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On the vanishing of twisted negative K-theory and homotopy invariance

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ABSTRACT. In this article, we revisit Weibel's conjecture for twisted K-theory. We also examine the vanishing of twisted negative K-groups for Prüfer domains. Furthermore, we observe that the homotopy invariance of twisted K-theory holds for (finite-dimensional) Prüfer domains.

CONTENTS

1. Introduction

It is well known that for a regular noetherian scheme X, the homotopy invariance of K-theory holds (i.e., the natural map $K_n(X) \to K_n(X \times \mathbb{A}^r)$ is an isomorphism for all $r \geq 0$ and $n \in \mathbb{Z}$) and $K_{-n}(X) = 0$ for all $n > 0$. This is not true for non-regular schemes in general. Therefore, it has been an interesting question to investigate certain classes of schemes for which homotopy invariance of algebraic K-theory holds and negative K-groups vanishes. In this direction, Weibel's conjectured in [20] that for a d-dimensional Noetherian scheme X, the following should hold:

(1) $K_{-n}(X) = 0$ for $n > d$;
(2) $K_{-n}(X) \cong K_{-n}(X \times \mathbb{A}^r)$ for $n \geq d$ and $r \geq 0$.

This conjecture was first proven for varieties over a field (see [6], [7] and [13]). For a finite-dimensional quasi-excellent Noetherian scheme, Kelly showed in [10] that the negative K-groups vanish (up to torsion) after dimension. In 2018,

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Kerz-Strunk-Tamme ultimately settled Weibel's conjecture (see Theorem B of [12]). A relative version of Weibel's conjecture is discussed in [17].

In this article, we are mainly interested in similar types of questions (i.e., homotopy invariance and vanishing of negative K-groups) in the context of the twisted K-theory. Given an Azumaya algebra $\mathcal{A}$ over a scheme S, one can define twisted K-group $K_n^{\mathcal{A}}(S)$ for $n \in \mathbb{Z}$ (see section 2). It is natural to ask Weibel's conjecture for $K_n^{\mathcal{A}}(S)$. In [19], J. Stapleton discussed Weibel's conjecture for $K_n^{\mathcal{A}}(S)$ and proved the first part, i.e., vanishing of twisted negative K-groups (see Corollary 4.2 of [19]). The second part of this conjecture has also been discussed in the same paper except the boundary case, i.e., $n = d$ (see Theorem 4.3 of [19]). In section 3, we revisit Weibel's conjecture for twisted K-theory and give proof that also takes care of the boundary case. Here is our result (see Theorem 3.4):

Theorem 1.1. *Let S be a Noetherian scheme of dimension d. Let $\mathcal{A}$ be an Azumaya algebra of rank q^2 over S. Then*

(1) $K_{-n}^{\mathcal{A}}(S) = 0$ for $n > d$;
(2) S is $K_{-n}^{\mathcal{A}}$-regular for $n \geq d$, i.e., the natural map $K_{-n}^{\mathcal{A}}(S) \to K_{-n}^{\mathcal{A}}(S \times \mathbb{A}^r)$ is an isomorphism for $n \geq d$ and $r \geq 0$.

A subring V of a field K is said to be *valuation ring* if for each $0 \neq a \in K$, either $a \in V$ or $a^{-1} \in V$. We say that an integral domain R is a *Prüfer* domain if it is locally a valuation domain, i.e., $R_{\mathfrak{p}}$ is a valuation domain for all prime ideals $\mathfrak{p}$ of R. In [11], Kelly and Morrow observed that algebraic K-theory is homotopy invariant and negative K-groups vanishes for valuation rings (see Theorem 3.3 of [11]). Later, Banerjee and Sadhu in [2] extended the above mentioned results for Prüfer domains (see Theorem 1.1 of [2]). In section 4, we investigate the same for twisted K-groups. More precisely, we show (see Example 4.1 and Corollary 4.5):

Theorem 1.2. *Let $\mathcal{A}$ be an Azumaya algebra of rank q^2 over a ring R and $SB(\mathcal{A})$ be the associated Severi Brauer variety. Assume that R is a Prüfer domain with finite krull dimension. Then*

(1) $K_{-n}^{\mathcal{A}}(R) = 0$ for $n > \dim(SB(\mathcal{A}))$;
(2) the natural map $K_n^{\mathcal{A}}(R) \to K_n^{\mathcal{A}}(R[t_1, t_2, \dots, t_r])$ is an isomorphism for all $n \in \mathbb{Z}$ and $r \geq 0$.

By Morita equivalence, for a ring R and $n \in \mathbb{Z}$, $K_n(R) \cong K_n^{\mathcal{A}}(R)$ in the case when $\mathcal{A}$ is a matrix algebra over R. This isomorphism may not hold for all Azumaya algebras. In section 5, we examine the relationship between $K_n(R)$ and $K_n^{\mathcal{A}}(R)$, assuming R is a valuation ring of characteristic p. We show that there is an injection from $K_n(R)$ to $K_n^{\mathcal{A}}(R)$ for all $n \geq 0$ provided the rank of $\mathcal{A}$ is p^2 (see Theorem 5.1).

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2. Twisted K-theory

Let A be an algebra (not necessarily commutative) over a commutative local ring R. The opposite algebra A^{op} of A is the algebra A with multiplication reversed. We say that A is an *Azumaya algebra* over R if it is free R-module of finite rank and the map $A \otimes_R A^{op} \to End_R(A), a \otimes a' \mapsto (x \mapsto axa')$ is an isomorphism. For example, the matrix algbera $M_n(R)$ is an Azumaya algebra over R. Let X be a scheme. An $\mathcal{O}_X$-algebra $\mathcal{A}$ is said to be an *Azumaya algebra* over X if it is coherent, locally free as an $\mathcal{O}_X$-module and $\mathcal{A}_x$ is an Azumaya algebra over $\mathcal{O}_{X,x}$ for any point $x \in X$. Equivalently, $\mathcal{A}$ is étale locally isomorphic to $M_n(\mathcal{O}_X)$ for some n. For details, see [14].

2.1. Twisted K-groups. Let $\mathcal{A}$ be an Azumaya algebra over a scheme S. Let $\mathbf{Vect}^{\mathcal{A}}(S)$ denote the category of vector bundles on S that are left modules for $\mathcal{A}$. The category $\mathbf{Vect}^{\mathcal{A}}(S)$ is exact. The twisted K-theory space is defined by $K^{\mathcal{A}}(S) := K(\mathbf{Vect}^{\mathcal{A}}(S))$. For $n \geq 0$, the n-th twisted K-group $K_n^{\mathcal{A}}(S)$ is defined as $\pi_n(K(\mathbf{Vect}^{\mathcal{A}}(S))$.

Write $S[t]$ for $S \times_{\mathbb{Z}} \mathbb{Z}[t]$ and $S[t, t^{-1}]$ for $S \times_{\mathbb{Z}} \mathbb{Z}[t, t^{-1}]$. Since the projection map $p : S[t] \to S$ is flat, it induces an exact functor $p^* : K^{\mathcal{A}}(S) \to K^{p^*\mathcal{A}}(S[t])$. Thus we have maps between twisted K-groups $K_n^{\mathcal{A}}(S) \to K_n^{p^*\mathcal{A}}(S[t])$. By abuse of notation, we write $K_n^{\mathcal{A}}(S[t])$ instead of $K_n^{p^*\mathcal{A}}(S[t])$. Similarly, we also have maps between $K_n^{\mathcal{A}}(S) \to K_n^{\mathcal{A}}(S[t, t^{-1}])$. Following Bass (see chapter XII of [3]), the twisted negative K-group $K_{-1}^{\mathcal{A}}(S)$ is defined as

$$Coker[K_0^{\mathcal{A}}(S[t]) \times K_0^{\mathcal{A}}(S[t^{-1}]) \xrightarrow{\pm} K_0^{\mathcal{A}}(S[t, t^{-1}])].$$

By iterating, we have

$$K_{-n}^{\mathcal{A}}(S) := Coker[K_{-n+1}^{\mathcal{A}}(S[t]) \times K_{-n+1}^{\mathcal{A}}(S[t^{-1}]) \xrightarrow{\pm} K_{-n+1}^{\mathcal{A}}(S[t, t^{-1}])].$$

There is a split exact sequence for $n \in \mathbb{Z}$ (see section 3 of [19])

$$0 \to K_n^{\mathcal{A}}(S) \xrightarrow{\Delta} K_n^{\mathcal{A}}(S[t]) \times K_n^{\mathcal{A}}(S[t^{-1}]) \xrightarrow{\pm} K_n^{\mathcal{A}}(S[t, t^{-1}]) \to K_{n-1}^{\mathcal{A}}(S) \to 0, \quad (2.1)$$

where $\Delta(a) = (a, a)$ and $\pm(a, b) = a - b$.

2.2. Quillen's generalized projective bundle formula. It is well-known that there is a natural bijection of sets

$$\{\mathrm{Severi-Brauer\ varieties\ of\ relative\ dimension\ (q-1)\ over\ S}\} \longleftrightarrow \{\mathrm{Azumaya\ algebras\ over\ S\ of\ rank\ q^2}\}.$$

Let $\mathcal{A}$ be an Azumaya algebra of rank q^2 over a scheme S. One can associate a Severi-Brauer variety $SB(\mathcal{A})$ of relative dimension $q - 1$ over S. The structure morphism $SB(\mathcal{A}) \to S$ is always smooth and projective. Quillen's generalized projective bundle formula state that there is a natural isomorphism for each $n \geq 0$ (see Theorem 4.1 of [15] or V.1.6.6 of [21]),

$$K_n(SB(\mathcal{A})) \cong \bigoplus_{i=0}^{q-1} K_n^{\mathcal{A}^{\otimes i}}(S). \tag{2.2}$$

We consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & & 0 & & 0 & & & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
K_0(S) & \xrightarrow{injects} & K_0(S[t]) \times K_0(S[t^{-1}]) & \to & K_0(S[t,t^{-1}]) & \to & K_{-1}(S) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
K_0(SB(\mathcal{A})) & \xrightarrow{injects} & K_0(SB(\mathcal{A})[t]) \times K_0(SB(\mathcal{A})[t^{-1}]) & \to & K_0(SB(\mathcal{A})[t,t^{-1}]) & \to & K_{-1}(SB(\mathcal{A})) & \to & 0. \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\bigoplus_{i=1}^{q-1} K_0^{\mathcal{A}^{\otimes i}}(S) & \xrightarrow{injects} & \bigoplus_{i=1}^{q-1} K_0^{\mathcal{A}^{\otimes i}}(S[t]) \times \bigoplus_{i=1}^{q-1} K_0^{\mathcal{A}^{\otimes i}}(S[t^{-1}]) & \to & \bigoplus_{i=1}^{q-1} K_0^{\mathcal{A}^{\otimes i}}(S[t,t^{-1}]) & \to & \bigoplus_{i=1}^{q-1} K_{-1}^{\mathcal{A}^{\otimes i}}(S) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & & & \\
0 & & 0 & & 0 & & & &
\end{array}$$

By the fundamental theorem of K-theory and (2.1), the rows are split exact. The first three columns are also split exact by (2.2). Finally, a diagram chase gives a natural isomorphism

$$K_{-1}(SB(\mathcal{A})) \cong \bigoplus_{i=0}^{q-1} K_{-1}^{\mathcal{A}^{\otimes i}}(S). \tag{2.3}$$

By iterating, we conclude that for each $n \in \mathbb{Z}$, there is a natural isomorphism

$$K_n(SB(\mathcal{A})) \cong \bigoplus_{i=0}^{q-1} K_n^{\mathcal{A}^{\otimes i}}(S). \tag{2.4}$$

Proposition 2.1. *Let $\mathcal{A}$ be an Azumaya algebra of rank q^2 over a Noetherian regular scheme S. Then $K_n^{\mathcal{A}}(S) = 0$ for $n < 0$ and $K_n^{\mathcal{A}}(S) \cong K_n^{\mathcal{A}}(S[t_1, \dots, t_r])$ for all n and $r \geq 0$.*

Proof. Since S is a Noetherian regular scheme, so is $SB(\mathcal{A})$. In this situation, we know $K_n(SB(\mathcal{A})) = 0$ for $n < 0$ and $K_n(SB(\mathcal{A})) \cong K_n(SB(\mathcal{A})[t_1, \dots, t_r])$ for all n and $r \geq 0$. By (2.4), we get the result. □

2.3. Brauer groups vs Twisted K-theory. We say that two unital rings A and B (possibly non-commutative) are *Morita equivalent* if the categories $\mathbf{Mod}_A$ and $\mathbf{Mod}_B$ of right modules are equivalent. For example, a unital ring R is Morita equivalent to $M_n(R)$ for $n \geq 0$.

Two Azumaya algebras $\mathcal{A}$ and $\mathcal{B}$ over a commutative ring R are Morita equivalent if and only if there exist finitely generated projective R-modules P and Q such that $\mathcal{A} \otimes_R End(P) \cong \mathcal{B} \otimes_R End(Q)$ (see Theorem 1.3.15 of [4]). However, this is not true for Azumaya algebras over scheme, for instance see Example

1.3.16 of [4]. If R is a commutative local ring then $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent if and only if $M_n(\mathcal{A}) \cong M_m(\mathcal{B})$ for $n, m > 0$. The Brauer group $\mathrm{Br}(R)$ of a commutative ring R consists of Morita equivalence classes of Azumaya algebras over R (see [14]). The group operation on $\mathrm{Br}(R)$ is $\otimes_R$. An element of $\mathrm{Br}(R)$ is represented by a class $[\mathcal{A}]$, where $\mathcal{A}$ is an Azumaya algebra over R. The inverse of $[\mathcal{A}]$ is given by $[\mathcal{A}^{op}]$.

Let R be a commutative ring with unity. For $n \in \mathbb{Z}$, we consider the set

$$\mathcal{F}_n = \{K_n^{\mathcal{A}}(R) | [\mathcal{A}] \in \mathrm{Br}(R)\}.$$

An equivalence relation $\sim$ on $\mathcal{F}_n$ is given by $K_n^{\mathcal{A}}(R) \sim K_n^{\mathcal{A}'}(R)$ if $K_n^{\mathcal{A}}(R) \cong K_n^{\mathcal{A}'}(R)$. Define $BK_n(R) := \mathcal{F}_n / \sim$. An element of $BK_n(R)$ is represented by a class $(K_n^{\mathcal{A}}(R))$.

Lemma 2.2. *For $n \in \mathbb{Z}$, $BK_n(R)$ is an abelian group with the operation*

$$(K_n^{\mathcal{A}}(R)) * (K_n^{\mathcal{A}'}(R)) = (K_n^{\mathcal{A} \otimes_R \mathcal{A}'}(R)).$$

Proof. If $\mathcal{A}$ and $\mathcal{B}$ both are Azumaya algebras over R then $\mathcal{A} \otimes_R \mathcal{B}$ is also an Azumaya algebra over R. Thus, $*$ is closed. Since $\otimes_R$ is associative and abelian, so is $*$. We know that algebraic K-theory is Morita invariant, i.e., for all $n \in \mathbb{Z}$, $K_n(R) \cong K_n(S)$ whenever R and S are Morita equivalent. This implies that $(K_n(R))$ is the identity element. The inverse of $(K_n^{\mathcal{A}}(R))$ is given by $(K_n^{\mathcal{A}^{op}}(R))$. □

We define a map $\psi_n : \mathrm{Br}(R) \to BK_n(R), [\mathcal{A}] \mapsto (K_n^{\mathcal{A}}(R))$ for each $n \in \mathbb{Z}$. Each ψ_n is a well defined map because K-theory is Morita invariant. Moreover, one can check the following:

Proposition 2.3. *For a commutative ring R, there is a short exact sequence*

$$0 \to \ker \psi_n \to \mathrm{Br}(R) \to BK_n(R) \to 0$$

of abelian group for each $n \in \mathbb{Z}$. Moreover,

$$\ker \psi_n = \{\mathcal{A} \in Az(R) | K_n^{\mathcal{A}}(R) \cong K_n(R)\}.$$

Remark 2.4. (1) If $\mathrm{Br}(R) = 0$ then there are no twisted K-groups.
(2) If $R = \mathbb{R}$ then $\mathrm{Br}(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z} = \{\mathbb{R}, \mathbb{H}\}$. We know $K_1^{\mathbb{H}}(\mathbb{R}) \ncong K_1(\mathbb{R})$ (see Table VI.3.1.1 of [21]). In this case, $\ker \psi_1 = 0$. By Proposition 2.1, $K_n^{\mathbb{H}}(\mathbb{R}) = 0$ for $n < 0$. So, $\ker \psi_n = \mathbb{Z}/2\mathbb{Z}$ for $n < 0$.

3. Twisted version of Weibel's Conjecture

Throughout, $\mathcal{A}$ is an Azumaya algebra of rank q^2 over a scheme S and $SB(\mathcal{A})$ is the associated Severi-Brauer variety. We would like to understand the K-theory of the structure map $\rho : SB(\mathcal{A}) \to S$.

Let $f : X \to S$ be a map of schemes. Let $K(f)$ denote the homotopy fibre of $K(S) \to K(X)$. Here $K(X)$ denotes the Bass non-connective K-theory spectrum of a scheme X. We have the associated long exact sequence

$$\cdots \to K_n(f) \to K_n(S) \to K_n(X) \to K_{n-1}(f) \to K_{n-1}(S) \to \ldots \tag{3.1}$$

Let F be a functor from category of rings (or schemes) to abelian groups. Let $NF(X) = \ker[F(X \times \mathbb{A}^1) \to F(X)]$. There is a natural decomposition $F(X \times \mathbb{A}^1) \cong F(X) \oplus NF(X)$. By iterating, one can define $N^tF(X)$. We have a natural decomposition $F(X \times \mathbb{A}^r) \cong (1+N)^rF(X)$. We say that X is F-regular if the natural map $F(X) \to F(X \times \mathbb{A}^r)$ is an isomorphism for $r \geq 0$. Equivalently, $N^rF(X) = 0$ for $r > 0$.

By comparing the exact sequences (3.1) for f and $f \times \mathbb{A}^1 : X \times \mathbb{A}^1 \to S \times \mathbb{A}^1$, a diagram chase gives a long exact sequence for NK_*. By iterating, we also have a long exact sequence for N^rK_* (some more details related to NK_*-groups can be found in section 3 of [17])

$$\cdots \to N^rK_n(f) \to N^rK_n(S) \to N^rK_n(X) \to N^rK_{n-1}(f) \to N^rK_{n-1}(S) \to \dots \tag{3.2}$$

The following result is due to Kerz-Strunk-Tamme.

Theorem 3.1. *Let X be a Noetherian scheme of dimension d. Then*

(1) $K_{-n}(X) = 0$ for $n > d$;
(2) X is K_{-n}-regular for $n \geq d$, i.e., the natural map $K_{-n}(X) \to K_{-n}(X \times \mathbb{A}^r)$ is an isomorphism for $n \geq d$ and $r \geq 0$.

Proof. See Theorem B of [12]. □

A relative version of the aforementioned theorem is as follows:

Theorem 3.2. *Let $f : X \to S$ be a smooth, quasi-projective map of noetherian schemes with S reduced. Assume that* $\dim S = d$. *Then $K_{-n}(f) = 0$ for $n > d+1$ and f is K_{-n}-regular for $n > d$, i.e., the natural map $K_{-n}(f) \to K_{-n}(f \times \mathbb{A}^r)$ is an isomorphism for $n > d$ and $r \geq 0$. Here $f \times \mathbb{A}^r$ denotes $X \times \mathbb{A}^r \to S \times \mathbb{A}^r$.*

Proof. See Theorem 3.8 of [17]. □

Let $\mathcal{K}^{\mathcal{A}}_{n,zar}$ denote the Zariski sheafification of the presheaf $U \mapsto K^{\mathcal{A}}_n(U)$. Similarly, $\mathcal{N}\mathcal{K}^{\mathcal{A}}_{n,zar}$ is the Zariski sheafification of the presheaf $U \mapsto NK^{\mathcal{A}}_n(U)$. More generally, one can define $\mathcal{N}^r\mathcal{K}^{\mathcal{A}}_{n,zar}$ for $r > 0$.

Lemma 3.3. *Let S be a Noetherian scheme of dimension d. Let $\mathcal{A}$ be an Azumaya algebra of rank q^2 over S. Then $K^{\mathcal{A}}_{-n}(S) \cong K^{\mathcal{A}}_{-n}(S_{red})$ and $N^rK^{\mathcal{A}}_{-n}(S) \cong N^rK^{\mathcal{A}}_{-n}(S_{red})$ for $n \geq d$ and $r > 0$.*

Proof. Given a commutative ring R, $K^{\mathcal{A}}_{-n}(R) \cong K^{\mathcal{A}}_{-n}(R_{red})$ for $n \geq 0$ (see Proposition 2.7 of [19]). Note $(R[t])_{red} = R_{red}[t]$. Thus, $N^rK^{\mathcal{A}}_{-n}(R) \cong N^rK^{\mathcal{A}}_{-n}(R_{red})$ for $n \geq 0$ and $r > 0$. The rest of the argument is based on comparing Zariski descent spectral sequences for S and S_{red} (see Corollary 2.8 of [19] and Lemma 3.4 of [17]). □

Theorem 3.4. *Let S be a Noetherian scheme of dimension d. Let $\mathcal{A}$ be an Azumaya algebra of rank q^2 over S. Then*

(1) $K^{\mathcal{A}}_{-n}(S) = 0$ for $n > d$;

(2) *S is $K_{-n}^{\mathcal{A}}$-regular for $n \geq d$, i.e., the natural map $K_{-n}^{\mathcal{A}}(S) \to K_{-n}^{\mathcal{A}}(S \times \mathbb{A}^r)$ is an isomorphism for $n \geq d$ and $r \geq 0$.*

Proof. We may assume that S is reduced (see Lemma 3.3). Let $SB(\mathcal{A})$ be the associated Severi-Brauer variety of relative dimension $q-1$ over S. Note $\rho : SB(\mathcal{A}) \to S$ is a smooth, projective morphism (hence also finite type). Since S is Noetherian, so is $SB(\mathcal{A})$. Then $K_{-n}(\rho) = 0$ for $n > d+1$ and $N^rK_{-n}(\rho) = 0$ for all $r \geq 0$ and $n > d$ by Theorem 3.2. The sequence (3.1) implies that $K_{-d-1}(S) \to K_{-d-1}(SB(\mathcal{A}))$ is surjective and $K_{-n}(S) \to K_{-n}(SB(\mathcal{A}))$ is an isomorphism for $n > d+1$. Similarly, the sequence (3.2) implies that $N^rK_{-d}(S) \to N^rK_{-d}(SB(\mathcal{A}))$ is surjective and $N^rK_{-n}(S) \to N^rK_{-n}(SB(\mathcal{A}))$ is an isomorphism for $n > d$. By Theorem 3.1, we get $K_{-n}(SB(\mathcal{A})) = 0$ for $n > d$ and $N^rK_{-n}(SB(\mathcal{A})) = 0$ for $n \geq d$. The natural decomposition (2.4) yields the result. □

4. Twisted K-theory of weakly regular stably coherent rings

Let R be a commutative ring. A finitely generated R-module M is called *coherent* if every finitely generated submodule of M is finitely presented. The ring R is *coherent* if it is a coherent module over itself, i.e., every finitely generated ideal of R is finitely presented. The ring R is said to be a *regular* ring if every finitely generated ideal of R has finite projective dimension.

Let M be a R-module. The *weak dimension* of M, denoted by $w.dim_R M$ is the least nonnegative integer n, for which there is an exact sequence

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$$

with F_i flat over R. The *weak dimension* of a ring R, denoted $w.dim(R)$, is defined by $w.dim(R) = Sup\{w.dim_R M | M \text{ is a } R\text{−module}\}$. If R is a coherent ring then $w.dim(R) = Sup\{Pd_R M | M \text{ is a finitely presented } R\text{−module}\}$ (see Corollary 2.5.6 of [8]). Clearly, $Pd_R(M) \leq w.dim(R)$ for all finitely presented R-modules M. We also have $w.dim(R_{\mathfrak{p}}) \leq w.dim(R)$ for all $\mathfrak{p} \in Spec(R)$ (see Theorem 1.3.13 of [8]).

A coherent ring R is called *weakly regular* if R has finite flat (or weak) dimension. A ring R is said to be *stably coherent* if every finitely presented R-algebra is coherent.

Example 4.1. Here is a list of weakly regular stably coherent rings:

(1) Noetherian regular local rings of finite krull dimension. In this case, global dimension coincides with weak dimension.
(2) Valuation rings (see Proposition 2.1 of [1]);
(3) Prüfer domains (see Lemma 3.1 of [2] and P.25 of [8]).

Let $\mathcal{K}_{n,zar}$ denote the Zariski sheafification of the presheaf $U \mapsto K_n(U)$. Similarly, $\mathcal{N}\mathcal{K}_{n,zar}$ is the Zariski sheafification of the presheaf $U \mapsto NK_n(U)$. More generally, one can define $\mathcal{N}^r\mathcal{K}_{n,zar}$ for $r > 0$.

Lemma 4.2. *Let X be a scheme and $X \to Spec(R)$ be a smooth map with R weakly regular stably coherent. Then the Zariski sheaves on X, $\mathcal{K}_{n,zar} = 0$ for $n < 0$ and $\mathcal{N}^r\mathcal{K}_{n,zar} = 0$ for $n \in \mathbb{Z}, r > 0$.*

Proof. Let $Spec(A) \hookrightarrow X$ be an affine open subset. Then $Spec(A) \to Spec(R)$ is smooth. Note A is weakly regular stably coherent (see Corollary 2.3 of [1]). Since any localization of stably coherent ring is stably coherent and

$$w.dim(A_{\mathfrak{p}}) \leq w.dim(A) < \infty,$$

$A_{\mathfrak{p}}$ is weakly regular stably coherent for all $\mathfrak{p} \in Spec(A)$. Each stalk of $\mathcal{K}_{n,zar}$ and $\mathcal{N}^r\mathcal{K}_{n,zar}$ are $K_n(A_{\mathfrak{p}})$ and $N^rK_n(A_{\mathfrak{p}})$, where $A_{\mathfrak{p}}$ is weakly regular stably coherent. By Proposition 2.4 of [1], algebraic K-theory is homotopy invariant, and negative K-groups vanish for weakly regular stably coherent rings. Hence the assertion. □

Lemma 4.3. *Let $X \to S$ be a projective morphism with S quasi-compact and quasi separated. Then X is also quasi-compact and quasi-separated.*

Proof. Projective morphisms are always quasi-compact and quasi separated morphism. Since S quasi-compact and quasi separated scheme, so is X. □

Theorem 4.4. *Let X be a finite dimensional quasi-compact and quasi-separated scheme and $X \to Spec(R)$ be a smooth map with R weakly regular stably coherent. Then*

1. *$K_{-n}(X) = 0$ for $n > d$ and $H^d_{Zar}(X, \mathbb{Z}) \cong K_{-d}(X)$, where $d = \dim(X)$;*
2. *The natural map $K_n(X) \to K_n(X \times \mathbb{A}^r)$ is an isomorphism for $n \in \mathbb{Z}$ and $r \geq 0$.*

Proof. (1) The scheme X has finite Krull dimension d. We have a descent spectral sequence (see Theorem 4.1 of [16] and Remark 3.3.1 of [5])

$$H^p_{Zar}(X, \mathcal{K}_{n,Zar}) \Longrightarrow K_{n-p}(X).$$

Here $\mathcal{K}_{n,Zar}$ is the Zariski sheaf on X. By Corollary 4.6 of [18], X_{Zar} has cohomological dimension at most $d = \dim(X)$. Moreover, $\mathcal{K}_{n,Zar} = 0$ for $n < 0$ (see Lemma 4.2). This implies that $K_{-n}(X) = 0$ for $n > d$ and $H^d_{Zar}(X, \mathbb{Z}) = K_{-d}(X)$.

(2) Consider the Zariski sheaf $\mathcal{N}^r\mathcal{K}_{n,Zar}$ on X. We have $\mathcal{N}^r\mathcal{K}_{n,Zar} = 0$ for $n \in \mathbb{Z}, r > 0$ (see Lemma 4.2). The following descent spectral sequence

$$H^p_{Zar}(X, \mathcal{N}^r\mathcal{K}_{n,Zar}) \Longrightarrow N^rK_{n-p}(X)$$

implies $N^rK_n(X) = 0$ for $n \in \mathbb{Z}, r > 0$. □

Corollary 4.5. *Let R be a finite dimensional weakly regular stably coherent ring. Let $\mathcal{A}$ be an Azumaya algebra over R of rank q^2 and $SB(\mathcal{A})$ be the associated Severi Brauer variety. Then*

1. *$K^{\mathcal{A}}_{-n}(R) = 0$ for $n > \dim(SB(\mathcal{A}))$.*
2. *the natural map $K^{\mathcal{A}}_n(R) \to K^{\mathcal{A}}_n(R[t_1, t_2, \ldots, t_r])$ is an isomorphism for all $n \in \mathbb{Z}$ and $r \geq 0$.*

Proof. The structure morphism $SB(\mathcal{A}) \to Spec(R)$ is smooth and projective (hence of finite type). The Severi Brauer variety $SB(\mathcal{A})$ has finite Krull dimension because R is finite dimensional. By Lemma 4.3, $SB(\mathcal{A})$ is a quasi-compact and quasi-separated scheme. The result now follows from Theorem 4.4 and the decomposition (2.4). □

5. An observation

Let $\mathcal{A}$ and $\mathcal{B}$ be Azumaya algebras over a scheme X. Assume that $\varphi : \mathcal{B} \to \mathcal{A}$ is an $\mathcal{O}_X$-algebra homomorphism and $\mathcal{A}$ is a flat $\mathcal{B}$-module. Then the functor

$$- \otimes_{\mathcal{B}} \mathcal{A} : \mathbf{Vect}^{\mathcal{B}}(X) \to \mathbf{Vect}^{\mathcal{A}}(X), P \mapsto P \otimes_{\mathcal{B}} \mathcal{A}$$

is exact and it induces a group homomorphism $\varphi_n : K_n^{\mathcal{B}}(X) \to K_n^{\mathcal{A}}(X)$ for each $n \geq 0$. We also have a restriction functor $res^{\mathcal{A}}_{\mathcal{B}} : \mathbf{Vect}^{\mathcal{A}}(X) \to \mathbf{Vect}^{\mathcal{B}}(X)$, which is exact. It induces a group homomorphism $\phi_n : K_n^{\mathcal{A}}(X) \to K_n^{\mathcal{B}}(X)$ for each $n \geq 0$.

If $\mathcal{B} = \mathcal{O}_X$ then $\mathcal{A}$ is a flat $\mathcal{O}_X$-module and $K_n^{\mathcal{O}_X}(X) = K_n(X)$. For $n \geq 0$, we get group homomorphisms

$$\varphi_n : K_n(X) \to K_n^{\mathcal{A}}(X)$$

and

$$\phi_n : K_n^{\mathcal{A}}(X) \to K_n(X).$$

The composition $\phi_n\varphi_n : K_n(X) \to K_n(X)$ is a map multiplication by $[\mathcal{A}] \in K_0(X)$.

Theorem 5.1. *Let V be a valuation ring of characteristic $p > 0$. Let $\mathcal{A}$ be a Azumaya algebra over V of rank q^2, where $q = p^r$ for some $r \geq 1$. Then the map $\varphi_n : K_n(V) \to K_n^{\mathcal{A}}(V)$ is injective for all $n \geq 0$.*

Proof. We have $[\mathcal{A}].\ker(\varphi_n) = 0$ for $n \geq 0$ (see the above discussion or Proposition 2 of [9]). Since V is local, $\mathcal{A}$ is free over V of rank q^2. Thus, $q^2.\ker(\varphi_n) = 0$. On the otherhand, $K_n(V)$ is p-torsion free for $n \geq 0$ (see Theorem 1.1 of [11]). So, $\ker(\varphi_n)$ is also p-torsion free for $n \geq 0$. This forces that $\ker(\varphi_n) = 0$ for $n \geq 0$. Hence the assertion. □

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