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Distributions as travelling waves in a classical conservation law

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ABSTRACT. The present paper concerns the propagation of distributional travelling waves in models ruled by the equation $u_t + [\phi(u)]_x = 0$, where ϕ stands for an entire function. Using the concept of α-solution defined in the setting of a distributional product, it is possible to establish a deeper insight about the propagation of such waves. The set of α-solutions contains all weak solutions for this equation and allows us to understand that, in the nonlinear case, the propagation is not possible for a large class of important profiles (all nonconstant continuous profiles are included). However some profiles that are not continuous functions, others that are not functions, and also others that are not measures may propagate. As a particular case, the characterization of all profiles that, locally, are bounded variation functions, is easily established. A brief survey of ideas and formulas used to compute $\phi(u)$ when u is a distribution is included.

CONTENTS

1. Introduction and contents

Let us consider the conservation law

$$u_t + [\phi(u)]_x = 0 \tag{1}$$

where $x \in \mathbb{R}$ is the space variable, $t \in I = [0, +\infty[$ is the time variable, $u = u(x,t)$, is the unknown state variable and $\phi : \mathbb{C} \to \mathbb{C}$ is an entire function that takes real values on the real axis.

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In the present paper we study the propagation of distributional travelling waves for equation (1). Such propagation is a relevant physical phenomenon. Discontinuous functions, Dirac-delta measures and their distributional derivatives are respectively idealizations for sharp jumps, localized high picks and single sharp localized oscillations.

The difficulties in evaluating $\phi(u)$ when u is a distribution will be afforded by our theory of distributional products. We stress that we do not assume any classical result about conservation laws; thus, an easier and more general framework to discover singular solutions to this and others equations is presented.

The main question is a necessary and sufficient condition for the propagation of a distributional profile as travelling wave with a certain speed $\gamma'(t)$, supposed a continuous function of t. This condition has a lot of consequences. One of them is the impossibility of propagation of nonconstant continuous profiles in the nonlinear case. Possibly, this is one of the reasons for the absence in the literature of a significant number of references to travelling waves for equation (1). For the same equation with source terms, see [3, 2, 33, 28] and references therein. Thus, we will seek for travelling waves among distributions that do not correspond to continuous functions. We present explicit particular cases including profiles that are singular measures and profiles that are not measures.

The condition for the propagation of a distributional profile also allows us to characterize all locally bounded variation functions that can be taken as profiles of travelling waves for equation (1). In all cases, the speed of propagation is computed.

For such purpose, we adopt a solution concept defined in the framework of a distributional product. This concept extends both the classical solution concept and the concept of weak solution (for this one see [22], p.868, Remark 13). Our distributional product is not defined by approximation; it is known that processes involving weak limits of sequences of continuous functions may not yield to mathematically consistent solutions (see [24], section II) and also frequently these solutions cannot be substituted in the equations owing to the difficulties in multiplying distributions or computing $\phi(u)$ when u is a distribution.

In our framework, the product of two distributions is a distribution that depends on the choice of a function α encoding the indeterminacy inherent to such products. As a consequence, the solutions of differential equations with such products may depend, or not, on α. We call such solutions α-solutions.

When the solutions depend on α, the future behavior of the system cannot be fully predicted. We will give an example at the end of example 6.6. This fact might be due to physical features omitted in the formulation of the model with the goal of simplifying it. Solutions not depending on α mean that the physical system can exhibit such solutions.

The α-solutions can be substituted directly in the equations and represent a significant advance in the study of singular solutions of nonlinear equations or systems. One of the key advantages of this method is the fact that, for many problems, the α-solutions remain independent of α. This feature has led to an increasing adoption of this method by several authors as evidenced by recent publications such as [30, 32, 5, 6, 7, 27, 29, 8, 9, 10, 1, 4, 31]).

Another reason regards to important simplifications: the Rankine-Hugoniot shock conditions and their multiple generalizations are not necessary and, as far as we know, the final result is the same (see for example, [13, 14, 6]).

Also interestingly, in a recent paper [8], Pang et al. studied a degenerate system of conservation laws and compared the exact α-solutions, computed by our methods, with the numerical solutions obtained by the Nessyahu-Tadmor scheme. It happens that the exact solutions coincide with the corresponding numerical solutions. For more problems where the concept of α-solution was also applied, the reader may see [23, 22, 16, 25, 19, 21] and other works of the author.

The present paper is organized as follows: in Sections 2, 3 and 4 we present some ideas about our product of distributions, and we define powers of distributions, all this with the goal of giving a sense to $\phi(u)$, the composition of an entire function with a distribution. In Section 5, the concept of α-solution for equation (1) is defined. Finally in Section 6, we give the necessary and sufficient conditions for the propagation of a distributional profile for the equation (1). In this section, we also establish several consequences of this result including the characterization of all profiles that locally are bounded variation functions and may emerge as travelling waves in models ruled by the mentioned equation.

2. The multiplication of distributions

The present section concerns some formulas from our theory of distributional products that are of interest in the sequel. To get a general view of our distributional products, see [20] Sections 2 and 3 and also [23] Sections 2, 3 and 4. Details are given in [11].

Let $\mathcal{D}$ be the space of indefinitely differentiable complex-valued functions defined on $\mathbb{R}$, with compact support, and let $\mathcal{D}'$ be the space of Schwartz distributions. In our theory, each function $\alpha \in \mathcal{D}$ with $\int_{-\infty}^{+\infty} \alpha = 1$ affords a general product $T \dot{\alpha} S \in \mathcal{D}'$ of $T \in \mathcal{D}'$ with $S \in \mathcal{D}'$. The exact definition of this general α-product can be seen in [22] Section 2, formula (2.2).

Our general α-product is bilinear and is transformed as usual by translations, that is,

$$\tau_a(T \dot{\alpha} S) = (\tau_a T) \dot{\alpha} (\tau_a S), \tag{2}$$

where τ_a denotes the usual translation operator in distributional sense. In general, associativity and commutativity do not hold. Recall that in the setting of the classical products of distributions, the commutative property is a convention inherent to the definition of such products and the associative property does not hold in general (see the monograph of Schwartz [26] pp. 117, 118, 121, where these products are defined). The usual differential rules are satisfied, including the Leibniz formula, which must be written in the form

$$D(T_{\dot{\alpha}}S) = (DT)_{\dot{\alpha}}S + T_{\dot{\alpha}}(DS), \tag{3}$$

where D is the derivative operator in distributional sense. This general α-product is not consistent with the Schwartz products of distributions with functions but it is possible to define certain α-products in order to recover that consistency. This happens with the α-product

$$T_{\dot{\alpha}}S = T\beta + (T * \alpha)f \tag{4}$$

for $T \in \mathcal{D}'^p$ and $S = \beta + f \in C^p \oplus \mathcal{D}'_\mu$, where $p \in \{0,1,2,\dots,\infty\}$, $\mathcal{D}'^p$ is the space of distributions of order $\le p$ in the sense of Schwartz ($\mathcal{D}'^\infty$ means $\mathcal{D}'$), $\mathcal{D}'_\mu$ is the space of distributions whose support has Lebesgue measure zero, $T\beta$ is the usual Schwartz product of a $\mathcal{D}'^p$-distribution with a C^p-function and $(T * \alpha)f$ is the usual product of a C^∞-function with a distribution. For instance, if β is a continuous function, we have for any α,

$$\delta_{\dot{\alpha}}\beta = \delta_{\dot{\alpha}}(\beta + 0) = \delta\beta + (\delta * \alpha)0 = \beta(0)\delta, \tag{5}$$

$$\beta_{\dot{\alpha}}\delta = \beta_{\dot{\alpha}}(0 + \delta) = \beta 0 + (\beta * \alpha)\delta = [(\beta * \alpha)(0)]\delta, \tag{6}$$

$$\delta_{\dot{\alpha}}\delta = \delta_{\dot{\alpha}}(0 + \delta) = \delta 0 + (\delta * \alpha)\delta = \alpha\delta = \alpha(0)\delta, \tag{7}$$

$$\delta_{\dot{\alpha}}(D\delta) = (\delta * \alpha)D\delta = \alpha D\delta = \alpha(0)D\delta - \alpha'(0)\delta \tag{8}$$

$$(D\delta)_{\dot{\alpha}}\delta = (D\delta * \alpha)\delta = \alpha'\delta = \alpha'(0)\delta, \tag{9}$$

$$H_{\dot{\alpha}}\delta = (H * \alpha)\delta = \left[\int_{-\infty}^{+\infty} \alpha(-\tau)H(\tau)\,d\tau\right]\delta = \left(\int_{-\infty}^{0} \alpha\right)\delta, \tag{10}$$

$$H_{\dot{\alpha}}(D\delta) = (H * \alpha)(D\delta) = \left(\int_{-\infty}^{0} \alpha\right)(D\delta) - \alpha(0)\delta, \tag{11}$$

$$(D\delta)_{\dot{\alpha}}(D\delta) = (D\delta * \alpha)(D\delta) = \alpha'(D\delta) = \alpha'(0)(D\delta) - \alpha''(0)\delta \tag{12}$$

The α-product (4) is consistent with all Schwartz products of $\mathcal{D}'^p$-distributions with C^p-functions if the C^p-functions are placed at the right-hand side. It also keeps the bilinearity and satisfies (2) and (3), this one clearly under certain natural conditions; for $T \in \mathcal{D}'^p$ we must suppose $S \in \mathcal{C}^{p+1} \oplus \mathcal{D}'_\mu$.

From Leibniz formula (3), it is possible to define new α-products. The following formula was constructed this way (for details see [15], Section 2),

$$T_{\dot{\alpha}}S = Tw + (T * \alpha)f, \tag{13}$$

for $T \in \mathcal{D}'^{-1}$ and $S = w + f \in L^1_{loc} \oplus \mathcal{D}'_\mu$, where $\mathcal{D}'^{-1}$ denotes the space of distributions $T \in \mathcal{D}'$ such that $DT \in \mathcal{D}'^0$, and Tw is the usual pointwise product of $T \in \mathcal{D}'^{-1}$ with $w \in L^1_{loc}$. Thus, locally, T can be read as a function of bounded variation and $\mathcal{D}'^{-1}$ as the space of locally bounded variation functions. For instance, since $H \in \mathcal{D}'^{-1}$ and $H = H + 0 \in L^1_{loc} \oplus \mathcal{D}'_\mu$, we have

$$H_{\dot{\alpha}}H = HH + (H * \alpha)0 = H, \tag{14}$$

for any α. More generally, if $T \in \mathcal{D}'^{-1}$ and $S \in L^1_{loc}$ then $T_{\dot\alpha}S = TS$, for any α, because by (13) we can write

$$T_{\dot\alpha}S = T_{\dot\alpha}(S+0) = TS + (T * \alpha)0 = TS.$$

We also use another α-product that is computed by the formula

$$T_{\dot\alpha}S = D(Y_{\dot\alpha}S) - Y_{\dot\alpha}(DS), \tag{15}$$

for $T, Y \in \mathcal{D}'$ such that $DY = T$. It is applied in two instances:

(N) for $T \in \mathcal{D}'^0 \cap \mathcal{D}'_\mu$ and $S, DS \in L^1_{loc} \oplus \mathcal{D}'_c$, where $\mathcal{D}'_c \subset \mathcal{D}'_\mu$ is the space of distributions whose support is at most countable;

(NP) for $T \in \mathcal{D}'^1 \cap \mathcal{D}'_\mu$ and $S, DS, D^2S \in L^1_{loc} \oplus \mathcal{D}'_c$.

In any case, the value of $T_{\dot\alpha}S$ is independent of the choice of Y such that $DY = T$ (see [15], p. 1004 and [17] Section 2, for details). The compatibility of (4), (13) and (15) is effective, that is, if the same α-product can be computed by two different formulas, they give the same result. For the proof see [12] Lemma 3.3, [15] Section 3 and [17] Theorem 1. For instance, using (15) in case (N), and (10), we have for any α,

$$\begin{aligned} \delta_{\dot\alpha}H &= D(H_{\dot\alpha}H) - H_{\dot\alpha}(DH) = DH - H_{\dot\alpha}\delta \\ &= \delta - \left(\int_{-\infty}^{0} \alpha\right)\delta = \left(\int_{0}^{+\infty} \alpha\right)\delta, \end{aligned} \tag{16}$$

so that

$$H_{\dot\alpha}\delta + \delta_{\dot\alpha}H = \delta,$$

for any α. Another example applied in the sequel can be obtained using (15) in case (NP),

$$\begin{aligned} (D\delta)_{\dot\alpha}H &= D(\delta_{\dot\alpha}H) - \delta_{\dot\alpha}(DH) = D\left[\left(\int_0^{+\infty} \alpha\right)\delta\right] - \delta_{\dot\alpha}\delta \\ &= \left(\int_0^{+\infty} \alpha\right)(D\delta) - \alpha(0)\delta. \end{aligned} \tag{17}$$

In general, $\operatorname{supp}(T_{\dot\alpha}S) \subset \operatorname{supp} S$ as it happens for the usual product of functions, but it may happen that $\operatorname{supp}(T_{\dot\alpha}S) \not\subset \operatorname{supp} T$. This takes place, for example, in the α-product (see (4)),

$$\delta_{\dot\alpha}(\tau_1\delta) = (\delta * \alpha)(\tau_1\delta) = \alpha(\tau_1\delta) = \alpha(1)(\tau_1\delta).$$

Remark 2.1. *We stress that the mentioned α-products are always considered in a global sense, that is, they are not defined for distributions on proper subsets of $\mathbb{R}$. For example, denoting by T_Ω the restriction of $T \in \mathcal{D}'$ to an open set Ω of $\mathbb{R}$, the α-product $(T_\Omega)_{\dot\alpha}(S_\Omega)$ possibly make sense but only if $\Omega = \mathbb{R}$. Clearly, also $(T_\Omega)(S_\Omega)$ may have a sense as a classical product of distributions. About the equality $(T_{\dot\alpha}S)_\Omega = (T_\Omega)(S_\Omega)$, the reader can see some delicate results in* [20]*, Section 3.*

3. Powers of distributions

Let $M \subset \mathcal{D}'$ be a set of distributions such that, if $T_1, T_2 \in M$ then $T_1 \dot{\alpha} T_2$ is well defined and $T_1 \dot{\alpha} T_2 \in M$. For each $T \in M$, we define the α-power of T, T_α^n, by the recurrence relation

$$T_\alpha^n = (T_\alpha^{n-1})_{\dot{\alpha}} T \text{ for } n \geq 1, \text{ with } T_\alpha^0 = 1 \text{ for } T \neq 0. \tag{18}$$

Naturally, if $0 \in M$, $0_\alpha^n = 0$ for all $n \geq 1$.

Since our distributional products are consistent with the Schwartz products of distributions with functions, when the functions are placed on the right-hand side, we have $\beta_\alpha^n = \beta^n$ for all $\beta \in C^0 \cap M$. Thus, this definition is consistent with the usual definition of powers of C^0-functions. Moreover, if M is such that $\tau_a T \in M$ for all $T \in M$ and all $a \in \mathbb{R}$, then we also have

$$(\tau_a T)_\alpha^n = \tau_a(T_\alpha^n).$$

Taking, for instance, $M = C^p \oplus (\mathcal{D}'^p \cap \mathcal{D}'_\mu)$ and supposing $T_1, T_2 \in M$, we have $T_1 = \beta_1 + f_1$, $T_2 = \beta_2 + f_2$ and by (4) we can write

$$\begin{aligned} T_1 \dot{\alpha} T_2 &= T_1\beta_2 + (T_1 * \alpha) f_2 = (\beta_1 + f_1)\beta_2 + [(\beta_1 + f_1) * \alpha] f_2 \\ &= \beta_1\beta_2 + f_1\beta_2 + [(\beta_1 + f_1) * \alpha] f_2 \in M. \end{aligned}$$

Therefore, we can define α-powers T_α^n of distributions $T \in C^p \oplus (\mathcal{D}'^p \cap \mathcal{D}'_\mu)$. For instance, using (7) we have $\delta_\alpha^1 = \delta$, and for $n \geq 2$, $\delta_\alpha^n = [\alpha(0)]^{n-1}\delta$.

Setting $M = \mathcal{D}'^{-1}$ and supposing $T_1, T_2 \in \mathcal{D}'^{-1}$, we have $T_2 = T_2 + 0 \in L^1_{loc} \oplus \mathcal{D}'_\mu$. Then, by applying (13) we have,

$$T_{1\dot{\alpha}} T_2 = T_{1\,\dot{\alpha}}(T_2 + 0) = T_1 T_2 + (T_1 * \alpha)0 = T_1 T_2 \in \mathcal{D}'^{-1}.$$

Thus, we also can define α-powers T_α^n of distributions $T \in \mathcal{D}'^{-1}$ by the recurrence relation (18) and clearly we get,

$$T_\alpha^n = T^n, \tag{19}$$

that is, in distributional sense the α-powers of functions that, locally, are of bounded variation coincide with the usual powers of these functions when considered as distributions. In the sequel and for short, we will write T^n instead of T_α^n, supposing α fixed. For instance, we will write $\delta^3 = [\alpha(0)]^2\delta$.

4. Composition of an entire function with a distribution

Let $\phi : \mathbb{C} \to \mathbb{C}$ be an entire function. Then we have,

$$\phi(z) = a_0 + a_1 z + a_2 z^2 + \cdots \tag{20}$$

for the sequence $a_n = \frac{\phi^{(n)}(0)}{n!}$ of complex numbers and for all $z \in \mathbb{C}$. Given α, if $T \in [C^p \oplus (\mathcal{D}'^p \cap \mathcal{D}'_\mu)] \cup \mathcal{D}'^{-1}$, we define the composition $\phi \circ T$ by formula

$$\phi \circ T = a_0 + a_1 T + a_2 T^2 + \cdots \tag{21}$$

whenever this series converge in $\mathcal{D}'$. Clearly, this definition is consistent with the usual meaning of $\phi\circ T$, when $T \in C^0 \cup \mathcal{D}'^{-1}$, and we have $\tau_a(\phi\circ T) = \phi\circ(\tau_a T)$, if $\phi\circ T$ or $\phi\circ(\tau_a T)$ are well defined. Remember that, such as T^n, $\phi\circ T$ may depend on α. For instance, we will use the following result:

Theorem 4.1. *Let $\phi : \mathbb{C} \to \mathbb{C}$ be an entire function. Then, given α, we have*

$$\phi\circ\delta = \begin{cases} \phi(0) + \phi'(0)\delta & \text{if } \alpha(0) = 0, \\ \phi(0) + \frac{\phi(\alpha(0))-\phi(0)}{\alpha(0)}\delta & \text{if } \alpha(0) \neq 0. \end{cases} \tag{22}$$

Proof. Let ϕ be defined by (20). Using definition (21) and the formula $\delta^n = [\alpha(0)]^{n-1}\delta$, we have,

$$\begin{aligned} \phi\circ\delta &= a_0 + a_1\delta + a_2\delta^2 + a_3\delta^3 + \cdots \\ &= a_0 + a_1\delta + a_2\alpha(0)\delta + a_3\alpha(0)^2\delta + \cdots \\ &= a_0 + [a_1 + a_2\alpha(0) + a_3\alpha(0)^2 + \cdots]\delta. \end{aligned} \tag{23}$$

Thus, if $\alpha(0) = 0$ we have $\phi\circ\delta = a_0 + a_1\delta$. If $\alpha(0) \neq 0$, setting

$$S = a_1 + a_2\alpha(0) + a_3\alpha(0)^2 + \cdots$$

we have

$$\alpha(0)S = a_1\alpha(0) + a_2\alpha(0)^2 + a_3\alpha(0)^3 + \cdots = \phi\big(\alpha(0)\big) - a_0,$$

and

$$S = \frac{\phi\big(\alpha(0)\big) - a_0}{\alpha(0)},$$

follows. Also from (23) we have $\phi\circ\delta = a_0 + S\delta$ and the result follows immediately, since $a_0 = \phi(0)$ and $a_1 = \phi'(0)$. □

5. The α-solution concept for equation (1)

Let I be an interval of $\mathbb{R}$ with more than one point and let $\mathcal{F}(I)$ be the space of continuously differentiable maps $\tilde{u} : I \to \mathcal{D}'$ in the sense of the usual topology of $\mathcal{D}'$. For $t \in I$, the notation $[\tilde{u}(t)](x)$ is sometimes used for emphasizing that the distribution $\tilde{u}(t)$ acts on functions $\xi \in \mathcal{D}$ depending on x.

Let $\Sigma(I)$ be the space of functions $u : \mathbb{R} \times I \to \mathbb{R}$ such that:

(a) for each $t \in I$, $u(x,t) \in L^1_{loc}(\mathbb{R})$;

(b) $\tilde{u} : I \to \mathcal{D}'$, defined by $[\tilde{u}(t)](x) = u(x,t)$ belongs to $\mathcal{F}(I)$.

The natural injection $u \mapsto \tilde{u}$ from $\Sigma(I)$ into $\mathcal{F}(I)$ identifies any function in $\Sigma(I)$ with a certain map in $\mathcal{F}(I)$. Since $C^1(\mathbb{R}\times I) \subset \Sigma(I)$, we can write the inclusions

$$C^1(\mathbb{R}\times I) \subset \Sigma(I) \subset \mathcal{F}(I),$$

the last one clearly in the sense of the mentioned identification. Thus, identifying u with $\tilde{u}$ the equation (1) reads as follows:

$$\frac{d\tilde{u}}{dt}(t) + D[\phi\circ\tilde{u}(t)] = 0. \tag{24}$$

Definition 5.1. *Given α, the function $\tilde{u} \in \mathcal{F}(I)$ will be called an α- solution of the equation (24) on I, if $\phi \circ \tilde{u}(t)$ is well defined, and this equation is satisfied for all $t \in I$.*

Thus, the equation (24) is seen as an evolution equation in the time interval I and we have the following results:

Theorem 5.2. *If u is a classical solution of (1) on $\mathbb{R} \times I$ then, for any α, $\tilde{u} \in \mathcal{F}(I)$ defined by $[\tilde{u}(t)](x) = u(x,t)$ is an α-solution of (24) on I.*

Note that by a classical solution of (1) on $\mathbb{R} \times I$, we mean a C^1-function $u(x,t)$ that satisfies (1) on $\mathbb{R} \times I$.

Theorem 5.3. *If $u : \mathbb{R} \times I \to \mathbb{R}$ is a C^1-function and, for a certain α, $\tilde{u} \in \mathcal{F}(I)$ defined by $[\tilde{u}(t)](x) = u(x,t)$ is an α-solution of (24) on I, then u is a classical solution of (1) on $\mathbb{R} \times I$.*

For the proof, it is enough to observe that the C^1-functions $u(x,t)$ can be read as continuously differentiable function $\tilde{u} \in \mathcal{F}(I)$ defined by $[\tilde{u}(t)](x) = u(x,t)$ and to use the consistency of the α-products with the classical Schwartz products as well as the consistency of the operation $\circ$ with the usual composition of C^1-functions.

Remark 5.4. *Recall that, for the α-products (4), (13) or (15), $T \dot{\alpha} S = TS$ for any α, if $T \in \mathcal{D}'^p$ and $S \in C^p$, that is, if TS is a classical Schwartz product. Also recall that $\phi \circ T$ has the usual meaning when T corresponds to a C^1-function (see the initial part of Section 4).*

As a consequence, an α-solution $\tilde{u}$ in this sense, read as a usual distributional solution u affords a general and consistent extension of the concept of a classical solution for the equation (1). In this sense, we also call u an α-solution of (1) on I. Thus, from now on, α-solutions are referred as according to (1) or (24).

6. Travelling waves

We introduce the following definition:

Definition 6.1. *Let $\gamma : I = [0, +\infty[\to \mathbb{R}$ be a C^1-function. Then, given α, the wave profile $U \in \mathcal{D}'$ α-propagates according to (1) or (24) with the movement $\gamma(t)$ (and speed $\gamma'(t)$) if the travelling wave $\tilde{u}(t) = \tau_{\gamma(t)} U$ is an α-solution of (24) on I.*

Note that any constant profile U α-propagates according to (1) or (24). Thus, from now on, we always consider $U \in \mathcal{D}'$ nonconstant, that is, such that $DU \neq 0$.

Theorem 6.2. *Let $U \in \mathcal{D}'$ such that $DU \neq 0$. Moreover suppose that, given α, the composition $\phi \circ U$ is well defined. Then if $\gamma : I \to \mathbb{R}$ is a C^1-function, the profile U α-propagate according to (24) with the movement $\gamma(t)$, if and only if the following two conditions are satisfied:*

(a) the speed $\gamma'(t) = c$ is a constant function;

(b) $cDU = D(\phi \circ U)$.

Remark 6.3. *This result can be seen as a particular case of Theorem 7 p. 8 in* [18] *or even as a particular case of Theorem 5 p. 249 in* [14]. *However, for the sake of completeness we will give the following*

Proof. Let us suppose that, given α, the profile U α-propagates, according to (24), with the movement $\gamma(t)$. Then $\tilde{u}(t) = \tau_{\gamma(t)}U$ is an α-solution of (24) and we have:

$$\frac{d\tilde{u}}{dt}(t) = -\gamma'(t)\tau_{\gamma(t)}DU,$$
$$\phi\circ\tilde{u}(t) = \phi\circ\tau_{\gamma(t)}U = \tau_{\gamma(t)}(\phi\circ U).$$

Thus, from (24) such α-propagation holds if and only if

$$-\gamma'(t)\tau_{\gamma(t)}DU + \tau_{\gamma(t)}D(\phi\circ U) = 0,$$

and applying the operator $\tau_{-\gamma(t)}$ to this equality we get

$$\gamma'(t)DU = D(\phi\circ U). \tag{25}$$

Since the right-hand side of (25) does not depend on t and $DU \neq 0$, we conclude that $\gamma'(t) = c$ is a constant function and (a), (b) follows immediately. □

Remember that in the equality (b) of this Theorem, U is always a global distribution and, in general, cannot be replaced with a distribution defined on a proper open set of $\mathbb{R}$. In fact, the definition of $\phi\circ U$ contains α-products of distributions. See (21) and Remark 1.

Example 6.4. *For short, we will take* $p = \int_{-\infty}^{0}\alpha$ *and* $q = \int_{0}^{+\infty}\alpha$. *Consider Burger's inviscid equation, that is, equation* (1) *with* $\phi(u) = \frac{1}{2}u^2$. *For the profile* $U = 1 - H + m\delta$, *with* $m \in \mathbb{R}$, *we have, using* (7), (10), (16) *and noting that* $p + q = 1$,

$$U^2 = U \dot{_\alpha} U = 1 - H + m\delta - H + H - mp\delta + m\delta - mq\delta + m^2\alpha(0)\delta$$
$$= 1 - H + m[1 + m\alpha(0)]\delta.$$

Then equation (b) of Theorem 6.2 *turns out to be*

$$-c\delta + cm(D\delta) = -\frac{1}{2}\delta + \frac{m}{2}[1 + m\alpha(0)](D\delta).$$

Hence, if $m = 0$ *it follows* $c = \frac{1}{2}$. *If* $m \neq 0$ *it follows* $c = \frac{1}{2}$ *and* $\alpha(0) = 0$. *We conclude that:*

(i) if $m = 0$ *the shock wave profile* $U = 1 - H$ α*-propagates with speed* $c = \frac{1}{2}$ *for any* α.

(ii) If $m \neq 0$ *the delta shock wave profile* $U = 1 - H + m\delta$ α*-propagates with speed* $c = \frac{1}{2}$ *for any* α *such that* $\alpha(0) = 0$.

Remember that within the classical framework of conservation laws the propagation of the profile $U = 1 - H$ *with speed* $c = \frac{1}{2}$ *is well known and can be obtained from the Rankine-Hugoniot shock conditions. However, despite the impossibility of applying these conditions to the profile* $U = 1 - H + m\delta$, *we get the same speed by applying the concept of* α*-solution which, as we also mentioned,* *can also be seen as an extension of the concept of weak solution. Thus, in this setting, we conclude that the perturbation $m\delta$ does not modify the motion of the profile $U = 1 - H$.*

It is now interesting to see what happens exchanging the perturbation $m\delta$ with $m(D\delta)$, which is not a measure.

Example 6.5. *Taking $U = 1 - H + m(D\delta)$ in Theorem (6.2) and $\phi(u) = \frac{1}{2}u^2$, as in Burger's conservative inviscid equation, we have, using (17), (11) and (12),*

$U^2 = U_{\dot\alpha}U = 1-H+m(D\delta)-H+H-m[p(D\delta)-\alpha(0)\delta]+mD\delta-m[q(D\delta)-\alpha(0)\delta]$
$+m^2[\alpha'(0)(D\delta)-\alpha''(0)\delta] = 1-H+m[2\alpha(0)-m\alpha''(0)]\delta+m[1+m\alpha'(0)]D\delta$,
and (b) of Theorem (6.2) turns out to be

$$c[-\delta + m(D^2\delta)] = -\frac{1}{2}\delta + \frac{m}{2}[2\alpha(0) - m\alpha''(0)](D\delta) + \frac{m}{2}[1 + m\alpha'(0)](D^2\delta).$$

Hence, if $m = 0$ it follows $c = \frac{1}{2}$. If $m \neq 0$ it follows $c = \frac{1}{2}$, $\alpha'(0) = 0$ and $2\alpha(0) - \alpha''(0) = 0$. We conclude that the profile U α-propagates, according to $u_t + \left(\frac{u^2}{2}\right)_x = 0$, with speed $c = \frac{1}{2}$, for any α such that $\alpha'(0) = 0$ and $2\alpha(0) - \alpha''(0) = 0$. Also in this case the perturbation $m(D\delta)$ does not modify the motion of the profile $U = 1 - H$.

In these examples the α-propagation is always possible and the speed $c = \frac{1}{2}$ of the wave does not depend on α. Therefore, if such propagation is physically observed, the speed of the wave is $\frac{1}{2}$. However, other situations may occur:

Example 6.6. *Taking $U = \delta$ in Theorem (6.2) and using Theorem 4.1, the equation $cDU = D(\phi\circ U)$ turns out to be*

$$c(D\delta) = \begin{cases} \phi'(0)(D\delta) & \text{if } \alpha(0) = 0 \\ \frac{\phi[\alpha(0)]-\phi(0)}{\alpha(0)}(D\delta) & \text{if } \alpha(0) \neq 0 \end{cases},$$

and $c = \phi'(0)$ if $\alpha(0) = 0$, or $c = \frac{\phi[\alpha(0)]-\phi(0)}{\alpha(0)}$ if $\alpha(0) \neq 0$, follows. For instance,

(i) taking $\phi(u) = \frac{u^2}{2}$ as in Burger's inviscid equation we have $c = 0$ if $\alpha(0) = 0$, or $c = \frac{\alpha(0)}{2}$ if $\alpha(0) \neq 0$. Hence, $c = \frac{\alpha(0)}{2}$ which means that the speed c of the profile $U = \delta$ depends on α and it is arbitrary;

(ii) taking $\phi(u) = u^3 + u$, we have $c = \phi'(0) = 1$ if $\alpha(0) = 0$, or $c = \frac{\phi[\alpha(0)]-\phi(0)}{\alpha(0)} = \alpha(0)^2 + 1$ if $\alpha(0) \neq 0$. Hence, $c = \alpha(0)^2 + 1$, which means that the speed c can take several values but it is not completely arbitrary; in fact, we always have $c \geq 1$.

This means that, if the α-solutions of certain differential equations depend on α, in certain cases a physical meaning can still be associated to such solutions.

When $\phi'' = 0$, the equation (1) is linear and applying Theorem 6.2 it is easy to conclude that any profile $U \in \mathcal{D}'$ α-propagates with speed $c = \phi'$, which is constant, and coincides with what is well known for the transport equation.

Now, suppose that $U \in C^1$ is a classical profile. Then, $DU = U' \neq 0$ and $\phi \circ U \in C^1$. Hence, the equality (b) of the preceding Theorem turns out to be $cU' = (\phi' \circ U)U'$, which is the equation we obtain when we seek for classical C^1-travelling waves solutions $u(x,t) = U(x - ct)$ for the equation (1). This means that Definition 6.1 is a consistent extension of the travelling wave classical concept for the equation (1).

The following result is an important consequence of the preceding Theorem.

Theorem 6.7. *Suppose $\phi'' \neq 0$ and $U \in \mathcal{D}'$ such that, given α, $\phi \circ U$ is well defined. In addition, suppose that there exists a nonempty open interval $J \subset \mathbb{R}$, such that U_J corresponds to a nonconstant continuous function and $(\phi \circ U)_J = \phi \circ U_J$. Then the profile U cannot α-propagates according to (24).*

Proof. Suppose that U α-propagate according to (24) with the movement $\gamma(t)$. Then from Theorem (6.2), (a) and (b) would be satisfied. Thus, from (b) we would have $cU = \phi \circ U + k$, where k stands for a constant distribution. By restriction to the interval J, we would have $cU_J = \phi \circ U_J + k_J$, where k_J can be identified with a constant. Then, we can write

$$cU_J(x) = \phi[U_J(x)] + k_J, \tag{26}$$

for all $x \in J$. On the other hand, taking $w(z) = cz - \phi(z) - k_J$, w is not a constant function (if $w = const.$ it would follow $\phi'' = 0$ which contradicts $\phi'' \neq 0$), w is an entire function and from (26) we get $w[U_J(x)] = 0$ for all $x \in J$. However, we know that the roots of a nonconstant entire function are isolated points, which means that, for all $x \in J$, $U_J(x)$ takes values on a set of isolated points, which is a contradiction because U_J is continuous and nonconstant on J. The statement is proved. □

Example 6.8. *Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^2$. Then, for $U = g + \delta$ and $\phi(u) = \frac{1}{2}u^2$, we have, using (5) and (6),*

$$U^2 = U \dot{_\alpha} U = g^2 + g \dot{_\alpha} \delta + \delta \dot{_\alpha} g + \delta \dot{_\alpha} \delta = g^2 + (g * \alpha)(0)\delta + \alpha(0)\delta,$$

and taking $J =]0,1[$ we have, for any α, $(\phi \circ U)_J = (\frac{1}{2}U^2)_J = \frac{1}{2}g_J^2$, and also $\phi \circ U_J = \frac{1}{2}g_J^2$ which is continuous and nonconstant. Then, the profile U cannot α-propagate according to Burger's conservative inviscid equation.

Since for distributions U corresponding to continuous functions, $\phi \circ U$ is consistent with the usual meaning, $\phi \circ U$ corresponds to a continuous function and taking $J = \mathbb{R}$ in the precedent Theorem, we conclude immediately that:

Corollary 6.9. *If $\phi'' \neq 0$ then, for any α, any continuous and nonconstant profile U cannot α-propagate according to (1).*

Thus, if we want to seek for nonconstant travelling wave profiles $U \in \mathcal{D}'$, for the nonlinear equation (1), we must seek them among distributions which do not correspond to continuous functions. For profiles U that, locally, correspond to bounded variation functions, since, for any α, $\phi \circ U$ is also consistent with the usual meaning, taking $J = \mathbb{R}$ in Theorem 6.7, we get:

Corollary 6.10. *Let $\phi'' \neq 0$ and $U \in \mathcal{D}'^{-1}$ such that there exists a nonempty open interval J such that U_J is continuous and nonconstant. Then the profile U cannot α-propagate according to (1) for any α.*

Example 6.11. *Taking $U : \mathbb{R} \to \mathbb{R}$ defined by $U(x) = x$ if $x \in]0,1[$ and $U(x) = 0$ if $x \in \mathbb{R}\backslash]0,1[$, and $J =]0,1[$, we conclude that the discontinuous profile $U \in \mathcal{D}'^{-1}$ cannot α-propagate, according to the nonlinear equation (1), for any α.*

Thus, if $\phi'' \neq 0$, the α-propagation of a profile $U \in \mathcal{D}'^{-1}$ is possible only for step functions. A necessary and sufficient condition for the α-propagation of a profile U that, locally, is a bounded variation function is given in the following result:

Theorem 6.12. *Suppose $\phi'' \neq 0$, U a locally bounded variation function defined on $\mathbb{R}$ satisfying $DU \neq 0$, and Z the set of points where U is discontinuous. Then, given any α, the profile U α-propagates according to (1) with the movement $\gamma(t)$ if and only if Z is not empty and the following two conditions are satisfied:*

(a) In each maximal open interval $J \subset \mathbb{R}$ where U is continuous, $U_J = \lambda_J$ is a constant function;

(b) there exist two constants c, k_0 such that

$$c\lambda_J = \phi(\lambda_J) + k_0, \tag{27}$$

for all λ_J.

In that case, U α-propagates with the speed $\gamma'(t) = c$.

Remark 6.13. *Recall that, from the theory of locally bounded variation functions, Z is a countable set.*

Proof. Suppose that U α-propagates, according to (1), with the movement $\gamma(t)$. Then Z is not empty because, if Z were empty, U would be continuous and nonconstant, which contradicts Corollary 6.9.

On the other hand, from Theorem 6.2, the mentioned α-propagation holds if and only if there exists a constant c such that $\gamma'(t) = c$ for all t and $cDU = D(\phi \circ U)$, that is,

$$cU = \phi \circ U + k,$$

being k a constant distribution. Since U is a distribution corresponding to a locally bounded variation function, $\phi \circ U$ corresponds to the usual meaning (as we have mentioned before Theorem 4.1) and we can write

$$cU(x) = \phi[U(x)] + k_0, \tag{28}$$

for almost all $x \in \mathbb{R}$, being k_0 the unique constant corresponding to the constant distribution k.

Let ψ be the function defined by

$$\psi(z) = cz - \phi(z) - k_0. \tag{29}$$

Then, ψ is not a constant function because, if ψ were constant, taken the second derivative of both sides of (29) we would get $\phi''(z) = 0$, for all z, which contradicts $\phi'' \neq 0$. Also from (28) we can write

$$\psi[U(x)] = 0, \tag{30}$$

for all $x \in \mathbb{R}\backslash Z$ (recall that Z has Lebesgue measure zero). However, we know that the zeros of an entire and nonconstant function are isolated points. Hence, from (30), in each maximal open interval J where U is continuous, and for each $x \in J$, $U(x)$ is a zero of ψ, which means that the values of $U(x)$, for $x \in J$, are isolated points. Since U is continuous in J, it follows that U_J is constant. Therefore, for each maximal open interval J where U is continuous, $U(x) = \lambda_J$, being λ_J a constant. Finally, from (28), it follows $c\lambda_J = \phi(\lambda_J) + k_0$, for all λ_J, and the statement is proved. □

Example 6.14. *According to equation (1), let us examine the possibility of the α-propagation of the profile*

$$U(x) = \begin{cases} a & \text{if } \; x < 0 \\ b & \text{if } \; x > 0 \end{cases},$$

with $a \neq b$. We have $Z = \{0\}$ not empty and two maximal open intervals $J_1 =]-\infty, 0[$ and $J_2 =]0, +\infty[$ where U is continuous with $U_{J_1} = \lambda_1 = a$ and $U_{J_2} = \lambda_2 = b$. Then we have two equations (27),

$$ca = \phi(a) + k_0$$
$$cb = \phi(b) + k_0,$$

and $c = \frac{\phi(b)-\phi(a)}{b-a}$, $k_0 = \frac{\phi(b)a-\phi(a)b}{b-a}$ follows. Hence, for any α, the profile U α-propagates according to (1) with speed $c = \frac{\phi(b)-\phi(a)}{b-a}$. As a particular case, and for Burger's inviscid equation, $\phi(u) = \frac{u^2}{2}$ and we get, for the profile speed $\gamma'(t) = c = \frac{a+b}{2}$, for any α.

Example 6.15. *Regarding the same equation (1), let us consider the profile*

$$U(x) = \begin{cases} a & \text{if } \quad x < 0 \\ b & \text{if } \; 0 < x < 1 \\ c & \text{if } \quad x > 1 \end{cases},$$

with $a \neq b$ and $b \neq d$. We have $Z = \{0, 1\}$ not empty and three maximal intervals $J_1 =]-\infty, 0[$, $J_2 =]0, 1[$ and $J_3 =]1, +\infty[$ where U is continuous, with $U_{J_1} = \lambda_1 = a$, $U_{J_2} = \lambda_2 = b$ and $U_{J_3} = \lambda_3 = d$.

Then we have three equations (27),

$$ca = \phi(a) + k_0,$$

$$cb = \phi(b) + k_0,$$
$$cd = \phi(d) + k_0,$$

and $c = \frac{\phi(b)-\phi(a)}{b-a}$, $k_0 = \frac{\phi(b)a-\phi(a)b}{b-a}$ follows, jointly with the compatibility condition

$$[\phi(b) - \phi(a)]d = [\phi(d) - \phi(a)]b + [\phi(b) - \phi(d)]a.$$

Thus, if $a = d$ the compatibility condition is always satisfied and the α-propagation of the profile U holds for any α, with the speed $\gamma'(t) = c = \frac{\phi(b)-\phi(a)}{b-a}$.

If $a \neq d$ the α-propagation can be effective or not. For instance, in Burger's inviscid equation, $\phi(u) = \frac{u^2}{2}$ and the profile U with $a = 0$, $b = 1$ and $d = 2$ does not α-propagate for any α because the compatibility condition is not satisfied.

Instead, for the equation $u_t + (u^3)_x = 0$, where $\phi(u) = u^3$ the profile U with $a = 0$, $b = 1$ and $d = -1$, α-propagate with speed $\gamma'(t) = c = \frac{\phi(1)-\phi(0)}{1-0} = 1$ for any α, because the compatibility condition is satisfied.

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Data availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Declarations

I declare that I do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

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