

On the sum of the inverses of element orders of $\text{PSL}(2, 7)$

Alireza Khalili Asboei and Habib Rahimi Esbo

ABSTRACT. For a finite group G , we write $\beta(G) = \sum_{k \in \pi_e(G)} \frac{m_k(G)}{k}$, where $m_k(G)$ denotes the number of elements of order k in G and $\pi_e(G)$ is the set of element orders of G . We show that the finite simple group $\text{PSL}(2, 7)$ is uniquely determined by the value of this invariant. In particular, we prove that if G is a finite non-solvable group with $\beta(G) = \frac{998}{21}$, then $G \cong \text{PSL}(2, 7)$.

CONTENTS

1. Introduction	1431
2. Preliminaries	1432
3. Proof of main theorem	1435
4. Acknowledgment	1438
References	1438

1. Introduction

Let G be a finite group, and $\pi(G)$ be the set of prime divisors of order of G . The study of numerical invariants associated with the element orders of finite groups has become an active topic in group theory. One such invariant is

$$\beta(G) = \sum_{k \in \pi_e(G)} \frac{m_k(G)}{k},$$

where $m_k(G)$ is the number of elements of order k in G and $\pi_e(G)$ is the set of element orders of G . This parameter was introduced and investigated in [4], where several classification and recognition results were obtained. In [2], the authors proved that A_5 is the only non-solvable group G satisfying $\beta(G) = \frac{599}{30}$.

In this paper, we continue this line of research and focus on characterizing the projective special linear group $\text{PSL}(2, 7)$. Our main result is the following theorem:

Received August 22, 2025.

2020 *Mathematics Subject Classification.* 20D60, 20D15, 20F16.

Key words and phrases. Finite groups, non-solvable group, solvable radical.

Theorem 1.1. *Let G be a finite non-solvable group. Then*

$$\beta(G) = \frac{998}{21} \iff G \cong \text{PSL}(2, 7).$$

The proof combines group-theoretic arguments with known classification results and relies on detailed calculations of $\beta(G)$ for candidate simple groups. This contributes to the general program of recognizing finite simple groups via numerical invariants.

2. Preliminaries

In order to establish our main theorem, we first collect several auxiliary results. These lemmas provide useful bounds for $\beta(G)$ and help restrict the possible simple groups that may appear.

Lemma 2.1. [4, Lemma 2.2] *If $H \leq G$, then $\beta(H) \leq \beta(G)$, with equality if and only if $H = G$.*

Lemma 2.2. [4, Lemma 2.3] *If $N \trianglelefteq G$, then $\beta(G/N) \leq \beta(G)$, with equality if and only if $N = 1$.*

Lemma 2.3. [5] *Let G be a finite group, and m be a positive integer dividing $|G|$. If $G_m = \{g \in G \mid g^m = 1\}$, then m divides $|G_m|$.*

Lemma 2.4. [3, Theorem 3.1] *Let G be a finite group with fewer than 15 involutions. Then solvable radical of G is nontrivial.*

Lemma 2.5. *Let G be a finite group whose Sylow 3-subgroups are all elementary abelian of order 9 (so $P \cong C_3 \times C_3$ for every $P \in \text{Syl}_3(G)$). Let $\mathcal{H}$ be the set of cyclic subgroups of order 3 in G . For $H \in \mathcal{H}$, let $r(H)$ denote the number of Sylow 3-subgroups of G that contain H . Then $r(H) \in \{1, 4\}$ for every such H , and $8n_3 = \sum_{H \in \mathcal{H}} 2r(H)$, $m_3 = 2|\mathcal{H}|$. Hence with $\bar{r} := \frac{1}{|\mathcal{H}|} \sum_{H \in \mathcal{H}} r(H)$, we have $m_3 = \frac{8n_3}{\bar{r}}$. In particular, $2n_3 \leq m_3 \leq 8n_3$, with equality on the left (resp. right) precisely when $r(H) = 4$ (resp. $r(H) = 1$) for all H .*

Proof. Define

$$\mathcal{S} = \{(g, P) \mid P \in \text{Syl}_3(G), g \in P, g \neq 1, |g| = 3\}.$$

Count by Sylows. Each $P \cong C_3 \times C_3$ has 8 nonidentity elements, all of order 3, so $|\mathcal{S}| = 8n_3$.

Count by elements. For $H = \langle g \rangle \in \mathcal{H}$, its two generators lie in exactly $r(H)$ Sylow 3-subgroups, giving $2r(H)$ pairs in $\mathcal{S}$. Summing over all H yields

$$|\mathcal{S}| = \sum_{H \in \mathcal{H}} 2r(H).$$

On the other hand each cyclic subgroup of order 3 has two generators, so $m_3 = 2|\mathcal{H}|$. Equating the two counts gives $8n_3 = \sum_{H \in \mathcal{H}} 2r(H)$, $m_3 = 2|\mathcal{H}|$, and therefore

$$m_3 = \frac{8n_3}{\bar{r}}, \quad \bar{r} = \frac{1}{|\mathcal{H}|} \sum_{H \in \mathcal{H}} r(H).$$

Determination of $r(H)$. Fix $H = \langle g \rangle$ of order 3 and let $C := C_G(H)$. A Sylow 3-subgroup of G contains H iff it is a Sylow 3-subgroup of C , so $r(H) = n_3(C)$. If $P \leq C$ is such a Sylow, conjugation by C induces a homomorphism $C \rightarrow \text{Aut}(P) \cong \text{GL}_2(3)$. Because $H \leq P$ is a 1-dimensional subspace of $P \cong \mathbb{F}_3^2$ (2-dimensional vector space over the field with 3 elements), the image lies in the stabilizer of a line in $\text{GL}_2(3)$, which has order 12. Thus $C/O_3(C)$ embeds into a group of order dividing 12, so $r(H) = n_3(C)$ divides $|C : O_3(C)|$, and by Sylow's theorem congruence $r(H) \equiv 1 \pmod{3}$. The only possibilities are $r(H) \in \{1, 4\}$. $\square$

We observe that if $L = \text{lcm}\{o(g) : g \in G\}$ is the exponent of group G , then $L\beta(G) = \sum_{g \in G} \frac{L}{o(g)} \in \mathbb{Z}$. In particular, if $\beta(G) = \frac{a}{b}$ in lowest terms with $\gcd(a, b) = 1$, then $b \mid L$.

Let C_n be cyclic of order n . Then (well-known) $\beta(C_n) = \sum_{d|n} \frac{\varphi(d)}{d}$, where φ is Euler's totient function. In particular $n\beta(C_n) \in \mathbb{Z}$.

Proposition 2.6. *If C_n is cyclic and $\beta(C_n) = \frac{998}{21}$, then $21 \mid n$.*

Proof. We have $n\beta(C_n) \in \mathbb{Z}$. Writing $\beta(C_n) = \frac{998}{21}$ in lowest terms and using $\gcd(998, 21) = 1$, we obtain $21 \mid n$. $\square$

Proposition 2.7. *Let G be a nontrivial finite p -group. Then $\beta(G) \neq \frac{998}{21}$.*

Proof. Let $L = \exp(G)$. For a p -group the exponent L is a power of p . We have $L\beta(G) \in \mathbb{Z}$. If $\beta(G) = \frac{998}{21}$ then

$$L \cdot \frac{998}{21} \in \mathbb{Z}.$$

Since $\gcd(998, 21) = 1$, this implies $21 \mid L$. But L is a power of a single prime p , while $21 = 3 \cdot 7$ has two distinct prime divisors; therefore no power of a single prime can be divisible by 21. This contradiction shows a nontrivial p -group cannot have $\beta(G) = \frac{998}{21}$. $\square$

If G is finite nilpotent then $G \cong \prod_p P_p$ the direct product of its Sylow p -subgroups P_p . When $\gcd(|A|, |B|) = 1$ by [4, Lemma 2.6]

$$\beta(A \times B) = \beta(A)\beta(B).$$

Thus for nilpotent G ,

$$\beta(G) = \prod_p \beta(P_p).$$

Proposition 2.8. *Let G be a nontrivial finite nilpotent group. Then $\beta(G) \neq \frac{998}{21}$.*

Proof. Since G is nilpotent we have a direct product decomposition $G \cong \prod_p P_p$ into its Sylow p -subgroups P_p , and for such coprime direct products β is multiplicative:

$$\beta(G) = \prod_p \beta(P_p).$$

For each prime p , write $\beta(P_p)$ in lowest terms as

$$\beta(P_p) = \frac{a_p}{p^{e_p}}, \quad a_p \in \mathbb{Z}_{\geq 1}, \quad e_p \in \mathbb{Z}_{\geq 0}, \quad \gcd(a_p, p) = 1.$$

(That p^{e_p} is a p -power follows from the standard fact that $p \cdot \beta(P_p) \in \mathbb{Z}$: m_{p^k} is divisible by $\varphi(p^k) = p^{k-1}(p-1)$, hence

$$p \beta(P_p) = p + \sum_{k \geq 1} \frac{m_{p^k}}{p^{k-1}} \in \mathbb{Z},$$

so in lowest terms the denominator of $\beta(P_p)$ divides a power of p .)

Assume, for contradiction, that

$$\prod_p \beta(P_p) = \frac{998}{21}.$$

Separate the $p = 3, 7$ factors from the rest and set

$$\beta(P_3) = \frac{A}{3^\alpha}, \quad \beta(P_7) = \frac{B}{7^\beta}, \quad T := \prod_{p \neq 3, 7} \beta(P_p) = \prod_{p \neq 3, 7} \frac{a_p}{p^{e_p}}.$$

Then

$$\frac{A}{3^\alpha} \cdot \frac{B}{7^\beta} \cdot T = \frac{998}{21}.$$

Since the only primes appearing in the denominator on the right are 3 and 7, all prime powers p^{e_p} with $p \neq 3, 7$ in the denominator of T must cancel *within* T itself. Therefore T is in fact an integer:

$$T \in \mathbb{Z}_{\geq 1}.$$

Moreover, for a nontrivial p -group P_p we have $\beta(P_p) > 1$; in particular, if any Sylow p -subgroup with $p \neq 3, 7$ is nontrivial then the corresponding factor $\beta(P_p) \geq 2$ can occur (indeed examples like C_4 give $\beta = 2$), so T is a positive integer ≥ 2 unless all such Sylow subgroups are trivial.

Next, observe that $\alpha, \beta \in \{0, 1\}$. Indeed, as above $p \beta(P_p) \in \mathbb{Z}$, so in lowest terms the denominator of $\beta(P_p)$ divides p (not a higher p -power); hence the total denominator contributed by $p = 3, 7$ on the left is $3^\alpha 7^\beta$. Comparing with the right-hand side, whose denominator is exactly $21 = 3 \cdot 7$, we must have $\alpha = \beta = 1$. Thus write

$$3 \beta(P_3) = A \in \mathbb{Z}_{\geq 1}, \quad 7 \beta(P_7) = B \in \mathbb{Z}_{\geq 1}.$$

Since P_3, P_7 are nontrivial, we also have the crude lower bounds

$$A = 3 \beta(P_3) \geq 3 + (3 - 1) = 5, \quad B = 7 \beta(P_7) \geq 7 + (7 - 1) = 13,$$

because there is at least one subgroup of order p in a nontrivial p -group, contributing $(p - 1)$ elements of order p and hence at least $(p - 1)/p$ to β .

Now multiply both sides of $\beta(G) = \frac{998}{21}$ by 21:

$$(3\beta(P_3)) \cdot (7\beta(P_7)) \cdot T = 998,$$

i.e.

$$A \cdot B \cdot T = 998 = 2 \cdot 499.$$

Here A, B, T are positive integers with $A \geq 5$ and $B \geq 13$. Since 998 has only prime factors 2 and 499, the only positive divisors of 998 are 1, 2, 499, 998. Therefore T must itself be one of $\{1, 2, 499, 998\}$, and consequently $A \cdot B$ must be the complementary divisor in $\{998, 499, 2, 1\}$ respectively. But $A \geq 5$ and $B \geq 13$ rule out all these possibilities: $A \cdot B \geq 65$ cannot be 1, 2, or 499, and $A \cdot B = 998$ would force $(A, B) \in \{(2, 499), (499, 2)\}$, which again contradicts $A \geq 5, B \geq 13$. Thus no such factorization is possible.

The only remaining way to avoid the contradiction would be $T = 1$, i.e. all Sylow subgroups with $p \neq 3, 7$ are trivial and hence $G \cong P_3 \times P_7$. But then we still need $A \cdot B = 998$ with $A \geq 5, B \geq 13$, which has no solution as above. Hence no nilpotent G satisfies $\beta(G) = \frac{998}{21}$. $\square$

3. Proof of main theorem

We first note that

$$\beta(\text{PSL}(2, 7)) = 1 + \frac{21}{2} + \frac{56}{3} + \frac{42}{4} + \frac{48}{7} = \frac{998}{21}.$$

Now suppose G is a finite non-solvable group with $\beta(G) = \frac{998}{21}$. From the definition of $\beta(G)$, all prime divisors appearing in denominators must divide $|G|$. Since $\beta(G) = \frac{998}{21}$, it follows that $\{3, 7\} \subseteq \pi(G)$. Since G is a non-solvable group, we have $2 \in \pi(G)$. Let N be the solvable radical of G , and choose a chief factor K/N of G . Then $K/N \cong S^t$ for some non-abelian simple group S and integer $t \geq 1$. By Lemmas 2.1 and 2.2 we have

$$\beta(K/N) \leq \beta(G) = \frac{998}{21}.$$

We will show that $K/N \cong \text{PSL}(2, 7)$. By Lemma 2.4, every non-abelian simple group S satisfies $m_2(S) \geq 15$. If $t \geq 2$, then the direct product structure forces $m_2(K/N) \geq 255$, whence $\beta(K/N) > \frac{998}{21}$, a contradiction. Therefore $t = 1$ and $K/N \cong S$.

If $3 \notin \pi(S)$, then $S \cong \text{Sz}(2^p)$ for some odd prime p . By [1, Proposition 3.3] one has

$$m_2(\text{Sz}(2^p)) = (q - 1)(q^2 + 1) \geq 455,$$

so again $\beta(S) > \frac{998}{21}$, a contradiction. Hence $3 \in \pi(S)$. Moreover, due to the simplicity of S , it has more than one cyclic subgroup for any prime $p \in \pi(S)$. On the other hand, $p \mid (1 + m_p(S))$. Therefore, we have the estimate $m_p(S) \geq p^2 - 1$, and in particular $m_3(S) \geq 8$.

If there exists a prime $p \geq 37$ in $\pi(S)$, then $\frac{m_p(S)}{p} \geq 36.9$, so

$$\beta(G) \geq \beta(S) \geq 1 + \frac{15}{2} + \frac{8}{3} + \frac{m_p(S)}{p} > \frac{998}{21},$$

a contradiction. Therefore every prime in $\pi(S) \setminus \{2, 3\}$ is at most 31. The finite simple groups with prime divisors contained in

$$\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31\}$$

are listed in [9, Table 1]. Checking these groups (the GAP computation) shows that, aside from A_5 and $\text{PSL}(2, 7)$, every other candidate satisfies $\beta(G) > \frac{998}{21}$. Hence $S \cong A_5$ or $S \cong \text{PSL}(2, 7)$.

If $S \cong \text{PSL}(2, 7)$, then the conclusion follows. Assume therefore that $S \cong A_5$ and derive a contradiction.

Let K denote the full preimage in G of $C_{G/N}(K/N)$, i.e. set

$$K = \{x \in G \mid xN \in C_{G/N}(K/N)\}.$$

Then $A_5 \trianglelefteq G/K \leq \text{Aut}(A_5) = S_5$.

Assume $N \neq K$. Since $N < K$ and N is the solvable radical, the subgroup K is non-solvable. Applying Lemmas 2.1 and 2.2 to K and repeating the reduction above, we obtain a normal subgroup T of K with

$$A_5 \trianglelefteq K/T \leq S_5.$$

By replacing K with K/T and N with N/T we may assume without loss of generality that T is the solvable radical of K .

Since $7 \mid |G| = |A_5||K|$ or $|S_5||K|$, we have $7 \mid |K|$, and hence $7 \mid |T|$. We claim that T is an elementary abelian 7-group. Indeed, because T is solvable we have $O^7(T) \neq T$. Put $\hat{T} = T/O^7(T)$ and $\hat{K} = K/O^7(T)$. The Frattini quotient $\hat{T}/\Phi(\hat{T})$ is elementary abelian of exponent 7. We have

$$K/T \cong \hat{K}/\hat{T} \cong \frac{\hat{K}/\Phi(\hat{T})}{\hat{T}/\Phi(\hat{T})}.$$

Replacing T by $\hat{T}/\Phi(\hat{T})$ we may assume T is elementary abelian of exponent 7.

Write $|T| = 7^t$. If $t \geq 3$ then $m_7(T) \geq 342$, so

$$\beta(G) \geq \beta(T) \geq 1 + \frac{m_7(T)}{7} > \frac{998}{21},$$

a contradiction. Hence $t \leq 2$.

If $t = 2$ then $|K| = 2940$ or 5880 . Note that by [6, Theorem 2.1],

$$n_3(K)n_3(G/K) \mid n_3(G).$$

Since K/T and G/K are A_5 or S_5 , we get $n_3(G) \geq 100$. If G has an element of order 9, then the number of elements of order 9 satisfies

$$m_9(G) = n_3(G)\phi(9) \geq 600,$$

whence $m_9(G)/9 > \beta(G)$, a contradiction. Therefore every Sylow 3-subgroup of G is elementary abelian $C_3 \times C_3$. By Lemma 2.5, we obtain $m_3 \geq 2n_3(G) \geq 200$, and thus $\beta(G) > \frac{998}{21}$, again a contradiction.

If $t = 1$ a similar argument leads to a contradiction; hence $N = K$ is the solvable radical of G . Thus

$$A_5 \trianglelefteq G/N \leq S_5,$$

with N solvable. Repeating the previous reduction for N we may assume N is an elementary abelian 7-group. As above $|N| = 7^t$ with $t \leq 2$.

If $t = 2$ then $|G| = 2940$ (when $G/N = A_5$) or $|G| = 5880$ (when $G/N = S_5$), and $m_7(G) = m_7(N) = 48$.

Assume $|G| = 5880$. Since $3 \nmid |N|$, by [6, Theorem 2.1] we have $n_3(G/N) = n_3(G)$. Hence $n_3(G) = 10 = \frac{m_3(G)}{\phi(3)}$ and so $m_3(G) = 20$. Similarly $n_5(G) = 6$ and $m_5(G) = 24$.

If G had no element of order 35 then a Sylow 7-subgroup would act fixed-point freely on the set of elements of order 5, forcing $|P_7|$ to divide m_5 and hence $m_5 \geq 49$, a contradiction. Therefore G has an element of order 35. By Lemma 2.3, we get $35 \mid (1 + m_5 + m_7 + m_{35})$, and since $24 \mid m_{35}$ we deduce $m_{35} \geq 312$.

On the other hand, $n_5(G) = 6 = \frac{|G|}{|N_G(P_5)|}$, so $|N_G(P_5)| = 4 \times 5 \times 49$. As $|N_G(P_5)/C_G(P_5)| \mid |\text{Aut}(P_5)| = 4$, we have $|C_G(P_5)| = 4 \times 5 \times 49$, $2 \times 5 \times 49$, or 5×49 . In each case $m_7(C_G(P_5)) = 48$. By [7, Lemma 2.3] we have $m_7(C_G(P_5))m_5 = 1152 \mid m_{35}$, hence $m_{35} \geq 1152$. Therefore

$$\beta(G) \geq 1 + \frac{20}{3} + \frac{24}{5} + \frac{48}{7} + \frac{1152}{35} > \frac{998}{21},$$

contradiction. Thus $|G| \neq 5880$. A similar calculation shows $|G| \neq 2940$.

If $t = 1$ then $|G| = 840$ or 420 . A GAP check shows the only non-solvable group of order 420 is $A_5 \times C_7$; but for this group $\beta(G) < \frac{998}{21}$, a contradiction. There are three non-solvable groups of order 840 with $P_7 = C_7 \times C_7$, namely $S_5 \times C_7 \times C_7$, $\text{SL}(2, 5) \times C_7 \times C_7$, and $A_5 \times C_2 \times C_7 \times C_7$, and in all these cases $\beta(G) > \frac{998}{21}$, a contradiction.

Combining the contradictions above, we conclude that $S \cong \text{PSL}(2, 7)$, and hence $K/N \cong \text{PSL}(2, 7)$. By Lemmas 2.1 and 2.2 this forces $G \cong \text{PSL}(2, 7)$, completing the proof.

Now the natural question arises: if G is a finite group, is it true that $\beta(G) = \frac{998}{21}$ if and only if $G \cong \text{PSL}(2, 7)$?

By our main theorem, the case of non-solvable groups is already settled. Proposition 2.8 provides a partial step in this direction (the case of nilpotent groups), but to fully resolve the above question we are led to the following:

Question. Does there exist a finite solvable group G such that $\beta(G) = \frac{998}{21}$?

4. Acknowledgment

We thank the referee for some extremely helpful remarks, which allowed us to improve the paper.

References

- [1] ALAVI, SEYED H.; DANESHKHAH, ASHRAF., MOSAED, HOSEIN P. Groups of the same type as Suzuki groups. *Int. J. Group Theory*. **8** (2019), no. 1, 35–42. MR3953262, Zbl 1443.20013, doi: 10.22108/ijgt.2017.21556. 1435
- [2] ANABANTI, CHIMERE. S.; ASBOEI, ALIREZA K. Another characterization of A_5 . Submitted to *Proceedings of the Japan Academy, Series A* (2025). 1431
- [3] ASBOEI, ALIREZA K. Groups with fewer than 15 involutions. *Commun Algebra* **51**(2023), no. 10, 4171–4175. MR4613261, Zbl 1521.20049, doi: 10.1080/00927872.2023.2198027. 1432
- [4] AZAD, MORTEZA B.; KHOSRAVI, BEHROOZ.; RASHIDI, HAMIDEH. On the sum of the inverses of the element orders in finite groups. *Commun. Algebra*. **51** (2023), no. 2, 694–698. MR4532818, Zbl 1517.20033, doi: 10.1080/00927872.2022.2109043. 1431, 1432, 1433
- [5] FROBENIUS, FERDINAND G. VERALLGEMEINERUNG DES SYLOWSCHEM SATZES. *Berlin* (1895), 981–993. JFM 26.0158.01, doi: 10.3931/e-rara-18880. 1432
- [6] HALL, MARSHAL J. On the number of Sylow subgroups in a finite group. *J. Algebra* **7** (1967), 363–371. MR0222159, Zbl 0178.02102, doi: 10.1016/0021-8693(67)90076-2 1436, 1437
- [7] SHAO, CHANGGUO.; JIANG, QINHUI. A new characterization of some linear groups by NSE. *J. Algebra Appl* **13** (2014), no. 2, 1350094 (9 pages). MR3119655, Zbl 1291.20013, doi: 10.1142/S0219498813500941. 1437
- [8] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.14.0 (2024). <https://www.gap-system.org>.
- [9] ZAVARNITSINE, ANDREI V. Finite simple groups with narrow prime spectrum. *Sib. Elektron. Mat. Izv* **6** (2009), 1–12. MR2586673, Zbl 1289.20021. 1436

(A. K. Asboei) DEPARTMENT OF MATHEMATICS EDUCATION, FARHANGIAN UNIVERSITY, P.O. BOX 14665-889, TEHRAN, IRAN
a.khalili@cfu.ac.ir

(H. R. Esbo) DEPARTMENT OF MATHEMATICS EDUCATION, BABOL, IRAN
hr192820@gmail.com

This paper is available via <http://nyjm.albany.edu/j/2025/31-55.html>.