New York Journal of Mathematics

New York J. Math. **31** (2025) 1583–1606.

# Dynamical properties of a sequence of cosine operators in weighted Orlicz spaces on hypergroup

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Abstract. In this paper, the dynamical properties for a sequence of cosine operators in weighted Orlicz spaces on hypergroups are investigated. Firstly, we generalize the equivalent conditions of the topological transitivity for a finite sequence of cosine operators from Orlicz spaces on locally compact group to weighted Orlicz spaces on hypergroup. Secondly, in the hypergroup setting we give some sufficient or almost necessary conditions for a sequence of cosine operators to be topologically recurrent, even topologically multiply recurrent on Orlicz spaces. Besides, we deduce that topological mixing is a sufficient or necessary condition for a sequence of cosine operators to be topologically recurrent on Orlicz spaces. Thirdly, we also obtain sufficient or necessary conditions for a sequence of cosine operators to be chaotic in Orlicz spaces on hypergroup.

Contents



## 1. Introduction

In this article, we focus on the dynamical properties of a sequence of cosine operators in weighted Orlicz spaces on hypergroup. To begin with, we recall some related notations and definitions of dynamic theory of linear operators and the weighted Orlicz space on hypergroup.

Let $X$ be a separable and infinite dimensional Banach space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), and the linear operator $T \in L(X)$ be bounded. The operator $T : X \to$

Received November 15, 2024.

2020 *Mathematics Subject Classification.* Primary 47A16; Secondary 46E30, 54H20.

*Key words and phrases.* Hypercyclicity, Topological transitivity, Topological mixing, Topologically multiple recurrence, Chaoticity, Cosine operator, Weighted Orlicz space, Locally compact hypergroup.

This work is supported by Natural Science Foundation of Ningxia (Grant No. 2023AAC03001) and Natural Science Foundation of China(Grant Nos. 12261068, 12571088 and 12301156).

ISSN 1076-9803/2025

$X$ is called to be hypercyclic, if there exists a vector $x \in X$ such that the set $\mathrm{Orb}(x, T) = \{T^n x \mid n \in \mathbb{N}\}$ is dense in $X$. A bounded linear operator $T$ on a separable Banach space $X$ is said to be topologically transitive if, for given two non-empty open sets $U, V \in X$, there exists $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. In addition, if $T^n(U) \cap V \neq \emptyset$ from some $n$ onwards, then $T$ is topological mixing. Moreover, if $T$ is topologically transitive (hypercyclic) together with the density in X of the periodic elements of $T$, then $T$ is said to be chaotic.

For a Banach space $X$, a sequence $(T_n)_{n\in\mathbb{Z}_+}$ of bounded linear operators from $X$ into $X$ is called to be hypercyclic if there exists an element $x$ in $X$ (the called hypercyclic vector) such that the set $\{T_0(= I_X)x, T_1 x, ...\}$ is dense in $X$. If given nonempty open subsets $U, V$ of $X$, $T_n(U) \cap V \neq \emptyset$ from some $n$ onwards, then $(T_n)_{n\in\mathbb{N}_0}$ is called topological mixing. Furthermore. a sequence of bounded linear operators $(T_n)_{n\in\mathbb{N}_0}$ on $X$ is chaotic if $(T_n)_{n\in\mathbb{N}_0}$ is topologically transitive and the set of its periodic elements is dense in $X$.

Next, recall some basic properties of hypergroups; refer to two monographs [28] and [5] for more details. For a locally compact Hausdorff space $X$, denote by $\mathcal{M}(X)$ the space of all Radon complex measures on $X$, by $L^\infty(K)$ the Banach space of all essentially bounded and measurable functions on $K$ and by $C_c(X)$ the set of all continuous compactly supported complex-valued functions on $X$. $\mathcal{N}(f, r)$ denotes a neighborhood of $f \in \mathcal{L}^\Phi_\omega(K, \mathbb{K})$ with radius $r > 0$. The point mass measure at $x \in X$ and the support of any measure $\mu \in \mathcal{M}(X)$ are denoted by $\delta_x$ and $\mathrm{supp}\,(\mu)$, respectively. For each $A \subseteq X$, denote by $\chi_A$ the characteristic function of $A$. From [33] we know that $K \equiv (K, *, ^-, e)$ equipped with convolution $*$ and involution $^-$ is called a locally compact hypergroup (or simply a hyper-group) with an identity $e$. Moreover, a hypergroup $K$ is called commutative group if for each $x, y \in K$, $\delta_x * \delta_y = \delta_y * \delta_x$ holds. Notice that any locally compact group is a hypergroup.

A non-zero non-negative Radon measure $\lambda$ on a hypergroup $K$ is called a right Haar measure if $\lambda * \delta_x = \lambda$ for each $x \in K$. For each Borel measurable function $f : K \to \mathbb{C}$, define the right translation of function $f$ at $x \in K$ with respect to an element $y \in K$ by

$$f_x(y) = f_y(x) = f(x * y) := \int_K f d(\delta_x * \delta_y)$$

for $x, y \in K$, whenever this integral exists. A locally compact group $K$ equipped with

$$\mu * \nu \mapsto \int_K \int_K \delta_{xy} d\mu(x) d\nu(y) \quad (\mu, \nu \in \mathcal{M}(K))$$

as convolution, and $x \mapsto x^{-1}$ from $K$ onto $K$ as involution is a hypergroup. Although any locally compact group is a hypergroup, in general there is no action between elements of a hypergroup; See [5] for several classes of hypergroups.

Further, define the convolution $f * g$ of any two measurable functions $f, g : K \to \mathbb{C}$ by

$$(f * g)(x) := \int_K f(y)g(y^- * x)d\lambda(y) \quad (x \in K)$$

whenever this integral exists, where

$$g(y^- * x) := \int_K g(t)d(\delta_{y^-} * \delta_x)(t).$$

Similarly, if $\mu \in \mathcal{M}(K)$ and $f$ is a Borel measurable function on $K$, the convolution $f * \mu$ is defined by

$$(f * \mu)(x) = \int_K f(x * y^-)d\mu(y) \quad (x \in K).$$

In particular, $(f * \delta_{z^-})(x) = f_z(x) =: (L_z f)(x)$ for $x, z \in K$.

For any Borel subsets $A, B \subseteq K$, define

$$A * B := \sum_{x \in A, y \in B} \operatorname{supp}(\delta_x * \delta_y).$$

For each $x \in K$, denote $\{x\} * A$ and $A * \{x\}$ simply by $x * A$ and $A * x$. Also, for each $n \in \mathbb{N}$ we put

$$\{x\}^n := \{x\} * \cdots * \{x\} \quad (n\ times).$$

For a hypergroup $K$, the center $Z(K)$ of $K$ is defined by

$$Z(K) := \{x \in K : \delta_x * \delta_{x^-} = \delta_{x^-} * \delta_x = \delta_e\}.$$

The center of a hypergroup $K$, as the maximal subgroup of $K$, was introduced and studied in [22] and [28] (see also [41]). For each $x \in Z(K)$ and $y \in K$, the $\operatorname{supp}(\delta_x * \delta_y)$ and $\operatorname{supp}(\delta_y * \delta_x)$ are two singletons, and their single elements are denoted by $xy$ and $yx$, respectively. Furthermore, for each $z \in Z(K)$ and $n \in \mathbb{N}$, letting $\delta_z^n := \delta_z * \delta_z * \cdots * \delta_z (n\ times)$, we know that $\delta_z^n = \delta_{z^n}$.

A Young function is called to be a continuous, even and convex function $\Phi : \mathbb{R} \to [0, +\infty)$ satisfying $\Phi(t) > 0$ for $t > 0$ with $\Phi(0) = 0$ and $\lim_{t\to\infty} \Phi(t) = \infty$. Furthermore, the complementary function of $\Phi$ is defined by

$$\Psi(y) = \sup\{x \mid y \mid -\Phi(x) : x \geq 0\} \quad (y \in \mathbb{R}),$$

which is also a Young function. Moreover, $\Phi$ and $\Psi$ satisfy the following Young inequality

$$xy \leq \Phi(x) + \Psi(y) \quad (x, y \geq 0).$$

For a locally compact hypergroup $K$ with identity $e$ and a right Haar measure $\lambda$, the Orlicz space $\mathcal{L}^\Phi(K, \mathbb{K})$ is defined by

$$\mathcal{L}^\Phi(K, \mathbb{K}) = \left\{ f : K \to \mathbb{K} : \int_K \Phi(\alpha \mid f \mid) d\lambda < \infty \text{ for some } \alpha > 0 \right\},$$

where $f$ is a Borel measurable function. Furthermore, $\mathcal{L}^{\Phi}(K,\mathbb{K})$ is a Banach space equipped with the Orlicz norm

$$\| f \|_{\Phi}= \sup\left\{\int_K | fg | \, d\lambda \, : \, \int_K \Psi(| g |)d\lambda \leq 1\right\} \text{ for each } f \in \mathcal{L}^{\Phi}(K,\mathbb{K}),$$

equivalently, with the Luxemburg norm denoted by

$$N_{\Phi}(f) = \inf\left\{k > 0 \, : \, \int_K \Phi\left(\frac{| f |}{k}\right) d\lambda \leq 1\right\} \text{ for each } f \in \mathcal{L}^{\Phi}(K,\mathbb{K}).$$

For a Borel set $B$ of $K$, let $\lambda(B) > 0$ and $\chi_B$ be the characteristic function of $B$. Then, we know that

$$N_{\Phi}(\chi_B) = \frac{1}{\Phi^{-1}\left(\frac{1}{\lambda(B)}\right)},$$

where $\Phi^{-1}(t)$ is the modulus of the preimage of a singleton $t$ under $\Phi$. For convenience, let $\Omega$ be the set of all Borel functions $g$ on $K$ satisfying

$$\int_K \Phi(| g |)d\lambda \leq 1.$$

Consequently, $\| f \|_{\Phi}= \sup_{g\in\Omega} \int_K | fg | \, d\lambda$.

A Young function $\Phi(t)$ is said to be $\Delta_2$-regular if there exist a constant $M > 0$ and $t_0 > 0$ such that $\Phi(2t) \leq M\Phi(t)$ for $t \geq t_0$ and $\Phi(2t) \leq M\Phi(t)$ for all $t > 0$ when $K$ is compact and non-compact, respectively. For example,

$$\Phi(t) := \frac{| t |^p}{p} \quad (1 \leq p < \infty) \quad \text{and} \quad \Phi(t) :=| t |^{\alpha} (1+ | \log | t ||) \quad (\alpha > 1)$$

are both $\Delta_2$-regular Young functions [40]. Another examples of Young functions include $e^x - x - 1$ and $\cosh x - 1$. If $\Phi$ is $\Delta_2$-regular, then the space $C_c(K)$ of all continuous functions on $K$ with compact support is dense in $\mathcal{L}^{\Phi}(K,\mathbb{K})$, and the dual space $(\mathcal{L}^{\Phi}(K,\mathbb{K}), \| \cdot \|_{\Phi})$ is $(\mathcal{L}^{\Psi}(K,\mathbb{K}), N_{\Psi}(\cdot))$, where $\Psi$ is the complementary function of $\Phi$, and $N_{\Psi}(\cdot)$ is the Luxemburg norm on $\mathcal{L}^{\Psi}(K,\mathbb{K})$.

A weight on $K$ is a continuous function $\omega : K \to (0,\infty)$, which satisfies

$$\omega(x * y) \leq \omega(x)\omega(y) \quad (x, y \in K).$$

For a weight $\omega$ on $K$, let the weighted Orlicz space $\mathcal{L}^{\Phi}_{\omega}(K,\mathbb{K})$ be defined by $\mathcal{L}^{\Phi}_{\omega}(K,\mathbb{K}) = \{f : f\omega \in \mathcal{L}^{\Phi}(K,\mathbb{K})\}$, endowed with the weighted Orlicz norm $\| f \|_{\Phi,\omega} :=\| f\omega \|_{\Phi}$. Note that $\mathcal{L}^{\Phi}_{\omega}(K,\mathbb{K})$ is Banach space with respect to weighted Orlicz norm. By Lemma 2.1 in [38], we know if $\Phi$ satisfies $\Delta_2$-condition, then $C_c(K)$ is dense in $\mathcal{L}^{\Phi}_{\omega}(K,\mathbb{K})$. Here, we point out that

$$N_{\Phi}(f\omega) \leq\| f \|_{\Phi,\omega}\leq 2N_{\Phi}(f\omega).$$

For $a \in K$, let $\delta_a$ be the unit point mass at $a$ and $u : K \to (0,\infty)$ be a bounded continuous function on $K$. A weighted translation on $K$ is a weighted convolution operator $T_{a,u} : \mathcal{L}^{\Phi}_{\omega}(K,\mathbb{K}) \to \mathcal{L}^{\Phi}_{\omega}(K,\mathbb{K})$ defined by

$$T_{a,u}(f) = uT_a(f) \quad (f \in \mathcal{L}^{\Phi}_{\omega}(K,\mathbb{K})),$$

where $T_a(f) = f * \delta_a \in \mathcal{L}^{\Phi}_{\omega}(K, \mathbb{K})$ is the left translation operator or convolution denoted by

$$(f * \delta_a)(x) = \int_{y \in K} f(x * y^{-1}) d\delta_a(y) = f(x * a^{-1}) \quad (x \in K).$$

If $u^{-1} \in L^{\infty}(K)$, then we can define a self-map $S_{a,u}$ on $\mathcal{L}^{\Phi}_{\omega}(K, \mathbb{K})$ by

$$S_{a,u}(f) = (u^{-1} f) * \delta_{a^{-1}} \quad (f \in \mathcal{L}^{\Phi}_{\omega}(K, \mathbb{K}))$$

satisfying

$$T_{a,u} S_{a,u}(f) = f \quad (f \in \mathcal{L}^{\Phi}_{\omega}(K, \mathbb{K})).$$

In what follows, we assume that $u, u^{-1} \in L^{\infty}(K)$.

For each $n \in \mathbb{Z}$, the single cosine operator $\mathcal{C}_n : \mathcal{L}^{\Phi}_{\omega}(K, \mathbb{K}) \to \mathcal{L}^{\Phi}_{\omega}(K, \mathbb{K})$ is defined by

$$\mathcal{C}_n := \frac{1}{2}(T^n_{a,u} + S^n_{a,u}),$$

which infers $\mathcal{C}_0 = I$ and $2\mathcal{C}_n \mathcal{C}_m = \mathcal{C}_{n+m} + \mathcal{C}_{n-m}$ for $n, m \in \mathbb{Z}$.

Over the last decades, linear dynamical properties of bounded operators have attracted many scholars to investigate; refer to some monographs and the survey articles [7, 26, 27, 31] for more detailed information. Specially, Salas [42] ever posed some conditions on the weight function and characterized hypercyclicity of bilateral weighted shifts on $\ell^2(\mathbb{Z})$. Then, León-Saavedra proved that the bilateral weighted shifts on $\ell^2(\mathbb{Z})$ are Cesàro hypercyclic. Also, Costakis and Sambarino [17] gave a sufficient and necessary condition for bilateral weighted shifts on $\ell^p(\mathbb{Z})$ to be mixing. Afterwards, linear dynamics of weighted translations on a Lebesgue space on locally compact group were intensely studied; refer to [13] for the existence of hypercyclic weighted translations on Lebegue space, [20] for chaotic operators on the hypergroup case and [21] topological transitivity on vector-valued version. In [1, 15] some linear dynamical properties of weighted translation operators in Orlicz spaces on locally compact groups have been studied. As well as Sobolev Spaces and Morrey spaces, Orlicz spaces also are generalizations of the usual Lebesgue spaces, which have been thoroughly investigated; refer to [40, 45] for Orlicz spaces. Initially, the linear dynamics for weighted translations on the Orlicz space are due to Azimi and Akbarbaglu [1] or Chen and Du [15]. Recently, Chen et al. [12, 14] characterized topological transitivity, topologically multiple recurrence and chaoticity of translation operators in a weighted Orlicz space on locally compact group. Moreover, Kumar et al. [34, 35, 36] investigated Orlicz spaces on locally compact hypergroup. For the disjoint dynamics on Orlicz spaces and weighted Orlicz spaces, we see [15, 16] for disjoint topological transitivity and mixing of translation operator and [47] for disjoint supercyclic dynamic of translation operators. Based on their works, Kumar and Tabatabie [33] introduced a sequence of bounded linear operators in the Orlicz space on hypergroup and obtained some necessary conditions for this sequence to be densely hypercyclic. Besides, they improved their results for the special case that the sequence of weighted translation operators corresponds to a sequence in the center of hypergroup, and gave an equivalent condition for a single weighted translation operator to be hereditarily hypercyclic in the Orlicz space.

In the dynamical system Poincaré [39] firstly introduced the notation of recurrence. However, a systematic study of recurrent operators went back to the works of Gottschalk and Hedlund [24], and also the work of Furstenberg [23] who investigated the recurrence in ergodic theory and combinatorial number theory. Later, Costakis, et al. [19] considered linear operators to be recurrent. Clearly, topologically multiple recurrence is stronger than recurrence. Costakis and Parissis [18] ever characterized topologically multiply recurrent weighted shifts on $\ell^p(\mathbb{Z})$ in terms of the weight sequence. For another related works, we may refer to Bonilla et al. [3] for the frequently recurrent operators, Galán, et al. [25] for the product recurrence for weighted backward shifts, and Yin and Wei [48] for the recurrence and topological entropy of translation operators.

For the cosine operator functions of semigroup, we pay attention to the works from Butzer and Koliha [6] and Shaw [44]. As for the cosine operators on Banach spaces, Bonilla and Miana [8] originally provided some sufficient conditions for the hypercyclicity and topological mixing of a strongly continuous cosine operator function. Afterwards, Kalmes [30] investigated the hypercyclicity of cosine operator functions on locally Lebesgue space generated by second order partial differential operators. He also showed that the hypercyclicity and weak mixing of these type of operators are equivalent. Later then, Kostić [32] studied the main structural properties of hypercyclic and chaotic integrated C-cosine functions. Chen et al. [9, 10] characterized topological transitivity, topologically multiple recurrence for cosine operator functions, generated by weighted translations on Lebesgue space. Immediately after that, Chen [11] obtained sufficient conditions for finite sequences of cosine operators to be disjoint topologically transitive and mixing in terms of the group elements and weights. In 2021, Tabatabaie and Ivković [46] deduced some sufficient and necessary conditions for discrete cosine operator functions on solid Banach function spaces to be chaotic or topologically transitive. Meanwhile, Akbarbaglu, Azimi and Kumar [2] also presented some conditions for the topological transitivity and topological mixing of a sequence of cosine operators.

Inspired by the statement above, we intend to consider the dynamical properties of a sequence of cosine operators in weighted Orlicz space on hypergroup setting. Firstly, according to [2] we characterize the equivalent conditions of the topological transitivity for a finite sequence of cosine operators from Orlicz spaces on locally compact group to weighted Orlicz spaces on hypergroup. Secondly, in light of [12], in the hypergroup setting we give some sufficient or necessary conditions for a sequence of cosine operators to be topologically recurrent and topologically multiply recurrent on Orlicz spaces. Besides, we infer that topological mixing is a sufficient or necessary condition for a sequence of cosine operators to be topologically recurrent on Orlicz spaces. Thirdly, we also obtain the sufficient or necessary conditions for a sequence of cosine operators to be chaotic in Orlicz spaces on hypergroup.

## 2. Some Preliminary Lemmas

Recall that an element $a \in K$ of finite order is called a torsion element. An element $a \in K$ is called to be periodic if the closed subgroup $K(a)$ generated by $a$ is compact. Further, an element $a \in K$ is aperiodic if it is not periodic. Equivalently, $a \in K$ is an aperiodic element if and only if for any compact subset $\widetilde{K} \subset K$, there exists an $N \in \mathbb{N}$ such that $\widetilde{K} \cap \widetilde{K}a^{-n} = \emptyset$ for all $n > N$. Note that $T_a$ cannot be transitive if $a$ is a torsion element. For discrete groups, the periodic element and the torsion element are identical.

In fact, since $(fg)^z \neq f^z g^z$ generally in hypergroup setting, we have to consider in the center $Z(K)$. For convenience, we are prepared for some preliminary lemmas below. Following the same method as the proof of Lemma 1.3 in [33], here we give the following result.

**Lemma 2.1.** *For $z \in Z(K)$ and $f \in \mathcal{L}^{\Phi}_{\omega}(K, \mathbb{C})$, we have $N_{\Phi}(f^z \omega) = N_{\Phi}(f \omega^{z^-})$.*

**Lemma 2.2** ([33]). *An element $z \in Z(K)$ is aperiodic if and only if for each compact subset $\widetilde{K} \subseteq K$ with $\lambda(\widetilde{K}) > 0$, there exists $N \in \mathbb{N}$ such that $(\widetilde{K} * \{z\}^{pn}) \cap (\widetilde{K} * \{z\}^{qn} = \emptyset$ for $n > N$ and $p, q \in \mathbb{N}$ with $p \neq q$, where $\{z\}^{-n} = \{z^-\}^n$.*

## 3. Main Results and Their Proofs

In this section we deal with the dynamical properties for a sequence of cosine operators in weighted Orlicz space on hypergroup. At first, we establish the next theorem for the topologically transitive in weighted Orlicz spaces on hypergroup.

**Theorem 3.1.** *Let $a \in K$ be an aperiodic element of $Z(K)$, $\Phi$ be a $\Delta_2$-regular Young's function and $u, u^{-1} \in L^{\infty}(K)$. If $\mathcal{C}_n := \frac{1}{2}(T^n_{a,u} + S^n_{a,u})$ is a cosine operator on $\mathcal{L}^{\Phi}_{\omega}(K, \mathbb{C})$, then the following statements are equivalent.*

(I) $(\mathcal{C}_n)_{n \in \mathbb{N}_0}$ *is topologically transitive in $\mathcal{L}^{\Phi}_{\omega}(K, \mathbb{C})$.*

(II) *For each non-empty compact subset $\widetilde{K} \subset K$ with $\lambda(\widetilde{K}) > 0$, there exist three sequences of Borel sets $(E_k)$, $(E^+_k)$ and $(E^-_k)$ in $\widetilde{K}$, and a sequence $(n_k)$ of positive numbers such that $E_k = E^+_k \cup E^-_k$ and*

$$\lim_{k \to \infty} \sup_{\nu \in \Omega} \int_{\widetilde{K} \setminus E_k} | \nu(x) | \omega(x) d\lambda(x) = 0.$$

*Moreover, two sequences*

$$\varphi_n = \omega * \delta^n_{a^{-1}} \cdot \left( \Pi^n_{j=1} u * \delta^j_{a^{-1}} \right) \quad and \quad \widetilde{\varphi}_n = \omega * \delta^n_a \cdot \left( \Pi^{n-1}_{j=0} u * \delta^j_a \right)^{-1}$$

*satisfy*

$$\lim_{k \to \infty} \sup_{\nu \in \Omega} \int_{E_k} \varphi_{n_k}(x) \mid \nu(x * a^{n_k}) \mid d\lambda(x) = 0,$$

$$\lim_{k\to\infty}\sup_{\nu\in\Omega}\int_{E_k}\widetilde{\varphi}_{n_k}(x)\mid\nu(x*a^{-n_k})\mid d\lambda(x)=0,$$

$$\lim_{k\to\infty}\sup_{\nu\in\Omega}\int_{E_k^+}\varphi_{2n_k}(x)\mid\nu(x*a^{2n_k})\mid d\lambda(x)=0$$

*and*

$$\lim_{k\to\infty}\sup_{\nu\in\Omega}\int_{E_k^-}\widetilde{\varphi}_{2n_k}(x)\mid\nu(x*a^{-2n_k})\mid d\lambda(x)=0,$$

*where $\Omega$ is the set of all Borel functions $\nu$ on $K$ satisfying $\int_K\Phi(\mid\nu\mid)d\lambda\le 1$.*

**Proof.** (I)$\Rightarrow$(II). Let $\widetilde{K}$ be a compact subset of $K$ such that $\lambda(\widetilde{K})>0$. Since $a\in K$ is an aperiodic element of $Z(K)$, there exists $N\in\mathbb{N}$ such that $\widetilde{K}\cap\widetilde{K}a^{\pm n}=\emptyset$ for $n>N$. Denote by $\chi_{\widetilde{K}}$ the characteristic function of $\widetilde{K}$ defined on $K$. Clearly $\chi_{\widetilde{K}}\in\mathcal{L}^{\Phi}_{\omega}(K,\mathbb{C})$. According to the definition of the topologically transitive for the sequence $(\mathcal{C}_n)_{n\in\mathbb{N}_0}$, choose $U=\mathcal{N}(\chi_{\widetilde{K}},\epsilon^2)$ and $V=\mathcal{N}(-\chi_{\widetilde{K}},\epsilon^2)$. Then, for each $\epsilon\in(0,1)$ there exist $f\in\mathcal{L}^{\Phi}_{\omega}(K,\mathbb{C})$ and $m\in\mathbb{N}$ such that

$$\parallel f-\chi_{\widetilde{K}}\parallel_{\Phi,\omega}<\epsilon^2\quad\text{and}\quad\parallel\mathcal{C}_mf+\chi_{\widetilde{K}}\parallel_{\Phi,\omega}<\epsilon^2.$$

Hence, we remark that

$$\parallel\Re(f)-\chi_{\widetilde{K}}\parallel_{\Phi,\omega}<\epsilon^2\quad\text{and}\quad\parallel\Re(\mathcal{C}_mf)+\chi_{\widetilde{K}}\parallel_{\Phi,\omega}=\parallel\mathcal{C}_m\Re(f)+\chi_{\widetilde{K}}\parallel_{\Phi,\omega}<\epsilon^2,$$

where $\Re(f)$ is the real part of the complex valued function $f$. Since the maps $\Re:\mathcal{L}^{\Phi}_{\omega}(K,\mathbb{C})\to\mathcal{L}^{\Phi}_{\omega}(K,\mathbb{R})$ and $f\to f^{\pm}:=\max\{0,\pm f\}$ from $\mathcal{L}^{\Phi}_{\omega}(K,\mathbb{R})$ to $\mathcal{L}^{\Phi}_{\omega}(K,\mathbb{R})$ are continuous and also commute with both $T_{a,u}$ and $S_{a,u}$, without loss of generality we may assume that the function $f$ is real valued such that for any Borel subset $F\subseteq K$, we may imply

$$\begin{aligned}\parallel\mathcal{C}_mf^+\chi_F\parallel_{\Phi,\omega}&\le\parallel(\mathcal{C}_mf)^+\parallel_{\Phi,\omega}=\parallel(\mathcal{C}_mf+\chi_{\widetilde{K}}-\chi_{\widetilde{K}})^+\parallel_{\Phi,\omega}\\&\le\parallel(C_mf+\chi_{\widetilde{K}})^+\parallel_{\Phi,\omega}+\parallel(-\chi_{\widetilde{K}})^+\parallel_{\Phi,\omega}\\&=\parallel(C_mf+\chi_{\widetilde{K}})^+\parallel_{\Phi,\omega}\le\parallel\mathcal{C}_mf+\chi_{\widetilde{K}}\parallel_{\Phi,\omega}<\epsilon^2.\end{aligned}$$

Denote $A=\{x\in\widetilde{K}:\mid f(x)-1\mid\ge\epsilon\}$. Then

$$\begin{aligned}\epsilon^2>\parallel f-\chi_{\widetilde{K}}\parallel_{\Phi,\omega}&=\sup_{\nu\in\Omega}\int_K\mid f(x)-\chi_{\widetilde{K}}(x)\mid\mid\nu(x)\mid\omega(x)d\lambda(x)\\&\ge\sup_{\nu\in\Omega}\int_{\widetilde{K}}\mid f(x)-1\mid\mid\nu(x)\mid\omega(x)d\lambda(x)\\&\ge\sup_{\nu\in\Omega}\int_A\epsilon\mid\nu(x)\mid\omega(x)d\lambda(x),\end{aligned}\tag{1}$$

which infers

$$\sup_{\nu\in\Omega}\int_A\mid\nu(x)\mid\omega(x)d\lambda(x)<\epsilon.$$

Denote $B_m = \{x \in \widetilde{K} :| \mathcal{C}_m f(x) + 1 |\geq \epsilon\}$. Similarly, we know

$$\sup_{\nu\in\Omega} \int_{B_m} | \nu(x) | \omega(x)d\lambda(x) < \epsilon.$$

Now, let

$$E_m := \{x \in \widetilde{K} :| f(x) - 1 |< \epsilon\} \cap \{x \in \widetilde{K} :| \mathcal{C}_m f(x) + 1 |< \epsilon\}.$$

Then, for $x \in E_m$ we get $f(x) > 1 - \epsilon > 0$ and $\mathcal{C}_m f(x) < \epsilon - 1 < 0$. Therefore

$$\begin{aligned}\sup_{\nu\in\Omega} \int_{\widetilde{K}\setminus E_m} | \nu(x) | \omega(x)d\lambda(x) &= \sup_{\nu\in\Omega} \int_{A\cup B_m} | \nu(x) | \omega(x)d\lambda(x)\\ &\leq \sup_{\nu\in\Omega} \int_{A} | \nu(x) | \omega(x)d\lambda(x) + \sup_{\nu\in\Omega} \int_{B_m} | \nu(x) | \omega(x)d\lambda(x) < 2\epsilon.\end{aligned}$$

Since Haar measure $\lambda$ is right invariant, in this sense $T^m_{a,u} f^+$ and $S^m_{a,u} f^+$ are positive. Then, from (1) we obtain that

$$\begin{aligned}2\epsilon^2 &> \| 2(\mathcal{C}_m f^+)\chi_{E_m * a^m} \|_{\Phi,\omega}\\ &= \| (T^m_{a,u} f^+ + S^m_{a,u} f^+)\chi_{E_m * a^m} \|_{\Phi,\omega} \geq \| T^m_{a,u} f^+ \chi_{E_m * a^m} \|_{\Phi,\omega}\\ &= \sup_{\nu\in\Omega} \int_{E_m * a^m} | T^m_{a,u} f^+(x) || \nu(x) | \omega(x)d\lambda(x)\\ &= \sup_{\nu\in\Omega} \int_{E_m * a^m} | u(x)u(x * a^{-1}) \cdots u(x * a^{-m+1}) f^+(x * a^{-m}) || \nu(x) | \omega(x)d\lambda(x)\\ &= \sup_{\nu\in\Omega} \int_{E_m} u(x * a^m)u(x * a^{m-1}) \cdots u(x * a) f^+(x) | \nu(xa^m) | \omega(x * a^m)d\lambda(x)\\ &= \sup_{\nu\in\Omega} \int_{E_m} \varphi_m(x) f^+(x) | \nu(x * a^m) | d\lambda(x)\\ &> (1 - \epsilon) \sup_{\nu\in\Omega} \int_{E_m} \varphi_m(x) | \nu(x * a^m) | d\lambda(x).\end{aligned}$$

Therefore

$$\sup_{\nu\in\Omega} \int_{E_m} \varphi_m(x) | \nu(x * a^m) | d\lambda(x) < \frac{2\epsilon^2}{1 - \epsilon}.$$

Similarly, we get

$$2\epsilon^2 >\| (S^m_{a,u} f^+)\chi_{E^m * a^m} \|_{\Phi,\omega} > (1 - \epsilon) \sup_{\nu\in\Omega} \int_{E_m} \widetilde{\varphi}_m(x) | \nu(x * a^{-m}) | d\lambda(x)$$

and thus

$$\sup_{\nu\in\Omega} \int_{E_m} \widetilde{\varphi}_m(x) | \nu(x * a^{-m}) | d\lambda(x) < \frac{2\epsilon^2}{1 - \epsilon}.$$

Hence, because $\epsilon$ is arbitrary, the first part of Condition (II) holds .

From now on, let $E_m^- = \{x \in E_m : T_{a,u}^m f(x) < \epsilon - 1\}$ and $E_m^+ = E_m \setminus E_m^-$. Then, for $x \in E_m^+$, we have

$$\epsilon - 1 > \mathcal{C}_m f(x) = \frac{1}{2}T_{a,u}^m f(x) + \frac{1}{2}S_{a,u}^m f(x) \geq \frac{1}{2}(\epsilon - 1) + \frac{1}{2}S_{a,u}^m f(x)$$

and so

$$S_{a,u}^m f(x) < \epsilon - 1, \quad x \in E_m^+.$$

Now, we infer that

$$\begin{aligned}
&(1-\epsilon)\sup_{\nu\in\Omega}\int_{E_m^+}\varphi_{2m}(x) \mid \nu(x * a^{2m}) \mid d\lambda(x)\\
&\leq \sup_{\nu\in\Omega}\int_{E_m^+} \mid \Pi_{k=1}^{2m} u(x * a^k) \mid\mid S_{a,u}^m f^-(x) \mid\mid \nu(x * a^{2m}) \mid \omega(x * a^{2m}) d\lambda(x)\\
&\leq \sup_{\nu\in\Omega}\int_{E_m^+ * a^{2m}} \mid \Pi_{k=0}^{2m-1} u(x * a^{-k}) \mid\mid S_{a,u}^m f^-(x * a^{-2m}) \mid\mid \nu(x) \mid \omega(x) d\lambda(x)\\
&= \sup_{\nu\in\Omega}\int_{E_m^+ * a^{2m}} \mid T_{a,u}^{2m} S_{a,u}^m f^-(x) \mid\mid \nu(x) \mid \omega(x) d\lambda(x)\\
&= \sup_{\nu\in\Omega}\int_{E_m^+ * a^{2m}} \mid T_{a,u}^{m} f^-(x) \mid\mid \nu(x) \mid \omega(x) d\lambda(x)\\
&\leq 2\sup_{\nu\in\Omega}\int_{E_m^+ * a^{2m}} \mid \mathcal{C}_m f^-(x) \mid\mid \nu(x) \mid \omega(x) d\lambda(x)\\
&= 2\sup_{\nu\in\Omega}\int_{K} \mid \mathcal{C}_m f^-(x)\chi_{E_m^+ * a^{2m}} \mid\mid \nu(x) \mid \omega(x) d\lambda(x)\\
&= 2\sup_{\nu\in\Omega}\int_{K} \mid \mathcal{C}_m (f^+ - f)(x)\chi_{E_m^+ * a^{2m}} \mid\mid \nu(x) \mid \omega(x) d\lambda(x)\\
&= 2\sup_{\nu\in\Omega}\int_{K} \mid (\mathcal{C}_m f^+)\chi_{E_m^+ * a^{2m}} - (\mathcal{C}_m f + \chi_{\widetilde{K}})\chi_{E_m^+ * a^{2m}} + \chi_{\widetilde{K}\cap E_m^+ * a^{2m}} \mid\mid \nu(x) \mid \omega(x) d\lambda(x)\\
&\leq 2\sup_{\nu\in\Omega}\int_{K} \mid (\mathcal{C}_m f^+)\chi_{E_m^+ * a^{2m}} \mid\mid \nu(x) \mid \omega(x) d\lambda(x)\\
&\quad + 2\sup_{\nu\in\Omega}\int_{K} \mid (\mathcal{C}_m f + \chi_K)\chi_{E_m^+ * a^{2m}} \mid\mid \nu(x) \mid \omega(x) d\lambda(x)\\
&\quad + 2\sup_{\nu\in\Omega}\int_{\widetilde{K}\cap E_m^+ * a^{2m}} \mid \nu(x) \mid \omega(x) d\lambda(x)\\
&\leq 2 \parallel (\mathcal{C}_m f^+ \chi_{E_m^+ * a^{2m}}) \parallel_{\Phi,\omega} + 2 \parallel (\mathcal{C}_m f + \chi_{\widetilde{K}}) \parallel_{\Phi,\omega}\\
&+ 2 \parallel \chi_{\widetilde{K}\cap E_m^+ * a^{2m}} \parallel_{\Phi,\omega} < 2\epsilon^2 + 2\epsilon^2 + 0 = 4\epsilon^2.
\end{aligned}$$

In the last inequality above, we apply the fact $\widetilde{K} \cap \widetilde{K}a^{\pm 2m} = \emptyset$. Therefore, we give

$$\sup_{\nu\in\Omega}\int_{E_m^+}\varphi_{2m}(x) \mid \nu(x * a^{2m}) \mid d\lambda(x) < \frac{4\epsilon^2}{(1-\epsilon)}.$$

Similarly, we can also be sure that

$$\sup_{\nu\in\Omega}\int_{E_m^-}\widetilde{\varphi}_{2m}(x)\mid \nu(x*a^{-2m})\mid d\lambda(x)<\frac{4\epsilon^2}{(1-\epsilon)}.$$

Since $\epsilon$ is arbitrary, the last two conditions of (II) part also is fulfilled.

(II)⇒(I). Let $U$ and $V$ be two non-empty open subsets of $\mathcal{L}^{\Phi}_{\omega}(K,\mathbb{C})$. Since $\Phi$ is $\Delta_2$-regular, we can choose two non-zero functions $f$ and $g$ in $C_c(K)$ such that $f\in U$ and $g\in V$. Let $\widetilde{K}=\operatorname{supp}(f)\cup\operatorname{supp}(g)$, where $\operatorname{supp}(f)$ and $\operatorname{supp}(g)$ are the supports of $f$ and $g$, respectively. Now, assume that $E_k\subset\widetilde{K}$ satisfies condition (II). However, $a\in K$ is an aperiodic element, then there exists $M\in\mathbb{N}$ such that $\widetilde{K}\cap\widetilde{K}a^{\pm n}=\emptyset$ for all $n>M$. Subsequently, for a fixed $\epsilon>0$, one can find $N\in\mathbb{N}$ such that for each $k>N$, there exists $n_k>M$ satisfying

$$\parallel g\parallel_\infty\cdot\sup_{\nu\in\Omega}\int_{E_k}\varphi_{n_k}(x)\mid\nu(x*a^{n_k})\mid d\lambda(x)<\epsilon$$

and

$$\parallel g\parallel_\infty\cdot\sup_{\nu\in\Omega}\int_{\widetilde{K}\setminus E_k}\mid\nu(x)\mid\omega(x)d\lambda(x)<\epsilon.$$

Therefore, we imply that

$$\begin{aligned}\parallel T^{n_k}_{a,u}(g\chi_{E_k})\parallel_{\Phi,\omega}&=\sup_{\nu\in\Omega}\int_K\mid T^{n_k}_{a,u}(g\chi_{E_k})(x)\nu(x)\mid\omega(x)d\lambda(x)\\
&=\sup_{\nu\in\Omega}\int_K\mid\Pi_{i=0}^{n_k-1}u(x*a^{-i})g(x*a^{-n_k})\chi_{E_k}(x*a^{-n_k})\nu(x)\mid\omega(x)d\lambda(x)\\
&=\sup_{\nu\in\Omega}\int_K\mid\Pi_{i=1}^{n_k}u(x*a^{i})g(x)\chi_{E_k}(x)\nu(x*a^{n_k})\mid\omega(x*a^{n_k})d\lambda(x)\\
&\leq\parallel g\parallel_\infty\cdot\sup_{\nu\in\Omega}\int_{E_k}\varphi_{n_k}(x)\mid\nu(x*a^{n_k})\mid d\lambda(x)<\epsilon.\end{aligned}$$

Since $E^+\subset E$, we remark that

$$\lim_{k\to\infty}\parallel T^{n_k}_{a,u}(g\chi_{E_k^+})\parallel_{\Phi,\omega}=\lim_{k\to\infty}\parallel T^{n_k}_{a,u}(g\chi_{E_k})\parallel_{\Phi,\omega}=0.$$

In addition

$$\begin{aligned}\parallel g-g\chi_{E_k}\parallel_{\Phi,\omega}&=\sup_{\nu\in\Omega}\int_K\mid g(x)-g(x)\chi_{E_k}(x)\mid\mid\nu(x)\mid\omega(x)d\lambda(x)\\
&\geq\sup_{\nu\in\Omega}\int_K\mid g(x)\chi_{\widetilde{K}\setminus E_k}(x)\mid\mid\nu(x)\mid\omega(x)d\lambda(x)\\
&=\int_{\widetilde{K}\setminus E_k}\mid g(x)\mid\mid\nu(x)\mid\omega(x)d\lambda(x)\\
&\leq\parallel g\parallel_\infty\cdot\int_{\widetilde{K}\setminus E_k}\mid\nu(x)\mid\omega(x)d\lambda(x)<\epsilon.\end{aligned}$$

Similarly, by condition (II) we get

$$\lim_{k\to\infty} \| S^{n_k}_{a,u}(g\chi_{E^-_k}) \|_{\Phi,\omega} = \lim_{k\to\infty} \| S^{n_k}_{a,u}(g\chi_{E_k}) \|_{\Phi,\omega} = 0$$

and

$$\lim_{k\to\infty} \| S^{2n_k}_{a,u}(g\chi_{E^-_k}) \|_{\Phi,\omega} = \lim_{k\to\infty} \| T^{2n_k}_{a,u}(g\chi_{E^+_k}) \|_{\Phi,\omega} = 0.$$

Consequently, if $f$ is exchanged with $g$, then we have

$$\lim_{k\to\infty} \| S^{n_k}_{a,u}(f\chi_{E_k}) \|_{\Phi,\omega} = \lim_{k\to\infty} \| T^{n_k}_{a,u}(f\chi_{E_k}) \|_{\Phi,\omega} = \lim_{k\to\infty} \| f - f\chi_{E_k} \|_{\Phi,\omega} = 0.$$

For each $k \in \mathbb{N}$, let

$$v_k = f\chi_{E_k} + 2T^{n_k}_{a,u}(g\chi_{E^+_k}) + 2S^{n_k}_{a,u}(g\chi_{E^-_k}).$$

Based on Lemma 2.2, we have $\widetilde{K} \cap \widetilde{K} * a^{(m_1-m_2)n_k} = \emptyset$ ($m_1, m_2 \in \mathbb{Z}$ and $m_1 \neq m_2$), and with Minkowski inequality we may yield that

$$\| v_k - f \|_{\Phi,\omega} \leq \| f - f\chi_{E_k} \|_{\Phi,\omega} + 2 \| T^{n_k}_{a,u}(g\chi_{E^+_k}) \|_{\Phi,\omega} + 2 \| S^{n_k}_{a,u}(g\chi_{E^-_k}) \|_{\Phi,\omega}$$

and

$$\begin{aligned} \| \mathcal{C}_{n_k} v_k - g \|_{\Phi,\omega} \leq & \| g - g\chi_{E_k} \|_{\Phi,\omega} + \frac{1}{2} \| T^{n_k}_{a,u}(f\chi_{E_k}) \|_{\Phi,\omega} \\ & + \frac{1}{2} \| S^{n_k}_{a,u}(f\chi_{E_k}) \|_{\Phi,\omega} + \| T^{2n_k}_{a,u}(g\chi_{E^+_k}) \|_{\Phi,\omega} + \| S^{2n_k}_{a,u}(g\chi_{E^-_k}) \|_{\Phi,\omega} . \end{aligned}$$

Therefore, we remark

$$\lim_{k\to\infty} \| v_k - f \|_{\Phi,\omega} = 0 \quad \text{and} \quad \lim_{k\to\infty} \| \mathcal{C}_{n_k} v_k - g \|_{\Phi,\omega} = 0,$$

which means that

$$\lim_{k\to\infty} v_k = f \quad \text{and} \quad \lim_{k\to\infty} \mathcal{C}_{n_k} v_k = g.$$

Then, $\mathcal{C}_{n_k}(U) \cap V \neq \emptyset$ for some $k$. Therefore, we conclude that the sequence $(\mathcal{C}_n)_{n\in\mathbb{N}_0}$ is topologically transitive. □

**Corollary 3.2.** *Let $a \in K$ be an aperiodic element of hypergroup $Z(K)$, $\Phi$ be a $\Delta_2$-regular Young's function and $u, u^{-1} \in L^\infty(K)$. If $\mathcal{C}_n := \frac{1}{2}(T^n_{a,u} + S^n_{a,u})$ is a cosine operator on $\mathcal{L}^\Phi_\omega(K, \mathbb{C})$, then the following statements are equivalent.*

(I) *$(\mathcal{C}_n)_{n\in\mathbb{N}_0}$ is topological mixing in $\mathcal{L}^\Phi_\omega(K, \mathbb{C})$.*

(II) *For each non-empty compact subset $\widetilde{K} \subset K$ with $\lambda(\widetilde{K}) > 0$, there exists a sequence of Borel sets $(E_n)$ in $\widetilde{K}$ such that we have*

$$\lim_{n\to\infty} \sup_{v\in\Omega} \int_{\widetilde{K}\setminus E_n} | v(x) | \omega(x) d\lambda(x) = 0.$$

*Moreover, the two sequences*

$$\varphi_n = \omega * \delta^n_{a^{-1}} \cdot \left(\Pi^n_{j=1} u * \delta^j_{a^{-1}}\right) \quad \text{and} \quad \widetilde{\varphi}_n = \omega * \delta^n_a \cdot \left(\Pi^{n-1}_{j=0} u * \delta^j_a\right)^{-1}$$

*satisfy*

$$\lim_{n\to\infty}\sup_{\nu\in\Omega}\int_{E_n}\varphi_n(x)\mid\nu(x*a^n)\mid d\lambda(x)=0$$

*and*

$$\lim_{n\to\infty}\sup_{\nu\in\Omega}\int_{E_n}\widetilde{\varphi}_n(x)\mid\nu(x*a^{-n})\mid d\lambda(x)=0,$$

*where $\Omega$ is the set of all Borel functions $\nu$ on $K$ satisfying $\int_K\Phi(\mid\nu\mid)d\lambda\le 1$.*

Further, if $K$ is a discrete hypergroup with the counting measure as its Haar measure, then the set $E_k$ is exactly $\widetilde{K}$ itself so that we reformulate the theorem above.

**Corollary 3.3.** *Let $a\in K$ be a non-torsion element of $Z(K)$, $\Phi$ be a $\Delta_2$-regular Young's function and $u,u^{-1}\in L^\infty(K)$. If $\mathcal{C}_n:=\frac{1}{2}(T^n_{a,u}+S^n_{a,u})$ is a cosine operator on $\mathcal{L}^\Phi_\omega(K,\mathbb{C})$, then the following statements are equivalent.*

(I) $(\mathcal{C}_n)_{n\in\mathbb{N}_0}$ *is topologically transitive in $\mathcal{L}^\Phi_\omega(K,\mathbb{C})$.*

(II) *For each non-empty finite subset $\widetilde{K}\subset K$ with $\lambda(\widetilde{K})>0$, there exist two sequences of Borel sets $(E^+_k)$ and $(E^-_k)$ in $\widetilde{K}$, and a sequence $(n_k)$ of positive numbers such that $\widetilde{K}=E^+_k\cup E^-_k$, and we know two sequences*

$$\varphi_n=\omega*\delta^n_{a^{-1}}\cdot\left(\Pi^n_{j=1}u*\delta^j_{a^{-1}}\right)\quad\text{and}\quad\widetilde{\varphi}_n=\omega*\delta^n_a\cdot\left(\Pi^{-n}_{j=0}u*\delta^j_a\right)^{-1}$$

*satisfy*

$$\lim_{k\to\infty}\sup_{\nu\in\Omega}\sum_{x\in\widetilde{K}}\varphi_{n_k}(x)\mid\nu(x*a^{n_k})\mid=0,\quad\lim_{k\to\infty}\sup_{\nu\in\Omega}\sum_{x\in\widetilde{K}}\widetilde{\varphi}_{n_k}(x)\mid\nu(x*a^{-n_k})\mid=0$$

*and*

$$\lim_{k\to\infty}\sup_{\nu\in\Omega}\sum_{x\in E^+_k}\varphi_{2n_k}(x)\mid\nu(x*a^{2n_k})\mid=0,\quad\lim_{k\to\infty}\sup_{\nu\in\Omega}\sum_{x\in E^-_k}\widetilde{\varphi}_{2n_k}(x)\mid\nu(x*a^{-2n_k})\mid=0,$$

*where $\Omega$ is the set of all Borel functions $\nu$ on $K$ satisfying $\int_K\Phi(\mid\nu\mid)d\lambda\le 1$.*

For a fixed $L\in\mathbb{N}$, let $(a_\ell)_{1\le\ell\le L}$ and $(u_\ell)_{1\le\ell\le L}$ be the sequences of aperiodic elements of hypergroup $K$ and positive weight, respectively. Then $(T_{a_\ell,u_\ell})_{1\le\ell\le L}$ and $(S_{a_\ell,u_\ell})_{1\le\ell\le L}$ are a sequence of weighted translation operators and a sequence of self-maps with weight, respectively. Now, we characterize the topological transitivity for a finite sequence of cosine operators.

**Theorem 3.4.** *Let $(a_\ell)_{1\le\ell\le L}$ and $(u_\ell)_{1\le\ell\le L}$ be a sequence of aperiodic elements of $Z(K)$ and a sequence of positive weights respectively such that $u_\ell,u_\ell^{-1}\in L^\infty(K)$. If $\mathcal{C}_{\ell,n}:=\frac{1}{2}(T^n_{a_\ell,u_\ell}+S^n_{a_\ell,u_\ell})$ is a cosine operator on $\mathcal{L}^\Phi_\omega(K,\mathbb{C})$ for $1\le\ell\le L$, then the following statements are equivalent.*

(I) $(\mathcal{C}_{1,n}\oplus\mathcal{C}_{2,n}\oplus\cdots\oplus\mathcal{C}_{L,n})_{n\in\mathbb{N}_0}$ *is topologically transitive.*

(II) *For each non-empty compact subset $\widetilde{K}\subset K$ with $\lambda(\widetilde{K})>0$, there has some sequence $(n_k)$ of positive integers such that for $1\le\ell\le L$, there exist three sequences of Borel sets $(E_{\ell,k})$, $(E^+_{\ell,k})$ and $(E^-_{\ell,k})$ of $\widetilde{K}$ such that $E_{\ell,k}=E^+_{\ell,k}\cup E^-_{\ell,k}$,* *and we have*

$$\lim_{k\to\infty}\sup_{\nu\in\Omega}\int_{\widetilde{K}\setminus E_{\ell,k}} \mid \nu(x)\mid \omega(x)d\lambda(x)=0$$

*and two sequences*

$$\varphi_{\ell,n_k}=\omega*\delta^{n_k}_{a_\ell^{-1}}\cdot\left(\Pi_{j=1}^{n_k}u_\ell*\delta^{j}_{a_\ell^{-1}}\right)\quad \textit{and}\quad \widetilde{\varphi}_{\ell,n_k}=\omega*\delta^{n_k}_{a_\ell}\cdot\left(\Pi_{j=0}^{n_k-1}u_\ell*\delta^{j}_{a_\ell}\right)^{-1}$$

*satisfy*

$$\lim_{k\to\infty}\sup_{\nu\in\Omega}\int_{E_{\ell,k}}\varphi_{\ell,n_k}(x)\mid\nu(x*a_\ell^{n_k})\mid d\lambda(x)=0,$$

$$\lim_{k\to\infty}\sup_{\nu\in\Omega}\int_{E_{\ell,k}}\widetilde{\varphi}_{\ell,n_k}(x)\mid\nu(x*a_\ell^{-n_k})\mid d\lambda(x)=0,$$

$$\lim_{k\to\infty}\sup_{\nu\in\Omega}\int_{E^+_{\ell,k}}\varphi_{\ell,2n_k}(x)\mid\nu(x*a_\ell^{2n_k})\mid d\lambda(x)=0$$

*and*

$$\lim_{k\to\infty}\sup_{\nu\in\Omega}\int_{E^-_{\ell,k}}\widetilde{\varphi}_{\ell,2n_k}(x)\mid\nu(x*a_\ell^{-2n_k})\mid d\lambda(x)=0,$$

*where* $\Omega$ *is the set of all Borel functions* $\nu$ *on* $K$ *satisfying* $\int_K\Phi(\mid\nu\mid)d\lambda\le 1$.

**Proof.** (I)⇒(II). Let $\widetilde{K}$ be a compact subset of $K$ such that $\lambda(\widetilde{K})>0$. Since $(\mathcal{C}_{1,n}\oplus\mathcal{C}_{2,n}\oplus\cdots\oplus\mathcal{C}_{L,n})_{n\in\mathbb{N}_0}$ is topologically transitive, for $\epsilon\in(0,1)$ there exist $f_\ell\in\mathcal{L}^\Phi_\omega(K,\mathbb{C})$ and $m\in\mathbb{N}$ such that for $1\le\ell\le L$, we have $\| f_\ell-\chi_{\widetilde{K}}\|_{\Phi,\omega}<\epsilon^2$ and $\|\mathcal{C}_{\ell,m}f_\ell+\chi_{\widetilde{K}}\|_{\Phi,\omega}<\epsilon^2$. Further, it relies on the proof of part (I)⇒(II) of Theorem 3.1 to get desired conditions on weights $u_\ell$ for each $\ell$.

(II)⇒(I). Let $U_\ell$ and $V_\ell$ be non-empty open subsets of $\mathcal{L}^\Phi_\omega(K,\mathbb{C})$. Since $\Phi$ is $\Delta_2$-regular, we can choose two non-zero functions $f_\ell$ and $h_\ell$ in $C_c(K)$ such that $f_\ell\in U_\ell$ and $h\in V_\ell$. Let $\widetilde{K}=\operatorname{supp}(f_\ell)\cup\operatorname{supp}(h_\ell)$ and $E_{\ell,k}\subset\widetilde{K}$ satisfy condition (II). From now on, following the proof of (II) ⇒ (I) of Theorem 3.1 we can conclude that $\mathcal{C}_{\ell,n_k}(U_\ell)\cap V_\ell\neq\emptyset$ for each $\ell(1\le\ell\le L)$. □

An operator $T$ is said to be topologically multiply recurrent on $X$, if for all $L\in\mathbb{N}$ and all nonempty open subset $U$ of $X$, there exists $n\in\mathbb{N}$ such that

$$U\cap T^{-n}(U)\cap T^{-2n}(U)\cap\cdots\cap T^{-Ln}(U)\neq\emptyset.$$

A vector $x\in X$ is called a topologically multiply recurrent vector for $T$ if there exist an increasing sequence $(n_k)$ of positive integers and $1\le\ell\le L$ for $L\in\mathbb{N}$ such that $T^{\ell n_k}x\to x$ as $k\to\infty$. The set of all topologically multiply recurrent vectors for $T$ is denoted by $\mathcal{RM}(T)$, and we know that $T$ is topologically multiply recurrent if and only if $\mathcal{RM}(T)$ is dense in $X$. Note that when $L=1$, the topologically multiple recurrence reduces to the topological recurrence, which is weaker than (closed to) the topological transitivity. In fact, every topologically transitive operator is topologically multiply recurrent on separable Banach spaces [18]. Besides, a sequence of operators $(T_n)$ is said to be topologically multiply recurrent on $X$, if for all $L \in \mathbb{N}$ and all nonempty open subset $U$ of $X$, there exists $n \in \mathbb{N}$ such that

$$U \cap T_n^{-1}(U) \cap T_{2n}^{-1}(U) \cap \cdots \cap T_{Ln}^{-1}(U) \neq \emptyset.$$

From now on, we consider the topologically multiple recurrence of a sequence of cosine operators in weighted Orlicz spaces on hypergroup.

**Theorem 3.5.** *Let $K$ be a second countable locally compact hypergroup with a right Haar measure $\lambda$ and $\Phi$ be a $\Delta_2$-regular Young function. Assume that $a \in K$ is an aperiodic element of $Z(K)$ and $\mathcal{C}_n := \frac{1}{2}(T_{a,u}^n + S_{a,u}^n)$ is a cosine operator on $\mathcal{L}_\omega^\Phi(K, \mathbb{C})$ generated by a weight function $u$ on $K$. Then* (I)$\Rightarrow$(II) *is true.*

(I) $(\mathcal{C}_n)_{n\in\mathbb{N}_0}$ *is topologically multiply recurrent on $\mathcal{L}_\omega^\Phi(K, \mathbb{C})$, and*

$$\lim_{n\to\infty} S_{a,u}^n f(x) = 0 \quad \text{or} \quad \lim_{n\to\infty} T_{a,u}^n f(x) = 0;$$

(II) *For each compact subset $\widetilde{K} \subset K$ with $\lambda(\widetilde{K}) > 0$ and $1 \leq \ell \leq L$ for $L \in \mathbb{N}$, there exist a sequence of Borel sets $(E_k)$ in $\widetilde{K}$, and a sequence $(n_k)$ of positive numbers such that $\lambda(\widetilde{K}) = \lim_{k\to\infty} \lambda(E_k)$ and the sequence*

$$\varphi_{\ell n} = \omega * \delta_{a^{-1}}^{\ell n} \cdot \left(\Pi_{j=1}^{\ell n} u * \delta_{a^{-1}}^{j}\right) \quad \text{or} \quad \widetilde{\varphi}_{\ell n} = \omega * \delta_a^{\ell n} \cdot \left(\Pi_{j=0}^{\ell n-1} u * \delta_a^{j}\right)^{-1}$$

*admits respectively subsequence $(\varphi_{\ell n_k})$ or $(\widetilde{\varphi}_{\ell n_k})$ satisfying*

$$\lim_{k\to\infty} \| \varphi_{\ell n_k} |_{E_k} \|_\infty = \lim_{k\to\infty} \| \varphi_{(\ell+L)n_k} |_{E_k} \|_\infty$$

*or*

$$\lim_{k\to\infty} \| \widetilde{\varphi}_{\ell n_k} |_{E_k} \|_\infty = \lim_{k\to\infty} \| \widetilde{\varphi}_{(\ell+L)n_k} |_{E_k} \|_\infty = 0.$$

**Proof.** Now we assume that $(\mathcal{C}_n)_{n\in\mathbb{N}_0}$ is topologically multiply recurrent. Let $\widetilde{K} \subseteq K$ be a compact set with $\lambda(\widetilde{K}) > 0$ so that

$$c := \inf_{x\in\widetilde{K}} \omega(x) > 0 \quad \text{and} \quad C := \sup_{x\in\widetilde{K}} \omega(x) < \infty$$

and $0 < \epsilon \ll c$. By the aperiodicity of $a \in Z(K)$, there exists some $N \in \mathbb{N}$ such that $\widetilde{K} \cap \widetilde{K} a^{\pm n} = \emptyset$ for $n > N$. Let $\chi_{\widetilde{K}} \in \mathcal{L}_\omega^\Phi(K, \mathbb{C})$ be the characteristic function of $\widetilde{K}$ and $U = \{g \in \mathcal{L}_\omega^\Phi(K, \mathbb{C}) : N_\Phi[(g - \chi_{\widetilde{K}})\omega] < \epsilon^2\}$. By the assumption of topologically multiple recurrence and the continuity of $(\mathcal{C}_n)_{n\in\mathbb{N}_0}$, for some fixed $L \in \mathbb{N}$ there exists $k > N$ such that

$$U \cap \mathcal{C}_{n_k}^{-1} U \cap \mathcal{C}_{2n_k}^{-1} U \cap \cdots \cap \mathcal{C}_{Ln_k}^{-1} U \neq \emptyset.$$

Hence, there exists $f \in \mathcal{L}_\omega^\Phi(K, \mathbb{C})$ such that $N_\Phi[(f - \chi_{\widetilde{K}})\omega] < \epsilon^2$ and $N_\Phi[(\mathcal{C}_{\ell n_k} f - \chi_{\widetilde{K}})\omega] < \epsilon^2$ for $\ell = 1, 2, \cdots, L$. By the same argument as in Theorem 3.1, let $f$ be a real valued function. Denote $A = \{x \in \widetilde{K} : | f(x) - 1 | \omega(x) \geq \epsilon\}$. Then

$$f(x)\omega(x) > \omega(x) - \epsilon \quad (x \in \widetilde{K} \setminus A), \quad \text{and} \quad \lambda(A) < 1/\Phi(\frac{1}{\epsilon})$$

by

$$\begin{aligned}\epsilon^2 &> N_\Phi[(f-\chi_{\widetilde{K}})\omega] \geq N_\Phi[\chi_{\widetilde{K}}(f-1)\omega] \geq N_\Phi[\chi_A(f-1)\omega] \\ &\geq N_\Phi(\chi_A\epsilon) = \frac{\epsilon}{\Phi^{-1}(1/\lambda(A))},\end{aligned}$$

which implies $\lambda(A) < 1/\Phi(\frac{1}{\epsilon})$. Let

$$B_{\ell,k} = \{x \in \widetilde{K} :| \mathcal{C}_{\ell n_k} f(x) - 1 \mid \omega(x) \geq \epsilon\}.$$

Then

$$\mathcal{C}_{\ell n_k} f(x)\omega(x) > \omega(x) - \epsilon \quad (x \in \widetilde{K} \setminus B_{\ell,k}) \text{ and } \lambda(B_{\ell,k}) < 1/\Phi(\frac{1}{\epsilon})$$

by the following procedure

$$\begin{aligned}\epsilon^2 &> N_\Phi[(\mathcal{C}_{\ell n_k} f - \chi_{\widetilde{K}})\omega] \geq N_\Phi\left[\chi_{\widetilde{K}}(\mathcal{C}_{\ell n_k} f - 1)\omega\right] \\ &\geq N_\Phi\left[\chi_{B_{\ell,k}}(\mathcal{C}_{\ell n_k} f - 1)\omega\right] \geq N_\Phi(\chi_{B_{\ell,k}}\epsilon) = \frac{\epsilon}{\Phi^{-1}\left(\frac{1}{\lambda(B_{\ell,k})}\right)}.\end{aligned}$$

Let $C_{\ell,k} = \{x \in \widetilde{K} \setminus (A \cup B_{\ell,k}) : \varphi_{\ell n_k}(x) \geq \epsilon\}$. Now we remark that

$$\varphi_{\ell n_k}(x) < \epsilon \quad (x \in \widetilde{K} \setminus (A \cup B_{\ell,k} \cup C_{\ell,k})),$$

and $\lambda(C_{\ell,k}) < 1/\Phi(\frac{c/2-\epsilon}{2C\epsilon})$. In fact, according to Lemma 2.1, $\widetilde{K} \cap \widetilde{K} a^{\pm \ell n_k} = \emptyset$ and the right invariance of Haar measure $\lambda$, due to the fact $f(x) > (c-\epsilon)/C > 0$ for $x \in \widetilde{K} \setminus A$ and $\lim_{n\to\infty} S^n_{a,u} f(x) = 0$ we take $n$ large enough so that $| S^n_{a,u} f(x) |< c\epsilon/2C$ and

$$\begin{aligned}2\epsilon^2 &> N_\Phi[2(\mathcal{C}_{\ell n_k} f - \chi_{\widetilde{K}})\omega] \geq N_\Phi\left(2\chi_{C_{\ell,k} * a^{\ell n_k}}(\mathcal{C}_{\ell n_k} f)\omega\right) \\ &\geq N_\Phi\left(\chi_{C_{\ell,k} * a^{\ell n_k}}\left(\Pi_{j=0}^{\ell n_k - 1} u * \delta_a^j\right)(f * \delta_a^{\ell n_k})\omega - \epsilon\chi_{C_{\ell,k} * a^{\ell n_k}}\omega/2\right) \\ &= N_\Phi\left(\chi_{C_{\ell,k}}\left(\Pi_{j=1}^{\ell n_k} u * \delta_{a^{-1}}^j\right) f\omega * \delta_{a^{-1}}^{\ell n_k} - \epsilon\chi_{C_{\ell,k}}\omega * \delta_{a^{-1}}^{\ell n_k}/2\right) \\ &= N_\Phi(\chi_{C_{\ell,k}}\varphi_{\ell n_k} f - \epsilon\chi_{C_{\ell,k}}\omega * \delta_{a^{-1}}^{\ell n_k}/2) > \frac{\epsilon(c/2-\epsilon)/C}{\Phi^{-1}\left(\frac{1}{\lambda(C_{\ell,k})}\right)},\end{aligned}$$

which implies $\lambda(C_{\ell,k}) < 1/\Phi(\frac{c/2-\epsilon}{2C\epsilon})$.

Similar to $C_{\ell,k}$, let $\widehat{C}_{\ell,k} = \{x \in \widetilde{K} \setminus (A \cup B_{\ell,k} \cup C_{\ell,k}) : \varphi_{(\ell+L)n_k}(x) \geq \epsilon\}$. Similarly, we imply that

$$\varphi_{(\ell+L)n_k}(x) < \epsilon \quad (x \in \widetilde{K} \setminus (A \cup B_{\ell,k} \cup C_{\ell,k} \cup \widehat{C}_{\ell,k})$$

and $\lambda(\widehat{C}_{\ell,k}) < 1/\Phi(\frac{c/2-\epsilon}{2C\epsilon})$.

Let $E_k = (\widetilde{K} \setminus A) \setminus \cup_{\ell=1}^{L}(B_{\ell,k} \cup C_{\ell,k} \cup \widehat{C}_{\ell,k})$. Then, we conclude $\lambda(\widetilde{K} \setminus E_k) < \frac{1+L}{\Phi(\frac{1}{\epsilon})} + \frac{2L}{\Phi(\frac{c/2-\epsilon}{2C\epsilon})}$ and $\| \varphi_{(\ell+L)n_k} |_{E_k}\|_\infty < \epsilon$ as well as $\| \varphi_{\ell n_k} |_{E_k}\|_\infty < \epsilon$, which implies the former of the condition (II) together with the fact $\lim_{t\to\infty} \Phi(t) =$

$\infty$. Similarly, if $\lim_{n\to\infty} S^n_{a,u} f(x) = 0$, then the latter of the condition (II) is obtained. □

**Theorem 3.6.** *Let $K$ be a second countable locally compact hypergroup with a right Haar measure $\lambda$ and $\Phi$ be a $\Delta_2$-regular Young function. Assume that $a \in K$ is an aperiodic element of $Z(K)$ and $\mathcal{C}_n := \frac{1}{2}(T^n_{a,u} + S^n_{a,u})$ is a cosine operator on $\mathcal{L}^\Phi_\omega(K, \mathbb{C})$ generated by a weight function $u$ on $K$. Then* (II)⇒(I) *holds.*

(I) $(\mathcal{C}_n)_{n\in\mathbb{N}_0}$ *is topologically multiply recurrent on $\mathcal{L}^\Phi_\omega(K, \mathbb{C})$;*

(II) *For each compact subset $\widetilde{K} \subset K$ with $\lambda(\widetilde{K}) > 0$ and $1 \le \ell \le L$ for $L \in \mathbb{N}$, there exist a sequence of Borel sets $(E_k)$ in $\widetilde{K}$, and a sequence $(n_k)$ of positive numbers such that $\lambda(\widetilde{K}) = \lim_{k\to\infty} \lambda(E_k)$ and two sequences*

$$\varphi_{\ell n} = \omega * \delta^{\ell n}_{a^{-1}} \cdot \left(\Pi^{\ell n}_{j=1} u * \delta^{j}_{a^{-1}}\right) \quad \textit{and} \quad \widetilde{\varphi}_{\ell n} = \omega * \delta^{\ell n}_{a} \cdot \left(\Pi^{\ell n-1}_{j=0} u * \delta^{j}_{a}\right)^{-1}$$

*admit respectively subsequences $(\varphi_{\ell n_k})$ and $(\widetilde{\varphi}_{\ell n_k})$ satisfying*

$$\begin{aligned}\lim_{k\to\infty} \| \varphi_{\ell n_k} |_{E_k} \|_\infty &= \lim_{k\to\infty} \| \widetilde{\varphi}_{\ell n_k} |_{E_k} \|_\infty \\ &= \lim_{k\to\infty} \| \varphi_{(\ell+L) n_k} |_{E_k} \|_\infty \\ &= \lim_{k\to\infty} \| \widetilde{\varphi}_{(\ell+L) n_k} |_{E_k} \|_\infty \\ &= 0.\end{aligned}$$

**Proof.** Now we show that $(\mathcal{C}_n)_{n\in\mathbb{N}_0}$ is topologically multiply recurrent. Let $U$ be a non-empty open subset of $\mathcal{L}^\Phi_\omega(K, \mathbb{C})$. Since the space $C_c(K)$ of continuous functions on $K$ with compact support is dense in $\mathcal{L}^\Phi_\omega(K, \mathbb{C})$, we can choose $f \in C_c(K)$ satisfying $f \in U$. Let $\widetilde{K}$ be the compact support of $f$. Given some $L \in \mathbb{N}$, let $E_k \subset \widetilde{K}$ and the sequences $(\varphi_{\ell n})$, $(\widetilde{\varphi}_{\ell n})$ satisfy the condition (II). By aperiodicity of $a \in Z(K)$, there exists $M \in \mathbb{N}$ such that $\widetilde{K} \cap \widetilde{K} a^{\pm n} = \emptyset$ for all $n > M$. By the condition (II), there exists $N \in \mathbb{N}$ such that $n_k > M$ and $\varphi_{\ell n_k} |_{E_k}, \widetilde{\varphi}_{\ell n_k} |_{E_k} < \frac{1/2^k}{\|f\|_\infty}$ for $k > N$. Hence,

$$N_\Phi[(f \chi_{\widetilde{K}\setminus E_k})\omega] \lesssim \| f \|_\infty \| \chi_{\widetilde{K}\setminus E_k} \|_{\Phi,\omega} \to 0$$

and by Lemma 2.1 we obtain

$$\begin{aligned}N_\Phi\left[\left(T^{\ell n_k}_{a,u}(f\chi_{E_k})\right)\omega\right] &= N_\Phi\left[\left(\Pi^{\ell n_k - 1}_{j=0} u * \delta^{j}_{a}\right)(f * \delta^{\ell n_k}_{a})(\chi_{E_k} * \delta_{a^{\ell n_k}})\omega\right] \\ &= N_\Phi\left[\left(\Pi^{\ell n_k}_{j=1} u * \delta^{j}_{a^{-1}}\right) f\chi_{E_k}(\omega * \delta^{\ell n_k}_{a^{-1}})\right] \\ &= N_\Phi\left(\varphi_{\ell n_k} f \chi_{E_k}\right) < \frac{\| f \|_\infty \frac{1/2^k}{\|f\|_\infty}}{\Phi^{-1}\left(\frac{1}{\lambda(E_k)}\right)} \to 0\end{aligned}$$

as $k \to \infty$ for $1 \le \ell \le L$. Similarly, we infer that

$$\lim_{k\to\infty} N_\Phi\left[S^{\ell n_k}_{a,w}(f\chi_{E_k})\omega\right] = \lim_{k\to\infty} N_\Phi\left[\left(\Pi^{\ell n_k}_{j=1} u * \delta^{j}_{a^{-1}}\right)^{-1} (f * \delta^{\ell n_k}_{a^{-1}})(\chi_{E_k} * \delta^{\ell n_k}_{a^{-1}})\omega\right]$$

$$\begin{aligned} &= \lim_{k\to\infty} N_\Phi \left[ \left( \Pi_{j=0}^{\ell n_k - 1} u * \delta_a^j \right)^{-1} f\chi_{E_k}(\omega * \delta_a^{\ell n_k}) \right] \\ &= \lim_{k\to\infty} N_\Phi \left( \widetilde{\varphi}_{\ell n_k} f\chi_{E_k} \right) = 0 \end{aligned}$$

for $l = 1, 2, \cdots, L$. Therefore

$$\lim_{k\to\infty} N_\Phi \left[ \mathcal{C}_{\ell n_k}(f\chi_{E_k})\omega \right] = 0$$

Similarly, for $1 \le \ell, m \le L$ we obtain

$$\begin{aligned} &\lim_{k\to\infty} N_\Phi \left[ \left( T_{a,u}^{(\ell+m)n_k}(f\chi_{E_k}) \right) \omega \right] = \lim_{k\to\infty} N_\Phi \left[ \left( S_{a,u}^{(\ell+m)n_k}(f\chi_{E_k}) \right) \omega \right] \\ = \; &\lim_{k\to\infty} N_\Phi \left[ \left( T_{a,u}^{\ell n_k} S_{a,u}^{mn_k}(f\chi_{E_k}) \right) \omega \right] = \lim_{k\to\infty} N_\Phi \left[ \left( S_{a,u}^{\ell n_k} T_{a,u}^{mn_k}(f\chi_{E_k}) \right) \omega \right] = 0. \end{aligned}$$

For each $k \in \mathbb{N}$, let

$$\nu_k = f\chi_{E_k} + \sum_{\ell=1}^{L} T_{a,u}^{\ell n_k}(f\chi_{E_k}) + \sum_{\ell=1}^{L} S_{a,u}^{\ell n_k}(f\chi_{E_k}).$$

Then

$$\begin{aligned} N_\Phi[(\nu_k - f)\omega] \;&\le\; N_\Phi[(f\chi_{\widetilde{K}\setminus E_k})\omega] + \sum_{\ell=1}^{L} N_\Phi \left[ \left( T_{a,w}^{\ell n_k}(f\chi_{E_k}) \right) \omega \right] \\ &+ \sum_{\ell=1}^{L} N_\Phi \left[ \left( S_{a,u}^{\ell n_k}(f\chi_{E_k}) \right) \omega \right] \end{aligned}$$

and

$$\begin{aligned} N_\Phi[(\mathcal{C}_{\ell n_k}\nu_k - f)\omega] \;&\le\; N_\Phi \left[ \left( \mathcal{C}_{\ell n_k}(f\chi_{E_k}) \right) \omega \right] + \frac{1}{2} \sum_{m=1}^{L} N_\Phi \left[ \left( T_{a,u}^{(\ell+m)n_k}(f\chi_{E_k}) \right) \omega \right] \\ &+ \frac{1}{2} \sum_{m=1}^{L} N_\Phi \left[ \left( S_{a,u}^{(\ell+m)n_k}(f\chi_{E_k}) \right) \omega \right] + N_\Phi \left[ (f\chi_{\widetilde{K}\setminus E_k})\omega \right] \\ &+ \frac{1}{2} \sum_{m=1, m\ne\ell}^{L} N_\Phi \left[ \left( T_{a,u}^{\ell n_k} S_{a,u}^{mn_k}(f\chi_{E_k}) \right) \omega \right] \\ &+ \frac{1}{2} \sum_{m=1, m\ne\ell}^{L} N_\Phi \left[ \left( S_{a,u}^{\ell n_k} T_{a,u}^{mn_k}(f\chi_{E_k}) \right) \omega \right], \end{aligned}$$

which implies

$$\lim_{k\to\infty} N_\Phi[(\nu_k - f)\omega] = \lim_{k\to\infty} N_\Phi[(\mathcal{C}_{\ell n_k}\nu_k - f)\omega] = 0.$$

Hence

$$U \cap \mathcal{C}_{n_k}^{-1} U \cap \mathcal{C}_{2n_k}^{-1} U \cap \cdots \cap \mathcal{C}_{Ln_k}^{-1} U \ne \emptyset.$$

□

Letting $L = 1$, immediately we can characterize the recurrence and topological mixing of a sequence of cosine operators as follows.

**Corollary 3.7.** *Let $K$ be a second countable locally compact hypergroup with a right Haar measure $\lambda$ and $\Phi$ be a $\Delta_2$-regular Young function. Assume that $a \in K$ is an aperiodic element of $Z(K)$ and $\mathcal{C}_n := \frac{1}{2}(T^n_{a,u} + S^n_{a,u})$ is a cosine operator on $\mathcal{L}^\Phi_\omega(K,\mathbb{C})$ generated by a weight function $u$ on $K$. Then* (I)$\Rightarrow$(II) *is true.*

(I) $(\mathcal{C}_n)_{n\in\mathbb{N}_0}$ *is topologically recurrent on $\mathcal{L}^\Phi_\omega(K,\mathbb{C})$, and $\lim_{n\to\infty} S^n_{a,u}f(x) = 0$ or $\lim_{n\to\infty} T^n_{a,u}f(x) = 0$;*

(II) *For each compact subset $\widetilde{K} \subset K$ with $\lambda(\widetilde{K}) > 0$, there exist a sequence of Borel sets $(E_k)$ in $\widetilde{K}$, and a sequence $(n_k)$ of positive numbers such that $\lambda(\widetilde{K}) = \lim_{k\to\infty} \lambda(E_k)$ and the sequence*

$$\varphi_n := \omega * \delta^n_{a^{-1}} \cdot \left(\Pi^n_{j=1} u * \delta^j_{a^{-1}}\right) \quad or \quad \widetilde{\varphi}_n := \omega * \delta^n_a \cdot \left(\Pi^{n-1}_{j=0} u * \delta^j_a\right)^{-1}$$

*admits respectively subsequence $(\varphi_{n_k})$ or $(\widetilde{\varphi}_{n_k})$ satisfying*

$$\lim_{k\to\infty} \| \varphi_{n_k} |_{E_k} \|_\infty = \lim_{k\to\infty} \| \varphi_{2n_k} |_{E_k} \|_\infty \ or \ \lim_{k\to\infty} \| \widetilde{\varphi}_{n_k} |_{E_k} \|_\infty = \lim_{k\to\infty} \| \widetilde{\varphi}_{2n_k} |_{E_k} \|_\infty = 0.$$

**Corollary 3.8.** *Let $K$ be a second countable locally compact hypergroup with a right Haar measure $\lambda$ and $\Phi$ be a $\Delta_2$-regular Young function. Assume that $a \in K$ is an aperiodic element of $Z(K)$ and $\mathcal{C}_n := \frac{1}{2}(T^n_{a,u} + S^n_{a,u})$ is a cosine operator on $\mathcal{L}^\Phi_\omega(K,\mathbb{C})$ generated by a weight function $u$ on $K$. Then* (II)$\Rightarrow$(I) *holds.*

(I) $(\mathcal{C}_n)_{n\in\mathbb{N}_0}$ *is recurrent on $\mathcal{L}^\Phi_\omega(K,\mathbb{C})$;*

(II) *For each compact subset $\widetilde{K} \subset K$ with $\lambda(\widetilde{K}) > 0$, there exist a sequence of Borel sets $(E_k)$ in $\widetilde{K}$, and a sequence $(n_k)$ of positive numbers such that $\lambda(\widetilde{K}) = \lim_{k\to\infty} \lambda(E_k)$ and two sequences*

$$\varphi_n := \omega * \delta^n_{a^{-1}} \cdot \left(\Pi^n_{j=1} u * \delta^j_{a^{-1}}\right) \quad and \quad \widetilde{\varphi}_n := \omega * \delta^n_a \cdot \left(\Pi^{n-1}_{j=0} u * \delta^j_a\right)^{-1}$$

*admit respectively subsequences $(\varphi_{n_k})$ and $(\widetilde{\varphi}_{n_k})$ satisfying*

$$\lim_{k\to\infty} \| \varphi_{n_k} |_{E_k} \|_\infty = \lim_{k\to\infty} \| \widetilde{\varphi}_{n_k} |_{E_k} \|_\infty = \lim_{k\to\infty} \| \varphi_{2n_k} |_{E_k} \|_\infty = \lim_{k\to\infty} \| \widetilde{\varphi}_{2n_k} |_{E_k} \|_\infty = 0.$$

**Corollary 3.9.** *Let $K$ be a second countable locally compact hypergroup with a right Haar measure $\lambda$ and $\Phi$ be a $\Delta_2$-regular Young function. Assume that $a \in K$ is an aperiodic element of $Z(K)$ and $\mathcal{C}_n := \frac{1}{2}(T^n_{a,u} + S^n_{a,u})$ is a cosine operator on $\mathcal{L}^\Phi_\omega(K,\mathbb{C})$ generated by a weight function $u$ on $K$. Then* (I)$\Rightarrow$(II) *is satisfied.*

(I) $(\mathcal{C}_n)_{n\in\mathbb{N}_0}$ *is topological mixing on $\mathcal{L}^\Phi_\omega(K,\mathbb{C})$, and $\lim_{n\to\infty} S^n_{a,u}f(x) = 0$ or $\lim_{n\to\infty} T^n_{a,u}f(x) = 0$;*

(II) *For each compact subset $\widetilde{K} \subset K$ with $\lambda(\widetilde{K}) > 0$, there exists a sequence of Borel sets $(E_n)$ in $\widetilde{K}$ such that $\lambda(\widetilde{K}) = \lim_{k\to\infty} \lambda(E_n)$ and the sequence*

$$\varphi_n := \omega * \delta^n_{a^{-1}} \cdot \left(\Pi^n_{j=1} u * \delta^j_{a^{-1}}\right) \quad or \quad \widetilde{\varphi}_n := \omega * \delta^n_a \cdot \left(\Pi^{n-1}_{j=0} u * \delta^j_a\right)^{-1}$$

*satisfies*

$$\lim_{n\to\infty} \| \varphi_n |_{E_n} \|_\infty = 0 \ or \ \lim_{n\to\infty} \| \widetilde{\varphi}_n |_{E_n} \|_\infty = 0.$$

**Corollary 3.10.** *Let $K$ be a second countable locally compact hypergroup with a right Haar measure $\lambda$ and $\Phi$ be a $\Delta_2$-regular Young function. Assume that $a \in K$ is an aperiodic element of $Z(K)$ and $\mathcal{C}_n := \frac{1}{2}(T^n_{a,u} + S^n_{a,u})$ is a cosine operator on $\mathcal{L}^{\Phi}_{\omega}(K, \mathbb{C})$ generated by a weight function $u$ on $K$. Then* (II)$\Rightarrow$(I) *holds.*

(I) $(\mathcal{C}_n)_{n\in\mathbb{N}_0}$ *is topological mixing on $\mathcal{L}^{\Phi}_{\omega}(K, \mathbb{C})$;*

(II) *For each compact subset $\widetilde{K} \subset K$ with $\lambda(\widetilde{K}) > 0$, there exists a sequence of Borel sets $(E_n)$ in $\widetilde{K}$ such that $\lambda(\widetilde{K}) = \lim_{k\to\infty} \lambda(E_n)$ and two sequences*

$$\varphi_n := \omega * \delta^n_{a^{-1}} \cdot \left(\Pi^n_{j=1} u * \delta^j_{a^{-1}}\right) \quad \text{and} \quad \widetilde{\varphi}_n := \omega * \delta^n_a \cdot \left(\Pi^{n-1}_{j=0} u * \delta^j_a\right)^{-1}$$

*satisfy*

$$\lim_{n\to\infty} \| \varphi_n |_{E_n} \|_\infty = \lim_{n\to\infty} \| \widetilde{\varphi}_n |_{E_n} \|_\infty = 0.$$

At last, we pay attention to the chaoticity for cosine operators on hypergroup and give some results.

**Theorem 3.11.** *Let $K$ be a second countable locally compact hypergroup with a right Haar measure $\lambda$ and $\Phi$ be a Young function. Assume that $a \in K$ is an aperiodic element of $Z(K)$, $\{\mathcal{C}_n\}_{n\in\mathbb{N}_0}$ is a cosine function on $\mathcal{L}^{\Phi}_{\omega}(K, \mathbb{C})$ and $\{\mathcal{P}(\mathcal{C}_n)\}_{n\in\mathbb{N}_0}$ is the set of periodic elements of $\{\mathcal{C}_n\}_{n\in\mathbb{N}_0}$ generated by a weight function $u$ on $K$. Then* (III)$\Rightarrow$(I)$\Rightarrow$(II) *holds true.*

(I) $(\mathcal{C}_n)_{n\in\mathbb{N}_0}$ *is chaotic on $\mathcal{L}^{\Phi}_{\omega}(K, \mathbb{C})$;*

(II) $(\mathcal{P}(\mathcal{C}_n))_{n\in\mathbb{N}_0}$ *is dense in $\mathcal{L}^{\Phi}_{\omega}(K, \mathbb{C})$;*

(III) *For each compact subset $\widetilde{K} \subseteq K$ with $\lambda(\widetilde{K}) > 0$, there exist a sequence of Borel sets $(E_k)$ in $\widetilde{K}$, and a sequence $(n_k)$ of positive numbers such that $\lambda(\widetilde{K}) = \lim_{k\to\infty} \lambda(E_k)$ and two sequences*

$$\varphi_n := \omega * \delta^n_{a^{-1}} \cdot \left(\Pi^n_{j=1} u * \delta^j_{a^{-1}}\right) \quad \text{and} \quad \widetilde{\varphi}_n := \omega * \delta^n_a \cdot \left(\Pi^{n-1}_{j=0} u * \delta^j_a\right)^{-1}$$

*admit respectively subsequences $(\varphi_{\ell n_k})$ and $(\widetilde{\varphi}_{\ell n_k})$ satisfying*

$$\lim_{k\to\infty} \| \left(\sum_{\ell=1}^{\infty} \varphi_{\ell n_k} + \sum_{\ell=1}^{\infty} \widetilde{\varphi}_{\ell n_k}\right) |_{E_k} \|_\infty = 0.$$

**Proof.** (III)$\Rightarrow$(I)$\Rightarrow$(II). Clearly, here we only prove (III)$\Rightarrow$(I). Based on Corollary 3.9, we see that the condition (III) follows the topological transitivity of $(\mathcal{C}_n)_{n\in\mathbb{N}_0}$. Then, we have to show the density of $(\mathcal{P}(\mathcal{C}_n))_{n\in\mathbb{N}_0}$. According to the density of $C_c(K)$ in $\mathcal{L}^{\Phi}_{\omega}(K, \mathbb{C})$, we take a non-zero function $f \in C_c(K)$ with compact support $\widetilde{K} \subseteq K$. Then, there exist a sequence $(E_k)$ of Borel sets in $\widetilde{K}$ and a sequence $(n_k)$ of positive numbers so that $\lambda(\widetilde{K}) = \lim_{k\to\infty} \lambda(E_k)$, $\widetilde{K}a^{pn_k} \cap \widetilde{K}a^{qn_k} \neq \infty$ for each $p, q \in \mathbb{Z}$ with $p \neq q$ and

$$\sum_{\ell=1}^{\infty} \varphi_{\ell n_k}(x) + \sum_{\ell=1}^{\infty} \widetilde{\varphi}_{\ell n_k}(x) < 1/2^k \quad (x \in E_k).$$

Moreover, we conclude that

$$N_\Phi[(f - f\chi_{\widetilde{K}})\omega] = N_\Phi[(f\chi_{\widetilde{K}\setminus E_k})\omega] \le \frac{C \parallel f \parallel_\infty}{\Phi^{-1}\left(1/\lambda(\widetilde{K} \setminus E_k)\right)} \to 0(k \to \infty).$$

For each $k \in \mathbb{N}$, put

$$v_k = f\chi_{E_k} + \sum_{\ell=1}^{\infty} T_{a,u}^{\ell n_k}(f\chi_{E_k}) + \sum_{\ell=1}^{\infty} S_{a,u}^{\ell n_k}(f\chi_{E_k}).$$

Then

$$\begin{aligned} T_{a,u}^{n_k} v_k &= T_{a,u}^{n_k}(f\chi_{E_k}) + \sum_{\ell=1}^{\infty} T_{a,u}^{n_k} T_{a,u}^{\ell n_k}(f\chi_{E_k}) + \sum_{\ell=1}^{\infty} T_{a,u}^{n_k} S_{a,u}^{\ell n_k}(f\chi_{E_k}) \\ &= f\chi_{E_k} + \sum_{\ell=1}^{\infty} T_{a,u}^{\ell n_k}(f\chi_{E_k}) + \sum_{\ell=1}^{\infty} S_{a,u}^{\ell n_k}(f\chi_{E_k}) = v_k \end{aligned}$$

and

$$\begin{aligned} S_{a,u}^{n_k} v_k &= S_{a,u}^{n_k}(f\chi_{E_k}) + \sum_{\ell=1}^{\infty} S_{a,u}^{n_k} T_{a,u}^{\ell n_k}(f\chi_{E_k}) + \sum_{\ell=1}^{\infty} S_{a,u}^{n_k} S_{a,u}^{\ell n_k}(f\chi_{E_k}) \\ &= f\chi_{E_k} + \sum_{\ell=1}^{\infty} T_{a,u}^{\ell n_k}(f\chi_{E_k}) + \sum_{\ell=1}^{\infty} S_{a,u}^{\ell n_k}(f\chi_{E_k}) = v_k, \end{aligned}$$

which imply $\mathcal{C}_{\ell n_k} v_k = v_k$, and so $v_k \in \mathcal{P}(\mathcal{C}_n)$ for every $n \in \mathbb{N}$. Besides, from Lemma 2.1, $\widetilde{K}a^{pn_k} \cap \widetilde{K}a^{qn_k} = \emptyset$ with $p \neq q$ and the right invariance of Haar measure $\lambda$, we infer that

$$\begin{aligned} & N_\Phi\left(\sum_{\ell=1}^{\infty} T_{a,u}^{\ell n_k}(f\chi_{E_k})\omega + \sum_{\ell=1}^{\infty} S_{a,u}^{\ell n_k}(f\chi_{E_k})\omega\right) \\ = \;& N_\Phi\left(\sum_{\ell=1}^{\infty}\left(\Pi_{j=0}^{\ell n_k - 1} u * \delta_a^j\right)(f * \delta_a^{\ell n_k})(\chi_{E_k} * \delta_{a^{\ell n_k}})\omega\right. \\ & \left. + \sum_{\ell=1}^{\infty}\left(\Pi_{j=1}^{\ell n_k} u * \delta_{a^{-1}}^j\right)^{-1}(f * \delta_{a^{-1}}^{\ell n_k})(\chi_{E_k} * \delta_{a^{-1}}^{\ell n_k})\omega\right) \\ = \;& N_\Phi\left(\sum_{\ell=1}^{\infty}\left(\Pi_{j=1}^{\ell n_k} u * \delta_{a^{-1}}^j\right) f\chi_{E_k}(\omega * \delta_{a^{-1}}^{\ell n_k})\right. \\ & \left. + \sum_{\ell=1}^{\infty}\left(\Pi_{j=0}^{\ell n_k - 1} u * \delta_a^j\right)^{-1} f\chi_{E_k}(\omega * \delta_a^{\ell n_k})\right) \\ = \;& N_\Phi\left(\sum_{\ell=1}^{\infty} \varphi_{\ell n_k} f\chi_{E_k} + \sum_{\ell=1}^{\infty} \widetilde{\varphi}_{\ell n_k} f\chi_{E_k}\right) \end{aligned}$$

$$< \frac{\| f \|_\infty \frac{1}{2^k}}{\Phi^{-1}\left(\frac{1}{\lambda(E_k)}\right)} \to 0 \quad (k \to \infty),$$

which means

$$N_\Phi[(\nu_k - f)\omega] \le N_\Phi[(f\chi_{\widetilde{K}\setminus E_k})\omega] + N_\Phi\left(\sum_{\ell=1}^{\infty} T_{a,u}^{\ell n_k}(f\chi_{E_k})\omega + \sum_{\ell=1}^{\infty} S_{a,u}^{\ell n_k}(f\chi_{E_k})\omega\right)$$
$$\to 0 \quad (k \to \infty).$$

Consequently, $\nu_k$ converges to $f$ as $k \to \infty$ such that $(\mathcal{P}(\mathcal{C}_n))_{n\in\mathbb{N}_0}$ is dense in $\mathcal{L}_\omega^\Phi(K,\mathbb{C})$. □

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This paper is available via http://nyjm.albany.edu/j/2025/31-63.html.