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An alternative geometric proof of a theorem of Horrocks

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ABSTRACT. We give a geometric proof of Horrocks' theorem, which states that a vector bundle on the affine line over a local ring, extending to the projective line, is trivial. Unlike Horrocks' original approach via formal cohomology, our method uses semicontinuity and cohomology and base-change to construct a nowhere-vanishing section.

1. Introduction

Jean-Pierre Serre in his article Faisceaux algébriques cohérents [6] raised the following question: Let k be a field. Are vector bundles over $\mathbb{A}^n_k$(affine n-space) trivial? Over twenty years, there was progress made towards solving this. Finally, Daniel Quillen [5] and Andrei Suslin [8] independently proved the result. A detailed exposition can be found in [4]. One of the key steps in the complete solution to Serre's problem is due to Horrocks [3].

Theorem 1.1. *(Horrocks) Let R be a local ring and $\mathcal{F}$ be a vector bundle on $\mathbb{A}^1_R$ which extends to a vector bundle on $\mathbb{P}^1_R$. Then $\mathcal{F}$ is trivial.*

The crucial step in the proof is to produce a nowhere-vanishing section of $\mathcal{F}$. This is achieved by Horrocks using formal cohomology, whereas we use semicontinuity and base-change theorem to produce such a section.

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2. Proof

The proof relies on the following results concerning base change and the semicontinuity of cohomology. We present the result as stated in [7, Tag 0BDM].

Theorem 2.1. *Let $f : X \to S$ be a flat, proper morphism of finite presentation. Let $\mathcal{F}$ be a $\mathcal{O}_X$-module of finite presentation, flat over S. For a fixed $i \in \mathbb{Z}$. Consider the function*

$$\beta_i : S \to \mathbb{N}, s \mapsto dim_{\kappa(s)} H^i(X_s, \mathcal{F}_s)$$

Then we have

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(1) *formation of β_i commutes with arbitrary base change*
(2) *The functions β_i are upper semi-continuous*

Lemma 2.2. *Let $f : X \to S$ be a flat, proper morphism of finite presentation. Let $\mathcal{F}$ be a $\mathcal{O}_X$-module of finite presentation, flat over S. The function*

$$s \mapsto \chi(X_s, \mathcal{F}_s) = \sum_p (-1)^p dim_{\kappa(s)} H^p(X_s, \mathcal{F}_s)$$

is locally constant on S.

The following fundamental result on cohomology and base change is well known when the base scheme is locally Noetherian. The statement, however, holds without this assumption, and the proof is obtained by reducing to the Noetherian case via “Noetherian approximation”. Although this argument can be found in several sources, we include it here for completeness.

Proposition 2.3. *Let $f : X \to S$ be a proper and finitely presented morphism of schemes. Let $\mathcal{F}$ be a finitely presented sheaf on X which is flat over S. Suppose that for a point $s \in S$ and integer i the map $\phi_s^i : R^i f_*(\mathcal{F})_s \otimes_{\mathcal{O}_{S,s}} \kappa(s) \to H^i(X_s, \mathcal{F} \otimes \kappa(s))$ is surjective. Then the following hold*

(1) *There is an open neighbourhood $U \subset S$ containing s such that ϕ_s^i is an isomorphism*
(2) *ϕ_s^{i-1} is surjective if and only if $R^i f_* \mathcal{F}$ is a vector bundle in an open neighbourhood of s*

Proof. The statement is local on S, we can assume S is affine and we can write $S = \lim_{\lambda \in \Lambda} S_\lambda$ where S_λ are affine schemes of finite type over $\mathbb{Z}$. Since f is finitely presented, there exists $0 \in \Lambda$ and a finitely presented morphism $f_0 : X_0 \to S_0$ such that $X \simeq X_0 \times_{S_0} S$ [7, Tag 01ZM]. For each $\lambda > 0$, we define $X_\lambda = X_0 \times_{S_0} S_\lambda$ and we have $X \simeq X_\lambda \times_{S_\lambda} S$. By [7, Tag 081C, Tag 05M5], $X_\lambda \to S_\lambda$ is proper for $\lambda \gg 0$. By [7, Tag 01ZR, Tag OB8W, Tag O5LY], there exists an index $\mu \in \Lambda$ and a coherent sheaf $\mathcal{F}_\mu$ on X_μ that pulls back to $\mathcal{F}$ under $X \to X_\mu$. For $\lambda > \mu$, set $\mathcal{F}$ to the pull back of $\mathcal{F}_\mu$ under $X_\lambda \to X_\mu$. By [7, Tag 01ZR, Tag OB8W, Tag O5LY], $\mathcal{F}_\lambda$ is flat for S_λ for $\lambda \gg 0$. We apply [1, 3.2.1] to the data $X_\lambda \to S_\lambda$ and $\mathcal{F}_\lambda$ for $\lambda \gg 0$ and we deduce the proposition for $X \to S$ and $\mathcal{F}$ under the base change $S \to S_\lambda$. □

We now state a key result, due to Dedekind–Weber, and independently, Birkhoff, which is instrumental to the proof. For a proof, the reader may refer to [2, Theorem 4.1].

Theorem 2.4. *Let $\mathcal{E}$ be a rank r vector bundle on $\mathbb{P}_k^1$. Then $\mathcal{E} = \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r)$ for a unique nonincreasing sequence of integers $a_1, a_2, \cdots, a_r$.*

Proof of Horrocks's theorem. Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}_R^1$ which is an extension of $\mathcal{F}$. Let $\mathcal{E}'$ be the restriction of $\mathcal{E}$ to the special fibre $\mathbb{P}_k^1$, where $k = R/\mathfrak{m}$. Then it follows from Theorem 2.4 that

$$\mathcal{E}' = \oplus_{i=0}^m \mathcal{O}_{\mathbb{P}_k^1}(a_i),$$

for some integers a_i. We may assume $a_i \geq 0$, for $1 \leq i \leq m$, and $a_i = 0$, for some i, by suitably twisting $\mathcal{E}^{'}$ and $\mathcal{E}$. Note that twisting(tensoring by line bundle) does not affect the hypothesis as the assertion is local, hence we get that $\mathcal{E}^{'}$ has a nowhere vanishing section and $H^1(\mathbb{P}^1_k, \mathcal{E}^{'}) = 0$.

We claim $\mathcal{E}$ has a nowhere vanishing section. Then we have an exact sequence given by a nowhere vanishing section of $\mathcal{E}$

$$0 \to \mathcal{O}_{\mathbb{P}^1_R} \to \mathcal{E} \to \mathcal{E}/\mathcal{O}_{\mathbb{P}^1_R} \to 0$$

Now, restriction of $\mathcal{E}$ to the $\mathbb{A}^1_R$ splits. In particular, $\mathcal{F} = \mathcal{F}^{'} \oplus \mathcal{O}_{\mathbb{A}^1_R}$ with the rank of $\mathcal{F}^{'}$ strictly less than the rank of $\mathcal{F}$. Then the proof follows by induction on the rank. Now we show that $\mathcal{E}$ has a nowhere vanishing section.

Since R is local every closed subset of Spec$R$ contains the unique maximal ideal $\mathfrak{m}$, using Theorem 2.1, we have the following inequality:

$$dim_{\kappa(y)} H^0(\mathbb{P}^1_{\kappa(y)}, \mathcal{E}_y) \leq dim_k H^0(\mathbb{P}^1_k, \mathcal{E}^{'})$$

for all $y \in$ Spec$R$.

Moreover, R being local, it follows from Lemma 2.2 that the function $y \mapsto \chi(\mathcal{E}_y)$ is constant and we conclude

$$dim_k H^0(\mathbb{P}^1_k, \mathcal{E}^{'}) \leq dim_{\kappa(y)} H^0(\mathbb{P}^1_{\kappa(y)}, \mathcal{E}_y)$$

Combining, we have $dim_k H^0(\mathbb{P}^1_k, \mathcal{E}^{'}) = dim_{\kappa(y)} H^0(\mathbb{P}^1_{\kappa(y)}, \mathcal{E}_y)$ and

$$H^1(\mathbb{P}^1_{\kappa(y)}, \mathcal{E}_y) = 0$$

for all $y \in$ Spec$R$.

Let $\pi : \mathbb{P}^1_R \to$ Spec$R$ be the structure map. Then, the pullback map $\varphi : H^0(\mathbb{P}^1_R, \mathcal{E}) \to H^0(\mathbb{P}^1_k, \mathcal{E}^{'})$ factors as

$$\begin{tikzcd} H^0(\mathbb{P}^1_R, \mathcal{E}) \arrow[r, "\varphi"] \arrow[d, two heads] & H^0(\mathbb{P}^1_k, \mathcal{E}^{'}) \\ \pi_* \mathcal{E} \otimes k \arrow[ur, two heads, "\phi^0"'] & \end{tikzcd}$$

where the surjectivity of ϕ^0 is given by Proposition 2.3 as $H^1(\mathbb{P}^1_{\kappa(y)}, \mathcal{E}_y) = 0$ for all $y \in$ Spec$R$. Hence, the pullback map φ is surjective.

Finally, we show the pullback of nowhere vanishing section is nowhere vanishing. Let s be a nowhere vanishing section of $\mathcal{E}^{'}$. Suppose $s = \varphi(t)$, where t is not a nowhere vanishing section of $\mathcal{E}$. Then as the map $\pi : \mathbb{P}^1_R \to$ Spec$R$ is proper the image of $V(t)$(the pullback of the locus where t vanishes is the locus where the pulled-back section s vanishes) is closed, R being local implies $V(t)$ contains $\mathfrak{m}$, which is absurd.

$$\begin{tikzcd} V(s) \arrow[r] \arrow[d, hook] & V(t) \arrow[d, hook] \\ \mathbb{P}^1_k \arrow[r] \arrow[d] & \mathbb{P}^1_R \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} R \end{tikzcd}$$

□

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