

The Gilmer-Masbaum map is not injective on the skein module

Edwin Kitaeff

ABSTRACT. In [13], Gilmer and Masbaum use Witten-Reshetikhin-Turaev (WRT) invariants to define a map from the Kauffman bracket skein module to a set of complex-valued functions defined on roots of unity in order to provide a lower bound for its dimension. We show that the restriction of the map to a certain homology class is not injective. We also provide a basis for the KBSM of mapping tori associated to a power of a Dehn twist on the 2-torus.

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1. Introduction

1.1. The Kauffman bracket skein module. Let M be a closed oriented 3-manifold. The Kauffman bracket skein module $K(M, \mathbb{Q}(A))$ (or $K(M)$ for short) over $\mathbb{Q}(A)$, or just skein module here, was introduced independently by Przytycki ([18]) and Turaev ([19]).

The skein module $K(M)$ is defined as the $\mathbb{Q}(A)$ -vector space spanned by the framed links in M over the ground field $\mathbb{Q}(A)$ modulo isotopies and the Kauffman skein relations:

$$\begin{aligned} & \text{Crossing} - A \text{Positive Twist} - A^{-1} \text{Negative Twist} \\ & L \sqcup \text{Circle} + A^2 L + A^{-2} L \end{aligned}$$

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For $\xi \in \mathbb{C}^*$, we use the notation $K_\xi(M) = K(M, \mathbb{Q}[A^{\pm 1}]) \otimes_{A=\xi} \mathbb{C}$.

It was shown in [11] that the bracket skein module over $\mathbb{Q}(A)$ is finite dimensional for every closed 3-manifold. Unfortunately, this proof is not constructive and cannot be used to compute the dimension of $K(M)$. Recently, another proof of this fact was provided in [1], in a more constructive and elementary way. However, even if [1] provide a set of generators, it does not give a basis for $K(M)$, nor its dimension.

The computation of $K(M)$ is therefore still a very active and open problem. So far, three main ideas stand out to compute the skein module. The first one, used for instance in [14] and in [15] for lens spaces, is to view M as a Heegaard splitting. Then, $K(M)$ is the skein module of a handlebody (for which we know a basis) quotiented by some relations called "slide relations". Unfortunately, the computations obtained in this way become very complicated as soon as the genus of the Heegaard splitting is more than 1.

The second method is to compare $K(M)$ to the better known $K_\xi(M)$. This method was developed in [5] and can be applied on 3-manifolds with some "tameness" property. However, the tameness property is not easy to check and does not always hold (an example where it does not hold would be when the character variety of M is infinite).

The third idea is to combinatorially compute a set of generator for $K(M)$ and prove that this set is free through representations of $K(M)$. In this vein, Gilmer and Masbaum introduced the evaluation map in [13], which they applied to exhibit a free family of $K(\Sigma_g \times \mathbb{S}^1)$. Later, in [6], it was found that this family was also a set of generators, providing a basis for $K(\Sigma_g \times \mathbb{S}^1)$.

In this paper, we prove that the (third) method of [13] cannot always detect that a given family of skein elements is free. It then appears that stronger methods must be developed to deal with the problem of the dimension of skein modules.

1.2. The Gilmer-Masbaum evaluation map. The technique of [13] is relying on the Witten-Reshetikhin-Turaev invariants $RT_\xi(M, L)$ of a framed link L in M associated to ξ a primitive root of even degree. The invariant $RT_\xi(M, L)$ only depends on the skein class of L . Below is a more detailed description of this method.

Denote by $\mathbb{U}_0 := \{e^{\frac{isr}{r}} \mid r > 1, \gcd(s, 2r) = 1\}$ the set of primitive roots of unity of even order, and by $\mathbb{C}_{a.e.}^{\mathbb{U}_0}$ the set of complex-valued functions that are defined almost everywhere on \mathbb{U}_0 . For given $\lambda_1, \dots, \lambda_n \in \mathbb{Q}(A)$ and $f_1, \dots, f_n \in \mathbb{C}_{a.e.}^{\mathbb{U}_0}$ one can define the element

$$\sum_{j=0}^n \lambda_j f_j = (\xi \mapsto \sum_{j=0}^n \lambda_j(\xi) f_j(\xi)) \in \mathbb{C}_{a.e.}^{\mathbb{U}_0}.$$

This is well-defined since λ_j is well-defined everywhere except at its poles. This gives $\mathbb{C}_{a.e.}^{\mathbb{U}_0}$ a structure of $\mathbb{Q}(A)$ -vector space.

Here is the evaluation map:

Definition 1.1 ([13]).

$$ev_M : \begin{array}{l} K(M) \longrightarrow \mathbb{C}_{a.e.}^{\cup_0} \\ \sum_{j=1}^n \lambda_j L_j \longmapsto \left(\begin{array}{l} \cup_0 \longrightarrow \mathbb{C} \\ \xi \longmapsto \sum_{j=1}^n \lambda_j(\xi) RT_\xi(M, L_j) \end{array} \right) \end{array}.$$

This application is slightly different from the original one since we are also looking at primitive roots of unity of order divisible by 4.

Since the skein relations are homogeneous, the skein module admits a decomposition into graded subspaces:

$$K(M) = \bigoplus_{\alpha \in H_1(M, \mathbb{Z}/2\mathbb{Z})} K_\alpha(M),$$

in which $K_\alpha(M)$ is the skein module of homology α .

One can use the evaluation map to check linear independence in $K(M)$. Indeed, if a family of skein elements have a free image in $\mathbb{C}_{a.e.}^{\cup_0}$, this family is itself free. Although difficult, showing that a family is free in $\mathbb{C}_{a.e.}^{\cup_0}$ is purely a matter of linear algebra.

In [13], this technique yielded linearly independent families for $K(\Sigma \times I)$. Historically, this map has been used previously in a similar fashion in [8] for quaternionic manifolds and in [10] for \mathbb{T}^3 . It has also been used in [4] to show the non-triviality of the empty link in rational homology spheres.

However, a key ingredient for this method to be optimal is that ev_M has to be injective on the graded subspace $K_\alpha(M)$ for each homological class α . The question of the injectivity of the evaluation map is then a very natural question, which was asked by Gilmer and Masbaum in [13].

In fact, it has already been answered in [9] for $SU(2)$ -invariants (that is with \cup_0 restricted to primitive roots of order $4r$) as Gilmer exhibited elements of the basis of the skein module of certain lens spaces for which all their $SU(2)$ -invariants are zero. However, since this work was done for only $SU(2)$ -invariants, one could hope the map introduced above to be stronger. In particular, we know some examples in the set given in [9] that are not detected by the $SU(2)$ -invariants but are detected by the $SO(3)$ -invariants.

Yet, we will show that this map is also not always injective.

1.3. The mapping tori. The manifolds that will interest us are from the following family:

Definition 1.2. For $B \in Mod(\mathbb{T}^2) \cong SL_2(\mathbb{Z})$, we define the mapping torus of the 2-torus \mathbb{T}^2 of monodromy B by:

$$M_B = \mathbb{T}^2 \times [0, 1] / (x, 0) \sim (B(x), 1).$$

Let $B_k \in \text{Mod}(\mathbb{T}^2)$ be the application of monodromy $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, which represent a power of a Dehn twist on \mathbb{T}^2 , and let $M_k := M_{B_k}$.

Let (α, β) be a basis of $H_1(\mathbb{T}^2, \mathbb{Z}/2\mathbb{Z})$ such that α is the class of the curve along which the Dehn twist is done. Denote by (x, y) the image of (α, β) by the map induced by the inclusion $\mathbb{T}^2 \rightarrow M_k$. Let $z \in H_1(M_k, \mathbb{Z}/2\mathbb{Z})$ be the class of the curve $\{pt\} \times \mathbb{S}^1$.

The easiest example of the non-injectivity of the evaluation map will be found for $k = 3$.

Lemma 1.3. *The first homology group of M_3 is given by:*

$$H_1(M_3, \mathbb{Z}/2\mathbb{Z}) = \{\emptyset, y, z, yz\}.$$

Proof. It is not hard to check (see [16, Lemma 2.3] for instance) that

$$H_1(M_3, \mathbb{Z}/2\mathbb{Z}) \sim (\mathbb{Z}/2\mathbb{Z})^2.$$

On the other hand, $H_1(M_3, \mathbb{Z}/2\mathbb{Z})$ is obviously generated by $\{x, y, z\}$, but by isotoping y all along the \mathbb{S}^1 factor, one get that $y = 3x + y = x + y$. Hence $H_1(M_3, \mathbb{Z}/2\mathbb{Z})$ is generated by $\{y, z\}$. \square

1.4. The horizontal part. Since the fibration over \mathbb{S}^1 induces a map $M_k \rightarrow \mathbb{S}^1$, we can define the *horizontal part* of M_k to be:

$$h_k := \text{Ker}(H_1(M_k, \mathbb{Z}/2\mathbb{Z}) \rightarrow H_1(\mathbb{S}^1, \mathbb{Z}/2\mathbb{Z})).$$

Similarly, the *horizontal part* of $K(M_k)$ will be:

$$\mathcal{H}_k := \bigoplus_{\alpha \in h_k} K_\alpha(M).$$

Kinnear computed in [16] the dimensions of the skein modules of the mapping tori of the 2-torus, including the dimension of \mathcal{H}_k :

Theorem 1.4 ([16]). *The dimension of \mathcal{H}_k is $\frac{k-1}{2} + 4$ if k is odd and $\frac{k}{2} + 5$ if k is even.*

According to [12, Corollary 1.7.], the skein module of a 3-manifold containing an embedded 2-torus is spanned by skein elements that can be represented by links that intersecting the torus at most once. In our case this implies that \mathcal{H}_k is spanned by skeins that can be represented by links in $\mathbb{T}^2 \times [\frac{1}{4}, \frac{3}{4}] \subset M_k$.

To facilitate the computations, we will use the Frohman-Gelca basis for $K(\mathbb{T}^2) := K(\mathbb{T}^2 \times I)$. We recall its description here:

As the skein module of a thickened surface, $K(\mathbb{T}^2)$ has an algebra structure induced by the operation $\alpha \star \beta$ of stacking α over β .

For coprime integers p, q and x, y as in Lemma 1.3, we define $\gamma_{(p,q)}$ to be the skein element represented by an oriented curve of homology class $px + qy$ on

$\mathbb{T}^2 \times \{\frac{1}{2}\} \subset M_k$. The multicurves $\gamma_{(p,q)}^n$, composed by n parallel copies of $\gamma_{(p,q)}$, together with the empty curve form a basis of $K(\mathbb{T}^2) \subset K(M_k)$.

We recall the definition of the Chebychev polynomials of the first kind $(T_n)_{n \geq 0}$:

$$\begin{cases} T_0 = 2, T_1 = X \\ \forall n \geq 3, T_n = XT_{n-1} - T_{n-2}. \end{cases} \quad (1)$$

Frohman and Gelca introduced the following basis of $K(\mathbb{T}^2)$, for which the product (stacking operation) satisfies the so-called product-to-sum formula:

Theorem 1.5. [7] *The family $\{(p, q)_T := T_d(\gamma_{(\frac{p}{d}, \frac{q}{d})})$, $d = \gcd(p, q)\}$ is a basis for $K(\mathbb{T}^2)$ for which we have the following:*

$$(p, q)_T \star (r, s)_T = A^{ps-qr}(p+r, q+s)_T + A^{qr-ps}(p-r, q-s)_T.$$

Remark 1.6. *Here we choose the convention $(0, 0)_T = 2 \cdot \emptyset$.*

1.5. The results.

In section 2, we compute a basis for \mathcal{H}_k :

Theorem 1.7.

- *If k is odd, $\{(p, 0)_T \mid 0 \leq p \leq \lfloor \frac{k}{2} \rfloor\} \cup \{(0, 1)_T, (0, 2)_T, (1, 2)_T\}$ is a basis of \mathcal{H}_k .*
- *If k is even, $\{(p, 0)_T \mid 0 \leq p \leq \lfloor \frac{k}{2} \rfloor\} \cup \{(0, 1)_T, (0, 2)_T, (1, 2)_T, (1, 1)_T\}$ is a basis of \mathcal{H}_k .*

Remark 1.8. *With the same notations for the homology classes of M_k as in 1.3, when k is even (resp. k is odd) the graded subspaces that are not in \mathcal{H}_k are*

$$K_z(M), K_{x+z}(M), K_{y+z}(M), K_{x+y+z}(M)$$

(resp. $K_z(M), K_{y+z}(M)$).

One can deduce from [16, Prop. 4.2.] that the dimension of each of them is 1 where the generators are the natural ones. Therefore, Theorem 1.7 provides a full basis of $K(M_k)$.

In Section 3.2, we provide some of the images of the elements of this basis through the evaluation map:

Proposition 1.9. *Let p, q be integers so that $q \neq 0$. For almost all $\xi \in \mathbb{U}_0$ of order $2r$:*

If q is even:

$$ev((p, q)_T)(\xi) = \begin{cases} 2(-1)^{p+1}(-\xi)^{k(\frac{q}{2})^2-1} & \text{if } r \text{ is even} \\ (-1)^{p+1}(-\xi)^{k(\frac{q}{2})^2-1} & \text{if } r \text{ is odd} \end{cases}.$$

And if q is odd:

$$ev((p, q)_T)(\xi) = \begin{cases} 0 & \text{if } r \text{ is even} \\ (-\xi)^{k(\frac{r+q}{2})^2-1} & \text{if } r \text{ is odd} \end{cases}.$$

Moreover, a direct consequence of Theorem 1.7 is that $(0, 2)_T + (1, 2)_T$ is a non-zero vector in $K(M_3)$. We then have the main Theorem:

Theorem 1.10. $(0, 2)_T + (1, 2)_T$ is a non-zero vector in $\text{Ker}(ev_{M_3}|_{K_\emptyset(M_3)})$.

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2. A basis for the horizontal part \mathcal{H}_k

According to Theorem 1.4, the set described in Theorem 1.7 has $\dim_{\mathbb{Q}(A)} \mathcal{H}_k$ elements. This is why it is sufficient to show that this set spans \mathcal{H}_k to prove that it is a basis. To do so, we first notice the following relations:

Lemma 2.1. Let $p, q \in \mathbb{Z}$. If $q \neq 0$, then

$$(p + 1, q)_T = (p - 1, q)_T \quad (\text{a})$$

and

$$A^p(p, q+1)_T + A^{-p}(p, q-1)_T = A^{kq-p}(p+k, q+1)_T + A^{p-kq}(p-k, q-1)_T. \quad (\text{b})$$

Proof. Isotoping any skein element of the form $(p, q)_T \in \mathcal{H}_k$ along the \mathbb{S}^1 factor gives the relation

$$(p, q)_T = B_k((p, q)_T) = (p + qk, q)_T.$$

Applying this to $(1, 0)_T$, which is invariant by B_k , we get:

$$(p, q)_T \star (1, 0)_T = (1, 0)_T \star (p, q)_T.$$

Using the product-to-sum formula of Theorem 1.5 it becomes:

$$A^{-q}(p + 1, q)_T + A^q(p - 1, q)_T = A^q(p + 1, q)_T + A^{-q}(p - 1, q)_T.$$

Thus

$$(A^{-q} - A^q)(p + 1, q)_T = (A^{-q} - A^q)(p - 1, q)_T.$$

If $q \neq 0$, we finally have that $(p + 1, q)_T = (p - 1, q)_T$.

On the other hand, since $B_k((0, 1)_T) = (k, 1)_T$:

$$(p, q)_T \star (0, 1)_T = (k, 1)_T \star (p, q)_T.$$

Applying the product-to-sum formula again, and because $(p, q)_T = (-p, -q)_T$:

$$A^p(p, q + 1)_T + A^{-p}(p, q - 1)_T = A^{kq-p}(p + k, q + 1)_T + A^{p-kq}(p - k, q - 1)_T.$$

□

From these relations, we can prove Theorem 1.7:

Proof of theorem 1.7. Suppose that k is odd.

During this proof, we will use several times the formula (a) of Lemma 2.1, to say that

$$\forall q > 0, \forall p \in \mathbb{Z}, (p, q)_T \in \text{Span}_{\mathbb{Q}(A)}\{(0, q)_T, (1, q)_T\}. \quad (\text{e1})$$

Because $(p, q)_T = (-p, -q)_T$, the set $\{(p, q)_T, q \geq 0\}$ spans \mathcal{H}_k . Thus,

$$\mathcal{H}_k \subset \text{Span}_{\mathbb{Q}(A)}\{(p, 0)_T \mid p \in \mathbb{Z}\} \cup \{(p, q)_T \mid q > 0, p \in \{0, 1\}\}.$$

Consider $p \in \{0, 1\}$ and $q > 1$. Since k is odd, injecting (e1) into (b) leads to the following equation:

$$A^p(p, q+1)_T = A^{kq-p}(1-p, q+1)_T + A^{p-kq}(1-p, q-1)_T - A^{-p}(p, q-1)_T. \quad (\text{e2})$$

Re-applying this equality for $(1-p, q+1)_T$, we get:

$$\begin{aligned} A^p(p, q+1)_T &= A^{2kq+p-2}(p, q+1)_T + A^{-p}(p, q-1)_T \\ &\quad - A^{kq+p-2}(1-p, q-1)_T + A^{p-kq}(1-p, q-1)_T - A^{-p}(p, q-1)_T. \end{aligned}$$

Put another way:

$$(1 - A^{2kq-2})(p, q+1)_T = (-A^{kq-2} + A^{-kq})(1-p, q-1)_T.$$

Since $q > 1$, we have that $2kq - 2 > 0$ and $1 - A^{2kq-2} \neq 0$. We end up with the fact that:

$$\mathcal{H}_k = \text{Span}_{\mathbb{Q}(A)}(\{(p, 0)_T \mid p \in \mathbb{Z}\} \cup \{(0, 1)_T, (1, 1)_T, (0, 2)_T, (1, 2)_T\}).$$

Applying Relation (b) with $q = 1$ gets

$$(p, 0)_T \in \text{Span}_{\mathbb{Q}(A)}(\{(p+k, 2)_T, (p, 2)_T, (p-k, 0)_T\}).$$

After enough use of this relation we have that

$$(p, 0)_T \in \text{Span}_{\mathbb{Q}(A)}\left(\left\{\{(l, 0)_T \mid -\left\lfloor \frac{k}{2} \right\rfloor \leq l \leq \left\lfloor \frac{k}{2} \right\rfloor\} \cup \{(l, 2)_T \mid l \in \mathbb{Z}\}\right\}\right).$$

Using $(-p, 0)_T = (p, 0)_T$ to the elements $(l, 0)_T$ with $l < 0$ and (e1) to the elements $(l, 2)_T$, we get that:

$$\forall p \in \mathbb{Z}, (p, 0)_T \in \text{Span}_{\mathbb{Q}(A)}(\{(l, 0)_T \mid 0 \leq l \leq \left\lfloor \frac{k}{2} \right\rfloor\} \cup \{(0, 2)_T, (1, 2)_T\}).$$

Since k is odd, because of the formula (a), $(1, 1)_T = B_k((1, 1)_T) = (k+1, 1)_T = (0, 1)_T$.

At the end,

$$\mathcal{H}_k = \text{Span}_{\mathbb{Q}(A)}(\{(p, 0)_T \mid 0 \leq p \leq \left\lfloor \frac{k}{2} \right\rfloor\} \cup \{(0, 1)_T, (0, 2)_T, (1, 2)_T\}).$$

The only two differences if k were even would be that $(1, 1)_T$ would have remain necessary in the generating set and we would have p instead of $1-p$ in (e2), which would end that step directly. \square

3. Computation of ev_{M_k}

We now focus on the computations of the images of the elements of the basis of Section 2. Since it is not much harder to compute the images of ev on K_\emptyset for a general k and any $(p, q)_T$ with $q \neq 0$, we will continue this section with this setting.

3.1. The Reshetikhin–Turaev TQFT. Let us start by recalling the definition of the category of extended cobordisms in dimension 2+1:

Its objects are pairs (Σ, L) where Σ is an oriented compact closed surface together with a Lagrangian $L \subset H_1(\Sigma, \mathbb{Q})$. Its morphisms are $(M, K, n) \in \text{Hom}((\Sigma_1, L_1), (\Sigma_2, L_2))$ where M is a 3-manifold equipped with a fixed homeomorphism $\partial M \simeq \Sigma_1 \sqcup \Sigma_2$, a framed link $K \subset M$, and $n \in \mathbb{Z}$. n is called the weight of M and morally represents a choice of signature of a 4-manifold with boundary M . Moreover, a TQFT is a monoidal functor from the category of extended cobordisms in dimension 2+1 to the category of finite dimensional \mathbb{C} -vector spaces.

In [3], the 3-manifold invariant RT_ξ is extended to a TQFT:

Theorem 3.1. [3] *Let $\xi = e^{\frac{i\pi}{r}} \in \mathbb{U}_0$ be a primitive root of unity of even order. Then there exists a TQFT functor RT_ξ in dimension 2+1 satisfying:*

- (1) *For any oriented closed surface Σ with a choice of Lagrangian $L \subset H_1(\Sigma, \mathbb{Q})$, $RT_\xi(\Sigma)$ is a finite-dimensional \mathbb{C} -vector space such that each extended 3-manifold (N, K, n) with an homeomorphism $\partial N \simeq \Sigma$, corresponds a vector in $RT_\xi(\Sigma)$. Moreover, $RT_\xi(\Sigma)$ is spanned by such vectors.*
- (2) *Recall that the gluing of two extended 3-cobordisms (M, K, n) , from (Σ_1, L_1) to (Σ_2, L_2) and (M', K', m) from (Σ_2, L_2) to (Σ_3, L_3) , is the extended closed 3-cobordism $((M, K) \sqcup_{\Sigma} (M', K'), n + m - \mu)$ where $\mu \in \mathbb{Z}$, the Maslov index, depends only on L_1, L_2, L_3 . Then:*

$$RT_\xi((M, K, n) \sqcup_{\Sigma} (M', K', m)) = \kappa^{n+m-\mu} RT_\xi(M, K, n) \circ RT_\xi(M', K', m).$$

Also $RT_\xi(M, K, n) = \kappa^n RT_\xi(M, K, 0)$ and κ is called the anomaly of the TQFT RT_ξ .

- (3) *The extended mapping class group $\widetilde{\text{Mod}}(\Sigma)$, acting on extended 3-manifolds with boundary (Σ, L) gives rise to a representation*

$$\widetilde{\text{Mod}}(\Sigma) \rightarrow \text{Aut}(RT_\xi(\Sigma, L)).$$

- (4) *For any oriented closed 3-manifold M and skein element L in M ,*

$$RT_\xi(M, L) \in \text{Hom}(\mathbb{C}, \mathbb{C}) \simeq \mathbb{C},$$

and $RT_\xi(M, L)$ is the topological invariant introduced in Section 1.

Remark 3.2. *Since we will only be interested in the linear independence of the family $\{RT_\xi\}$, we will ignore the anomaly κ and fix the choice of Lagrangian in $H_1(\mathbb{T}^2, \mathbb{Q})$ to always be the subspace generated by the class of the meridian in \mathbb{T}^2 and therefore no longer make reference to the choice of Lagrangian.*

Let $z \in K(\mathbb{S}^1 \times \mathbb{D}^2)$ be represented by the core of $\mathbb{S}^1 \times \mathbb{D}^2$. Recall that $K(\mathbb{S}^1 \times \mathbb{D}^2)$ has a $\mathbb{Q}(A)[z]$ algebra structure defined by the operation of stacking on the boundary surface and define the Chebychev polynomials of the second kind:

$$\begin{cases} S_0 = 0, S_1 = 1 \\ \forall n \in \mathbb{Z}, S_{n+2} = XS_{n+1} - S_n \end{cases} \quad (2)$$

We have the following basis for $RT_\xi(\mathbb{T}^2)$:

Theorem 3.3. [3, Corollary 4.10] *For $j \in \mathbb{Z}$, set $e_j := (\mathbb{S}^1 \times \mathbb{D}^2, S_j(z)) \in RT_\xi(\mathbb{T}^2)$. Let ξ be a primitive $2r$ -root of unity, then:*

- *If r is odd, $\{e_j \mid 1 \leq j \leq \frac{r-1}{2}\}$ is a basis of $RT_\xi(\mathbb{T}^2)$.*
- *If r is even, $\{e_j \mid 1 \leq j \leq r-1\}$ is a basis of $RT_\xi(\mathbb{T}^2)$.*

Moreover, $e_r = 0$.

Remark 3.4. *Let $j \in \mathbb{Z}$. One can deduce from the induction formula and the fact that $e_r = 0$ that $e_{r+j} = -e_{r-j}$. Moreover, it was established in [2, Lemma 6.3.] that if r is odd, then $e_{\frac{r-1}{2}+j} = e_{\frac{r-1}{2}+1-j}$.*

Using this set, we compute the actions of the different cobordism applications involved.

3.2. Computations. The action of the mapping cylinder of B_k on the basis of Theorem 3.3 can be deduced from Theorem 3.1:

Lemma 3.5. *Let $\rho(B_k)$ be the representation of the mapping cylinder of B_k in $\text{Aut}(RT_\xi(\mathbb{T}^2))$. The action of $\rho(B_k)$ on the basis $\{e_j\}$ in $RT_\xi(\mathbb{T}^2)$ is:*

$$\rho(B_k)(e_j) = (-\xi)^{k(j^2-1)}e_j.$$

Proof. By noticing that $\rho(B_1)(z)$ is the effect of a simple Dehn twist on z , one can recognize the formula from [2, p.690-691]: $\rho(B_1)(e_j) = (-\xi)^{j^2-1}e_j$. Thus,

$$\rho(B_k)(e_j) = \rho(B_1)^k(e_j) = ((-\xi)^{j^2-1})^k e_j = (-\xi)^{k(j^2-1)}e_j.$$

□

The second part of the cobordism application that we will consider is the following:

Definition 3.6. *For $p, q \in \mathbb{Z}$, let $Z((p, q)_T)$ be the cobordism application associated to $(p, q)_T \in \mathcal{H}_k$.*

For instance, if $m = (1, 0)$ is the meridian of the 2-torus, $Z(m)e_j$ is the operation of stacking m on the boundary of e_j .

And then one can compute its action on e_j :

Lemma 3.7. *Let p, q be integers. The action of $Z((p, q)_T)$ on the basis $\{e_j\}$ is:*

$$Z((p, q)_T)e_j = (-1)^p(\xi^{2pj+pq}e_{j+q} + \xi^{-2pj+pq}e_{j-q}).$$

Proof. If one considers the morphism

$$Z : K(\mathbb{T}^2, \mathbb{Q}(A)) \rightarrow \text{Hom}(RT_\xi(\mathbb{T}^2), RT_\xi(\mathbb{T}^2)),$$

since $(0, 1)_T$ and $(1, 0)_T$ span $K(\mathbb{T}^2, \mathbb{Q}(A))$ as an algebra, it suffices to compute the actions of $Z((0, 1)_T)$, $Z((1, 0)_T)$ and their compositions on the basis $\{e_j \mid 1 \leq j \leq n\}$ to show that it coincides with the above formula.

First it is proven in [2, p.690-691] that

$$Z((1, 0)_T)e_j = -(\xi^{2j} + \xi^{-2j})e_j.$$

Moreover, $Z((0, 1)_T)e_j = ze_j = e_{j+1} + e_{j-1}$. And now, if, for some $p, q, s, t \in \mathbb{Z}$ and for all j ,

$$Z((s, t)_T)e_j = (-1)^s(\xi^{2sj+rt}e_{j+t} + \xi^{-2sj+st}e_{j-t})$$

and

$$Z((p, q)_T)e_j = (-1)^p(\xi^{2pj+pq}e_{j+q} + \xi^{-2pj+pq}e_{j-q}).$$

Then a direct computation gives that:

$$\begin{aligned} & Z((p, q)_T) \circ Z((s, t)_T)e_i \\ &= (-1)^{p+s}(\xi^{pt-sq}\xi^{2(p+s)j+(p+s)(q+t)}e_{j+t+q} + \xi^{pt-sq}\xi^{-2(p+s)j+(p+s)(q+t)}e_{j-t-q}) \\ &+ (-1)^{p-s}(\xi^{sq-pt}\xi^{-2(p-s)j+(p-s)(q-t)}e_{j+t-q} + \xi^{sq-pt}\xi^{-2(p-s)j+(p-s)(q-t)}e_{j-t+q}), \end{aligned}$$

which is what is expected of

$$Z((p, q)_T \star (s, t)_T)e_j = \xi^{pt-qs}Z((p+s, q+t)_T)e_j + \xi^{qs-pt}Z((p-s, q-t)_T)e_j.$$

□

We now can prove Proposition 1.9:

Proof of Proposition 1.9. Since our obstruction will come from the case q even, we only do the computations for this case. When q is odd, the proof is similar.

Let $r > 2q$. Theorem 3.1 implies that (see for instance [3, §1.2]),

$$RT_\xi(M_k, Z((p, q)_T)) = \text{Tr}(\rho(B_k) \circ Z((p, q)_T)),$$

and because of Lemmas 3.5 and 3.7,

$$\begin{aligned} & \rho(B_k) \circ Z((p, q)_T)e_j = \\ & (-1)^{p+k(j+q-1)}(\xi^{p(2j+q)+k((j+q)^2-1)}e_{j+q} + \xi^{p(-2j+q)+k((j-q)^2-1)}e_{j-q}). \quad (3) \end{aligned}$$

When r is even, we know from Remark 3.4 how e_{j+q} and e_{j-q} are expressed in terms of the basis $\{e_j\}_{1 \leq j \leq r-1}$ given in Theorem 3.3. Then, since q is even and $r > 2q$, the only contribution to the trace are coming from the case when j is such that $e_{j-q} = -e_j$ (when $j = \frac{q}{2}$) and $e_{j+q} = -e_j$ (when $j = r - \frac{q}{2}$).

Thus (still when r is even):

$$\begin{aligned} ev((p, q)_T)(\xi) &= (-1)^{p+1}(-\xi)^{k(\frac{q}{2})^2-1} \\ &\quad - (-1)^{p+k((r-\frac{q}{2})+q-1)}\xi^{p(2(r-\frac{q}{2})+q)+k((r-\frac{q}{2})+q)^2-1} \\ &= (-1)^{p+1}(-\xi)^{k(\frac{q}{2})^2-1}(1 + \xi^{2pr+kr^2-2kr\frac{q}{2}}). \end{aligned}$$

The first (resp. second) term corresponds to $j = \frac{q}{2}$ (resp. $r - \frac{q}{2}$). Since ξ is a root of unity of order $2r$, we have that $\xi^{2pr+kr^2-2kr\frac{q}{2}} = 1$ which gives the expected result. When r is odd, we would only have the contribution corresponding to $j = \frac{q}{2}$. \square

As said before, the vector considered in Theorem 1.10 is non-zero because of Theorem 1.7 and it is easy to check from Lemma 1.9 that it is in the Kernel of the evaluation map.

4. A few words on the general setting

The study of ev_{M_k} for a generic k was done in a previous version of this paper (available on ArXiv: [17] v2). As in [9], we find that it is highly related to the generalised quadratic Gauss sums and the dimension of its image depends on the number of squares modulo k . However, the conclusions were not exhaustive and required a very technical study of the linear independence of the generalised quadratic Gauss sums. Here are the conclusions of the previous version:

4.1. When $k = 2p$ with p a prime number (including 2). ev_{M_k} is injective on each of its graded subspaces.

4.2. When k is odd. $(0, 2)_T + (1, 2)_T \in Ker(ev_{M_k} |_{K_{\emptyset}(M_k)})$.

4.3. When $k \equiv 2 \pmod{4}$ has at least 3 different prime divisors. This case is more complicated. We prove the existence of an even divisor d of k , two number l, l' coprimes with $\frac{k}{d}$ and a number m such that $l^2 = l'^2 + m\frac{k}{d}$. We then have:

$$(dl, 0)_T - A^{-md}(dl', 0)_T - A^{-k}(1 - A^{-md})(0, 2)_T \in Ker(ev_{M_k} |_{K_{\emptyset}(M_k)}).$$

4.4. When $k \neq 4$ is a multiple of 4. Applying the same method as in 4.3 gets the same result. However, in the aforementioned previous version, we claimed that $ev((\frac{k}{2}, 0)_T)$ and $ev((0, 0)_T)$ were collinear in this case, which was an error.

4.5. When $k = 2p^\alpha$ with $p > 2$ prime and $\alpha > 1$. We do not have a conclusion in this case.

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(Edwin Kitaeff) UNIVERSITÉ BOURGOGNE EUROPE, CNRS, IMB UMR 5584, 21000 DIJON, FRANCE
edwin.kitaeff@ube.fr

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