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Hypercontractive inequalities and Nikol'skiĭ-type inequalities on weighted Bergman spaces

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ABSTRACT. In this note, we obtain hypercontractive inequalities between different weighted Bergman spaces. In addition, we establish Nikol'skiĭ-type inequalities for weighted Bergman spaces with optimal constants.

CONTENTS

1. Introduction

Let $\mathbb{T}$ be the unit circle and we identify the circle $\mathbb{T}$ with the interval $[0, 2\pi)$. For $1 \leqslant p < \infty$, $L^p(\mathbb{T})$ is the Banach space of L^p-integrable functions on the unit circle. Recall the Fourier coefficients $\hat{f}(n), n \in \mathbb{Z}$ of $f \in L^p(\mathbb{T})$ are defined by

$$\hat{f}(n) = \int_0^{2\pi} f(e^{i\theta})e^{-in\theta}d\theta,$$

where $d\theta$ is the normalized Lebesgue measure on $[0, 2\pi)$. Then the Hardy space $H^p(\mathbb{T})$ is the closed subspace of $L^p(\mathbb{T})$:

$$H^p(\mathbb{T}) = \left\{f \in L^p(\mathbb{T}) \,\middle|\, \hat{f}(n) = 0, \text{for each negative integer } n\right\}.$$

Let $f \in L^2(\mathbb{T})$, then f has a Fourier expansion

$$f(e^{i\theta}) = \sum_{n=-\infty}^{+\infty} \hat{f}(n)e^{in\theta}.$$

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For $0 \leqslant r < 1$, define the dilation operator

$$(T_r f)(e^{i\theta}) := \sum_{n=-\infty}^{+\infty} \hat{f}(n) r^{|n|} e^{in\theta}.$$

Observe that $\{T_r\}_{0 \leqslant r < 1}$ is a contractive operator semigroup on $L^p(\mathbb{T})$. The hypercontractive inequalities for T_r are to determine the optimal range of r such that

$$T_r : L^p(\mathbb{T}) \to L^q(\mathbb{T})$$

is a contraction. In his celebrated work, Weissler [15] established the following sharp criterion:

$$\|T_r f\|_{L^q(\mathbb{T})} \leqslant \|f\|_{L^p(\mathbb{T})}$$

if and only if $r^2 \leqslant (p-1)/(q-1)$. Moreover,

$$\|T_r f\|_{H^q(\mathbb{T})} \leqslant \|f\|_{H^p(\mathbb{T})}$$

if and only if $r^2 \leqslant p/q$.

Let $\mathbb{D}$ be the unit disk and $dA = dxdy/\pi$ be the normalized area measure on $\mathbb{D}$. Suppose that $0 < p < \infty$ and $1 < \alpha < \infty$, the standard weighted Bergman space $A^p_\alpha(\mathbb{D})$ consists of analytic functions on the unit disk such that

$$||f||_{A^p_\alpha(\mathbb{D})} := \left(\int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) \right)^{\frac{1}{p}} < \infty,$$

where $dA_\alpha(z) = (\alpha - 1)(1 - |z|^2)^{\alpha - 2} dA(z)$. Recall each analytic function f on $\mathbb{D}$ has the following Taylor expansion at 0:

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n, \quad \forall z \in \mathbb{D}.$$

Then the dilation operator can be formally written as

$$T_r f(z) := \sum_{n=0}^{+\infty} a_n r^n z^n = f(rz), \quad \forall z \in \mathbb{D}.$$

In a recent work Melentijević [10], the author obtained an optimal hypercontractive constant $r_0 = \sqrt{p/q}$ such that

$$||T_r f||_{A^q_\alpha(\mathbb{D})} \leqslant ||f||_{A^p_\alpha(\mathbb{D})}, \quad \forall 0 < r \leqslant r_0$$

for $0 < p \leqslant q < \infty$, $1 < \alpha < \infty$ and $q \geqslant 2$. Melentijević's work is motivated by Janson's hypercontractive problem of dilation operators on spaces of analytic functions. For background and recent developments on Janson's problem, one can consult [7] and [10].

The first goal of this paper is to extend Melentijević's work to the case of two distinct weighted Bergman spaces $A^p_\alpha(\mathbb{D})$ and $A^q_\beta(\mathbb{D})$.

Theorem 1.1. *For $1 < \alpha, \beta < \infty$ and $0 < p \leqslant q < \infty$, let $q \geqslant 2$ and $\beta p \leqslant \alpha q$. Then we have*

$$||T_r f||_{A^q_\beta(\mathbb{D})} \leqslant ||f||_{A^p_\alpha(\mathbb{D})} \tag{1}$$

for any $f \in A^p_\alpha(\mathbb{D})$ if and only if $r^2 \leqslant \beta p/\alpha q$.

Remark 1.2. It is known that $\lim\limits_{\alpha\to 1} \|f\|_{A^p_\alpha} = \|f\|_{H^p}$ for $f \in H^p$. Hence, Theorem 1.1 is a generalization of Janson's strong hypercontractive inequalities between Hardy spaces in [6] for the case $0 < p \leqslant q < \infty$ and $q \geqslant 2$. The proofs of Theorem 1.1 are inspired by some ideas and methods from [7] and [10]. After our paper was uploaded to arXiv, Huang Yi and Zhang Jianyang informed us that they had also obtained similar results.

Remark 1.3. We further note that in the recent work by Bao-Ma-Yan-Zhu [1], the authors have formulated a highly general conjecture concerning contractive embeddings, grounded in a new notion termed "tight fitting".

Let $\mathbb{C}[z_1, \dots, z_n]$ be the space of all complex polynomials on $\mathbb{C}^n$ and write $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$. For a m-homogeneous polynomial $P(\mathbf{z}) \in \mathbb{C}[z_1, \dots, z_n]$, as a corollary of Weissler's hypercontractive work [15], we have

$$||P||_{H^q(\mathbb{T}^n)} \leqslant \left(\sqrt{\frac{q}{p}}\right)^m ||P||_{H^p(\mathbb{T}^n)},$$

where

$$||P||_{H^p(\mathbb{T}^n)} = \left(\int_{\mathbb{T}^n} |P(\mathbf{z})|^p dm(\mathbf{z})\right)^{\frac{1}{p}}$$

and $dm(\mathbf{z})$ is the Haar measure on $\mathbb{T}^n$. For a complex polynomial P in $\mathbb{C}[z_1, \dots, z_n]$, we write

$$P(\mathbf{z}) = \sum_{\gamma\in\mathbb{N}^n} c_\gamma \mathbf{z}^\gamma,$$

where $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{N}^n$ is a multi-index and $\mathbf{z}^\gamma = z_1^{\gamma_1} \cdots z_n^{\gamma_n}$. Let $|\gamma| = \sum_{j=1}^n \gamma_j$. Then the degree of the polynomial P is defined by

$$\deg(P) := \max\{|\gamma| : c_\gamma \neq 0\}.$$

Let m be the degree of the polynomial P. The famous Nikol'skiĭ-type inequality [11, 12] states that for any complex polynomial P in $\mathbb{C}[z_1, \dots, z_n]$ and $0 < q < p < \infty$,

$$\|P\|_{H^q(T^n)} \leqslant C\|P\|_{H^p(\mathbb{T}^n)},$$

where the constant $C = C(n, m, p, q)$.

By hypercontractive inequalities of the Poisson transform between Hardy spaces [15], Defant and Mastyło [4] obtained a dimension-free estimate of

Nikol'skiĭ-type inequality for Hardy spaces. In particular, for $0 < p < q < \infty$, the optimal constant in Nikol'skiĭ-type inequality for Hardy spaces is

$$C(m, p, q) = \left(\sqrt{\frac{q}{p}}\right)^m .$$

For $n \geqslant 1, 0 < p < \infty$ and $1 < \alpha < \infty$, recall the weighted Bergman space $A^p_\alpha(\mathbb{D}^n)$ on the polydisc $\mathbb{D}^n$ is

$$A^p_\alpha(\mathbb{D}^n) = \left\{ f \in H(\mathbb{D}^n) \,\middle|\, \|f\|^p_{A^p_\alpha(\mathbb{D}^n)} := \int_{\mathbb{D}^n} |f(z_1, \dots, z_n)|^p dA_\alpha(z_1) \cdots dA_\alpha(z_n) < \infty \right\},$$

where $H(\mathbb{D}^n)$ is the set of analytic functions on $\mathbb{D}^n$. Inspired by ideas from [4], we have the following dimension-free Nikol'skiĭ-type inequality for the weighted Bergman spaces with optimal constants.

Theorem 1.4. *For $1 < \alpha, \beta < \infty$ and $0 < p \leqslant q < \infty$, let $q \geqslant 2$ and $\beta p \leqslant \alpha q$. Then for any positive integer m, n and a complex polynomial $P \in \mathbb{C}[z_1, \dots, z_n]$ with degree m*

$$||P||_{A^q_\beta(\mathbb{D}^n)} \leqslant \left(\sqrt{\frac{\alpha q}{\beta p}}\right)^m ||P||_{A^p_\alpha(\mathbb{D}^n)}. \tag{2}$$

The constant

$$C(\alpha, \beta, p, q) = \sqrt{\frac{\alpha q}{\beta p}}$$

is the best possible such that for each positive integers n, m and $P \in \mathbb{C}[z_1, \dots, z_n]$ with degree m, the following inequality holds

$$||P||_{A^q_\beta(\mathbb{D}^n)} \leqslant C(\alpha, \beta, p, q)^m ||P||_{A^p_\alpha(\mathbb{D}^n)}.$$

2. The proof of Theorem 1.1

2.1. The Necessity. We start with a simple but useful lemma in [6] to prove the necessity for Theorem 1.1.

Lemma 2.1. *For $1 < \alpha < \infty, 0 < p < \infty$ and a real number ε, let $f(z) = 1 + \varepsilon z$. As the real number ε approaches 0, we have*

$$||f||_{A^p_\alpha(\mathbb{D})} = ||1 + \varepsilon z||_{A^p_\alpha(\mathbb{D})} = 1 + \frac{p}{4\alpha}\varepsilon^2 + O(\varepsilon^3).$$

Proof. Fix $z \in \mathbb{C}$, then

$$\begin{aligned} |1 + \varepsilon z|^p &= \left(1 + 2\varepsilon \mathrm{Re} z + \varepsilon^2 |z|^2\right)^{\frac{p}{2}} \\ &= 1 + \frac{p}{2}\left(2\varepsilon \mathrm{Re} z + \varepsilon^2 |z|^2\right) + \frac{1}{2} \cdot \frac{p}{2}\left(\frac{p}{2} - 1\right)\left(2\varepsilon \mathrm{Re} z + \varepsilon^2 |z|^2\right)^2 + O\left(\varepsilon^3\right). \end{aligned}$$

Integrating on the unit disk with respect to the measure dA_α, one gets

$$\begin{aligned}&\left(\int_{\mathbb{D}} |1+\varepsilon z|^p dA_\alpha(z)\right)^{\frac{1}{p}}\\ =&1+\frac{1}{2}\varepsilon^2\int_{\mathbb{D}}|z|^2dA_\alpha(z)+\frac{p-2}{8}\int_{\mathbb{D}}\left(2\varepsilon\mathrm{Re}z+\varepsilon^2|z|^2\right)^2dA_\alpha(z)+O(\varepsilon^3)\\ =&1+\frac{1}{2}\varepsilon^2\left(||z||^2_{A^2_\alpha(\mathbb{D})}+(p-2)||\mathrm{Re}z||^2_{A^2_\alpha(\mathbb{D})}\right)+O(\varepsilon^3)\\ =&1+\frac{p}{4\alpha}\varepsilon^2+O(\varepsilon^3).\end{aligned}$$

This completes the proof of Lemma 2.1. □

For any $f\in A^p_\alpha(\mathbb{D})$ and $r\in[0,1)$, recall that

$$T_rf(z)=f(rz),\quad z\in\mathbb{D}.$$

Then, by Lemma 2.1,

$$||1+\varepsilon rz||_{A^q_\beta(\mathbb{D})}=1+\frac{q}{4\beta}\varepsilon^2r^2+O(\varepsilon^3).$$

Assume the hypercontractive inequality (1) holds, let $f(z)=1+\varepsilon z$, we have

$$\|1+\varepsilon rz\|_{A^q_\beta(\mathbb{D})}\leqslant\|1+\varepsilon z\|_{A^p_\alpha(\mathbb{D})},$$

Hence,

$$1+\frac{q}{4\beta}\varepsilon^2r^2\leqslant 1+\frac{p}{4\alpha}\varepsilon^2+O(\varepsilon^3).$$

As ε approaches 0, it follows that

$$r^2\leqslant\frac{\beta p}{\alpha q}.$$

This achieves the necessary part of Theorem 1.1.

2.2. The Sufficiency. For $0<p\leqslant q<\infty$, let

$$\beta'=\frac{q}{p}\cdot\alpha,$$

then

$$\frac{\beta'}{\alpha}=\frac{q}{p}\geqslant 1.$$

By Kulikov's inequality [7, Corollary 1.3], it is easy to see

$$||f||_{A^q_{\beta'}(\mathbb{D})}\leqslant||f||_{A^p_\alpha(\mathbb{D})}.$$

Since the $A^q_\beta(\mathbb{D})$ norm of T_rf decreases as r decreases, it suffices to prove that there holds

$$||T_rf||_{A^q_\beta(\mathbb{D})}\leqslant||f||_{A^q_{\beta'}(\mathbb{D})}$$

for $r = \sqrt{\beta p/\alpha q}$. It is equivalent to

$$(\beta-1)\int_{\mathbb{D}} |f(rz)|^q(1-|z|^2)^{\beta-2}dA(z) \leqslant (\beta'-1)\int_{\mathbb{D}} |f(z)|^q(1-|z|^2)^{\beta'-2}dA(z).$$

By the polar coordinates, we need to prove

$$\begin{aligned}(\beta-1)\int_0^1 \rho(1-\rho^2)^{\beta-2}\left(\int_0^{2\pi} |f(r\rho e^{i\theta})|^q d\theta\right)d\rho\\ \leqslant(\beta'-1)\int_0^1 \rho(1-\rho^2)^{\beta'-2}\left(\int_0^{2\pi} |f(\rho e^{i\theta})|^q d\theta\right)d\rho.\end{aligned}$$

Denote $\rho = \sqrt{y}$ and

$$\Phi(y) = \int_0^{2\pi} |f(\sqrt{y}e^{i\theta})|^q d\theta,$$

it is sufficient to verify

$$(\beta-1)\int_0^1 (1-y)^{\beta-2}\Phi(r^2y)dy \leqslant (\beta'-1)\int_0^1 (1-y)^{\beta'-2}\Phi(y)dy$$

when $r = \sqrt{\beta p/\alpha q} = \sqrt{\beta/\beta'}$. Since $q \geqslant 2$, by [10, Lemma 1], Φ is C^2 smooth and $\Phi''(y) \geqslant 0$. Using integration by parts, we have

$$\begin{aligned}(\beta-1)\int_0^1 (1-y)^{\beta-2}\Phi(r^2y)dy &= \Phi(0) + r^2\int_0^1 (1-y)^{\beta-1}\Phi'(r^2y)dy\\ &= \Phi(0) + \frac{r^2}{\beta}\Phi'(0) + \frac{r^4}{\beta}\int_0^1 (1-y)^{\beta}\Phi''(r^2y)dy\\ &= \Phi(0) + \frac{1}{\beta'}\Phi'(0) + \frac{r^2}{\beta'}\int_0^1 (1-y)^{\beta}\Phi''(r^2y)dy\\ &= \Phi(0) + \frac{1}{\beta'}\Phi'(0) + \frac{1}{\beta'}\int_0^{r^2} \left(1-\frac{y}{r^2}\right)^{\beta}\Phi''(y)dy,\end{aligned}$$

and

$$\begin{aligned}(\beta'-1)\int_0^1 (1-y)^{\beta'-2}\Phi(y)dy &= \Phi(0) + \int_0^1 (1-y)^{\beta'-1}\Phi'(y)dy\\ &= \Phi(0) + \frac{1}{\beta'}\Phi'(0) + \frac{1}{\beta'}\int_0^1 (1-y)^{\beta'}\Phi''(y)dy.\end{aligned}$$

Then it suffices to prove

$$\int_0^{r^2} \left(1-\frac{y}{r^2}\right)^{\beta}\Phi''(y)dy \leqslant \int_0^1 (1-y)^{\beta'}\Phi''(y)dy.$$

Note that $\beta'/\beta = \alpha q/\beta p \geqslant 1$, the function $g(y) = (1-y)^{\beta'/\beta}$ is convex. Hence, for all $0 \leqslant y \leqslant \beta/\beta' = r^2$, we have

$$g(y) \geqslant g(0) + g'(0)y = 1 - \frac{y}{r^2},$$

Note that $\Phi''(y) \geqslant 0$, we have

$$\begin{aligned}\int_0^{r^2} \left(1 - \frac{y}{r^2}\right)^{\beta} \Phi''(y)dy &\leqslant \int_0^{r^2} (1-y)^{\beta'} \Phi''(y)dy \\ &\leqslant \int_0^{1} (1-y)^{\beta'} \Phi''(y)dy.\end{aligned}$$

This completes the whole proof.

3. Proof of Theorem 1.4

Before proving Nikol'skiĭ-type inequality for the weighted Bergman spaces, we need a hypercontractive inequality for high dimensions. Let n be a positive integer and $f \in A^p_\alpha(\mathbb{D}^n)$. Assume that $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{D}^n$ and $\mathbf{r} = (r_1, \dots, r_n)$ with $0 \leqslant r_i < 1$ for $i = 1, \dots, n$. Define an operator

$$(\mathbf{T_r} f)(\mathbf{z}) = f(r_1 z_1, \dots, r_n z_n), \quad \mathbf{z} \in \mathbb{D}^n.$$

The following lemma is a corollary of Theorem 1.1.

Lemma 3.1. *For $1 < \alpha, \beta < \infty$ and $0 < p \leqslant q < \infty$, let $q \geqslant 2$, $\beta p \leqslant \alpha q$, and $\boldsymbol{r} = (r_1, \dots, r_n)$ with $0 \leqslant r_i < 1$ for $i = 1, \dots, n$. Then we have*

$$||\boldsymbol{T_r} f||_{A^q_\beta(\mathbb{D}^n)} \leqslant ||f||_{A^p_\alpha(\mathbb{D}^n)} \tag{3}$$

for any $f \in A^p_\alpha(\mathbb{D}^n)$ if and only if $r_i^2 \leqslant \beta p/\alpha q$, for all $i = 1, 2, \dots, n$.

Proof. We prove the necessity first. Assume that

$$||\mathbf{T_r} f||_{A^q_\beta(\mathbb{D}^n)} \leqslant ||f||_{A^p_\alpha(\mathbb{D}^n)}$$

for any $f \in A^p_\alpha(\mathbb{D}^n)$. For each $1 \leqslant i \leqslant n$, let $g \in A^p_\alpha(\mathbb{D})$ and

$$f(\mathbf{z}) = f(z_1, \dots, z_i, \dots, z_n) := g(z_i),$$

then $f \in A^p_\alpha(\mathbb{D}^n)$ and

$$\|f\|_{A^p_\alpha(\mathbb{D}^n)} = \|g\|_{A^p_\alpha(\mathbb{D})}.$$

Moreover,

$$(\mathbf{T_r} f)(\mathbf{z}) = (T_{r_i} g)(z_i).$$

Then we have

$$||T_{r_i} g||_{A^q_\beta(\mathbb{D})} \leqslant ||g||_{A^p_\alpha(\mathbb{D})}.$$

By Theorem 1.1, we get $r_i^2 \leqslant \beta p/\alpha q$ for each $i = 1, 2, \dots, n$.

We proceed by induction on n to prove the sufficiency. Suppose that $r_i^2 \leqslant \beta p/\alpha q$ for all $i = 1, 2, \dots, n$. First, the inequality (3) is true for $n = 1$ by Theorem

1.1. Assume that the lemma is valid for $n = k-1$. We will prove the inequality (3) still holds when $n = k$. Recall that

$$||\mathbf{T_r} f||_{A^q_\beta(\mathbb{D}^k)} = \left(\int_{\mathbb{D}^k} |f(r_1z_1, \dots, r_{k-1}z_{k-1}, r_kz_k)|^q dA_\beta(z_1)\cdots dA_\beta(z_{k-1})dA_\beta(z_k)\right)^{\frac{1}{q}}.$$

By Fubini's Theorem and Theorem 1.1, we have

$$\begin{aligned}
||\mathbf{T_r} f||_{A^q_\beta(\mathbb{D}^k)} &= \left[\int_{\mathbb{D}^{k-1}}\left(\int_{\mathbb{D}} |f(r_1z_1, \dots, r_{k-1}z_{k-1}, r_kz_k)|^q dA_\beta(z_k)\right) dA_\beta(z_1)\cdots dA_\beta(z_{k-1})\right]^{\frac{1}{q}} \\
&\leqslant \left[\int_{\mathbb{D}^{k-1}}\left(\int_{\mathbb{D}} |f(r_1z_1, \dots, r_{k-1}z_{k-1}, z_k)|^p dA_\alpha(z_k)\right)^{\frac{q}{p}} dA_\beta(z_1)\cdots dA_\beta(z_{k-1})\right]^{\frac{1}{q}}.
\end{aligned}$$

Since $0 < p \leqslant q < \infty$, by Minkowski's inequality and the induction hypothesis, we have

$$\begin{aligned}
||\mathbf{T_r} f||_{A^q_\beta(\mathbb{D}^k)} &\leqslant \left[\int_{\mathbb{D}}\left(\int_{\mathbb{D}^{k-1}} |f(r_1z_1, \dots, r_{k-1}z_{k-1}, z_k)|^q dA_\beta(z_1)\cdots dA_\beta(z_{k-1})\right)^{\frac{p}{q}} dA_\alpha(z_k)\right]^{\frac{1}{p}} \\
&\leqslant \left[\int_{\mathbb{D}}\left(\int_{\mathbb{D}^{k-1}} |f(z_1, \dots, z_{k-1}, z_k)|^p dA_\alpha(z_1)\cdots dA_\alpha(z_{k-1})\right) dA_\beta(z_k)\right]^{\frac{1}{p}} \\
&= ||f||_{A^p_\alpha(\mathbb{D}^k)}.
\end{aligned}$$

This completes the proof of Lemma 3.1. □

As a direct corollary of Lemma 3.1, the inequality (2) holds for m-homogeneous polynomials. For given $n \in \mathbb{N}$, we define a new norm for

$$Q(\mathbf{z}, w) \in \mathbb{C}[z_1, \dots, z_n, z_{n+1}].$$

Denote

$$||Q||^p_{A^p_\alpha(\mathbb{D}^n, \mathbf{z})\times H^p(\mathbb{T}, w)} = \int_{\mathbb{T}}\int_{\mathbb{D}^n} |Q(\mathbf{z}, w)|^p dA_\alpha(\mathbf{z})dw,$$

where $dA_\alpha(\mathbf{z}) = dA_\alpha(z_1)\cdots dA_\alpha(z_n)$ and dw is the normalized arc length on the unit circle $\mathbb{T}$. The following homogeneous technique of general complex polynomials is known. However, we cannot locate a suitable reference. For the sake of completeness, we include a short proof.

Lemma 3.2. *For a given $n \in \mathbb{N}$, let $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{D}^n$, $P(\mathbf{z}) \in \mathbb{C}[z_1, \dots, z_n]$ be a complex polynomial with degree m, then there exists an m-homogeneous polynomial $Q(\mathbf{z}, w) \in \mathbb{C}[z_1, \dots, z_n, z_{n+1}]$ such that*

$$||P||_{A^p_\alpha(\mathbb{D}^n)} = ||Q||_{A^p_\alpha(\mathbb{D}^n, \mathbf{z})\times H^p(\mathbb{T}, w)}.$$

Proof. For a polynomial P in $\mathbb{C}[z_1, \dots, z_n]$, write

$$P(\mathbf{z}) = \sum_{\gamma\in\mathbb{N}^n} c_\gamma \mathbf{z}^\gamma,$$

where $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{N}^n$ is a multi-index. Let $\deg P = m$ and

$$Q(\mathbf{z}, w) = \sum_{\gamma\in\mathbb{N}^n} c_\gamma \mathbf{z}^\gamma w^{m-|\gamma|} \in \mathbb{C}[z_1, \dots, z_n, z_{n+1}],$$

where $|\gamma| = \gamma_1 + \cdots + \gamma_n$. Then $Q(\mathbf{z}, w)$ is m-homogeneous. For $0 < p < \infty$, recall that

$$\begin{aligned}||Q||^p_{A^p_\alpha(\mathbb{D}^n,\mathbf{z})\times H^p(\mathbb{T},w)} &= \int_{\mathbb{T}}\int_{\mathbb{D}^n} |Q(\mathbf{z}, w)|^p dA_\alpha(\mathbf{z})dw \\ &= \int_{\mathbb{T}}\int_{\mathbb{D}^n} \left|\sum_{\gamma\in\mathbb{N}^n} c_\gamma \mathbf{z}^\gamma w^{m-|\gamma|}\right|^p dA_\alpha(\mathbf{z})dw.\end{aligned}$$

Since $w \in \mathbb{T}$, let $\xi = (z_1 w^{-1}, \dots, z_n w^{-1})$, we have

$$\begin{aligned}||Q||^p_{A^p_\alpha(\mathbb{D}^n,\mathbf{z})\times H^p(\mathbb{T},w)} &= \int_{\mathbb{T}}\int_{\mathbb{D}^n} \left|\sum_{\gamma\in\mathbb{N}^n} c_\gamma (z_1 w^{-1})^{\gamma_1} \dots (z_n w^{-1})^{\gamma_n}\right|^p dA_\alpha(\mathbf{z})dw \\ &= \int_{\mathbb{T}}\int_{\mathbb{D}^n} \left|\sum_{\gamma\in\mathbb{N}^n} c_\gamma \xi_1^{\gamma_1} \dots \xi_n^{\gamma_n}\right|^p dA_\alpha(\xi)dw \\ &= ||P||^p_{A^p_\alpha(\mathbb{D}^n)}.\end{aligned}$$

This completes the proof. □

Now we are ready to prove Theorem 1.4. Assume that

$$P(\mathbf{z}) = \sum_{\gamma\in\mathbb{N}^n} c_\gamma \mathbf{z}^\gamma \in \mathbb{C}[z_1, \dots, z_n]$$

is a polynomial. We divide the problem into two cases. For the case $1 < \alpha < \beta < \infty$. Let

$$p' = \frac{\beta}{\alpha} p,$$

then $0 < p < p' \leqslant q < \infty$, $p'/\beta = p/\alpha$. By Kulikov's inequality [7, Corollary 1.3], one gets

$$||P||_{A^{p'}_\beta(\mathbb{D}^n)} \leqslant ||P||_{A^p_\alpha(\mathbb{D}^n)}.$$

Then it suffices to prove

$$||P||_{A^q_\beta(\mathbb{D}^n)} \leqslant \left(\sqrt{\frac{\alpha q}{\beta p}}\right)^m ||P||_{A^{p'}_\beta(\mathbb{D}^n)} = \left(\sqrt{\frac{q}{p'}}\right)^m ||P||_{A^{p'}_\beta(\mathbb{D}^n)}$$

Let $r_0 = \sqrt{p'/q} < 1$, $\mathbf{r}_0 = (r_0, \dots, r_0)$ and

$$Q(\mathbf{z}, w) = \sum_{\gamma\in\mathbb{N}^n} c_\gamma \mathbf{z}^\gamma w^{m-|\gamma|} \in \mathbb{C}[z_1, \dots, z_{n+1}]$$

be an m-homogeneous polynomial. Since for each $w \in \mathbb{T}$, $Q(\mathbf{z}, w) \in A^p_\alpha(\mathbb{D}^n)$. By Lemma 3.1 and Minkowski's inequality, we have

$$
\begin{aligned}
||\mathbf{T}_{\mathbf{r}_0}Q||_{A^q_\alpha(\mathbb{D}^n,\mathbf{z})\times H^q(\mathbb{T},w)} &= \left(\int_{\mathbb{T}}\int_{\mathbb{D}^n}|Q(\mathbf{r}_0\mathbf{z}, r_0w)|^q dA_\beta(\mathbf{z})dw\right)^{\frac{1}{q}}\\
&\leqslant \left(\int_{\mathbb{T}}\left(\int_{\mathbb{D}^n}|Q(\mathbf{z}, r_0w)|^{p'} dA_\beta(\mathbf{z})\right)^{\frac{q}{p'}} dw\right)^{\frac{1}{q}}\\
&\leqslant \left(\int_{\mathbb{D}^n}\left(\int_{\mathbb{T}}|Q(\mathbf{z}, r_0w)|^q dw\right)^{\frac{p'}{q}} dA_\beta(\mathbf{z})\right)^{\frac{1}{p'}}
\end{aligned}
$$

Since for each $\mathbf{z} \in \mathbb{D}$, $Q(\mathbf{z}, w) \in H^p(\mathbb{T})$. By Weissler's work [15], one gets

$$
\begin{aligned}
||\mathbf{T}_{\mathbf{r}_0}Q||_{A^q_\alpha(\mathbb{D}^n,\mathbf{z})\times H^q(\mathbb{T},w)} &\leqslant \left(\int_{\mathbb{T}}\int_{\mathbb{D}^n}|Q(\mathbf{z}, w)|^{p'} dA_\beta(\mathbf{z})dw\right)^{\frac{1}{p'}}\\
&= ||Q||_{A^{p'}_\alpha(\mathbb{D}^n,\mathbf{z})\times H^{p'}(\mathbb{T},w)}.
\end{aligned}
$$

Note that Q is m-homogeneous, by Lemma 3.2, we complete the proof of the case $1 < \alpha < \beta < \infty$.

For the case $1 < \beta < \alpha < \infty$, let

$$
r_0 = \sqrt{\frac{\beta p}{\alpha q}} < \sqrt{\frac{p}{q}} < 1.
$$

With similar arguments in the last case, we have

$$
\begin{aligned}
||\mathbf{T}_{\mathbf{r}_0}Q||_{A^q_\alpha(\mathbb{D}^n,\mathbf{z})\times H^q(\mathbb{T},w)} &= \left(\int_{\mathbb{T}}\int_{\mathbb{D}^n}|Q(\mathbf{r}_0\mathbf{z}, r_0w)|^q dA_\beta(\mathbf{z})dw\right)^{\frac{1}{q}}\\
&\leqslant \left(\int_{\mathbb{T}}\left(\int_{\mathbb{D}^n}|Q(\mathbf{z}, r_0w)|^p dA_\alpha(\mathbf{z})\right)^{\frac{q}{p}} dw\right)^{\frac{1}{q}}\\
&\leqslant \left(\int_{\mathbb{D}^n}\left(\int_{\mathbb{T}}|Q(\mathbf{z}, r_0w)|^q dw\right)^{\frac{p}{q}} dA_\alpha(\mathbf{z})\right)^{\frac{1}{p}}\\
&\leqslant \left(\int_{\mathbb{T}}\int_{\mathbb{D}^n}|Q(\mathbf{z}, w)|^p dA_\alpha(\mathbf{z})dw\right)^{\frac{1}{p}}\\
&= ||Q||_{A^p_\alpha(\mathbb{D}^n,\mathbf{z})\times H^p(\mathbb{T},w)}.
\end{aligned}
$$

This completes the proof of the case $1 < \beta < \alpha < \infty$.

Now we show that the constant

$$C(\alpha,\beta,p,q)=\sqrt{\frac{\alpha q}{\beta p}}$$

is sharp. Let n be a positive integer. If one endows the polydisk $\mathbb{D}^n$ with Borel σ-field $\mathcal{B}(\mathbb{D}^n)$, then we get a probability space $(\mathbb{D}^n,\mathcal{B}(\mathbb{D}^n),dA_\alpha(z_1)\dots dA_\alpha(z_n))$. Define

$$z_i:\mathbb{D}^n\to\mathbb{C},\quad \mathbf{z}\mapsto z_i,\quad 1\leqslant i\leqslant n,$$

then $\{z_i|1\leqslant i\leqslant n\}$ are independent identically distributed random variables. Consider the 1-homogeneous polynomial

$$p_{\alpha,n}(\mathbf{z})=\sqrt{\frac{\alpha}{n}}\sum_{i=1}^{n}z_i$$

for given $n\in\mathbb{N}$ and $\alpha\geqslant 1$. By the central limit theorem [8], the sequence of polynomials $p_{\alpha,n}$ converges to the normal complex Gaussian random variable G in distribution as n approaches ∞. Then for a fixed integer m

$$\lim_{n\to\infty}\left\|p_{\alpha,n}^m\right\|_{A_\alpha^p(\mathbb{D}^n)}^p=\Gamma\left(\frac{pm}{2}+1\right).$$

Note that $p_{1,n}=p_{\alpha,n}/\sqrt{\alpha}$. Therefore,

$$\lim_{n\to\infty}\left\|p_{1,n}^m\right\|_{A_\alpha^p(\mathbb{D}^n)}=\left(\frac{1}{\sqrt{\alpha}}\right)^m\Gamma\left(\frac{pm}{2}+1\right)^{\frac{1}{p}}.$$

Similarly,

$$\lim_{n\to\infty}\left\|p_{1,n}^m\right\|_{A_\beta^q(\mathbb{D}^n)}=\left(\frac{1}{\sqrt{\beta}}\right)^m\Gamma\left(\frac{qm}{2}+1\right)^{\frac{1}{q}}.$$

Hence,

$$\lim_{n\to\infty}\frac{||p_{1,n}^m||_{A_\beta^q(\mathbb{D}^n)}}{||p_{1,n}^m||_{A_\alpha^p(\mathbb{D}^n)}}=\left(\sqrt{\frac{\alpha}{\beta}}\right)^m\frac{\Gamma\left(\frac{qm}{2}+1\right)^{\frac{1}{q}}}{\Gamma\left(\frac{pm}{2}+1\right)^{\frac{1}{p}}}.$$

By the Stirling's formula

$$\sqrt{2\pi}x^{x+\frac{1}{2}}e^{-x}<\Gamma(x+1)<\sqrt{2\pi}x^{x+\frac{1}{2}}e^{-x+\frac{1}{12x}},$$

we have

$$\frac{(2\pi)^{\frac{1}{2qm}}(\frac{qm}{2})^{\frac{1}{2}+\frac{1}{2qm}}e^{-\frac{1}{2}}}{(2\pi)^{\frac{1}{2pm}}(\frac{pm}{2})^{\frac{1}{2}+\frac{1}{2pm}}e^{-\frac{1}{2}+\frac{1}{6(pm)^2}}}<\frac{\Gamma\left(\frac{qm}{2}+1\right)^{\frac{1}{qm}}}{\Gamma\left(\frac{pm}{2}+1\right)^{\frac{1}{pm}}}<\frac{(2\pi)^{\frac{1}{2qm}}(\frac{qm}{2})^{\frac{1}{2}+\frac{1}{2qm}}e^{-\frac{1}{2}+\frac{1}{6(qm)^2}}}{(2\pi)^{\frac{1}{2pm}}(\frac{pm}{2})^{\frac{1}{2}+\frac{1}{2pm}}e^{-\frac{1}{2}}}$$

Note that

$$\frac{(2\pi)^{\frac{1}{2qm}}(\frac{qm}{2})^{\frac{1}{2}+\frac{1}{2qm}}e^{-\frac{1}{2}}}{(2\pi)^{\frac{1}{2pm}}(\frac{pm}{2})^{\frac{1}{2}+\frac{1}{2pm}}e^{-\frac{1}{2}+\frac{1}{6(pm)^2}}} = e^{-\frac{1}{6(pm)^2}}\frac{(qm\pi)^{\frac{1}{2qm}}}{(pm\pi)^{\frac{1}{2pm}}}\sqrt{\frac{q}{p}},$$

and

$$\lim_{m\to\infty}(qm)^{\frac{1}{qm}} = \lim_{m\to\infty} e^{\frac{1}{qm}\ln qm} = 1.$$

It follows that

$$\lim_{m\to\infty}\frac{\Gamma\left(\frac{qm}{2}+1\right)^{\frac{1}{qm}}}{\Gamma\left(\frac{pm}{2}+1\right)^{\frac{1}{pm}}} = \sqrt{\frac{q}{p}}.$$

Consequently,

$$\lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{||p_{1,n}^m||_{A^q_\beta(\mathbb{D}^n)}}{||p_{1,n}^m||_{A^p_\alpha(\mathbb{D}^n)}}\right)^{\frac{1}{m}} = \sqrt{\frac{\alpha q}{\beta p}}.$$

Therefore, the constant

$$C(\alpha,\beta,p,q) = \sqrt{\frac{\alpha q}{\beta p}}$$

is sharp. Then the whole proof is complete now.

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