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On certain Fourier expansions for the Riemann zeta function

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Abstract. We build on a recent paper on Fourier expansions for the Riemann zeta function. We establish Fourier expansions for certain L-functions, and offer series representations involving the Whittaker function $W_{\gamma,\mu}(z)$ for the coefficients. Fourier expansions for the reciprocal of the Riemann zeta function are also stated. A new expansion for the Riemann xi function is presented in the third section by constructing an integral formula using Mellin transforms for its Fourier coefficients.

Contents

1. Introduction and main results

The measure

$$\mu(B) := \frac{1}{2\pi}\int_B \frac{dy}{\frac{1}{4}+y^2},$$

for each B in the Borel set $\mathfrak{B}$, has been applied in the work of [7] as well as Coffey [3], providing interesting applications in analytic number theory. For the measure space $(\mathbb{R},\mathfrak{B},\mu)$,

$$||g||_2^2 := \int_{\mathbb{R}} |g(t)|^2 d\mu, \tag{1}$$

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is the $L^2(\mu)$ norm of $f(x)$. Here (1.1) is finite, and $f(x)$ is measurable [10, pg.326, Definition 11.34]. In a recent paper by Elaissaoui and Guennoun [7], an interesting Fourier expansion was presented which states that, if $f(x) \in L^2(\mu)$, then

$$f(x) = \sum_{n\in\mathbb{Z}} a_n e^{-2in\arctan(2x)} = \sum_{n\in\mathbb{Z}} a_n \left(\frac{\frac{1}{2} - ix}{\frac{1}{2} + ix}\right)^n, \tag{2}$$

where

$$a_n = \frac{1}{2\pi}\int_{\mathbb{R}} f(y)e^{2in\arctan(2y)}\frac{dy}{\frac{1}{4}+y^2}. \tag{3}$$

By selecting $x = \frac{1}{2}\tan(\phi)$, we return to the classical Fourier expansion, since $f(\frac{1}{2}\tan(\phi))$ is periodic in π. The main method applied in their paper to compute the constants a_n is the Cauchy residue theorem. However, it is possible (as noted therein) to directly work with the integral

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\frac{1}{2}\tan(\frac{\phi}{2}))e^{in\phi}d\phi. \tag{4}$$

Many remarkable results were extracted from the Fourier expansion (1.2)–(1.3), including criteria for the Lindelöf Hypothesis (Theorem 4.6 of [7]). In fact, the Lindelöf Hypothesis is part of the motivation for selecting the probability measure μ [7].

Let ρ denote any nontrivial zeros of $\zeta(s)$ in the critical region $\alpha \in (0,1)$, where $\Re(\rho) = \alpha$, $\Im(\rho) = \beta$. The goal of this paper is to offer some more applications of (1.2)–(1.3), including a criteria for the Riemann hypothesis. Recall that the Riemann Hypothesis is the statement that all ρ have $\alpha = \frac{1}{2}$. Equivalently, the functional equation says that this would mean all ρ are such that $\alpha \notin (\frac{1}{2}, 1)$.

Theorem 1.1. *For $\sigma > 1$, $x \in \mathbb{R}$,*

$$\frac{1}{\zeta(\sigma+ix)} = \frac{1}{\zeta(\sigma+\frac{1}{2})} + \sum_{n\geq 1}\bar{a}_n e^{-2in\arctan(2x)},$$

where

$$\bar{a}_n = \frac{1}{n!}\sum_{n\geq k>0}\binom{n}{k}\frac{(-1)^n(n-1)!}{(k-1)!}\lim_{s\to 0}\frac{\partial^k}{\partial s^k}\frac{1}{\zeta(\sigma+\frac{1}{2}-s)}.$$

Moreover, if the zeros of $\zeta(s)$ are simple, we have

$$\frac{1}{\zeta(\sigma-ix)} = \sum_{n\in\mathbb{Z}}\hat{a}_n e^{-2in\arctan(2x)},$$

where for $n \geq 1$,

$$\hat{a}_n = \frac{1}{n!} \sum_{n \geq k \geq 0} \binom{n}{k} \frac{(-1)^n (n-1)!}{(k-1)!} \lim_{s \to 0} \frac{\partial^k}{\partial s^k} \frac{1}{\zeta(\sigma - \frac{1}{2} + s)} - S(n, \sigma),$$

where

$$S(n,\sigma) = \sum_{\beta : \zeta(\rho) = 0} \left(\frac{\sigma - i\beta}{1 - \sigma + i\beta} \right)^n \frac{1}{\zeta'(\rho)(1 - \sigma + i\beta)(\sigma - i\beta)}$$

$$+ \sum_{k \geq 1} \left(\frac{\frac{1}{2} + \sigma + 2k}{\frac{1}{2} - \sigma - 2k} \right)^n \frac{1}{\zeta'(-2k)(\frac{1}{2} - \sigma - 2k)(\frac{1}{2} + \sigma + 2k)},$$

and $\hat{a}_n = -S(n,\sigma)$ *for* $n < 0$, $\hat{a}_0 = 1/\zeta(\sigma + \frac{1}{2})$.

Corollary 1.2. *For* $\sigma > 1$,

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{d\mu}{|\zeta(\sigma + iy)|^2} = \frac{1}{\zeta^2(\sigma + \frac{1}{2})} + \sum_{k \geq 1} |\bar{a}_k|^2,$$

where the $\bar{a}_n$ *are as defined in the previous theorem. Furthermore, even assuming the Riemann Hypothesis, this integral diverges for* $\frac{1}{2} < \sigma < 1$.

Next we consider a Fourier expansion with coefficients expressed as a series involving the Whittaker function $W_{\gamma,\mu}(z)$, which is a solution to the differential equation [8, pg.1024, eq.(9.220)]

$$\frac{d^2 W}{dz^2} + \left(-\frac{1}{4} + \frac{\gamma}{z} + \frac{1 - 4\mu^2}{4z^2} \right) W = 0.$$

This function also has the representation [8, pg.1024, eq.(9.220)]

$$W_{\gamma,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2} - \mu - \gamma)} M_{\gamma,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2} + \mu - \gamma)} M_{\gamma,-\mu}(z).$$

Here the other Whittaker function $M_{\gamma,\mu}(z)$ is given by

$$M_{\gamma,\mu}(z) = z^{\mu + \frac{1}{2}} e^{-z/2} {}_1F_1(\mu - \gamma + \frac{1}{2}; 2\mu + 1; z),$$

where ${}_1F_1(a;b;z)$ is the well-known confluent hypergeometric function.

Theorem 1.3. *Let* v *be a complex number which is not an even integer. Then for* $1 > \sigma > \frac{1}{2}$, *we have the expansion*

$$\zeta(\sigma + ix) \cos^v(\arctan(2x)) = \frac{1}{2}\zeta(\sigma + \frac{1}{2}) + \sum_{n \in \mathbb{Z}} \tilde{a}_n e^{-2in \arctan(2x)},$$

where

$$\tilde{a}_n = \frac{(2\sigma^2 - 4\sigma + \frac{5}{2})}{2(\sigma - \frac{1}{2})^2(\frac{3}{2} - \sigma)^2}\left(\frac{\frac{3}{2} - \sigma}{\sigma - \frac{1}{2}}\right)^n$$

for $n < 0$ *and*

$$\tilde{a}_n = \frac{2\Gamma(\upsilon + 1)}{\Gamma(\frac{\upsilon}{2} + n + 1)\Gamma(\frac{\upsilon}{2} - n + 1)} + \frac{\pi}{2^{\upsilon/2+1}} \sum_{k>1} k^{-\sigma} \left(\frac{\log(k)}{2}\right)^{\upsilon/2} \frac{W_{n,-\frac{\upsilon+1}{2}}(\log(k))}{\Gamma(1 + \frac{\upsilon}{2} + n)}$$

for $n \geq 1$.

2. Proof of main theorems

In our proof of Corollary 1.2, we will require a well-known result [13, pg.331, Theorem 11.45] on functions in $L^2(\mu)$.

Lemma 2.1. *Define the coefficients* $a_n = \int_X f\kappa_n d\mu$, *where* $\{\kappa_n\}$ *is a complete orthonormal set. If* $f(x) \in L^2(\mu)$, *and* $f(x)$ *has the representation* $\sum_{n=1} a_n\kappa_n$, *then*

$$\int_X |f(x)|^2 d\mu = \sum_{n=1} |a_n|^2.$$

Proof of Theorem 1.1. First note that for $\sigma > 1$

$$\begin{aligned}\bar{a}_n &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{2in\arctan(2y)} \frac{dy}{\zeta(\sigma + iy)(\frac{1}{4} + y^2)} dy \\ &= \frac{1}{2\pi i}\int_{(\frac{1}{2})} \left(\frac{s}{1-s}\right)^n \frac{ds}{\zeta(\sigma - \frac{1}{2} + s)s(1-s)}.\end{aligned} \tag{5}$$

We replace s by $1 - s$ and apply the residue theorem by moving the line of integration to the left. By the Leibniz rule, we compute the residue at the pole $s = 0$ of order $n + 1$, $n \geq 0$, as

$$\begin{aligned}&\frac{1}{n!}\lim_{s\to 0}\frac{d^n}{ds^n} s^{n+1}\left(\left(\frac{1-s}{s}\right)^n \frac{1}{\zeta(\sigma + \frac{1}{2} - s)s(1-s)}\right) \\ &= \frac{1}{n!}\lim_{s\to 0}\frac{d^n}{ds^n}\frac{(1-s)^{n-1}}{\zeta(\sigma + \frac{1}{2} - s)} \\ &= \frac{1}{n!}\sum_{n\geq k\geq 0}\binom{n}{k}\frac{(-1)^n(n-1)!}{(k-1)!}\lim_{s\to 0}\frac{\partial^k}{\partial s^k}\frac{1}{\zeta(\sigma + \frac{1}{2} - s)}\end{aligned} \tag{6}$$

The residue at $s = 0$ if $n = 0$ is $-1/\zeta(\sigma + \frac{1}{2})$. There are no additional poles when $n < 0$. Since the sum in (2.2) is zero for $k = 0$ it reduces to the one stated in the theorem.

Next we consider the second statement. The integrand in

$$\frac{1}{2\pi i}\int_{(\frac{1}{2})}\left(\frac{1-s}{s}\right)^n \frac{ds}{\zeta(\sigma-\frac{1}{2}+s)s(1-s)} \tag{7}$$

has simple poles at $s = 1-\sigma+i\beta$, where $\Im(\rho)=\beta$. The integrand in (2.3) also has simple poles at $s=\frac{1}{2}-\sigma-2k$, and a pole of order $n+1$, $n>0$, at $s=0$. We compute,

$$\begin{aligned}
&\frac{1}{n!}\lim_{s\to 0}\frac{d^n}{ds^n}s^{n+1}\left(\left(\frac{1-s}{s}\right)^n\frac{1}{\zeta(\sigma-\frac{1}{2}+s)s(1-s)}\right)\\
&\quad=\frac{1}{n!}\lim_{s\to 0}\frac{d^n}{ds^n}\frac{(1-s)^{n-1}}{\zeta(\sigma-\frac{1}{2}+s)}\\
&\quad=\frac{1}{n!}\sum_{n\geq k\geq 0}\binom{n}{k}\frac{(-1)^n(n-1)!}{(k-1)!}\lim_{s\to 0}\frac{\partial^k}{\partial s^k}\frac{1}{\zeta(\sigma-\frac{1}{2}+s)}.
\end{aligned}$$

The residue at the pole $s=1-\sigma+i\beta$, is

$$\sum_{\beta:\zeta(\rho)=0}\left(\frac{\sigma-i\beta}{1-\sigma+i\beta}\right)^n\frac{1}{\zeta'(\rho)(1-\sigma+i\beta)(\sigma-i\beta)},$$

and at the pole $s=\frac{1}{2}-\sigma-2k$ is

$$\sum_{k\geq 1}\left(\frac{\frac{1}{2}+\sigma+2k}{\frac{1}{2}-\sigma-2k}\right)^n\frac{1}{\zeta'(-2k)(\frac{1}{2}-\sigma-2k)(\frac{1}{2}+\sigma+2k)},$$

The residue at the pole $n=0$, $s=0$, is $-1/\zeta(\sigma-\frac{1}{2})$. □

Proof of Corollary 1.2. This result readily follows from application of Theorem 1.1 to Lemma 2.1 with $X=\mathbb{R}$. In the first part of the theorem, note from [14, pg.191, Theorem 8.7], if $\sigma>1$,

$$\left|\frac{1}{\zeta(s)}\right|\leq\frac{\zeta(\sigma)}{\zeta(2\sigma)}.$$

Hence

$$\frac{1}{|\zeta(s)|^2(t^2+\frac{1}{4})}=O\left(\frac{1}{|t|^2}\right),$$

as $t\to\infty$, and $1/\zeta(\sigma+it)\in L^2(\mu)$, for $\sigma>1$. The convergence of the series $\sum_k|\bar{a}_n|^2$ follows by applying [10, pg.580, Lemma 12.6]. In the second part of the theorem, note from [14, pg.377] or [14, pg.372]

$$\frac{1}{\zeta(s)}=O\left(\frac{|s|}{\sigma-\frac{1}{2}}\right).$$

Hence

$$\frac{1}{\left|\zeta(s)\right|^2(t^2+\frac{1}{4})}=O\left(\frac{|s|^2}{|t|^2}\right)=O(1),$$

as $t\to\infty$, and $1/\zeta(\sigma+it)\notin L^2(\mu)$, for $\frac{1}{2}<\sigma<1$. □

Proof of Theorem 1.3. It is clear that

$$\cos^{\upsilon}(2\arctan(2y))=\left(\frac{1-4y^2}{1+4y^2}\right)^{\upsilon}=O(1).$$

Comparing with [7, Theorem 1.2] we see our function belongs to $L^2(\mu)$. We compute that

$$\begin{aligned}
\tilde{a}_n&=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\frac{1}{2}\tan(\frac{\phi}{2}))e^{in\phi}d\phi\\
&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\zeta(\sigma+\frac{i}{2}\tan(\frac{\phi}{2}))\cos^{\upsilon}(\frac{\phi}{2})e^{in\phi}d\phi\\
&=\frac{1}{2\pi}\left(\int_{0}^{\pi}\zeta(\sigma+\frac{i}{2}\tan(\frac{\phi}{2}))\cos^{\upsilon}(\frac{\phi}{2})e^{in\phi}d\phi+\int_{-\pi}^{0}\zeta(\sigma+\frac{i}{2}\tan(\frac{\phi}{2}))\cos^{\upsilon}(\frac{\phi}{2})e^{in\phi}d\phi\right)\\
&=\frac{1}{2\pi}\left(\int_{0}^{\pi}\zeta(\sigma+\frac{i}{2}\tan(\frac{\phi}{2}))\cos^{\upsilon}(\frac{\phi}{2})e^{in\phi}d\phi+\int_{0}^{\pi}\zeta(\sigma-\frac{i}{2}\tan(\frac{\phi}{2}))\cos^{\upsilon}(\frac{\phi}{2})e^{-in\phi}d\phi\right)\\
&=\frac{1}{2\pi}\left(\int_{0}^{\pi}\zeta(\sigma+\frac{i}{2}\tan(\frac{\phi}{2}))\cos^{\upsilon}(\frac{\phi}{2})e^{in\phi}d\phi+\int_{0}^{\pi}\zeta(\sigma-\frac{i}{2}\tan(\frac{\phi}{2}))\cos^{\upsilon}(\frac{\phi}{2})e^{-in\phi}d\phi\right)\\
&=\frac{1}{\pi}\int_{0}^{\pi}\cos^{\upsilon}(\frac{\phi}{2})\sum_{k\geq1}k^{-\sigma}\cos\left(\frac{1}{2}\tan(\frac{\phi}{2})\log(k)-n\phi\right)d\phi\\
&=\frac{1}{\pi}\int_{0}^{\pi}\cos^{\upsilon}(\frac{\phi}{2})\cos(n\phi)\,d\phi+\frac{1}{\pi}\int_{0}^{\pi}\cos^{\upsilon}(\frac{\phi}{2})\sum_{k>1}k^{-\sigma}\cos\left(\frac{1}{2}\tan(\frac{\phi}{2})\log(k)-n\phi\right)d\phi\\
&=\frac{2}{\pi}\int_{0}^{\pi/2}\cos^{\upsilon}(\phi)\cos(n2\phi)\,d\phi\\
&\quad+\frac{2}{\pi}\int_{0}^{\pi/2}\cos^{\upsilon}(\phi)\sum_{k>1}k^{-\sigma}\cos\left(\frac{1}{2}\tan(\phi)\log(k)-n2\phi\right)d\phi.
\end{aligned}$$

Now by [8, pg.397] for $\mathfrak{R}(\upsilon)>0$, we have

$$\int_{0}^{\pi/2}\cos^{\upsilon-1}(y)\cos(by)dy=\frac{\pi\Gamma(\upsilon)}{\Gamma(\frac{\upsilon+b+1}{2})\Gamma(\frac{\upsilon-b+1}{2})}. \tag{8}$$

Let $\mathbb{Z}^-$ denote the set of negative integers. Then, by [8, pg.423] with $a > 0$, $\Re(v) > -1$, $\frac{v+\gamma}{2} \neq \mathbb{Z}^-$,

$$\int_0^{\pi/2} \cos^v(y)\cos(a\tan(y) - \gamma y)dy = \frac{\pi a^{v/2}}{2^{v/2+1}} \frac{W_{\gamma/2,-\frac{v+1}{2}}(2a)}{\Gamma(1+\frac{v+\gamma}{2})}. \tag{9}$$

Hence, if we put $b = 2n$ and replace v by $v+1$ in (2.4), and select $a = \frac{1}{2}\log(k)$ and $\gamma = 2n$ in (2.5), we find

$$\tilde{a}_n = \frac{2\Gamma(v+1)}{\Gamma(\frac{v}{2}+n+1)\Gamma(\frac{v}{2}-n+1)} + \frac{\pi}{2^{v/2+1}} \sum_{k>1} k^{-\sigma} \left(\frac{\log(k)}{2}\right)^{v/2} \frac{W_{n,-\frac{v+1}{2}}(\log(k))}{\Gamma(1+\frac{v}{2}+n)}. \tag{10}$$

Hence v cannot be a negative even integer.

The interchange of the series and integral is justified by absolute convergence for $\sigma > \frac{1}{2}$. To see this, note that [8, pg.1026, eq.(9.227), eq.(9.229)]

$$W_{\gamma,\mu}(z) \sim e^{-z/2} z^{\gamma},$$

as $|z| \to \infty$, and

$$W_{\gamma,\mu}(z) \sim (\frac{4z}{\gamma})^{1/4} e^{-\gamma+\gamma\log(\gamma)} \sin(2\sqrt{\gamma z} - \gamma\pi - \frac{\pi}{4}),$$

as $|\gamma| \to \infty$. Here the notation $f(x) \sim g(x)$ means that $\lim_{x\to\infty} f(x)/g(x) = 1$. Using (2.6) as coefficients for $n < 0$ is inadmissible, due to the resulting sum over n being divergent. On the other hand, it can be seen that

$$\begin{aligned}
\tilde{a}_n &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{2in\arctan(2y)} \frac{\zeta(\sigma+iy)\cos^v(\arctan(2y))dy}{(\frac{1}{4}+y^2)} \\
&= \frac{1}{2\pi i} \int_{(\frac{1}{2})} \frac{\zeta(\sigma-\frac{1}{2}+s)2(2s^2-2s+1)}{(2s(1-s))^2} \left(\frac{s}{1-s}\right)^n ds \\
&= \frac{1}{2\pi i} \int_{(\frac{1}{2})} \frac{\zeta(\sigma+\frac{1}{2}-s)2(2s^2-2s+1)}{(2s(1-s))^2} \left(\frac{1-s}{s}\right)^n ds.
\end{aligned}$$

We will only use the residues at the pole $s = 0$ when $n < 0$ and $s = \sigma - \frac{1}{2}$, and outline the details to obtain an alternative expression for the $\tilde{a}_n$ for $n \geq 0$. The integrand has a simple pole at $s = \sigma - \frac{1}{2}$, a pole of order $n+2$ at $s = 0$, and when $n < 0$ there is a simple pole when $n = -1$, at $s = 0$. The residue at the pole $s=0$ for $n\geq 0$ is computed as

$$\begin{aligned}&\frac{1}{n!}\lim_{s\to 0}\frac{d^{n+1}}{ds^{n+1}}s^{n+2}\left(\frac{\zeta(\sigma+\frac{1}{2}-s)2(2s^2-2s+1)}{(2s(1-s))^2}\left(\frac{1-s}{s}\right)^n\right)\\&=\frac{1}{n!2}\lim_{s\to 0}\frac{d^{n+1}}{ds^{n+1}}\left(\zeta(\sigma+\frac{1}{2}-s)(2s^2-2s+1)(1-s)^{n-2}\right).\end{aligned}\tag{11}$$

And because the resulting sum is a bit cumbersome, we omit this form in our stated theorem. The residue at the simple pole when $n=-1$, at $s=0$ is $\frac{1}{2}\zeta(\sigma+\frac{1}{2})$. Collecting our observations tells us that if $n<0$,

$$\tilde{a}_n=\frac{(2\sigma^2-4\sigma+\frac{5}{2})}{2(\sigma-\frac{1}{2})^2(\frac{3}{2}-\sigma)^2}\left(\frac{\frac{3}{2}-\sigma}{\sigma-\frac{1}{2}}\right)^n.$$

□

3. Riemann xi function

The Riemann xi function is given by

$$\xi(s):=\frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s),$$

and $\Xi(y)=\xi(\frac{1}{2}+iy)$. In many recent works [4],[5], Riemann xi function integrals have been shown to have interesting evaluations. (See also [11] for an interesting expansion for the Riemann xi function.) The classical application is in the proof of Hardy's theorem that there are infinitely many non-trivial zeros on the line $\Re(s)=\frac{1}{2}$.

We will need to utilize Mellin transforms to prove our theorems. By Parseval's formula [12, pg.83, eq.(3.1.11)], we have

$$\int_0^\infty f(y)g(y)dy=\frac{1}{2\pi i}\int_{(r)}\mathfrak{M}(f(y))(s)\mathfrak{M}(g(y))(1-s)ds,\tag{12}$$

provided that r is chosen so that the integrand is analytic, and

$$\int_0^\infty y^{s-1}f(y)dy=:\ \mathfrak{M}(f(y))(s).$$

From [12, pg.95, eq.(3.3.27)] with $n\geq 0$, $x>1$, $c>0$, we have

$$\frac{1}{2\pi i}\int_{(c)}\frac{x^s}{s^{n+1}}ds=\frac{(\log(x))^n}{n!}.\tag{13}$$

Now it is known [6, pg.207–208] that for any $\Re(s)=u\in\mathbb{R}$,

$$\Theta(y)=\frac{1}{2\pi i}\int_{(u)}\xi(s)y^{-s}ds,\tag{14}$$

where

$$\Theta(y) := 2y^2 \sum_{n\geq 1} (2\pi^2 n^4 y^2 - 3\pi n^2) e^{-\pi n^2 y^2}, \tag{15}$$

for $y > 0$. Define the operator $\mathfrak{D}_{n,y}(f(y)) := \underbrace{y\frac{\partial}{\partial y} \dots y\frac{\partial}{\partial y}}_{n}(f(y))$.

Theorem 3.1. *For real numbers $x \in \mathbb{R}$,*

$$\Xi(x) = \frac{1}{(\frac{1}{4} + x^2)} \sum_{n\in\mathbb{Z}} \breve{a}_n e^{-2in \arctan(2x)},$$

where $\breve{a}_0 = 0$, and for $n \geq 1$,

$$\breve{a}_n = \frac{(-1)^n}{(n-1)!} \int_0^1 \log^{n-1}(y) \mathfrak{D}_{n,y}(\Theta(y)) dy,$$

and

$$\breve{a}_{-n} = -\frac{(-1)^n}{(n-1)!} \sum_{n-1\geq k\geq 0} \binom{n-1}{k} \frac{n!}{(k+1)!} \xi^{(k)}(0).$$

Proof. Applying the operator $\mathfrak{D}_{n,y}$ to (3.3)–(3.4), then applying the resulting Mellin transform with (3.2) to (3.1), we have for $c < 1$, $n \geq 1$,

$$\frac{(-1)^n}{(n-1)!} \int_0^1 \log^{n-1}(y) \mathfrak{D}_{n,y}(\Theta(y)) dy = \frac{1}{2\pi i} \int_{(c)} \left(\frac{s}{1-s}\right)^n \xi(s) ds. \tag{16}$$

On the other hand,

$$\begin{aligned} \breve{a}_n &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{2in \tan^{-1}(2y)} \frac{(\frac{1}{4} + y^2)\Xi(y)dy}{(\frac{1}{4} + y^2)} dy = \frac{1}{2\pi i} \int_{(\frac{1}{2})} \left(\frac{s}{1-s}\right)^n \xi(s) ds \\ &= \frac{1}{2\pi i} \int_{(\frac{1}{2})} \left(\frac{s}{1-s}\right)^n \pi^{-s/2} \frac{s}{2}(s-1)\zeta(s)\Gamma(\frac{s}{2}) ds. \end{aligned} \tag{17}$$

This gives the coefficients for $n \geq 1$. If we place n by $-n$ in the integrand of (3.6), we see that there is a pole of order n, $n > 0$, at $s = 0$. These residues are computed in the same way as before, and so we leave the details to the reader. Hence, for $n > 0$, $-2 < r' < 0$,

$$\begin{aligned} \breve{a}_{-n} &= \frac{1}{2\pi i} \int_{(\frac{1}{2})} \left(\frac{1-s}{s}\right)^n \xi(s) ds \\ &= \frac{(-1)^n}{(n-1)!} \sum_{n-1\geq k\geq 0} \binom{n-1}{k} \frac{n!}{(k+1)!} \xi^{(k)}(0) + \frac{1}{2\pi i} \int_{(r')} \left(\frac{1-s}{s}\right)^n \xi(s) ds \\ &= \frac{(-1)^n}{(n-1)!} \sum_{n-1\geq k\geq 0} \binom{n-1}{k} \frac{n!}{(k+1)!} \xi^{(k)}(0). \end{aligned} \tag{18}$$

In the third line we implemented the fact that the remaining residue from the poles of $\Gamma(\frac{s}{2})$ at negative even integers is zero due to the trivial zeros of $\zeta(s)$. $\square$

Now according to Coffey [1, pg.527], $\xi^{(n)}(0) = (-1)^n\xi^{(n)}(1)$, which may be used to recast Theorem 3.1 in a slightly different form. The integral formulae obtained in [2, pg.1152, eq.(28)], and another form in [9, pg.11106, eq.(12)], bear some resemblance to the integral contained in (3.5). It would be interesting to obtain a relationship to the coefficients $\breve{a}_n$. Next we give a series evaluation for a Riemann xi function integral.

Corollary 3.2. *If the coefficients $\breve{a}_n$ are as defined in Theorem 3.1., then*

$$\int_{\mathbb{R}}(\frac{1}{4}+y^2)^2\Xi^2(y)d\mu = \sum_{n\in\mathbb{Z}}|\breve{a}_n|^2.$$

Proof. This is an application of Theorem 3.1 to Lemma 2.1 with $X = \mathbb{R}$. $\square$

4. On the partial Fourier series

Here we make note of some interesting consequences of our computations related to the partial sums of our Fourier series. First, we recall [10, pg.69] that

$$\sum_{n=-N}^{N} a_n e^{inx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)D_N(y)dy, \tag{19}$$

where

$$D_N(x) = \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})}.$$

Now making the change of variable $y = 2\arctan(2y)$, we find (4.1) is equal to

$$\frac{1}{2\pi}\int_{\mathbb{R}} f(x-2\arctan(2y))\frac{D_N(2\arctan(2y))}{\frac{1}{4}+y^2}dy.$$

Recall [10, pg.71] that $K_N(x)$ is the Fejér kernel if

$$K_N(x) = \frac{1}{N+1}\sum_{n=0}^{N} D_n(x).$$

Theorem 4.1. *Let $K_N(x)$ denote the Fejér kernel. Then, assuming the Riemann hypothesis,*

$$\lim_{N\to\infty}\frac{1}{2\pi}\int_{\mathbb{R}}\frac{K_N(x_0-2\arctan(2y))}{\zeta(\sigma+iy)(\frac{1}{4}+y^2)}dy = \frac{1}{\zeta(\sigma+\frac{i}{2}\tan(\frac{x_0}{2}))},$$

for $x_0 \in (-\pi,\pi)$, $\frac{1}{2} < \sigma < 1$.

Proof. Notice that $1/\zeta(\sigma + \frac{i}{2}\tan(\frac{y}{2}))$ is continuous for $y \in (-\pi, \pi)$ if there are no singularities for $\frac{1}{2} < \sigma < 1$. Hence, we may apply [10, pg.29, Theorem 1.26] to find $1/\zeta(\sigma + \frac{i}{2}\tan(\frac{y}{2}))$ would then be Riemann integrable on $(-\pi, \pi)$ if there are no singularities for $\frac{1}{2} < \sigma < 1$. It is also periodic in π. Applying Fejér's theorem [10, pg.73, Theorem 1.59] with $f(y) = 1/\zeta(\sigma + \frac{i}{2}\tan(\frac{y}{2}))$ implies the result. □

Note that if $1/\zeta(\sigma + \frac{i}{2}\tan(\frac{y}{2}))$ has even finitely many points of discountinuity for $\frac{1}{2} < \sigma < 1$, we would not be able to apply Fejér's theorem. This is because the function is unbounded by Montgomery's omega result [14, pg.209], and therefore not Riemann integrable by [10, pg.31, Proposition 1.29].

5. Concluding remarks

The Fourier series for the Riemann zeta function contained herein, just like those in [7], are pointwise convergent. Seeing as how there exists a Fourier series for $\zeta(\sigma + it)$ in the region $\frac{1}{2} < \sigma < 1$, that is pointwise convergent, it would be interesting if one existed that were absolutely convergent. Wiener's result [15, pg.14, Lemma IIe] says the following:

Lemma 5.1. *(Wiener [15]) Suppose $f(x)$ has an absolutely convergent Fourier series and $f(x) \neq 0$ for all $x \in \mathbb{R}$. Then its reciprocal $1/f(x)$ also has an absolutely convergent Fourier series.*

Therefore, assuming the Riemann Hypothesis, if $\zeta(\sigma + it)$ has an absolutely convergent Fourier series for $\frac{1}{2} < \sigma < 1$, then there exists an absolutely convergent Fourier series for $1/\zeta(\sigma + it)$ in the same region.

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