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Invertibility of Bergman Toeplitz operators

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ABSTRACT. In this paper, we establish the invertibility of the Berezin transform of the symbol as a necessary and sufficient condition for the invertibility of the Toeplitz operator on the Bergman space $L^2_a(\mathbb{D})$. More precisely, if $\varphi = cg + d\bar{g}$, where $c, d \in \mathbb{C}$ and $g \in H^\infty(\mathbb{D})$, the space of all bounded analytic functions, then T_φ is invertible on $L^2_a(\mathbb{D})$ if and only if $\inf_{z\in\mathbb{D}} |\widetilde{\varphi}(z)| = \inf_{z\in\mathbb{D}} |\varphi(z)| > 0$, where $\widetilde{\varphi}$ is the Berezin transform of φ.

CONTENTS

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc and $\mathbb{T}$ be the unit circle in the complex plane $\mathbb{C}$. Let dA denote the normalized Lebesgue area measure on $\mathbb{D}$. The space $L^2(\mathbb{D}, dA)$ stands for the complex Hilbert space of square integrable functions on $\mathbb{D}$ with the inner product defined by

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}\, dA(z) \quad (f, g \in L^2(\mathbb{D}, dA)).$$

The *Bergman space*, denoted by $L^2_a(\mathbb{D})$, is the closed subspace of all analytic functions on $\mathbb{D}$ in $L^2(\mathbb{D}, dA)$. Moreover, the space $L^2_a(\mathbb{D})$ is a reproducing kernel Hilbert space (RKHS) with respect to the kernel function K_z given by

$$K_z(w) = K(w, z) = \frac{1}{(1 - \bar{z}w)^2} \quad (z, w \in \mathbb{D}).$$

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Let $H^2(\mathbb{D})$ be the Hardy space of analytic functions f on $\mathbb{D}$ such that

$$\|f\| = \sup_{0\le r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 \, dt \right)^{\frac{1}{2}} < \infty.$$

We denote by $L^\infty(\mathbb{D})$ (or $L^\infty(\mathbb{T})$) the Banach space of essentially bounded functions on $\mathbb{D}$ (or $\mathbb{T}$) equipped with the sup-norm $\|\cdot\|_\infty$. For $\varphi \in L^\infty(\mathbb{D})$, the Toeplitz operator T_φ on the *Bergman space* is defined by

$$T_\varphi f = P(\varphi f), \quad (f \in L_a^2(\mathbb{D})),$$

where P is the orthogonal projection of $L^2(\mathbb{D}, dA)$ onto $L_a^2(\mathbb{D})$. For $\varphi \in L^\infty(\mathbb{D})$, the Berezin transform of φ on $\mathbb{D}$, denoted by $\widetilde{\varphi}$, is defined as follows

$$\widetilde{\varphi}(z) = \langle T_\varphi k_z, k_z \rangle = \int_{\mathbb{D}} \varphi(w) \frac{(1-|z|^2)^2}{|1-\bar{z}w|^4} \, dA(w) \quad (z \in \mathbb{D}),$$

where k_z is the normalized reproducing kernel, i.e., $k_z = \dfrac{K_z}{\|K_z\|}$. On the other hand, for $\varphi \in L^\infty(\mathbb{T})$, the harmonic extension of the function φ in $\mathbb{D}$, denoted by $\breve{\varphi}$, is given by the Poisson integral formula:

$$\breve{\varphi}(z) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{it}) \frac{1-|z|^2}{|1-ze^{-it}|^2} \, dt \quad (z \in \mathbb{D}).$$

It is easy to see that the harmonic extension of a function $\varphi \in L^\infty(\mathbb{T})$ in $\mathbb{D}$ coincides with the Berezin transform of the Toeplitz operator T_φ with symbol φ on the Hardy space $H^2(\mathbb{D})$ over the unit disc. The main purpose of this note is to study the invertibility criteria for the certain Toeplitz operators on the Bergman space.

One of the key problems in operator theory and function theory is to determine the invertibilty of Toeplitz operators. In this direction, the complete characterization of invertible Toeplitz operators on $H^2(\mathbb{D})$ was studied by Devinatz [1]. Douglas [2] characterized the invertibility of Toeplitz operators on $H^2(\mathbb{D})$ with continuous symbols by showing that for $\varphi \in C(\mathbb{T})$, the space of continuous functions on $\mathbb{T}$, the Toeplitz operator T_φ is invertible on $H^2(\mathbb{D})$ if there exists $\delta > 0$ such that $|\breve{\varphi}(z)| \ge \delta$ for all $z \in \mathbb{D}$. Thereby, Douglas raised the following problem: Let $\varphi \in L^\infty(\mathbb{T})$. Suppose there exists a $\delta > 0$ such that $|\breve{\varphi}(z)| \ge \delta$ for all $z \in \mathbb{D}$. Is the Toeplitz operator T_φ invertible on $H^2(\mathbb{D})$?

Many researchers including Chang and Tolokonnikov [14] investigated the above problem and obtained a sufficient condition that there exists a positive $\delta_0\,(< 1)$ such that if $\delta_0 < |\breve{\varphi}(z)| < 1$ for all $z \in \mathbb{D}$, then T_φ is invertible on $H^2(\mathbb{D})$ (see also [10]). On the other hand, Wolff [15] constructed a function in $L^\infty(\mathbb{T})$ that serves as an elegant counterexample for a negative answer to Douglas's question. Thus, in the Bergman space setting the question reads:

Question 1.1. Is the invertibility of Bergman Toeplitz operator T_φ for $\varphi \in L^\infty(\mathbb{D})$ equivalent to the invertibility of the Berezin transform $\widetilde{\varphi}$ in $L^\infty(\mathbb{D})$?

The above question is incredibly challenging and even for a bounded harmonic function φ on $\mathbb{D}$, the problem is still unsolved. However, the affirmative answers are available in the literature in the case when $\varphi \in L^\infty(\mathbb{D})$ is either 'real harmonic', or 'non-negative', or 'analytic', or 'coanalytic'. For a detailed survey, see [8, 9, 17]. The invertibility and spectral properties of Toeplitz operators with harmonic symbols on the Bergman space have been discussed in detail by many authors, for example, see [3, 4, 13, 18]. In general, Karaev [6] established certain sufficient conditions for the invertibility of bounded linear operators on RKHS using the Berezin transform and atomic decomposition. Recently, Taskinen and Čučković [12] have studied the invertibility problem for Bergman Toeplitz operators under a geometric condition on the image of the symbols. The problem of invertibility for Bergman Toeplitz operators with harmonic symbols had also been explored by Yoneda [16]. He obtained the invertibility of Berezin transform of the harmonic symbol of the form $\varphi = cg + d\overline{g}$, where $c, d \in \mathbb{C}$ and $g \in H^\infty(\mathbb{D})$ as a sufficient condition for the invertibility of the Bergman Toeplitz operator T_φ. Moreover, for such a φ if $g \in H^\infty(\mathbb{D}) \cap C(\overline{\mathbb{D}})$, then he obtained an affirmative answer to the Question 1.1 (see [16, Theorem 2.6, Theorem 2.8]). For a thorough discussion and properties of Bergman Toeplitz operators, one can see [5, 7, 19].

In this paper we have obtained an affirmative answer to Question 1.1 for harmonic symbols of the form $cg + d\overline{g}$, where $g \in H^\infty(\mathbb{D})$ which relaxes the condition that g be continuous on $\overline{\mathbb{D}}$ unlike Yoneda's result. Along the way, we prove some characterization results in the general setting of Hilbert space. In turn, it precisely extends the results concerning the invertibility of T_φ in terms of the invertibility of the Berezin transform of its symbol when φ is either real harmonic, analytic, coanalytic, or T_φ is normal provided φ is bounded harmonic.

The rest of the paper is structured as follows. Section 2 focuses on establishing the necessary background, and along the way, we fix some notations and definitions. In Section 3, we prove the invertibility of the Berezin transform of certain harmonic symbols determines the invertibility of the Bergman Toeplitz operators which extends Yoneda's result.

2. Preliminaries

This section compiles notations, definitions, and some basic results that are used in this paper. We denote $\mathcal{H}$ as a Hilbert space over the field of complex numbers $\mathbb{C}$ and $\mathcal{B}(\mathcal{H})$ as the C^*-algebra of all bounded linear operators on $\mathcal{H}$. A linear operator T is said to be bounded below if there exists $c > 0$ such that $\|Th\| \geq c\|h\|$ for all $h \in \mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be hyponormal if $T^*T - TT^*$ is a positive operator, equivalently, $\|Th\| \geq \|T^*h\|$ for all $h \in \mathcal{H}$.

Let $H(\mathbb{D})$ be a reproducing kernel Hilbert space (e.g. $H^2(\mathbb{D})$ or $L_a^2(\mathbb{D})$) on $\mathbb{D}$ and the map $K : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ defined by

$$K(w, z) = K_z(w) = \langle K_z, K_w \rangle$$

be the reproducing kernel function of $H(\mathbb{D})$. We denote $k_z = \dfrac{K_z}{\|K_z\|}$ for $z \in \mathbb{D}$ as the normalized reproducing kernel of $H(\mathbb{D})$ and the set $\{k_z : z \in \mathbb{D}\}$ is a total set in $H(\mathbb{D})$. For more details and references on RKHS, see [11]. For a bounded operator T on $H(\mathbb{D})$, the *Berezin transform* of T, denoted by $\widetilde{T}$, is a complex-valued function on $\mathbb{D}$ defined as

$$\widetilde{T}(z) = \langle Tk_z, k_z \rangle \quad \text{for all } z \in \mathbb{D}.$$

Thus, every bounded linear operator T on $H(\mathbb{D})$ induces a bounded function $\widetilde{T}$ on $\mathbb{D}$. Indeed, $\|\widetilde{T}\|_\infty \leq \|T\|$. Therefore, the *Berezin transform* of an operator yields crucial information about the operator. In this note we mainly focus on the *Berezin transform* of Toeplitz operators on the Bergman space to determine the invertibility criterion of the Toeplitz operators. Throughout this article, $H^\infty(\mathbb{D})$ stands for the algebra of bounded analytic functions on $\mathbb{D}$ and it is also a multiplier algebra for the Hardy space as well as the Bergman space, and $C(\mathbb{T})$ is the space of all continuous functions on $\mathbb{T}$.

We will be using the following known results very often.

Theorem 2.1 (cf. [9]). *For $\varphi \in H^\infty(\mathbb{D})$, let $T_\varphi : L^2_a(\mathbb{D}) \to L^2_a(\mathbb{D})$ be the Toeplitz operator. Then the following are equivalent:*

(1) *T_φ is invertible.*
(2) *$T_{\overline{\varphi}}$ is invertible.*
(3) $\inf_{z\in\mathbb{D}} |\varphi(z)| > 0$.

Theorem 2.2 (cf. [17]). *Let $\varphi \in L^\infty(\mathbb{D})$ be a bounded harmonic function on $\mathbb{D}$ such that T_φ is a normal operator on $L^2_a(\mathbb{D})$. Then T_φ is invertible if and only if $\inf_{z\in\mathbb{D}} |\widetilde{\varphi}(z)| = \inf_{z\in\mathbb{D}} |\varphi(z)| > 0$.*

3. Toeplitz operator on the Bergman space

In this section we discuss the invertibility of Bergman Toeplitz operators with the help of the Berezin transform of the symbol. More precisely, Theorem 3.1 gives an affirmative answer to the Problem 2.3 stated in [16] in a more general setting. Consequently, we obtain a refined characterization of invertible Bergman Toeplitz operators for some certain bounded harmonic functions.

Theorem 3.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be a hyponormal operator and $s \in \mathbb{C}$ with $|s| < 1$. Then the following two are equivalent:*

(1) *T^* is bounded below.*
(2) *$sT + T^*$ is bounded below.*

Proof. Suppose T^* is bounded below on $\mathcal{H}$. Since T is hyponormal on $\mathcal{H}$, $\|Th\| \geq \|T^*h\|$ for all $h \in \mathcal{H}$, and hence T is also bounded below. It follows that T is invertible. If $f \in \mathcal{H}$ and $f \neq 0$, then we choose $0 \neq h \in \mathcal{H}$ such that $f = Th$. For $s \in \mathbb{C}$ with $|s| < 1$, consider

$$\|(sT + T^*)f\| = \|(sT + T^*)Th\|$$

$$
\begin{aligned}
&= \sup\{|\langle (sT + T^*)Th, h'\rangle| : h' \in \mathcal{H}, \|h'\| = 1\} \\
&\geq \left|\left\langle (sT + T^*)Th, \frac{h}{\|h\|}\right\rangle\right| \\
&= \frac{1}{\|h\|}|\langle sT^2h + T^*Th, h\rangle| \\
&= \frac{1}{\|h\|}|s\langle Th, T^*h\rangle + \|Th\|^2| \\
&\geq \frac{1}{\|h\|}\left(\|Th\|^2 - |s||\langle Th, T^*h\rangle|\right) \\
&\geq \frac{1}{\|h\|}(\|Th\|^2 - |s|\|Th\|\|T^*h\|) \\
&\qquad\qquad\qquad\qquad \text{(by the Cauchy-Schwarz inequality)} \\
&\geq \frac{1}{\|h\|}(\|Th\|^2 - |s|\|Th\|^2) \qquad \text{(by the hyponormality of } T\text{)} \\
&= \frac{(1 - |s|)}{\|h\|}\|Th\|^2.
\end{aligned}
$$

Now $h = T^{-1}f$. Thus, $0 < \|h\| \leq \|T^{-1}\|\|f\|$. Therefore

$$
\|(sT + T^*)f\| \geq \frac{(1 - |s|)}{\|T^{-1}\|\|f\|}\|f\|^2 = \frac{(1 - |s|)}{\|T^{-1}\|}\|f\| \qquad (f \in \mathcal{H}).
$$

Hence $sT + T^*$ is bounded below.

Conversely, assume that $sT+T^*$ is a bounded below operator on $\mathcal{H}$. Suppose, if possible, that T^* is not bounded below. Then there exists a sequence $(f_n)_{n\geq 1}$ in $\mathcal{H}$ with $\|f_n\| = 1$ for every $n \geq 1$ and $\|T^*f_n\| \to 0$ as $n \to \infty$. Recall that an operator A on $\mathcal{H}$ is bounded below if and only if A^* is surjective. Therefore, for every $n \geq 1$, there exists $h_n \in \mathcal{H}$ such that

$$
(\bar{s}T^* + T)h_n = f_n.
$$

Also,

$$
\begin{aligned}
\|T^*f_n\| &= \|T^*(\bar{s}T^* + T)h_n\| \\
&= \|\bar{s}T^{*2}h_n + T^*Th_n\| \\
&\geq \left|\left\langle \bar{s}T^{*2}h_n + T^*Th_n, \frac{h_n}{\|h_n\|}\right\rangle\right| \\
&= \frac{1}{\|h_n\|}\left|\bar{s}\langle T^*h_n, Th_n\rangle + \|Th_n\|^2\right| \\
&\geq \frac{1}{\|h_n\|}(\|Th_n\|^2 - |s|\|Th_n\|\|T^*h_n\|) \\
&\qquad\qquad\qquad\qquad \text{(by the Cauchy-Schwarz inequality)} \\
&\geq \frac{1}{\|h_n\|}(\|Th_n\|^2 - |s|\|Th_n\|^2) \qquad \text{(by the hyponormality of } T\text{)}
\end{aligned}
$$

$$= \frac{(1-|s|)}{\|h_n\|}\|Th_n\|^2$$
$$= (1-|s|)\|h_n\| \left\| T\left(\frac{h_n}{\|h_n\|}\right)\right\|^2.$$

Now

$$1 = \|f_n\| = \|(\bar{s}T^* + T)h_n\| \leq \|\bar{s}T^* + T\|\|h_n\| \text{ for every } n \geq 1.$$

Thus,

$$\|T^* f_n\| \geq \frac{(1-|s|)}{\|\bar{s}T^* + T\|}\left\| T\left(\frac{h_n}{\|h_n\|}\right)\right\|^2 \geq \frac{(1-|s|)}{\|\bar{s}T^* + T\|}\left\| T^*\left(\frac{h_n}{\|h_n\|}\right)\right\|^2 \qquad (n \geq 1).$$

Thus it follows that

$$\left\| T\left(\frac{h_n}{\|h_n\|}\right)\right\| \to 0 \text{ and } \left\| T^*\left(\frac{h_n}{\|h_n\|}\right)\right\| \to 0 \text{ as } n \to \infty.$$

Since $sT + T^*$ is bounded below, we can choose $M > 0$ such that $M\|h_n\| \leq \|(sT + T^*)h_n\|$ for every $n \geq 1$. Therefore

$$0 < M \leq \left\| sT\left(\frac{h_n}{\|h_n\|}\right) + T^*\left(\frac{h_n}{\|h_n\|}\right)\right\| \to 0 \text{ as } n \to \infty,$$

which leads to a contradiction that $sT + T^*$ is bounded below. This establishes (2) $\implies$ (1).

□

Theorem 3.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be a hyponormal operator and $s \in \mathbb{C}$ with $|s| > 1$. The following two are equivalent:*

(1) *T is bounded below.*
(2) *$sT + T^*$ is bounded below.*

Proof. For $h \in \mathcal{H}$,

$$\begin{aligned}
(|s|+1)\|Th\| &= |s|\|Th\| + \|Th\| \\
&\geq |s|\|Th\| + \|T^*h\| \qquad \text{(by the hyponormality of } T) \\
&\geq \|sTh + T^*h\| = \|(sT + T^*)h\| \\
&\geq |s|\|Th\| - \|T^*h\| \\
&\geq (|s| - 1)\|Th\|.
\end{aligned}$$

Now $|s|-1 > 0$ as $|s| > 1$. Therefore, the above chain of inequalities establishes that (1) $\iff$ (2). □

Example 3.3. Consider the shift operator T_z on the Hardy space $H^2(\mathbb{D})$. It is an analytic Toeplitz operator so is hyponormal. Also, T_z is bounded below, in fact, $\|T_z h\| = \|h\|$ for all $h \in H^2(\mathbb{D})$. If $|s| > 1$ then it follows from Theorem 3.2 that $sT_z + T_z^*$ is also bounded below. Note that T_z is not surjective and hence

T_z^* is not bounded below. This shows that T^* may not be bounded below while $sT + T^*$ is bounded below for $|s| > 1$.

From the above theorems, we have the following result.

Proposition 3.4. *Let $T \in \mathcal{B}(\mathcal{H})$ be hyponormal on $\mathcal{H}$ and $s \in \mathbb{C}$ with $|s| \neq 1$. Then the following are equivalent:*

(1) *T is invertible on $\mathcal{H}$.*
(2) *$sT + T^*$ is invertible on $\mathcal{H}$.*

Proof. It suffices to show that both T and T^* are bounded below if and only if both $sT + T^*$ and $\bar{s}T^* + T$ are bounded below. First we note that if $s = 0$, then it is trivial as T is invertible if and only if T^* is invertible. Now assume that $s \neq 0$.

Case (i): Suppose $|s| > 1$.

Let both T and T^* be bounded below. It follows from Theorem 3.2 that $sT+T^*$ is bounded below. Moreover,

$$(sT + T^*)^* = \bar{s}T^* + T = \bar{s}\left(\frac{1}{\bar{s}}T + T^*\right) \text{ with } \left|\frac{1}{\bar{s}}\right| = \frac{1}{|s|} < 1.$$

Thus, Theorem 3.1 implies that $\bar{s}T^* + T$ is bounded below as well.

On the other hand, if both $sT + T^*$ and $\bar{s}T^* + T$ are bounded below, then T is bounded below follows from Theorem 3.2. However, the relation $\frac{1}{\bar{s}}T + T^* = \frac{1}{\bar{s}}(\bar{s}T^* + T)$ along with Theorem 3.1 yields T^* to be bounded below too.

Case (ii): Suppose $0 < |s| < 1$.

Using Theorems 3.1 and 3.2 and the relation

$$\bar{s}T^* + T = \bar{s}\left(\frac{1}{\bar{s}}T + T^*\right) \text{ with } \left|\frac{1}{\bar{s}}\right| = \frac{1}{|s|} > 1,$$

yields that both T and T^* are bounded below if and only if both $sT + T^*$ and $\bar{s}T^* + T$ are bounded below. □

We are now in a position to present the main result of this section.

Theorem 3.5. *Let $\varphi = cg + d\bar{g}$, where $g \in H^\infty(\mathbb{D})$ and the constants $c, d \in \mathbb{C}$, then the following are equivalent:*

(1) *T_φ is invertible on $L_a^2(\mathbb{D})$.*
(2) *There exists $\delta > 0$ such that $|\varphi(z)| \geq \delta$ for all $z \in \mathbb{D}$, i.e., $\inf_{z\in\mathbb{D}} |\widetilde{\varphi}(z)| = \inf_{z\in\mathbb{D}} |\varphi(z)| > 0$.*

Proof. If either $c = 0$ or $d = 0$, then the equivalence of (1) and (2) follows from Theorem 2.1. Assume now that $c \neq 0$ and $d \neq 0$ and let $s = c/d$, then $T_\varphi = dT_{sg+\bar{g}}$.

Case (i): For $|s| = 1$.

Firstly, T_φ is normal on $L_a^2(\mathbb{D})$. Indeed,

$$T_\varphi^* T_\varphi = |d|^2(\bar{s}T_{\bar{g}} + T_g)(sT_g + T_{\bar{g}})$$

$$
\begin{aligned}
&= |d|^2(T_{\overline{g}}T_g + sT_gT_g + \bar{s}T_{\overline{g}}T_{\overline{g}} + T_gT_{\overline{g}}) \\
&= |d|^2(sT_g + T_{\overline{g}})(\bar{s}T_{\overline{g}} + T_g) \\
&= T_\varphi T_\varphi^*.
\end{aligned}
$$

Hence, Theorem 2.2 yields the equivalence of (1) and (2).

Case (ii): When $|s| \neq 1$.

Note that for $g \in H^\infty(\mathbb{D})$, T_g is hyponormal because for $f \in L_a^2(\mathbb{D})$,

$$\|T_g^* f\| = \|T_{\overline{g}} f\| = \|P(\overline{g}f)\| \leq \|\overline{g}f\| = \|gf\| = \|T_g f\|.$$

Also, for $|s| \neq 1$ we have

$$||s|-1||g(z)| = \left||sg(z)| - \left|\overline{g(z)}\right|\right| \leq \left|sg(z) + \overline{g(z)}\right| \leq (|s|+1)|g(z)|, \qquad (z \in \mathbb{D}).$$

Thus

$$||s|-1| \left(\inf_{z\in\mathbb{D}} |g(z)|\right) \leq \inf_{z\in\mathbb{D}} \left|sg(z) + \overline{g(z)}\right| \leq (|s|+1) \left(\inf_{z\in\mathbb{D}} |g(z)|\right). \tag{3.1}$$

Let T_φ be invertible on $L_a^2(\mathbb{D})$. Thus, $T_{sg+\overline{g}} = sT_g + T_g^*$ is invertible. Employing Proposition 3.4, we have T_g is invertible on $L_a^2(\mathbb{D})$. It now follows from Theorem 2.1 that $\inf_{z\in\mathbb{D}} |g(z)| > 0$. Hence the inequality (3.1) implies that

$$\inf_{z\in\mathbb{D}} |\varphi(z)| = |d| \inf_{z\in\mathbb{D}} \left|sg(z) + \overline{g(z)}\right| > 0.$$

Conversely, assume that $\inf_{z\in\mathbb{D}} |\varphi(z)| > 0$. Again, the inequality (3.1) and Theorem 2.1 infers that T_g is invertible on $L_a^2(\mathbb{D})$. Finally, Proposition 3.4 yields that T_φ is invertible on $L_a^2(\mathbb{D})$. This completes the proof. □

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