetilde{ko}_*(B\mathbb{Z}/m)$$

is surjective.

Proof. The strategy of the proof is similar to [DL13, Lemma 2.2]. By Lemma 2.5 (i), it is enough to prove that

$$\widetilde{D} : \widetilde{\Omega}^{Spin}_*(B\mathbb{Z}/p^s) \to \widetilde{ko}_*(B\mathbb{Z}/p^s)$$

is surjective for any odd prime p.

Let M be a $\mathbb{Z}/p^s$-module. By the standard resolution of $\mathbb{Z}$ as a trivial $\mathbb{Z}[\mathbb{Z}/p^s]$-module, for any $i \geq 1$, the localization of the homology groups with coefficients in M,

$$H_i(\mathbb{Z}/p^s, M)\left[\frac{1}{p}\right] = 0$$

holds.

It follows that for $i \geq 1$ the maps

$$H_i(\mathbb{Z}/p^s, M) \to H_i(\mathbb{Z}/p^s, M)_{(p)} \to H_i(\mathbb{Z}/p^s, M_{(p)}) \tag{6}$$

are all isomorphisms. Consider the Atiyah-Hirzebruch spectral sequences converging to $\Omega^{Spin}_*(B\mathbb{Z}/p^s)$, $\Omega^{Spin}_*(B\mathbb{Z}/p^s)_{(p)}$, $ko_*(B\mathbb{Z}/p^s)$ and $ko_*(B\mathbb{Z}/p^s)_{(p)}$. By the comparison lemma for spectral sequences [Wei94, Theorem 5.2.12] and isomorphism (6), we have a commutative diagram

$$\begin{array}{ccc} \widetilde{\Omega}_m(B\mathbb{Z}/p^s) & \xrightarrow{\widetilde{D}} & \widetilde{ko}_m(B\mathbb{Z}/p^s) \\ \downarrow \cong & & \downarrow \cong \\ \widetilde{\Omega}_m(B\mathbb{Z}/p^s)_{(p)} & \xrightarrow{\widetilde{D}_{(p)}} & \widetilde{ko}_m(B\mathbb{Z}/p^s)_{(p)} \end{array}$$

Then it is enough to prove the surjectivity of $\widetilde{D}_{(p)}$

The Atiyah-Hirzebruch spectral sequence for computing p-local Spin bordism and ko-homology collapse at the E^2 term, yielding isomorphisms

$$E^\infty_{i,j} = \tilde{H}_i(\mathbb{Z}/p^s) \otimes \left(\Omega^{\text{Spin}}_j\right)_{(p)}$$

$$E^\infty_{i,j} = \tilde{H}_i(\mathbb{Z}/p^s) \otimes \left(\text{ko}_j\right)_{(p)}$$

Taking a look at the map on the (p)-localized coefficients

$$D_{(p)} : \left(\Omega_j^{\mathrm{Spin}}\right)_{(p)} \to \left(\mathrm{ko}_j\right)_{(p)},$$

which are non -zero only for j a multiple of four, we see that the map is surjective at the level of coefficients, and thus surjective at the E^∞- term.

□

We now recall the following consequence of Theorem 9.4 in page 415 of [DL13].

Lemma 4.4. *Let p be an odd prime number. The real K-theory of the real group C^*-algebra for the cyclic group $\mathbb{Z}/p^s$ is as follows:*

$KO_0(\mathbb{R}[\mathbb{Z}/p^s])$	$\mathbb{Z}^{1+\frac{p^s-1}{2}}$
$KO_1(\mathbb{R}[\mathbb{Z}/p^s])$	$\mathbb{Z}/2$
$KO_2(\mathbb{R}[\mathbb{Z}/p^s])$	$\mathbb{Z}/2 \oplus \mathbb{Z}^{\frac{p^s-1}{2}}$
$KO_3(\mathbb{R}[\mathbb{Z}/p^s])$	0
$KO_4(\mathbb{R}[\mathbb{Z}/p^s])$	$\mathbb{Z}^{1+\frac{p^s-1}{2}}$
$KO_5(\mathbb{R}[\mathbb{Z}/p^s])$	0
$KO_6(\mathbb{R}[\mathbb{Z}/p^s])$	$\mathbb{Z}^{\frac{p^s-1}{2}}$
$KO_7(\mathbb{R}[\mathbb{Z}/p^s])$	0

Theorem 4.5. *Let M^{2r+1} be a $2r+1$-dimensional smooth spin manifold, where $r \geq 2$ is odd, and its fundamental group is isomorphic to Γ, where Γ satisfies condition 1.2. Denote by $f_M : M \to B\Gamma$ the classifying map for the fundamental group. Assume that $\alpha(M) = 0$. Then M admits a metric of positive scalar curvature.*

Proof. Consider the diagram from Theorem 2.1 together with the additional left column given by $A \circ p_{BN}$.

$$\begin{array}{ccccc}
\bigoplus_{(N)\in\mathcal{N}} \widetilde{ko}_{2r+1}(BN) & \xrightarrow{\varphi_{2r+1}} & ko_{2r+1}(B\Gamma) & \xrightarrow{\beta} & ko_{2r+1}(\underline{B}\Gamma) \\
\downarrow{\scriptstyle A\circ p_{BN}} & & \downarrow{\scriptstyle A\circ p_{B\Gamma}} & & \downarrow{\scriptstyle p_{\underline{B}\Gamma}} \\
\bigoplus_{N\in(\mathcal{N})} \widetilde{KO}_{2r+1}(\mathbb{R}[\mathbb{Z}/p^s]) & & KO_{2r+1}(C_r^*(\Gamma;\mathbb{R})) & \longrightarrow & KO_{2r+1}(\underline{B}\Gamma)
\end{array}$$

Recall that from 4.2, the group $ko_{2r+1}(\underline{B}\Gamma)$ does not contain m-torsion, and from 2.2, ker $p_{\underline{B}\Gamma}$ only consists of m-torsion. It follows that $\beta(D_{[F_M]}) = 0$. By the exactness of the upper line, we can find $X_N \in \widetilde{ko}_{2r+1}(BN)$ in the preimage under the map φ_{2r+1}. By the surjectivity lemma 4.3, we can find classes $[M_N, F_N] \in \Omega_{2r+1}^{Spin}(BN)$ such that $D[M_N \xrightarrow{F_N} BN] = X_N$. By surgery, we can assume that F_M is 2-connected.

Now, recall that by the computation of the real K-theory of the group C^* algebra of $\mathbb{Z}/p^s$ 4.4, the reduced KO-theory groups are zero in odd degrees. By the proof of the Gromov-Lawson -Rosenberg Conjecture for groups with periodic cohomology [BGS97], the classes $[M_N, F_N]$ admit representatives $[M_N^+, F_M]$, where M_N^+ has positive scalar curvature.

We consider now the class

$$D[F_M : M \to B\Gamma] = D[\bigsqcup_{N\in\mathcal{N}} M_N^+ \longrightarrow \bigsqcup_{N\in\mathcal{N}} BN \longrightarrow B\Gamma]$$

and notice that it admits a representative of positive scalar curvature. This finishes the proof of theorem 4.1.

□

5. Construction of Counterexamples

In this section, we will give a condition on the action of $\mathbb{Z}/m$ on $\mathbb{Z}^n$ that implies that the group $\mathbb{Z}^n \rtimes \mathbb{Z}/m$ is a counterexample for the GLR conjecture.The assumption will be that m is odd and square-free to use results from [SV24a].

Let p be a prime that divides m. Then by Prop. 4.5 in [SV24a] there is $\mathbb{Z}/m$-submodule $N \subseteq \mathbb{Z}^n$ of finite index such that this index is coprime with p and such that there is a decomposition of $\mathbb{Z}/m$-modules

$$N \cong \mathbb{Z}^r \oplus (\mathbb{Z}[\mathbb{Z}/p])^s \oplus I^t.$$

Here $\mathbb{Z}$ has the trivial $\mathbb{Z}/p$-action, $I \subseteq \mathbb{Z}[\mathbb{Z}/p]$ is the augmentation ideal and both I and $\mathbb{Z}[\mathbb{Z}/p]$ are endowed with the canonical $\mathbb{Z}/p$-action and moreover $\mathbb{Z}^r$, $(\mathbb{Z}[\mathbb{Z}/p])^s$ and I^t are itself $\mathbb{Z}/m$-modules. We say the N (and also $\mathbb{Z}^n$) is a $\mathbb{Z}/p$-module of type (r, s, t). By Lemma 5.3 in [SV24a], for every ℓ, we have a canonical isomorphism

$$H^\ell(N \rtimes \mathbb{Z}/m)_{(p)} \cong H^\ell(\mathbb{Z}^n \rtimes \mathbb{Z}/m)_{(p)}.$$

Then we do not lose information about p-torsion in cohomology if we suppose that $\mathbb{Z}^n$ is a module of type (r, s, t). We have the following result, whose proof is a direct generalization of the main result in [Sch98].

Theorem 5.1. *Suppose m is square-free. Let $\mathbb{Z}^n$ be a $\mathbb{Z}/m$-module, and suppose that there exists a prime $p \mid m$ such that in the (r, s, t)-decomposition, we have that $r \geq 4$, then $\mathbb{Z}^n \rtimes \mathbb{Z}/m$ is a counter-example for the GLR conjecture.*

Proof. First note that we can suppose that $\mathbb{Z}^n$ is a $\mathbb{Z}/m$-module of type (r, s, t), for some prime $p \mid m$. Notice that by Corollary. 4.2 in [AGPP08] the Lyndon-Hochschild-Serre spectral sequence for (co-)homology associated to the extension (1) collapses. In particular, the group $H^1(\mathbb{Z}^n \rtimes \mathbb{Z}/m)$ contains as a subgroup $H^1(\mathbb{Z}^r \times \mathbb{Z}/m)$, then let $a_1 \dots, a_4$ be some generators as $\mathbb{Z}$-module of $H^1(\mathbb{Z}^r \times \mathbb{Z}/m)$, viewed as elements in the group $H^1(\mathbb{Z}^n \rtimes \mathbb{Z}/m)$. For each a_i, we have dual elements $\hat{x}_i \in H_1(\mathbb{Z}^n \times \mathbb{Z}/m)$, let $x_i = \iota_*(\hat{x}_i)$, where $\iota : \mathbb{Z}^r \times \mathbb{Z}/m \to \mathbb{Z}^n \rtimes \mathbb{Z}/m$ is the inclusion, and let $\hat{y}$ be an element of p-torsion in $H_1(\mathbb{Z}^r \times \mathbb{Z}/m)$. Now we follow [Sch98]. Let $w = \iota_*(\hat{x}_1 \times \dots \hat{x}_4 \times \hat{y}) \in H_5(\mathbb{Z}^n \rtimes \mathbb{Z}/m)$. Let us prove $a_1 \cap (a_2 \cap (a_3 \cap w)) \neq 0$.

First note that by the Kunneth formula applied to the decomposition as (r, s, t)-modules, the map $\iota_* : H_5(\mathbb{Z}^r \times \mathbb{Z}/m) \to H_5(\mathbb{Z}^n \rtimes \mathbb{Z}/m)$ is injective. On the other hand, by the naturality of the cap product we have

$$\iota_*(a_3 \cap (\widehat{x}_1 \times \ldots \widehat{x}_4 \times \widehat{y})) = \iota_*(\iota^*(a_3) \cap (\widehat{x}_1 \times \ldots \widehat{x}_4 \times \widehat{y})) = a_3 \cap w.$$

But $a_3 \cap (\widehat{x}_1 \times \ldots \widehat{x}_4 \times \widehat{y}) \neq 0$, and similarly $a_1 \cap (a_2 \cap (a_3 \cap (w))) \neq 0$ and now the same argument in Example 2.2 in [Sch98] applies. Then $\mathbb{Z}^n \rtimes \mathbb{Z}/m$ is a counterexample for the GLR conjecture. □

References

[AGPP08] Alejandro Adem, Jianquan Ge, Jianzhong Pan, and Nansen Petrosyan. Compatible actions and cohomology of crystallographic groups. *J. Algebra*, **320** (2008) No. 1, 341–353. MR2417992 (2009k:20115) Zbl 1163.20032 677, 688

[BGS97] Boris Botvinnik, Peter Gilkey, and Stephan Stolz. The Gromov-Lawson-Rosenberg conjecture for groups with periodic cohomology. *J. Differential Geom.*, **46** (1997) No. 3, 374–405. MRMR1484887 (98i:58227) Zbl 0929.58011 688

[DK01] James F. Davis and Paul Kirk. *Lecture notes in algebraic topology*, volume 35 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. MR1841974 (2002f:55001) Zbl 1018.55001 681

[DL13] James F. Davis and Wolfgang Lück. The topological K-theory of certain crystallographic groups. *J. Noncommut. Geom.*, **7** (2013) No. 2, 373–431. MR3054301 Zbl 1279.19003 677, 678, 680, 683, 685, 686, 687

[DP03] James F. Davis and Kimberly Pearson. The Gromov-Lawson-Rosenberg conjecture for cocompact Fuchsian groups. *Proc. Am. Math. Soc.*, **131** (2003) No. 11, 3571–3578. MR1991770 (2004f:53039) Zbl 1050.19004 677

[DSS03] William Dwyer, Thomas Schick, and Stephan Stolz. Remarks on a conjecture of Gromov and Lawson. In *High-dimensional manifold topology*, pages 159–176. World Sci. Publ., River Edge, NJ, 2003. MR2048721 (2005f:53043) Zbl 1052.57035 677

[Hug21] Sam Hughes. On the equivariant K- and KO-homology of some special linear groups. *Algebr. Geom. Topol.*, **21** (2021) No. 7, 3483–3512. MR4357611 Zbl 1509.19010 677

[LL12] Martin Langer and Wolfgang Lück. Topological K-theory of the group C^*-algebra of a semi-direct product $\mathbb{Z}^n \rtimes \mathbb{Z}/m$ for a free conjugation action. *J. Topol. Anal.*, **4** (2012) No. 2, 121–172. MR2949238 Zbl 1253.19004 677, 683, 684

[LW12] Wolfgang Lück and Michael Weiermann. On the classifying space of the family of virtually cyclic subgroups. *Pure Appl. Math. Q.*, **8** (2012) No. 2, 497–555. MR2900176 Zbl 1258.55011 679, 681

[Sch98] Thomas Schick. A counterexample to the (unstable) Gromov-Lawson-Rosenberg conjecture. *Topology*, **37** (1998) No. 6, 1165–1168. MR1632971 (99j:53049) Zbl 0976.53052 677, 679, 688, 689

[SV24a] Luis Jorge Sánchez Saldaña and Mario Velásquez. On the group cohomology of groups of the form $\mathbb{Z}^n \rtimes \mathbb{Z}/m$ with m free of squares, 2024. https://arxiv.org/abs/2410.09263 677, 679, 688

[SV24b] Luis Jorge Sánchez Saldaña and Mario Velásquez. On the K-theory of groups of the form $\mathbb{Z}^n \rtimes \mathbb{Z}/m$ with m square-free, 2024. https://arxiv.org/abs/2403.14569 681

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994. MR1269324 (95f:18001) Zbl 0834.18001 686

(Noé Bárcenas) Centro de Ciencias Matemáticas. UNAM, Ap.Postal 61-3 Xangari. Morelia, Michoacán 58089, MEXICO
barcenas@matmor.unam.mx

(Mario Velásquez) Universidad Nacional de Colombia Sede Bogotá Departamento de Matemáticas, Facultad de Ciencias Cra. 30 cll 45 - Ciudad Universitaria, Bogotá, Colombia
mavelasquezme@unal.edu.co

This paper is available via http://nyjm.albany.edu/j/2026/32-27.html.