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Coloring spheres in 3–manifolds

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ABSTRACT. The sphere graph of M_r, a connect sum of r copies of $S^1 \times S^2$ was introduced by Hatcher as an analog of the curve graph of a surface to study the outer automorphism group of a free group F_r. Bestvina, Bromberg, and Fujiwara proved that the chromatic number of the curve graph is finite; bounds were subsequently improved by Gaster, Greene, and Vlamis. Motivated by the analogy, we provide upper and lower bounds for the chromatic number of the sphere graph of M_r. As a corollary to the prime decomposition of 3–manifolds, this gives bounds on the chromatic number of the sphere graph for any orientable 3–manifold.

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1. Introduction

The sphere graph of a connect sum of r copies of $S^1 \times S^2$ was introduced by Hatcher to investigate homological stability of the automorphism groups of free groups; analogous to Harvey's use of the curve graph of a surface to investigate mapping class groups of surfaces [Hat95]. Subsequently, the large scale geometry of both graphs have been extensively studied on their respective sides of the long-running analogy between automorphisms of free groups and mapping class groups of surfaces [MM99, MM00, BBF15, HM13, HM19, BF14a, BF14b]. In addition to their above mentioned role, both graphs are rich combinatorial objects. The graph theory of both the curve graph [Iva97, AL13, AL16, GGV18, MRT14, DKG20] and the sphere graph [AS11, BL24] have seen a similar development of parallel results.

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We will focus our attention on the chromatic number. Bestvina, Bromberg, and Fujiwara first proved that the chromatic number of the curve graph is finite, on the way to determining the asymptotic dimension of the mapping class group of a surface [BBF15, Lemma 5.6]. Their initial bound was doubly exponential in the genus of the surface. Subsequently, Gaster, Greene, and Vlamis produced a lower bound super-linear in the genus and a singly exponential upper bound [GGV18, Theorem 1.5]. Motivated by the guiding analogy, we obtain upper and lower bounds for the sphere graph of M_r a connect sum of r copies of $S^1 \times S^2$.

Theorem 1.1.

$$r \log r \preceq \chi(\mathcal{S}(M_r)) \preceq 2^{2r2^r}.$$

The proof of Theorem 1.1 is given in two lemmas: Lemma 2.3 for the lower bound and Lemma 3.2 for the upper bound.

Prime decomposition of 3–manifolds [Hem04, Theorem 3.15] allows us to apply Theorem 1.1 to an orientable 3–manifold M. Let $M = N_1 \sharp N_2 \sharp \cdots \sharp N_k \sharp M_r$ be the prime decomposition of M into a connect sum of irreducible 3–manifolds N_i and r copies of $S^1 \times S^2$ collected in the M_r term. An essential embedded sphere in M is isotopic to either a connect-sum sphere or an essential sphere in M_r. Thus $\mathcal{S}(M)$ is the join of $\mathcal{S}(M_r)$ and k vertices, and we obtain the corollary:

Corollary 1.2. *If M is an orientable 3–manifold with prime decomposition*

$$N_1 \sharp N_2 \sharp \cdots \sharp N_k \sharp M_r$$

then

$$r \log(r) + k \preceq \chi(\mathcal{S}(M)) \preceq 2^{2r2^r} + k.$$

The proof of the upper bound in Theorem 1.1 involves a coloring constructed using double covers and $\mathbb{Z}/2\mathbb{Z}$ homology. Bestvina and Feighn define a coloring on the *free factor* graph of a free group [BF14b, Definition 4.11, Example 4.12], which can be characterized in terms of $\mathbb{Z}/2\mathbb{Z}$ homology. While not explicitly calculated, the coloring provided by Bestvina and Feighn is also doubly-exponential in the rank. Coloring the free factor graph is insufficient to easily recover Theorem 1.1: there are well-studied *coarse projections* from the sphere graph to the free factor graph, but coarse projections do not interact well with graph colorings.

In addition to the motivation from mapping class groups, Gaster, Greene, and Vlamis describe a connection between the chromatic number of the curve graph and interesting open problems in the theory of combinatorial designs. We are unaware of any analogous investigation for spheres in connect sums, but the parallel questions are natural and intriguing.

Notational conventions. A *vertex coloring* of a graph G is a function $\phi : V(G) \to X$ such that for every edge $(u, v) \in E(G)$, $f(u) \neq f(v)$. The cardinality $|X|$ is the size of the coloring. The *chromatic number* of G, denoted $\chi(G)$, is the minimal size of a vertex coloring of G.

Given two functions $f, g : \mathbb{N}^d \to \mathbb{R}_{\geq 0}$, define $f \preceq g$ if there exists an absolute constant C such that $f \leq C \cdot g$ for sufficiently large inputs (often written $f \in O(g)$ elsewhere in the literature), and $f \sim g$ if $f \preceq g$ and $g \preceq f$ (often $f \in \Theta(g)$).

For $r \geq 0$ we define $M_r = M_{r,0} = \sharp_r S^1 \times S^2$ to be the connect sum of r copies of $S^1 \times S^2$ with the convention $M_0 = S^3$. For $n \geq 0$ we define $M_{r,n} = M_r \setminus \sqcup_n B$ to be M_r with n disjoint open balls removed. A sphere embedded in a 3–manifold is *essential* if it does not bound a ball and *non-peripheral* if it is not isotopic to the boundary. The *sphere graph* of a 3–manifold M is the graph $\mathcal{S}(M)$ with vertex set the isotopy classes of embedded essential non-peripheral spheres in M where two classes are joined by an edge if they have disjoint representatives. For brevity, we will refer to vertices of $\mathcal{S}(M)$ as *spheres in M* when no ambiguity occurs. A *cut-system* in $M_{r,n}$ is a disjoint union of embedded spheres C such that $\overline{M_{r,n} \setminus C} = M_{0,n+2r}$.

Acknowledgments. The authors thank the anonymous referee for their careful reading and suggested improvements to the upper bound, particularly the point of view presented in Remark 3.3.

2. Kneser graphs and the lower bound

We establish the lower bound using Gaster, Greene, and Vlamis' computation of the chromatic number for the *total Kneser graph* [GGV18]. Given a pair of positive integers n, k with $n \geq 2k$ the *Kneser graph* $KG(n, k)$ is the graph with vertices the k element subsets of $\{1, \ldots, n\}$ and edges joining disjoint subsets. The *total Kneser graph* is the graph whose vertices are unordered partitions of $\{1, \ldots, n\}$ into two non-empty subsets, and two partitions are joined by an edge if they are *nested*, that is (A, B) is joined to (C, D) if one of A or B is a subset of C or D. Note this condition is symmetric. Gaster, Greene, and Vlamis compute the asymptotics of the chromatic number, which we record.

Theorem 2.1 ([GGV18, Theorem 1.1])**.**

$$\chi(KG(n)) \sim n \log n.$$

The partition definition of the Kneser graph is intimately related to the structure of $\mathcal{S}(M_{0,n})$.

Lemma 2.2.

$$\mathcal{S}(M_{0,n}) = KG(n) \setminus KG(n, 1).$$

Proof. A sphere $s \in \mathcal{S}(M_{0,n})$ is necessarily separating, and thus partitions the connected components of $\partial M_{0,n}$. In fact, this partition determines s up to isotopy [BL24, Lemma 9]. Moreover, since each sphere of $\mathcal{S}(M_{0,n})$ is non-peripheral, both pieces of the partition have more than one component. Fix a bijection between components of $\partial M_{0,n}$ and $\{1, \ldots, n\}$. The map that sends a partition of $\{1, \ldots, n\}$ to its corresponding sphere is a bijection between the vertices of $KG(n) \setminus KG(n, 1)$ and $\mathcal{S}(M_{0,n})$.

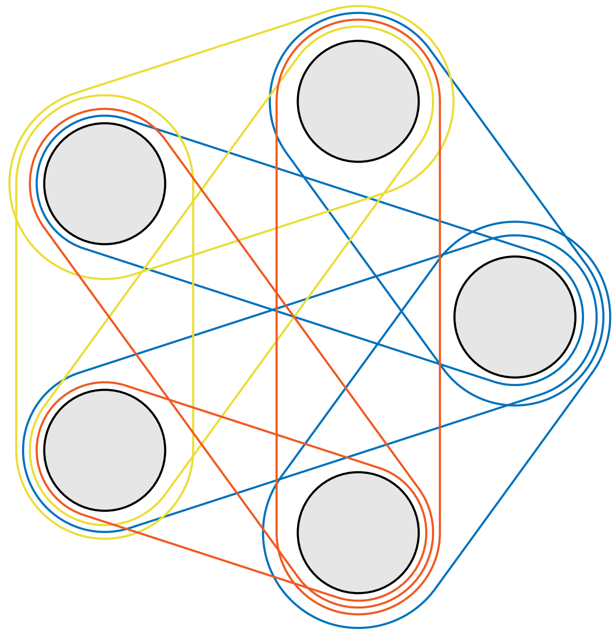

FIGURE 1. The stereographic projection of the midsphere of $M_{0,5}$ with the deleted balls shown as removed disks. Each sphere of $\mathcal{S}(M_{0,5})$ is shown as a simple closed curve of intersection with the midsphere. A pleasant exercise is to verify the intersection pattern of these spheres is the Petersen graph, 3-colored by the indicated colors.

It remains to verify that this assignment preserves the edge relation. Since the boundary partition determines each $s \in \mathcal{S}(M_{0,n})$ up to isotopy [BL24, Lemma 9] it suffices to produce disjoint spheres realizing each nested partition. Let (A, A'), and (B, B') be nested partitions of $\{1, \dots, n\}$ such that, without loss of generality, $A \subset B$. Let X_A be the subset of $M_{0,n}$ obtained by joining the boundary spheres corresponding to A by arcs. Then, the boundary of a regular neighborhood $a = \partial N(X_a)$ is a sphere which by construction induces the partition (A, A') on the boundary. Let X_B be the subset of $M_{0,n}$ obtained by joining a to the spheres corresponding to $B \setminus A$ by arcs. The boundary of a regular neighborhood $\partial N(X_B)$ has two connected components: one isotopic to a and the other a sphere b. By construction b induces the partition (B, B') and is disjoint from a. □

As a specific example, Fig. 1 illustrates $\mathcal{S}(M_{0,5})$, and the picture can be used to verify

$$\mathcal{S}(M_{0,5}) = KG(5) \setminus KG(5, 1) = KG(5, 2);$$

note that $KG(5, 2)$ is the well-known Petersen graph.

Lemma 2.3.

$$\chi(\mathcal{S}(M_{r,n})) \geq (n + 2r)\log(n + 2r).$$

Proof. Fix a cut system C in $M_{r,n}$. Consider the cut manifold

$$\overline{M_{r,n} \setminus C} = M_{0,n+2r}.$$

Gluing along C induces a graph map $\mathcal{S}(M_{0,n+2r}) \to \mathcal{S}(M_{r,n})$. Indeed, if an embedded sphere $a \subset M_{0,n+2r}$ bounds a ball in the glued manifold, then up to isotopy C can be made disjoint from this ball, which implies a is not essential. Moreover, if $a, b \subset M_{0,n+2r}$ are disjoint spheres, their images in the glued manifold are as well. Thus $\chi(\mathcal{S}(M_{r,n})) \geq \chi(\mathcal{S}(M_{0,n+2r}))$, and the conclusion follows from Lemma 2.2 and Theorem 2.1. □

3. Double covers and the upper bound

We exhibit an explicit coloring, using a double-cover construction analogous to that used by Bestvina, Bromberg, and Fujiwara in the curve complex setting [BBF15, Lemma 5.6].

Define $T(M)$ as the set of all of connected double covers of M. Define $X(M)$ to be the disjoint union of $H_2(\tilde{M}, \mathbb{Z}/2\mathbb{Z})$ over all $\tilde{M} \in T(M)$. Finally, let $F(M)$ be the set of all functions $f : T(M) \to X(M)$. The set $F(M)$ will serve as our coloring set.

Lemma 3.1. *For the manifold* M_r,

$$|F(M_r)| \leq 2^{2r2^r}.$$

Proof. First, since $\pi_1(M_r)$ is the free group of rank r and the index-2 subgroups are parameterized by non-trivial homomorphisms to $\mathbb{Z}/2\mathbb{Z}$, we have $|T(M_r)| = 2^r - 1$.

Next, for each cover $\tilde{M} \in T(M_r)$, $\pi_1(\tilde{M})$ is a free gorup of rank $2r - 1$, so the homology has rank $2r - 1$, thus

$$|H_2(\tilde{M}, \mathbb{Z}/2\mathbb{Z})| = 2^r - 1.$$

Taking a disjoint union over the $2^r - 1$ connected covers we find $|X(M_r)| \leq 2^{2r}$, from which we conclude $|F(M_r)| \leq 2^{2r2^r}$. □

Given a sphere $a \in \mathcal{S}(M_r)$ and a double cover $\tilde{M}$ let $\tilde{a}, \tilde{a}'$ denote the two lifts of a to $\tilde{M}$. For this sphere, define a function $f_a \in F(M_r)$ by

$$f_a(\tilde{M}) = [\tilde{a}]$$

that is, f_a assigns to a double cover $\tilde{M}$ a choice of $\mathbb{Z}/2\mathbb{Z}$ homology class of one of the lifts of a to $\tilde{M}$. We will show in the course of the proof of Lemma 3.2 that the ambiguity of choice in the definition of f_a does not matter. If the reader would like a concrete f_a without the axiom of choice, fix an ordered basis for each $H_2(\tilde{M}, \mathbb{Z}/2\mathbb{Z})$ (these are finite so this does not require choice) and select the class $[\tilde{a}], [\tilde{a}']$ that is lexicographically least in coordinates with respect to the fixed basis.

Lemma 3.2.

$$\chi(\mathcal{S}(M_r)) \leq 2^{2r2^r}.$$

Proof. While our result is asymptotic, we will prove that the desired inequality holds for all r. We first treat the case $r = 2$. Consider the function $\phi : \mathcal{S}(M_2) \to H_2(M_2, \mathbb{Z}/2\mathbb{Z})$. We claim this is a 4-coloring. Indeed, suppose $a, b \in \mathcal{S}(M_2)$ are adjacent spheres. If $a \cup b$ is non-separating, then a and b are not homologous. If a is separating, then the closure of both connected components of $M_2 \setminus \{a\}$ is $M_{1,1}$, and b is contained in one component. Since b is not parallel to a it is essential and non-peripheral in the $M_{1,1}$ component it is contained in, and therefore non-separating, so a and b are not homologous. If a is non-separating, then the closure of $M_2 \setminus \{a\}$ is $M_{1,2}$ and b is essential and non-peripheral in $M_{1,2}$. Either b is non-separating in $M_{1,2}$, so that $a \cup b$ is non-separating, or b is separating—in either possibility a is not homologous to b. Thus $[a] \neq [b]$ in all cases, and $\chi(\mathcal{S}(M_2)) \leq 4$.

Now suppose $r \geq 3$ and consider the function $\phi : \mathcal{S}(M_r) \to F(M_r)$ defined $\phi(a) = f_a$. We will prove ϕ is a coloring; the bound then follows from Lemma 3.1.

Let a, b be adjacent spheres in $\mathcal{S}(M_r)$. We will construct a cover $\tilde{M} \in T(M_r)$ such that $f_a(\tilde{M}) \neq f_b(\tilde{M})$. First, observe that if a and b are not $\mathbb{Z}/2\mathbb{Z}$ homologous in M_r then for any cover $\tilde{M}$ their lifts are also not homologous, so $f_a \neq f_b$. It remains to consider the case when $[a] = [b]$.

Since a, b are disjoint, there are at most three components of $M_r \setminus (a \cup b)$. Label the closures of the components W_1, W_2, W_3 and notice that each one is of the form M_{k_i,p_i} with $1 \leq p_i \leq 4$. Observe that for $k \geq 1$, $M_{k,p}$ contains a non-separating sphere. We claim that for each W_i, $k_i \geq 1$. To verify this we consider the four possible cases with $k_i = 0$.

- $p_i = 1$ In this case, W_i is a ball bounded by exactly one of a or b. Thus either a or b is non-essential in M_r, contradicting our hypotheses.
- $p_i = 2$ In this case, $\partial W_i = a \cup b$, and $W_i = M_{0,2}$, which is homeomorphic to $S^2 \times I$. This implies that a and b are isotopic, contradicting our hypotheses.
- $p_i = 3$ In this case, ∂W_i is three spheres. Without loss of generality, suppose two boundary components are identified with b and one with a. We claim there is some curve γ in M_r that passes through b exactly once and has $a \cap \gamma = \emptyset$. Given a neighborhood N of b, select a pair of points p_1, p_2 in the two components of $N \setminus b$. Since N is path-connected and b separates N, there exists some path P_N between p_1 and p_2 that intersects b exactly once. Since W_i is path-connected and $N \setminus b \subset W_i$, there exists a path $P_c \subset W_i$ connecting p_1, p_2. By construction the union $\gamma = P_N \cup P_c$ is the desired curve. By Poincaré duality we conclude a and b are not homologous in M_r, contradicting our hypotheses.
- $p_i = 4$ In this case, there is only one component $W_1 = M_{0,4}$. This implies $r = 2$. However, we are only considering $r \geq 3$.

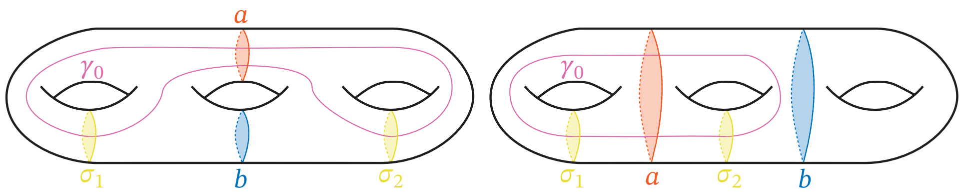

FIGURE 2. Two possible cases for the construction of γ_0 in M_3, here depicted as one-half of a Heegaard splitting. The spheres a, b, σ_1, σ_2 are the doubles of the indicated disks.

Thus, each W_i contains some non-separating sphere σ_i. Construct a double cover $\tilde{M}$ by cutting open two copies of M_r along the collection of σ_i and gluing crosswise.

We will construct a closed loop γ in $\tilde{M}$ that intersects both lifts of a, $\tilde{a}$ and $\tilde{a}'$ exactly once and disjoint from both lifts of b. Thus, regardless of the choice in the definition of f_a, f_b, $f_a(\tilde{M}) \neq f_b(\tilde{M})$.

Notice that if $a \cup b$ does not separate M_r, then we can show that $[a] \neq [b]$ via a similar curve construction. Therefore $a \cup b$ separates M_r. As noted in the case $p_i = 3$, if a appears twice in the boundary of one complementary component, $[a] \neq [b]$. So, without loss of generality suppose a is in the boundary of W_1 and W_2. Since each W_i is path connected and the spheres σ_i are non-separating, there is a loop γ_0 that intersects σ_1 and σ_2 exactly once, intersecting a twice. By construction, there is a loop γ in $\tilde{M}$ covering γ_0, disjoint from both lifts of b which intersects $\tilde{a}$ and $\tilde{a}'$ exactly once (see Fig. 2). □

Remark 3.3. We are thankful to the anonymous referee for pointing out the following.

In the proof above, $M_r \setminus (a \cup b)$ has either two or three components. In the former case, with components W_1, W_2, a Meyer-Vietoris argument shows that

$$H_1(M_r, \mathbb{Z}/2\mathbb{Z}) = H_1(W_1, \mathbb{Z}/2\mathbb{Z}) \oplus H_1(W_2, \mathbb{Z}/2\mathbb{Z}) \oplus V$$

for a 1-dimensional V; in the latter case, with components W_1, W_2, W_3,

$$H_1(M_r, \mathbb{Z}/2\mathbb{Z}) = H_1(W_1, \mathbb{Z}/2\mathbb{Z}) \oplus H_1(W_2, \mathbb{Z}/2\mathbb{Z}) \oplus H_1(W_3, \mathbb{Z}/2\mathbb{Z}).$$

The cover $\tilde{M}$ used to show that f_a, f_b are not equal is a cover that restricts to a connected double cover of the components W_1, W_2, W_3. There is a one-to-one correspondence between connected double covers of M_r and codimension one subspaces of $H_1(M_r, \mathbb{Z}/2\mathbb{Z})$. A cover $\tilde{M}$ corresponding to a subspace U restricts to a connected cover of W_1, W_2, W_3 if the intersection with $H_1(W_i, \mathbb{Z}/2\mathbb{Z})$ is codimension one for each i.

The bound in Lemma 3.2 is doubly-exponential because we require all $2^r - 1$ covers to separate f_a and f_b. The bound could be improved with a better asymptotic estimate on the minimum size of a collection $\mathcal{U}$ of codimension one subspaces of a $\mathbb{Z}/2\mathbb{Z}$ vector space V such that for every direct sum decomposition $V = V_1 \oplus V_2 \oplus V_3$ there is some $U \in \mathcal{U}$ such that $V_i \cap U$ is codimension one in V_i for all i. A subexponential upper bound for the size of such a collection has eluded us.

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