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The complete Nevanlinna-Pick property for sub-Bergman Hilbert spaces

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ABSTRACT. In this paper, we show that for non-constant $\varphi \in H_1^\infty(\mathbb{D}^n)$, the sub-Bergman Hilbert space $\mathcal{H}(K_n^\varphi)$ over the n-disk with $n \geq 2$ does not have the complete Nevanlinna-Pick property. Furthermore, we prove by different methods that if φ is an inner function on $\mathbb{D}$, the corresponding sub-Bergman Hilbert space on $\mathbb{D}$ has the complete Nevanlinna-Pick property if and only if φ is a Möbius map ([6]).

CONTENTS

1. Introduction

Let X be a set and let $k : X \times X \to \mathbb{C}$ be a function of two variables. We call k a positive semi-definite function on X if k is a self-adjoint ($k(z,w) = \overline{k(w,z)}$), and for any finite set $\{\lambda_1, \lambda_2, \cdots, \lambda_N\} \subseteq X$, the matrix $[k(\lambda_i, \lambda_j)]_{i,j=1}^N$ is positive semi-definite, i.e. for every $\alpha_1, \alpha_2, \cdots, \alpha_N \in \mathbb{C}$, we have that

$$\sum_{i,j=1}^{N} \alpha_i \bar{\alpha}_j k(\lambda_i, \lambda_j) \geq 0.$$

We will use the notation $k \succeq 0$ to denote that k is positive semi-definite. We can define an ordering on the set $\{k : X \times X \to \mathbb{C}\}$ by $k_1 \succeq k_2$ if and only if $k_1 - k_2 \succeq 0$.

A reproducing kernel Hilbert space is a Hilbert space $\mathcal{H}$ of functions on a set X with the property that evaluation at each $x \in X$ is a bounded linear functional

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on $\mathcal{H}$. By the Riesz Representation theorem, there exists for each $z \in X$ and function $k_z \in \mathcal{H}$ such that

$$\langle f, k_z \rangle = f(z)$$

for all $f \in \mathcal{H}$. See more details on the reproducing kernel Hilbert spaces in [1, 7].

The reproducing kernel Hilbert space corresponding to k is denoted by $\mathcal{H}(k)$. A function $\varphi : X \to \mathbb{C}$ is a multiplier on $\mathcal{H}(k)$ if $\varphi f \in \mathcal{H}(k)$ for each $f \in \mathcal{H}(k)$. Let $\mathcal{M}(\mathcal{H}(k))$ be the set of all multipliers of $\mathcal{H}(k)$. By an application of closed graph theorem, for each $\varphi \in \mathcal{M}(\mathcal{H}(k))$ the multiplication operator

$$M_\varphi : \mathcal{H}(k) \to \mathcal{H}(k), f \mapsto \varphi f$$

is a bounded operator on $\mathcal{H}(k)$, and we define $\|\varphi\|_{\mathcal{M}(\mathcal{H}(k))} = \left\|M_\varphi\right\|$. We set

$$\mathcal{M}_1(\mathcal{H}(k)) = \{\varphi \in \mathcal{M}(\mathcal{H}(k)) : \left\|M_\varphi\right\| \leq 1\}.$$

Complete Nevanlinna-Pick kernels are related to the solution of Nevanlinna-Pick interpolation problems. If for a set X, whenever $\{\lambda_1, \cdots, \lambda_n\} \subseteq X$, and $W_1, \cdots, W_n$ are s-by-t matrices such that

$$(I - W_i W_j^*) k(\lambda_i, \lambda_j) \geq 0,$$

we can find a Φ in the closed unit ball of

$$\mathcal{M}(\mathcal{H}(k) \otimes \mathbb{C}^t, \mathcal{H}(k) \otimes \mathbb{C}^s) = \{\Phi : X \to M_{s,t} : \Phi f \in \mathcal{H}(k) \otimes \mathbb{C}^s \text{ for } f \in \mathcal{H}(k) \otimes \mathbb{C}^t\}$$

such that

$$\Phi(\lambda_i) = W_i,\ i = 1, 2, \cdots, N,$$

then k has $s \times t$-Pick property. If k has $s \times t$-Pick property for all positive integers s and t, then we say k has the complete Nevanlinna-Pick property. For example, the Hardy space $H^2(\mathbb{D})$ with Szegö kernel $s(z, w) = \frac{1}{1 - \bar{w} z}$, and the Drury-Arveson space H_d^2 with kernel $k(z, w) = \frac{1}{1 - \langle z, w \rangle_{\mathbb{C}^d}}$ are the complete Nevanlinna-Pick space.

We will introduce the equivalent characterizations of complete Nevanlinna-Pick kernel from [2], which will be used in this paper.

Assume $k(z, w) \neq 0$ for all $z, w \in X$. Then $\mathcal{H}(k)$ is a complete Nevanlinna-Pick space if and only if for some $z_0 \in X$,

$$F_{z_0} : X \times X \to \mathbb{C},\ (z, w) \longmapsto 1 - \frac{k(z_0, w) k(z, z_0)}{k(z_0, z_0) k(z, w)}$$

is positive semi-definite. In this case, F_{z_0} is positive for all $z_0 \in X$. A kernel $k : X \times X \to \mathbb{C}$ is said to be normalized at $z_0 \in X$ if $k(z, z_0) = 1$ for all $z \in X$. If k is normalized, then k is a complete Nevanlinna-Pick kernel if and only if

$$1 - \frac{1}{k(z, w)} \geq 0.$$

For a kernel k on X and $\varphi \in \mathcal{M}_1(\mathcal{H}(k))$, the de Branges-Rovnyak space $\mathcal{H}(k^\varphi)$ associated to φ is the reproducing kernel Hilbert space on X with kernel k^φ which is defined as

$$k^\varphi(z,w) = (1 - \overline{\varphi(w)}\varphi(z))k(z,w)\ (z,w \in X).$$

When k is the Szegö kernel, the $\mathcal{H}(k^\varphi)$ is called sub-Hardy Hilbert space. When k is the Bergman kernel, the $\mathcal{H}(k^\varphi)$ is called sub-Bergman Hilbert space.

The de Branges-Rovnyak spaces are introduced by de Branges and Rovnyak in [5]. The initial motivation of introducing the de Branges-Rovnyak spaces was to study the invariant subspace problem. Sarason's book [10] presented most of the main developments. It was realized that these spaces have numerous connections with other topics in complex analysis and operator theory, in particular through Toeplitz operators. Subsequently, Zhu introduced and studied the sub-Bergman Hilbert spaces in [13, 14]. In this paper, we main focus on the complete Nevanlinna-Pick property of the sub-Bergman Hilbert space over the polydisk.

The following are known examples of the de Branges-Rovnyak spaces with complete Nevanlinna-Pick property. In 2020, Chu presented the characterization of the sub-Hardy Hilbert spaces with complete Nevanlinna-Pick property in [4]. In 2022, a similar characterization in the Drury-Arveson space was obtained by Sautel in [11]. In 2023, Luo and Zhu proved that when did the sub-Bergman Hilbert spaces have complete Nevanlinna-pick property in [6]. In 2024, we determined the complete Nevanlinna-Pick property for Beurling type quotient module over the bidisk ([12]). Generally, Ahmed, Das and Panja consider de Branges-Rovnyak spaces of a considerably large class of reproducing kernel Hilbert spaces and gave a characterization for them to be complete Nevanlinna-Pick spaces in [3].

Our initial motivation was to consider Schur product and tensor product of the Szegö kernels. A natural question is whether the corresponding de Branges-Rovnyak space satisfies the complete Nevanlinna-Pick property.

The remainder of this paper is organized as follows. In Section 2, we show that for non-constant holomorphic function $\varphi \in H_1^\infty(\mathbb{D}^n)$, the sub-Bergman Hilbert space

$$\mathcal{H}\left(\frac{1 - \overline{\varphi(w)}\varphi(z)}{(1-\bar{w}_1 z_1)^2(1-\bar{w}_2 z_2)^2 \cdots (1-\bar{w}_n z_n)^2}\right)$$

over the polydisk does not have the complete Nevanlinna-Pick property. In Section 3, we prove by a method different that the sub-Bergman Hilbert space

$$\mathcal{H}\left(\frac{1 - \overline{\varphi(w)}\varphi(z)}{(1-\bar{w}z)^2}\right)$$

over the unit disk has the complete Nevanlinna-Pick property if and only if φ is a Möbius map, there we assume that $\varphi \in H_1^\infty(\mathbb{D})$ is an inner function ([6]).

2. The sub-Bergman Hilbert spaces over the polydisk

First, we introduce the Schur product and Tensor product. For two kernels k_1 and k_2 on X, we define the Schur product of k_1 and k_2 by

$$(k_1 \circ k_2)(z, w) = k_1(z, w) k_2(z, w),$$

where $z, w \in X$. The positive semi-definition property of $k_1 \circ k_2$ follows from Schur's theorem, see [7]. The Schur product of two Szegö kernels on $\mathbb{D}$ generates the kernel of Bergman space $L_a^2(\mathbb{D})$.

For two kernels k_1 and k_2 on X, we define the tensor product of k_1 and k_2 by

$$(k_1 \otimes k_2)((z_1, z_2), (w_1, w_2)) = k_1(z_1, w_1) k_2(z_2, w_2),$$

where $z_1, z_2, w_1, w_2 \in X$. By Theorem 5.11 in [7], we know that $\mathcal{H}(k_1 \otimes k_2)$ is isometrically isomorphic to $\mathcal{H}(k_1) \otimes \mathcal{H}(k_2)$, which is the reason why we call the kernel as the tensor product of two kernels. The tensor product of two Szegö kernels on $\mathbb{D}$ generates the kernel of Hardy space $H^2(\mathbb{D}^2)$.

In [3], Ahmed, Das and Panja considered the Schur products, tensor products of n ($n \geq 2$) Szegö kernels and when is the corresponding de Branges-Rovnyak space a complete Nevanlinna-Pick space. A natural question is: When the Szegö kernel is subjected to both Schur product and tensor product, which de Branges-Rovnyak space has the complete Nevanlinna-Pick property? In this case, the resulting reproducing kernel is that of the Bergman space L_a^2 over the bidisk, see as follows.

$$\begin{tikzcd}
\frac{1}{1-\bar{w}z} \arrow[r, "\text{Schur product}"] \arrow[d, "\text{tensor product}"'] & \frac{1}{(1-\bar{w}z)^2} \arrow[d, "\text{tensor product}"] \\
\frac{1}{(1-\bar{w}_1 z_1)(1-\bar{w}_2 z_2)} \arrow[r, "\text{Schur product}"] & \frac{1}{(1-\bar{w}_1 z_1)^2(1-\bar{w}_2 z_2)^2}
\end{tikzcd}$$

In this paper, we denote by K_n the reproducing kernel for the Bergman space $L_a^2(\mathbb{D}^n)$. For $\varphi \in H_1^\infty(\mathbb{D}^n)$, the reproducing kernel of the corresponding sub-Bergman Hilbert spaces over polydisk is

$$K_n^\varphi(z, w) = \frac{1 - \overline{\varphi(w)}\varphi(z)}{(1 - \bar{w}_1 z_1)^2 (1 - \bar{w}_2 z_2)^2 \cdots (1 - \bar{w}_n z_n)^2}$$

where $z = (z_1, z_2, \cdots, z_n)$, $w = (w_1, w_2, \cdots, w_n) \in \mathbb{D}^n$. We make use of the corresponding theory of complete Nevanlinna-Pick spaces in [3]. The following notation is used in the theorem

$$H_1^\infty(\mathbb{D}, \mathcal{B}(\mathcal{E}, \mathcal{F})) = \left\{\varphi : \mathbb{D} \to \mathcal{B}(\mathcal{E}, \mathcal{F}) \mid \varphi \text{ is holomorphic on } \mathbb{D}, \|\varphi\|_\infty \leq 1\right\}.$$

Lemma 2.1 ([3], Theorem 3.1)**.** *Let k be a non-vanishing kernel on X such that k is normalized at $z_0 \in X$ and*

$$1 - \frac{1}{k(z, w)} = g(z)g(w)^* - f(z)f(w)^* \ (z, w \in X),$$

for some function $f : X \to \mathcal{B}(\mathcal{E}, \mathbb{C})$ and a non-zero function $g : X \to \mathcal{B}(\mathcal{F}, \mathbb{C})$, where $\mathcal{E}$ and $\mathcal{F}$ are Hilbert spaces. Suppose that $\varphi : X \to \mathbb{D}$ with $\varphi(z_0) = 0$ is a non-constant function in $\mathcal{M}_1(\mathcal{H}(k))$. Then the de Branges-Rovnyak space $\mathcal{H}(k^{\varphi})$ is a complete Nevanlinna-Pick space if and only if there exists $\Psi \in H_1^{\infty}(\mathbb{D}, \mathcal{B}((\mathcal{E} \oplus \mathbb{C}), \mathcal{F}))$ such that

$$f(z) = g(z)\psi_1(\varphi(z)) \text{ and } \varphi(z) = g(z)\psi_2(\varphi(z)) \ (z \in X),$$

where for all $\zeta \in \mathbb{D}$, $\Psi(\zeta) = [\psi_1(\zeta), \psi_2(\zeta)]$, $\psi_1(\zeta) \in \mathcal{B}(\mathcal{E}, \mathcal{F})$, and $\psi_2(\zeta) \in \mathcal{B}(\mathbb{C}, \mathcal{F})$.

By using Lemma 2.1, we can obtain the characterization on the bidisk.

Theorem 2.2. *Let $\varphi(z_1, z_2)$ be a non-constant holomorphic function in $H_1^{\infty}(\mathbb{D}^2)$, then the sub-Bergman Hilbert space $\mathcal{H}(K_2^{\varphi})$ over the bidisk does not have complete Nevanlinna-Pick property.*

Proof. It is not hard to check that

$$\begin{aligned}1 - \frac{1}{K_2(z,w)} &= 1 - (1 - z_1\bar{w}_1)^2(1 - z_2\bar{w}_2)^2 \\ &= 2\bar{w}_1 z_1 + 2\bar{w}_2 z_2 - \bar{w}_1^2 z_1^2 - \bar{w}_2^2 z_2^2 \\ &+ 2\bar{w}_1^2 z_1^2 \bar{w}_2 z_2 + 2\bar{w}_1 z_1 \bar{w}_2^2 z_2^2 - \bar{w}_1^2 z_1^2 \bar{w}_2^2 z_2^2 - 4\bar{w}_1 z_1 \bar{w}_2 z_2 \\ &= g(z_1, z_2) g(w_1, w_2)^* - f(z_1, z_2) f(w_1, w_2)^*,\end{aligned}$$

where $f, g : \mathbb{D}^2 \to \mathcal{B}(\mathbb{C}^4, \mathbb{C})$ are given by

$$g(z_1, z_2) = (\sqrt{2}z_1, \sqrt{2}z_2, \sqrt{2}z_1^2 z_2, \sqrt{2}z_1 z_2^2),$$

and

$$f(z_1, z_2) = (z_1^2, z_2^2, z_1^2 z_2^2, 2z_1 z_2).$$

Then by Lemma 2.1, $\mathcal{H}(K_2^{\varphi})$ is a complete Nevanlinna-Pick space if and only if there exists $\Psi \in H_1^{\infty}(\mathbb{D}, \mathcal{B}((\mathbb{C}^4 \oplus \mathbb{C}), \mathbb{C}^4))$ such that

$$f(z) = g(z)\psi_1(\varphi(z)),$$

and

$$\varphi(z) = g(z)\psi_2(\varphi(z)),$$

where $z = (z_1, z_2) \in \mathbb{D}^2$, i.e.

$$(z_1^2, z_2^2, z_1^2 z_2^2, 2z_1 z_2) = (\sqrt{2}z_1, \sqrt{2}z_2, \sqrt{2}z_1^2 z_2, \sqrt{2}z_1 z_2^2)\psi_1(\varphi(z_1, z_2)), \tag{2.1}$$

$$\varphi(z_1, z_2) = (\sqrt{2}z_1, \sqrt{2}z_2, \sqrt{2}z_1^2 z_2, \sqrt{2}z_1 z_2^2)\psi_2(\varphi(z_1, z_2)). \tag{2.2}$$

For $\zeta \in \mathbb{D}$, write

$$\Psi(\zeta) = [\psi_1(\zeta), \psi_2(\zeta)],$$

where

$$\psi_1(\zeta) = \begin{pmatrix} \psi_{11}^{(1)}(\zeta) & \psi_{12}^{(1)}(\zeta) & \psi_{13}^{(1)}(\zeta) & \psi_{14}^{(1)}(\zeta) \\ \psi_{21}^{(1)}(\zeta) & \psi_{22}^{(1)}(\zeta) & \psi_{23}^{(1)}(\zeta) & \psi_{24}^{(1)}(\zeta) \\ \psi_{31}^{(1)}(\zeta) & \psi_{32}^{(1)}(\zeta) & \psi_{33}^{(1)}(\zeta) & \psi_{34}^{(1)}(\zeta) \\ \psi_{41}^{(1)}(\zeta) & \psi_{42}^{(1)}(\zeta) & \psi_{43}^{(1)}(\zeta) & \psi_{44}^{(1)}(\zeta) \end{pmatrix} \in \mathcal{B}(\mathbb{C}^4, \mathbb{C}^4),$$

and

$$\psi_2(\zeta) = \begin{pmatrix} \psi_{11}^{(2)}(\zeta) \\ \psi_{21}^{(2)}(\zeta) \\ \psi_{31}^{(2)}(\zeta) \\ \psi_{41}^{(2)}(\zeta) \end{pmatrix} \in \mathcal{B}(\mathbb{C}, \mathbb{C}^4).$$

We now consider the slice functions $\varphi(0, z_2)$ and $\varphi(z_1, 0)$.

Suppose that one of $\varphi(0, z_2)$ and $\varphi(z_1, 0)$ is identically zero. If $\varphi(z_1, 0) = 0$, by (2.1) we have

$$\begin{aligned} z_1^2 = &\sqrt{2} z_1 \psi_{11}^{(1)}(\varphi(z_1, z_2)) + \sqrt{2} z_2 \psi_{21}^{(1)}(\varphi(z_1, z_2)) \\ &+ \sqrt{2} z_1^2 z_2 \psi_{31}^{(1)}(\varphi(z_1, z_2)) + \sqrt{2} z_1 z_2^2 \psi_{41}^{(1)}(\varphi(z_1, z_2)). \end{aligned}$$

Let $z_2 = 0$, we obtain that

$$z_1 = \sqrt{2} \psi_{11}^{(1)}(\varphi(z_1, 0)) = \sqrt{2} \psi_{11}^{(1)}(0),$$

which is a contradiction. If $\varphi(0, z_2) = 0$, it is also contradiction. Therefore, we know that $\varphi(0, z_2)$, $\varphi(z_1, 0)$ are not identically zero. However, by (2.1) we have

$$\begin{aligned} 2z_1 z_2 = &\sqrt{2} z_1 \psi_{14}^{(1)}(\varphi(z_1, z_2)) + \sqrt{2} z_2 \psi_{24}^{(1)}(\varphi(z_1, z_2)) \\ &+ \sqrt{2} z_1^2 z_2 \psi_{34}^{(1)}(\varphi(z_1, z_2)) + \sqrt{2} z_1 z_2^2 \psi_{44}^{(1)}(\varphi(z_1, z_2)). \end{aligned}$$

Let $z_1 = 0$, we obtain that

$$0 = \sqrt{2} z_2 \psi_{24}^{(1)}(\varphi(0, z_2))$$

holds for all $z_2 \in \mathbb{D}$, and hence $\psi_{24}^{(1)} \equiv 0$. On the other hand, let $z_2 = 0$, we have $\psi_{14}^{(1)} \equiv 0$. Thus

$$2z_1 z_2 = \sqrt{2} z_1^2 z_2 \psi_{34}^{(1)}(\varphi(z_1, z_2)) + \sqrt{2} z_1 z_2^2 \psi_{44}^{(1)}(\varphi(z_1, z_2))$$

i.e.

$$\sqrt{2} = z_1 \psi_{34}^{(1)}(\varphi(z_1, z_2)) + z_2 \psi_{44}^{(1)}(\varphi(z_1, z_2)),$$

which is a contradiction as the right hand side vanishes at $(0, 0)$.

Therefore, there does not exist a Ψ satisfying the above conditions, which implies that the sub-Bergman Hilbert space $\mathcal{H}(K_2^{\varphi})$ does not have the complete Nevanlinna-Pick property. □

We suspect that why $\mathcal{H}(K_2^{\varphi})$ lacks the complete Nevanlinna-Pick property is that the reproducing kernel of the space $L_a^2(\mathbb{D}^2)$ is the tensor product of two non-complete Nevanlinna-Pick kernels.

Remark. *The Bergman space $L_a^2(\mathbb{D}^2)$ does not have the complete Nevanlinna-Pick property.*

Proof. By definition, we need to check $1-\frac{1}{K_2(z,w)}$ is not positive semi-definite. Let

$$\begin{aligned} F(z,w) &= 1-\frac{1}{K_2(z,w)} \\ &= 1-(1-\bar{w}_1 z_1)^2(1-\bar{w}_2 z_2)^2 \\ &= 2\bar{w}_1 z_1 + 2\bar{w}_2 z_2 - \bar{w}_1^2 z_1^2 - \bar{w}_2^2 z_2^2 \\ &+ 2\bar{w}_1^2 z_1^2 \bar{w}_2 z_2 + 2\bar{w}_1 z_1 \bar{w}_2^2 z_2^2 - \bar{w}_1^2 z_1^2 \bar{w}_2^2 z_2^2 - 4\bar{w}_1 z_1 \bar{w}_2 z_2. \end{aligned}$$

For some $0<|a|<1$, we let

$$\lambda_1 = (a,0), \lambda_2 = (0,a), \lambda_3 = (a,a) \in \mathbb{D}^2,$$

then the matrix $F_{3\times 3} = [F(\lambda_i,\lambda_j)]_{i,j=1}^3$ is

$$\begin{pmatrix} 2|a|^2-|a|^4 & 0 & 2|a|^2-|a|^4 \\ 0 & 2|a|^2-|a|^4 & 2|a|^2-|a|^4 \\ 2|a|^2-|a|^4 & 2|a|^2-|a|^4 & 4|a|^2+4|a|^6-6|a|^4-|a|^8 \end{pmatrix}.$$

A straightforward computation shows that $\det(F_{3\times 3})<0$, and this means that $F(z,w)$ is not positive. $\square$

In [4], Chu proved that there exist many $\varphi \in H_1^\infty(\mathbb{D})$ such that $\mathcal{H}(\frac{1-\overline{\varphi(w)}\varphi(z)}{1-\bar{w}z})$ have the complete Nevanlinna-Pick property. However, for $\varphi \in H_1^\infty(\mathbb{D}^2)$, it was proved in [3] that

$$\mathcal{H}(\frac{1-\overline{\varphi(w)}\varphi(z)}{(1-\bar{w}_1 z_1)(1-\bar{w}_2 z_2)})$$

have the complete Nevanlinna-Pick property if and only if φ is Möbius map, and for $n \geq 3$, [3] proved that no exists $\varphi \in H_1^\infty(\mathbb{D}^n)$ for which

$$\mathcal{H}(\frac{1-\overline{\varphi(w)}\varphi(z)}{(1-\bar{w}_1 z_1)(1-\bar{w}_2 z_2)\cdots(1-\bar{w}_n z_n)})$$

has the complete Pick property. In the case of Bergman space on $\mathbb{D}$, Luo and Zhu proved that for $\varphi \in H_1^\infty(\mathbb{D})$, the $\mathcal{H}(K_1^\varphi)$ have the complete Nevanlinna-Pick property if and only if φ is Möbius map, see [6]. We suspect that when the kernel k tensor product or Schur product with an additional factor, the set of functions φ for which corresponding the de Branges-Rovnyak space have the complete Nevanlinna-Pick property will diminish. In the following, we demonstrate this conjecture in the setting of Bergman space over the polydisk.

Theorem 2.3. *Let $n \geq 2$, and let $\varphi(z)$ be a non-constant holomorphic function in $H_1^\infty(\mathbb{D}^n)$, then the sub-Bergman Hilbert space $\mathcal{H}(K_n^\varphi)$ does not have complete Nevanlinna-Pick property.*

Proof. Given $\varphi \in H_1^\infty(\mathbb{D}^n)$, if $\varphi(0) = a \neq 0$, let $\Phi(z) = \frac{a-\varphi(z)}{1-\bar{a}\varphi(z)}$, then $\varphi(0) = 0$, and

$$\begin{aligned}
K_n^\Phi(z,w) &= \frac{1-\overline{\Phi(w)}\Phi(z)}{(1-\bar{w}_1 z_1)^2(1-\bar{w}_2 z_2)^2 \cdots (1-\bar{w}_n z_n)^2} \\
&= \frac{(1-|a|^2)(1-\overline{\varphi(w)}\varphi(z))}{\prod_{i=1}^{n}(1-\bar{w}_i z_i)^2(1-a\overline{\varphi(w)})(1-\bar{a}\varphi(z))} \\
&= \overline{g(w)}K_n^\varphi(z,w)g(z),
\end{aligned}$$

where $g(z) = \frac{\sqrt{1-|a|^2}}{1-\bar{a}\varphi(z)}$. Then by the definition, we know that K_n^Φ has the complete Nevanlinna-Pick property if and only if K_n^φ has the complete Nevanlinna-Pick property. Therefore, we can assume that $\varphi(0) = 0$.

The proof is by induction on n. If $n = 2$, then Theorem 2.2 shows that $\mathcal{H}(K_2^\varphi)$ does not have complete Nevanlinna-Pick property. Let $n \geq 3$, and suppose that the result has been shown for $n-1$. If $\mathcal{H}(K_n^\varphi)$ has complete Nevanlinna-Pick property, then

$$\begin{aligned}
F(z,w) &= 1 - \frac{1}{K_n^\varphi(z,w)} \\
&= 1 - \frac{(1-\bar{w}_1 z_1)^2(1-\bar{w}_2 z_2)^2 \cdots (1-\bar{w}_n z_n)^2}{1-\overline{\varphi(w)}\varphi(z)}
\end{aligned}$$

is positive. This implies that for any finite set $\{\lambda_1, \lambda_2, \cdots, \lambda_N\}$ in $\mathbb{D}^n$ such that

$$\begin{aligned}
\lambda_1 &= (\lambda_1^{(1)}, \lambda_1^{(2)}, \cdots, \lambda_1^{(n-1)}, 0), \\
\lambda_2 &= (\lambda_2^{(1)}, \lambda_2^{(2)}, \cdots, \lambda_2^{(n-1)}, 0), \\
&\cdots \\
\lambda_N &= (\lambda_N^{(1)}, \lambda_N^{(2)}, \cdots, \lambda_N^{(n-1)}, 0),
\end{aligned}$$

the matrix $F_{N\times N} = [F(\lambda_i, \lambda_j)]_{i,j=1}^N$ is positive, i.e.

$$\begin{aligned}
&F(z_1, z_2, \cdots, z_{n-1}, 0, w_1, w_2, \cdots, w_{n-1}, 0) \\
=&1 - \frac{(1-\bar{w}_1 z_1)^2(1-\bar{w}_2 z_2)^2 \cdots (1-\bar{w}_{n-1} z_{n-1})^2}{1-\overline{\varphi(w_1, w_2, \cdots, w_{n-1}, 0)}\varphi(z_1, z_2, \cdots, z_{n-1}, 0)}
\end{aligned}$$

is positive. We write

$$\varphi_n : \mathbb{D}^{n-1} \to \mathbb{D},\ \varphi_n(z_1, z_2, \cdots, z_{n-1}) = \varphi(z_1, z_2, \cdots, z_{n-1}, 0).$$

A necessary condition for $\mathcal{H}(K_n^\varphi)$ to have complete Nevanlinna-Pick property is that $1 - \frac{1}{K_{n-1}^{\varphi_n}}$ is positive. By the inductive hypothesis, we have $1 - \frac{1}{K_{n-1}^{\varphi_n}}$ is not positive. Therefore, there is no non-constant $\varphi \in H_1^\infty(\mathbb{D}^n)$ such that $\mathcal{H}(K_n^\varphi)$ has the complete Nevanlinna-Pick property. □

3. The sub-Bergman Hilbert spaces over the unit disk

In this section, we assume that $\varphi \in H_1^\infty(\mathbb{D})$ is an inner function and prove that the sub-Bergman Hilbert space $\mathcal{H}(K_1^\varphi)$ has the complete Nevanlinna-Pick property if and only if φ is a Möbius map ([6]).

The following lemmas can be obtained from [7], which will be used in our proof.

Lemma 3.1 ([7], Theorem 5.1)**.** *Let $\mathcal{H}(k_1)$ and $\mathcal{H}(k_2)$ be the reproducing kernel Hilbert spaces on set X. Then*

$$\mathcal{H}(k_1) \subseteq \mathcal{H}(k_2)$$

if and only if there exists a constant $c > 0$, such that

$$k_1 \preceq c^2 k_2.$$

Moreover, $\|f\|_{\mathcal{H}(k_2)} \leq c\,\|f\|_{\mathcal{H}(k_1)}$ for all $f \in \mathcal{H}(k_1)$.

Lemma 3.2 ([7], Theorem 5.4)**.** *Let $\mathcal{H}(k_1)$ and $\mathcal{H}(k_2)$ be the reproducing kernel Hilbert spaces on set X, and let $k = k_1 + k_2$, then*

$$\mathcal{H}(k) = \{f = f_1 + f_2 : f_1 \in \mathcal{H}(k_1), f_2 \in \mathcal{H}(k_2)\}.$$

For every $f \in \mathcal{H}(k)$,

$$\|f\|^2_{\mathcal{H}(k)} = \min\left\{\|f_1\|^2_{\mathcal{H}(k_1)} + \|f_2\|^2_{\mathcal{H}(k_2)} : f = f_1 + f_2, f_1 \in \mathcal{H}(k_1), f_2 \in \mathcal{H}(k_2)\right\}.$$

Lemma 3.3 ([7],Theorem 5.7)**.** *Let $\varphi : S \to X$ be a function and let $k : X \times X \to \mathbb{C}$ be a positive kernel. Then*

$$\mathcal{H}(k \circ \varphi) = \{f \circ \varphi : f \in \mathcal{H}(k)\}.$$

Moreover, for $u \in \mathcal{H}(k \circ \varphi)$,

$$\|u\|_{\mathcal{H}(k\circ\varphi)} = \min\left\{\|f\|_{\mathcal{H}(k)} : u = f \circ \varphi\right\}.$$

Lemma 3.4. *Suppose $\varphi(z) \in H_1^\infty(\mathbb{D})$ is a non-constant inner function. A necessary condition for $\mathcal{H}(K_1^\varphi)$ to have complete Nevanlinna-Pick property is that φ is a Möbius map on $\mathbb{D}$.*

Proof. Without loss of generality, we can assume that $\varphi(0) = 0$. If $K_1^\varphi(z,w)$ has complete Nevanlinna-Pick property, then $1 - \frac{1}{K_1^\varphi(z,w)} \geq 0$, i.e.

$$1 - \frac{(1-\bar{w}z)^2}{1-\overline{\varphi(w)}\varphi(z)} = \frac{2\bar{w}z - \bar{w}^2 z^2 - \overline{\varphi(w)}\varphi(z)}{1-\overline{\varphi(w)}\varphi(z)} \geq 0. \tag{3.1}$$

Let

$$k_1(z,w) = \frac{2\bar{w}z}{1-\overline{\varphi(w)}\varphi(z)},$$
$$k_2(z,w) = \frac{\bar{w}^2z^2}{1-\overline{\varphi(w)}\varphi(z)},$$
$$k_\varphi(z,w) = \frac{\overline{\varphi(w)}\varphi(z)}{1-\overline{\varphi(w)}\varphi(z)}.$$

By Lemma 3.1 and Lemma 3.2, the (3.1) holds if and only if

$$\mathcal{H}(k_1) \supseteq \mathcal{H}(k_2+k_\varphi) \text{ and } \|f\|_{\mathcal{H}(k_1)} \le \|f\|_{\mathcal{H}(k_2+k_\varphi)}$$

for all $f \in \mathcal{H}(k_2+k_\varphi)$. Using Lemma 3.3, the corresponding reproducing kernel Hilbert space of reproducing kernel $\frac{1}{1-\overline{\varphi(w)}\varphi(z)}$ is

$$\mathcal{H}(s\circ\varphi) = \{f\circ\varphi : f \in H^2(\mathbb{D})\} = \left\{h(z) = \sum_{n=0}^{\infty} a_n(\varphi(z))^n, \sum_{n=0}^{\infty} |a_n|^2 < \infty\right\}.$$

Thus

$$\mathcal{H}(k_1) = \sqrt{2}z\mathcal{H}(s\circ\varphi),$$
$$\mathcal{H}(k_2) = z^2\mathcal{H}(s\circ\varphi),$$
$$\mathcal{H}(k_\varphi) = \varphi(z)\mathcal{H}(s\circ\varphi).$$

Since $\mathcal{H}(k_1) \supseteq \mathcal{H}(k_2+k_\varphi)$, then for every $f \in H^2(\mathbb{D})$, there exists $g \in H^2(\mathbb{D})$ such that

$$(z^2+\varphi(z))f(\varphi(z)) = \sqrt{2}zg(\varphi(z)).$$

By taking $f = 1$, then we also have $g \in H^2(\mathbb{D})$ such that

$$z^2+\varphi(z) = \sqrt{2}zg(\varphi(z)). \tag{3.2}$$

We claim that φ has to be injective. Suppose that $\varphi(\lambda_1) = \varphi(\lambda_2)$ for some pair $\lambda_1 \ne \lambda_2$, $\lambda_1, \lambda_2 \in \mathbb{D}$. Putting $z = \lambda_1$ and $z = \lambda_2$ in (3.2), we obtain that

$$\begin{cases} \lambda_1^2 + \varphi(\lambda_1) = \sqrt{2}\lambda_1 g(\varphi(\lambda_1)), \\ \lambda_2^2 + \varphi(\lambda_2) = \sqrt{2}\lambda_2 g(\varphi(\lambda_2)). \end{cases}$$

Then we have

$$g(\varphi(\lambda_1)) = \frac{\lambda_1+\lambda_2}{\sqrt{2}}.$$

Moreover, we assume that $\varphi(0) = 0$, thus we can let $\varphi(z) = zh(z)$. By $\varphi(z)$ is an inner function, we know that $h(z)$ is also inner function, and

$$g(\varphi(z)) = \frac{z+h(z)}{\sqrt{2}},$$

thus

$$h(\lambda_1) = \lambda_2,\ h(\lambda_2) = \lambda_1.$$

But if $\varphi(\lambda_1) = \varphi(\lambda_2)$, then for each $\tilde{\lambda}_1$ near to λ_1, there exists $\tilde{\lambda}_2$ near to λ_2 such that $\varphi(\tilde{\lambda}_1) = \varphi(\tilde{\lambda}_2)$, and hence we have

$$h(\tilde{\lambda}_1) = \tilde{\lambda}_2,\ h(\tilde{\lambda}_2) = \tilde{\lambda}_1.$$

This gives that

$$\varphi(\tilde{\lambda}_1) = \varphi(\tilde{\lambda}_2) = \varphi \circ h(\tilde{\lambda}_1)$$

for each $\tilde{\lambda}_1$ near to λ_1, then

$$\varphi(z) = \varphi \circ h(z),\ \forall z \in N(\lambda_1),$$

there $N(\lambda_1)$ is any domain of λ_1. This forces $\varphi \equiv \varphi \circ h$, and this gives that

$$zh(z) = h(z) h \circ h(z),$$

i.e. $h \circ h(z) = z$, we obtain that h is injective on $\mathbb{D}$, and $h^{-1} = h$. Therefore, $h(z)$ is a Möbius map. Let

$$h(z) = \beta \frac{a - z}{1 - \bar{a} z}$$

for some $|a| < 1$, $|\beta| = 1$, by $h^{-1} = h$ we have $\beta = 1$, thus

$$\varphi(z) = z \frac{a - z}{1 - \bar{a} z}.$$

Indeed, if $\varphi(z)$ is as described above, then

$$1 - \frac{1}{k^{\varphi}(z,w)} = \frac{2\bar{w}z - \bar{w}^2 z^2 - \bar{w}z \frac{\bar{a} - \bar{w}}{1 - a\bar{w}} \frac{a - z}{1 - \bar{a}z}}{1 - \bar{w}z \frac{\bar{a} - \bar{w}}{1 - a\bar{w}} \frac{a - z}{1 - \bar{a}z}}$$

is not positive (see e.g. Lemma 2.9 in [12]), which is a contradiction. Therefore φ must be injective, and φ is a Möbius map. □

Lemma 3.5. *Suppose $\varphi(z) = \beta \frac{a-z}{1-\bar{a}z}$, where $|a| < 1$, $|\beta| = 1$. Then $\mathcal{H}(K_1^{\varphi})$ is a complete Nevanlinna-Pick space.*

Proof. If $\varphi(z) = \beta \frac{a-z}{1-\bar{a}z}$ for all $z \in \mathbb{D}$, then

$$K_1^{\varphi}(z,w) = \frac{1 - \overline{\varphi(w)}\varphi(z)}{(1 - \bar{w}z)^2} = \frac{1 - |a|^2}{(1 - \bar{w}z)(1 - a\bar{w})(1 - \bar{a}z)},$$

it is easy to check that $K_1^{\varphi}(z,w)$ is a complete Nevanlinna-Pick kernel. □

Thus, combining Lemma 3.4 and Lemma 3.5, we obtain the following theorem.

Theorem 3.6. *Let $\varphi \in H_1^{\infty}(\mathbb{D})$ is a non-constant inner function. Then the sub-Bergman Hilbert space $\mathcal{H}(K_1^{\varphi})$ over the unit disk has the complete Nevanlinna-Pick property if and only if $\varphi(z)$ is a Möbius map.*

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