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# Another invariant for AT actions

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Abstract. We construct a collection of numerical invariants for approximately transitive (AT) actions (of $\mathbb{Z}$). We use them (sometimes supplemented by other invariants) to show that members of various one-parameter families of AT actions are mutually non-isomorphic. In particular, we construct continua of AT systems that are not conjugate to their inverses.

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## Introduction

Let $(X, \mu)$ be a measure space, and let $T: X \to X$ be a measurable invertible ergodic transformation.The classification of $(T, X, \mu)$ with respect to measurable conjugacy was shown [CW] to be equivalent to the classification (up to isomorphism) of their von Neumann algebra crossed products, $L^\infty(X) \rtimes_T \mathbb{Z}$, and in turn of matrix-valued random walks in terms of a boundary (called the Poisson boundary). This was subsequently [GH] shown to be equivalent to classification of a measure-theoretic version of dimension groups.

Approximately transitive (AT) actions are those that can be expressed in either of the latter two formulations as the $1 \times 1$ matrix case or, respectively, rank one (not in the ergodic sense, which is quite restrictive). In particular, AT actions correspond to direct limits, subject to an equivalence relation (that corresponds to conjugacy, isomorphism, etc., in the other formulations). We now describe this.

Let $(P_m)_{m \in \mathbb{N}}$ be a sequence of Laurent polynomials or absolutely summable Laurent series in one variable, with only nonnegative coefficients, and such that

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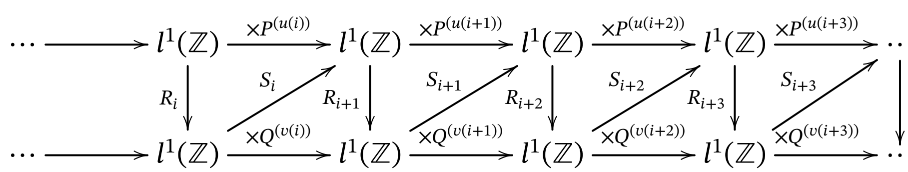
$P_m(1) = 1$ for all $m$. This describes an AT (approximately transitive) action, although there is no guarantee in this generality that it is nontrivial.

Let $(P_m)$ be a sequence of members of $l^1(\mathbb{Z})$ (which we usually view as functions on the unit circle) such that $P$ has only nonnegative coefficients and $P_m(1) = 1$. We may form the direct limit

$$l^1(\mathbb{Z}) \xrightarrow{\times P_1} l^1(\mathbb{Z}) \xrightarrow{\times P_2} l^1(\mathbb{Z}) \xrightarrow{\times P_3} l^1(\mathbb{Z}) \xrightarrow{\times P_4} \cdots,$$

where $\times P_m : l^1(\mathbb{Z}) \to l^1(\mathbb{Z})$, sending the $m$th copy of $l^1(\mathbb{Z})$ to the $m+1$st, is given by multiplication of functions, that is, $f \mapsto P_m \cdot f$ (equivalently, by convolution with the distribution corresponding to $P_i$). We also assume that the infinite product of the $P_m$ with any translation by powers of $x$, does *not* exist (this is to guarantee nontrivial ergodicity in this context). There is a lot of structure preserved by the maps (such as positivity), and then there is a completion process.

Fortunately, we do not have to go over this, because isomorphism is described by a relatively simple equivalence relation involving almost commuting diagrams. Let $(P_m)$ and $(Q_m)$ be sequences as above. The following diagram describes the equivalence relation. There exist telescopings, $P^{(u(i))} := P_{u(i)}P_{u(i)+1}\cdots P_{u(i+1)-1}$ and $Q^{v(i)} := Q_{v(i)}Q_{v(i)+1}\cdots Q_{v(i+1)-1}$ (the functions $u$ and $v$ demarcate the telescopings):




Here $R_i, S_j$ are elements of $l^1(\mathbb{Z})$ with only nonnegative coefficients, and $R_i(1) = S_j(1) = 1$ (we can reduce to the case that additionally, $R_i$ and $S_j$ are Laurent polynomials—the corresponding distributions are finite), and multiplication by $R_i$ sends the $u(i)$th copy of $l^1(\mathbb{Z})$ in the top row to the $v(i)$th copy of $l^1(\mathbb{Z})$ in the bottom row, while $S_j$ sends the $v(j)$ copy in the bottom row to the $u(j+1)$st copy in the top row. The functions $u$ and $v$ serve as index functions.

In particular, $R_{j+1}S_j$ sends the $v(j)$th copy on the bottom row to the $v(j+1)$st (on the same row). But we have another obvious map that does this, specifically, the product $Q_{v(j+1)-1}Q_{v(j+1)-2}\cdots Q_{v(j)}$, which we have denoted $Q^{(v(j))}$. Then we require the summability condition

$$\sum_j \left\| R_{j+1}S_j - Q^{(j)} \right\| < \infty.$$

(The $l^1(\mathbb{Z})$-norm is used.)

Similarly, $S_iR_i$ sends the $u(i)$th copy of $l^1(\mathbb{Z})$ on the top row to the $u(i+1)$. Set $P^{(i)}$ to be the corresponding product, $P_{u(i+1)-1}P_{u(i+1)-2}\ldots P_{u(i)+1}P_{u(i)}$. Then we require that

$$\sum_i \left\| R_i S_i - P^{(u(i))} \right\| < \infty.$$

The existence of $R_i, S_j$ satisfying all these conditions is equivalent to there being an isomorphism between the von Neumann algebras, or conjugacy of the corresponding ergodic transformations, etc. [GH, Theorem 3.1]. (So we don't have to know how to define the Poisson boundary, for example.)

If $\mathcal{M}$ is the (isomorphism class of) von Neumann algebra corresponding to the sequence $(P_m)$ (or any sequence equivalent to it), we typically write, $\mathcal{M}$ *corresponds to the system* $(P_m)$ or vice versa.

While this yields the correct notion of isomorphism, it is rarely easy to decide on isomorphism or nonisomorphism of two systems using it. Invariants have been developed. The best known and earliest is the $T$-set, corresponding to eigenvalues of the transformation on a suitable Banach space. This is relatively easy to calculate, but only coarsely separates systems (algebras). A massive family of numerical invariants was introduced in [GH] and used there and in [H], which we will call *mass-cancellation invariants* (they will be defined and used in section 3). An unpublished result of Giordano, Handelman, and Munteanu asserts that the $T$-set invariants can be recovered from the mass-cancellation invariants.

Mass cancellation invariants are often useful, but very often, are difficult to calculate. In this paper, we introduce a family of numerical invariants that are generally easier to calculate. We use them to show that for many natural one-parameter families of AT actions (more precisely, their von Neumann algebras), $(\mathcal{M}(r))_{\{r\in\mathbb{R}|r>0\}}$, the members, $\mathcal{M}(r)$ are mutually non-isomorphic.

The model for this is the one-parameter family of systems given by $\mathcal{M}(r) = (\operatorname{Exp} rx^{2^i})$ (here Exp is the normalized version of the exponential function, $\operatorname{Exp} h(x) = e^{-h(1)} \exp h(x) = \exp(h(x) - h(1))$, when $h$ has no negative coefficients). With the new invariants, it is easy to check for this example, the $\mathcal{M}(r)$ are mutually nonisomorphic (something that was already known); but we extend this type of result considerably.

On the other hand, mass cancellation invariants can distinguish (in many cases) an AT transformation from its inverse—which the new invariants cannot.

Section 1 describes the new invariant, and presents some elementary properties. Section 2 contains applications to divisible systems (where the $P_m$ are compound Poisson), culminating in the nonisomorphism Theorem 2.12. Section 3 contains applications to not necessarily divisible systems. Here, there are more complications, necessitating that mass cancellation invariants assist in distinguishing systems. Section 4 contains results about nonisomorphism between (tensor) powers of systems.

We describe a numerical invariant for equivalence (measure-theoretic isomorphism) that often allows to distinguish members of one-parameter families of these.

## 1. The invariant

It is not one invariant, but an uncountable collection of numerical invariants, analogous to those in [GH, H]. Let $(w_k)_{k\in\mathbb{N}}$ be a sequence of complex numbers of modulus 1. To the sequence, we wish to associate a number, denoted $\mathcal{S}((w_k),(P_m))$, in $[0,1]$, so that the assignment $(P_m) \mapsto \mathcal{S}((w_k),(P_m))$ is an isomorphism invariant for $(P_m)$. In other words, each sequence $(w_k)$ yields an isomorphism invariant for AT actions. Most sequences of elements of the circle yield uninteresting or simply uncomputable invariants, but for the examples we have in mind, there are natural choices of sequences which yield nonisomorphism results.

Fix the sequences $(w_k)$ and $(P_m)$. In direct analogy with the invariant discussed in [GH], [H], define for each $l \in \mathbb{N}$, the number $S_{k,l}$ defined as

$$S_{k,l} = \lim_{d\to\infty} \left| \prod_{m=l}^{m=l+d} P_m(w_k) \right| \tag{1}$$

That the limit exists follows from $|P(z)| \le 1$ when $P$ has no negative coefficients, $P(1) = 1$, and $|z| = 1$. Now define

$$S_l = \inf_{k\in\mathbb{N}} S_{k,l}.$$

Finally, set $\mathcal{S}((w_k),(P_m)) = \lim_l S_l$. That the limit exists follows from $S_l \le S_{l+1}$. This is a number in the unit interval, but there is no guarantee that it is nonzero or not 1.

Our first task is to show that this is indeed an isomorphism invariant. This is routine, but a proof is presented to convince the skeptics (and is more convincing than Arthur Cayley's proof of the Cayley-Hamilton theorem).

**Proposition 1.1.** *Suppose that $\mathcal{M}_1$ corresponds to the system $(P_m)$ and $\mathcal{M}_2$ corresponds to $(Q_m)$. Let $(w_k)$ be a sequence of elements of the unit circle. If $\mathcal{M}_1 \cong \mathcal{M}_2$, then $\mathcal{S}((w_k),(P_m)) = \mathcal{S}((w_k),(Q_m))$.*

**Proof.** We are given the almost commuting diagram given above. Given $\epsilon > 0$, there exists $j'$ such that for all $j \ge j$,

$$\begin{aligned} &\left\| R_j S_{j-1} R_{j-1} S_{j-2} \cdots R_{j'+1} S_{j'} - Q^{(v(j))} Q^{(v(j-1))} \cdots Q^{(v(j'))} \right\| \\ &< \epsilon \left\| S_{j-1} R_{j-1} S_{j-2} \cdots S_{j'+1} R_{j'+1} - P^{(u(j))} P^{(u(j-1))} \cdots Q^{(u(j'))} \right\| \quad < \epsilon \end{aligned}$$

Now $|R_j S_j \cdots S_{j'}(w_k)| \le |S_j \cdots R_{j'+1}(w_k)|$ (the second product is simply the first with the first and last terms deleted). As

$$|Q^{(v(j))} Q^{(v(j-1))} \cdots Q^{(v(j'))}(w_k) - R_j S_{j-1} R_{j-1} S_{j-2} \cdots R_{j'+1} S_{j'}(w_k)| < \epsilon$$

and similarly with the truncated version, we obtain $S_{k,u(j')} \le S^Q_{k,v(j')} + \epsilon$ for all sufficiently large $j'$. It follows that $S_{u(j')} \le S^Q_{v(j')} + \epsilon$ for infinitely many $j'$, and thus $\mathcal{S}((w_k),(P_m)) \le \mathcal{S}((w_k),(Q_m)) + \epsilon$. As this is true for all $\epsilon$, we have $\mathcal{S}((w_k),(P_m)) \le \mathcal{S}((w_k),(Q_m))$. Reversing the roles of $P_m$ and $Q_m$, we obtain the opposite inequality, so $\mathcal{S}((w_k),(P_m)) = \mathcal{S}((w_k),(Q_m))$. □

So we can write $\mathcal{S}((w_k),\mathcal{M})$ for $\mathcal{S}((w_k),(P_m))$ if $(P_m)$ corresponds to $\mathcal{M}$. If we replace inf by sup in the definition, we obtain another invariant—the *upper* version; so the initial invariant is the *lower* one associated to $(w_k)$; but in most of the examples here, the upper and lower ones agree. We never use the upper invariant.

As noted before, it is similar to the mass cancellation invariants introduced in [GH]. It has one advantage over these, in that it is usually easier to compute with, at least if we make the appropriate choice for the sequence $(w_k)$.

**Tensor products of actions.** If $\mathcal{M}$ and $\mathcal{N}$ are the von Neumann algebra crossed products associated to two actions, then we may form their von Neumann algebra tensor product, $\mathcal{M}\otimes\mathcal{N}$. If the actions are AT, say corresponding to $(P_m)$ and $(Q_m)$ respectively, then the action arising from the sequence of products $(P_mQ_m)$ corresponds to $\mathcal{M}\otimes\mathcal{N}$. However, it is not clear what the dynamical interpretation should be; that is, if $T$ and $U$ are ergodic transformations, how should $T\otimes U$ be defined? A first guess is $T\times U$, but this need not be ergodic. An attempt to resolve this, intended for the topological, rather than measure-theoretic, setting (minimal replacing ergodic) is given in [BH, Appendix A].

The invariant is frequently (but not always) multiplicative with respect to tensor products, that is, if $\mathcal{M}$ and $\mathcal{M}'$ are AT actions, then $\mathcal{S}((w_j),\mathcal{M}\otimes\mathcal{M}')=\mathcal{S}((w_j),\mathcal{M})\cdot\mathcal{S}((w_j),\mathcal{M}')$ often occurs. The $j$-fold tensor product of $j$ copies of $\mathcal{M}$ is denoted $\otimes^j\mathcal{M}$.

**Lemma 1.2.** *Let $\mathcal{M}$ and $\mathcal{M}'$ be AT, and let $(w_k)$ be a sequence of elements of the unit circle.*

(a) $\min\left\{\mathcal{S}((w_k),\mathcal{M}),\mathcal{S}((w_k),\mathcal{M}')\right\}$
$\geq\mathcal{S}((w_k),\mathcal{M}\otimes\mathcal{M}')\geq\mathcal{S}((w_k),\mathcal{M})\cdot\mathcal{S}((w_k),\mathcal{M}')$;

(b) *If* $\mathcal{S}((w_k),\mathcal{M})\neq 0$*, then* $\mathcal{S}((w_k),\mathcal{M}\otimes\mathcal{M}')=0$ *iff* $\mathcal{S}((w_k),\mathcal{M}')=0$;

(c) $\mathcal{S}((w_k),\otimes^j\mathcal{M})=\mathcal{S}((w_k),\mathcal{M})^j$;

(d) *If* $\mathcal{S}((w_k),\mathcal{M})=1$*, then* $\mathcal{S}((w_k),\mathcal{M}\otimes\mathcal{M}')=\mathcal{S}((w_k),\mathcal{M}')$.

**Proof.** If $\mathcal{M}$ is given by the sequence $(P_m)$ and $\mathcal{M}'$ is given by $(P'_m)$, then $\mathcal{M}\otimes\mathcal{M}'$ is given by $(P_mP'_m)$ (the sequence of products). If $S_{k,l}$, $S'_{k,l}$, and $S''_{k,l}$ represent the moduli of the products for $(P_m)$, $(P'_m)$ and $(P_mP'_m)$ respectively, then we have $S''_{k,l}=S_{k,l}\cdot S'_{k,l}$. Taking infima over $k$, we obtain $S''_l\geq S_l\cdot S'_l$. Taking limits as $l\to\infty$, the right inequality of (a) follows.

Next, we see that $S''_{k,l}=S_{k,l}\cdot S'_{k,l}\leq\min\left\{S_{k,l},S'_{k,l}\right\}$, and the left side of (a) follows.

(b) If $\mathcal{S}((w_k),\mathcal{M}')=0$, then $\mathcal{S}((w_k),\mathcal{M}\otimes\mathcal{M}')=0$ follows from the left side of (a). If $\mathcal{S}((w_k),\mathcal{M}')\neq 0$, then $\mathcal{S}((w_k),\mathcal{M}\otimes\mathcal{M}')\neq 0$ follows from the right side of (a).

(c) Follows from $|P_m(w_k)^j|=|P_m(w_k)|^j$.

(d) Follows from both parts of (a). □

An immediate consequence of Lemma 1.2(c) is the following.

**Lemma 1.3.** *Let $\mathcal{M}$ be an AT system, and suppose there exists a sequence $(w_k)$ of elements of the unit circle such that $\mathcal{S}((w_k),\mathcal{M}) \notin \{0,1\}$. Then the $j$-fold tensor products, $\{\otimes^j\mathcal{M}\}_{j=1,2,\dots}$ are mutually non-isomorphic.*

The following construction is obvious, but is given here for completeness.

**Example 1.4.** *Two AT systems, $\mathcal{M}_1$ and $\mathcal{M}_2$, and a sequence of roots of unity, $(w_k)$, such that $\mathcal{S}((w_k),\mathcal{M}_1\otimes\mathcal{M}_2) \neq \mathcal{S}((w_k),\mathcal{M}_1)\cdot\mathcal{S}((w_k),\mathcal{M}_2)$.*

**Proof.** Let $\tau$ be a positive real number less than $\frac{1}{2}$, and let $w_k = \exp(2i\pi/2^k)$. Define

$$P_m = \begin{cases} \frac{1+2x^{2^m}}{3} & \text{if } m \text{ is even} \\ \frac{1+\tau x^{2^m}}{1+\tau} & \text{if } m \text{ is odd.} \end{cases}$$

Let $\mathcal{M}_1$ denote the AT action determined by $(P_m)$. If we interchange *even* with *odd* in the definition of $P_m$, we obtain $\mathcal{M}_2$. It is routine to verify that, for the infinite products,

$$\left(1-\frac{8}{9}\right)\left(1-\frac{4\tau}{(1+\tau)^2}\cdot\frac{1}{2}\right)\left(1-\frac{8}{9}\sin^2\frac{2\pi}{16}\right)\left(1-\frac{4\tau}{(1+\tau)^2}\sin^2\frac{2\pi}{32}\right)\cdot\dots$$
$$<\left(1-\frac{4\tau}{(1+\tau)^2}\right)\left(1-\frac{8}{9}\cdot\frac{1}{2}\right)\left(1-\frac{4\tau}{(1+\tau)^2}\sin^2\frac{2\pi}{16}\right)\cdot\dots.$$

This implies that for every $l$, $S_l$ (for $\mathcal{M}_1$) is the square root of the top product, hence this is $\mathcal{S}((w_k),\mathcal{M})$. The same inequality yields that this is also $\mathcal{S}((w_k),\mathcal{M}_2)$.

However, $\mathcal{M}_1\otimes\mathcal{M}_2$ is given by the sequence $(Q_m = (1+2x^{2^m})(1+\tau x^{2^m})/3(1+\tau))$, and it is easy to check that

$$\mathcal{S}((w_k)),\mathcal{M}_1\otimes\mathcal{M}_2) = \left(\prod\left(1-\frac{8}{9}\sin^2\frac{2\pi}{2^m}\right)^{1/2}\cdot\prod\left(1-\frac{4\tau}{(1+\tau)^2}\sin^2\frac{2\pi}{2^m}\right)\right)^{1/2},$$

which is not $\mathcal{S}((w_k),\mathcal{M}_1)\cdot\mathcal{S}((w_k),\mathcal{M}_2) = \mathcal{S}((w_k),\mathcal{M}_1)^2$. □

If $\mathcal{M}\otimes\mathcal{M}\cong\mathcal{M}$ (as occurs for many odometers), then the values of the new invariants can only be 0 or 1, no matter what the choice of sequence $(w_k)$. A little more generally, if $\mathcal{M}(r)$ (for $r\in\mathbb{R}^{++}$) is a one-parameter family of AT actions such that $\mathcal{M}(r)\otimes\mathcal{M}(r')\cong\mathcal{M}(r+r')$ and the invariants $\mathcal{S}((w_k),\cdot)$ are multiplicative on $\{\mathcal{M}(r)\}$ (the model example above, $(\operatorname{Exp} rx^{2^i})$, satisfies these properties), then for fixed $(w_k)$, the map $\phi:\ \mathbb{R}^{++}\to[0,1]$ given by $r\mapsto\mathcal{S}((w_k),\mathcal{M}(r))$ satisfies $\phi(r+r') = \phi(r)\phi(r')$ (e.g., $\phi(r)=\gamma^r$ for some $\gamma\le 1$). In this case, if $\phi(r)\notin\{0,1\}$ for some $r$, then $\mathcal{M}(r)\cong\mathcal{M}(r')$ entails $r=r'$ (Corollary 2.2), which is precisely the conclusion we want.

## 2. Bounded AT actions

Here we give reasonably general sufficient conditions so that one-parameter families of AT actions, $\mathcal{M}(r)$ (for $r$ a positive real number), given by, for example, $(P_{m,r} = \exp\left(r(h_m(x^{n^m})-1)\right)$ (for some positive integer $n\ge 2$) where each

$h_m \in l^1(\mathbb{Z})$ has no negative coefficients and $h_m(1) = 1$, satisfy $\mathcal{M}(r') \cong \mathcal{M}(r)$ implies $r = r'$. The corresponding distributions are compound Poisson, and therefore divisible.

Let $f = \sum_{t\in\mathbb{Z}} a_t x^t$ be an element of $l^1(\mathbb{Z})$, that is, $\sum |a_t| < \infty$, such that all the coefficients, $a_t$, are nonnegative. In that case, $\|f\| = f(1)$. In general, for $f \in l^1(\mathbb{Z}))$, $\|f\| \geq \sup_{z\in\mathbb{T}} |f(z)|$. We say $f$ has *finite second moment,* if $f$ has only nonnegative coefficients, $f(1) = 1$, and $\sum a_t t^2 < \infty$. This implies that the real and imaginary parts of $f$ (as a function on the unit circle) are twice differentiable. When $f$ has finite second moment, we define $\mu_2(f) := \sum a_t t^2$; this is $f'(1) + f''(1)$. The *first moment*, $\mu_1(f) = \sum a_t t = f'(1)$, is defined if merely $\sum a_t |t| < \infty$, but need not be positive.

First we discuss a fairly general divisible AT situation. Let $H_m$ be elements of $l^1(\mathbb{Z})$ with no negative coefficients. Define $P_m = \operatorname{Exp} H_m := \exp(H_m - H_m(1))$, and let $\mathcal{M}$ denote the system corresponding to the sequence $(P_m)$. In this generality, there is no guarantee that the action is nontrivial.

For real $r > 0$, define $\mathcal{M}(r)$ to be the action obtained from the sequence $\left(P_{m,r} := \operatorname{Exp}(rH_m)\right)$ $(= (\exp(r(H_m - H_m(1)))$. The following is elementary.

**Proposition 2.1.** *Let $\mathcal{M}$ be the action determined by* $(\operatorname{Exp} H_m)$*, with corresponding $\mathcal{M}(r) = (\operatorname{Exp} rH_m)$ for positive real r. Suppose that $\mathcal{S}((w_k), \mathcal{M}) = S$ for some choice of sequence of elements of the unit circle, $(w_k)$. Then for all r, we have $\mathcal{S}((w_k), \mathcal{M}(r)) = S^r$.*

*Remark.* Our convention is that $S^r$ is $\exp(r \ln S)$ if $S > 0$, and 0 if $S = 0$.

**Proof.** If $w$ is of absolute value 1, then

$$\begin{aligned} \left|P_{m,r}(w)\right| &= \exp\left(\operatorname{Re}(r(H_{m,r}(w) - H(1)))\right) \\ &= \left(\exp(\operatorname{Re}(H_m(w) - H(1))\right)^r . \end{aligned}$$

It follows that $S_{k,l,r} := \lim_{d\to\infty} \prod_{j=l}^{l+d} |P_{j,r}(w_k)|$ is the exponential (in $r$) of $S_{k,l,1}$. Thus for fixed $r$, the infimum over $k$ of $S_{k,l,r}$ is just the exponential of the corresponding number with $r = 1$ (because $r \mapsto \gamma^r$ is order preserving). Now taking limits (as $l \to \infty$), the result follows. □

Recall that for $H \in l^1(\mathbb{Z})$ with only nonnegative coefficients and $H(1) = 1$, $\operatorname{Exp} H$ denotes $\exp(H - 1) \in l^1(\mathbb{Z})$.

**Corollary 2.2.** *Let $(H_m)$ be a sequence of elements of $l^1(\mathbb{Z})$ with only nonnegative coefficients, and such that $H_m(1) = 1$ for all m. Let $r > 0$, and let $\mathcal{M}(r)$ be the AT action determined by* $(\operatorname{Exp}(r(H_m))$*. If for some choice of sequence of elements of the unit circle, $(w_k)$, and some $r' > 0$, we have $\mathcal{S}((w_k), \mathcal{M}(r')) \notin \{0, 1\}$, then $\mathcal{M}(r)$ are mutually non-isomorphic.*

**Proof.** Without loss of generality, we may assume $r' = 1$. Let $S = \mathcal{S}((w_k),$ $\mathcal{M}(1))$; then by the preceding, $\mathcal{S}((w_k), \mathcal{M}(r)) = S^r$, and of course $r \mapsto S^r$ is one to one. □

Thus, to determine whether $\mathcal{M}(r) \cong \mathcal{M}(r')$ implies $r = r'$, it is sufficient to show that $S := \mathcal{S}((w_k), (\mathcal{M}))$ is neither zero or one. This is useful, as computing $S$ (an infinite product) *exactly* is problematic.

Let $f \in l^1(\mathbb{Z})$ be such that $f(1) = 1$ and $f$ is $C^2$, that is, $\mu_2(f) < \infty$. Define $V(f)$ to be $f''(1) + f'(1) - (f'(1))^2$. It is straightforward that for $f_1$ and $f_2$, we have $V(f_1 f_2) = V(f_1) + V(f_2)$, and if $f = x^n$ for some integer $n$, then $V(f) = 0$. In particular, $V(x^n f) = V(f)$, that is, $V$ is shift invariant.

Of course, $V$ is well-known. Let $(a_j)_{i\in\mathbb{Z}}$ be a sequence of nonnegative real numbers such that $\sum a_j = 1$, and $\sum a_j j^2 < \infty$, and let $f = \sum a_j x^j$; then $V(f)$ is simply the *variance* of the distribution $(a_j)$, or equivalently, of the corresponding integer-valued random variable $X$ defined by $\Pr(X = j) = a_j$. In our definition of $V$, there is no requirement that the $a_j$ be nonnegative or even real. We can similarly define (unnormalized) skewness and kurtosis (provided the third and fourth moments respectively are finite), as well as third and fourth cumulants; the latter convert multiplication (of functions) to sums (as variance does), but it is unlikely they will be of any use here.

The following three elementary results (Lemmas 2.3, 2.4, and Corollary 2.5) are probably known, but I could not find references for them. The variance results will be useful in section 3.

**Lemma 2.3.** *Let $f = \sum a_j x^j$ have finite second moment, with $f(1) = 1$. Then*

$$V(f) = \sum_{j<j'} a_j a_{j'} (j' - j)^2.$$

**Proof.** Everything is absolutely summable here, so there will be no problem with the infinite sums. We observe

$$\begin{aligned}
\mu_2(f) = 1 \cdot \mu_2(f) &= \left(\sum_j a_j\right) \cdot \left(\sum_t a_t t^2\right) \\
&= \sum a_j^2 + \sum_{j\neq j'} a_j a_j' (j')^2 \\
&= \sum a_j^2 + \frac{1}{2} \sum_{j\neq j'} a_j a_{j'} (j^2 + (j')^2) \\
&= \sum a_j^2 + \sum_{j<j'} a_j a_{j'} (j^2 + (j')^2). \\
V(f) &= f''(1) + f'(1) - (f'(1))^2 \\
&= \mu_2(f) - \left(\sum a_j j\right)^2 \\
&= \mu_2(f) - \sum a_j^2 j^2 - 2 \sum_{j<j'} a_j a_{j'} j j'.
\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{j<j'} a_j a_{j'} (j'-j)^2 &= \sum_{j<j'} a_j a_{j'} (j^2 + (j')^2) - 2 \sum_{j<j'} a_j a_{j'} j j' \\ &= \left(\mu_2(f) - \sum a_j^2 j^2\right) + V(f) - \left(\mu_2(f) - \sum a_j^2 j^2\right) \\ &= V(f).\end{aligned}$$

□

The following generalizes the fact that for $\theta > 0$, we have $\sin\theta < \theta$. The proof is by double induction, observing that the derivative transforms the partial sums into shorter versions of their ilk.

**Lemma 2.4.** *Suppose $\theta > 0$. Then*

(a) *for all nonnegative integers s,*

$$\sum_{t=0}^{2s+1} (-1)^t \frac{\theta^{2t+1}}{(2t+1)!} < \sin\theta < \sum_{t=0}^{2s} (-1)^t \frac{\theta^{2t+1}}{(2t+1)!};$$

(b) *for all $s \geq 1$,*

$$\sum_{t=0}^{2s+1} (-1)^t \frac{\theta^{2t}}{(2t)!} < \cos\theta < \sum_{t=0}^{2s} (-1)^t \frac{\theta^{2t}}{(2t)!}.$$

In the following, $K(h) = \sum_{j<j'} a_j a_{j'} (j'-j)^4/12$. I don't know whether this has any statistical significance.

**Corollary 2.5.** *Let $h \in l^1(\mathbb{Z})$ have only nonnegative coefficients, and $h(1) = 1$; assume the fourth moment exists. Let $\theta > 0$ and set $z = e^{i\theta}$.*

(a) $-\frac{\mu_2(h)}{2}\theta^2 + \frac{\mu_4(h)}{24}\theta^4 > \mathrm{Re}(h(z) - 1) > -\frac{\mu_2(h)}{2}\theta^2$

(b) $1 - V(h)\theta^2 + K(h)\theta^4 > |h(z)|^2 > 1 - V(h)\theta^2$.

**Proof.** Write $h = \sum a_j x^j$ with $a_j \geq 0$, $\sum a_j = 1$, and $\sum a_j j^4 < \infty$. Then

$$\begin{aligned}\mathrm{Re}(h(z) - 1) &= \sum a_j \cos j\theta - \sum a_j \\ &= \sum a_j (\cos j\theta - 1); \quad \text{by Lemma 2.4,}\end{aligned}$$

$$-\sum a_j \left(\frac{(j\theta)^2}{2} - \frac{(j\theta)^4}{24}\right) > \mathrm{Re}(h(z) - 1) > -\sum a_j \frac{(j\theta)^2}{2}; \quad \text{so}$$

$$-\theta^2 \frac{\mu_2(h)}{2} + \theta^4 \frac{\mu_4(h)}{24} > \mathrm{Re}(h(z) - 1) > -\theta^2 \frac{\mu_2(h)}{2}.$$

We also have

$$\begin{aligned}|h(z)|^2 &= \left|\sum a_j(\cos j\theta + i\sin\theta)\right|^2\\ &= \sum a_j^2 + 2\sum_{j<j'} a_j a_{j'}\left((\cos j\theta)\cdot(\cos j'\theta) + (\sin j\theta)\cdot(\sin j'\theta)\right)\\ &= \sum a_j^2 + 2\sum_{j<j'} a_j a_{j'}(\cos(j'-j)\theta); \quad \text{since } 1 = \sum a_j^2 + 2\sum_{j<j'} a_j a_{j'},\\ &= 1 - 2\sum_{j<j'} a_j a_{j'}(1-\cos(j'-j)\theta);\end{aligned}$$

thus by Lemma 2.4,

$$\begin{aligned}&1 - \sum_{j<j'} a_j a_{j'}(j'-j)^2\theta^2 + 2\sum_{j<j'} a_j a_{j'}\frac{((j'-j)\theta)^4}{24}\\ &> |h(z)|^2 > 1 - \sum_{j<j'} a_j a_{j'}(j'-j)^2\theta^2; \quad \text{so}\\ &1 - V(h)\theta^2 + K(h)\theta^4 > |h(z)|^2 > 1 - V(h)\theta^2.\end{aligned}$$

□

If in this lemma, $h$ is a Laurent polynomial, say with $M$ as maximal exponent and $m$ as minimal one, we can use the Bhatia-Davis inequality [BD], $V(h) \leq (M-h'(1))(h'(1)-m)$, to bound $V(h)$ without going to the trouble of calculating it.

Let $(n(m))_{m=1,2,\dots}$ be a sequence of positive integers, and form

$$T(m) = \prod_{l=1}^{m} n(l).$$

In the simplest case, $n(m) = n$ for all $m$, and then $T(m) = n^m$. Now let $(h_m)$ be a sequence of elements of elements of $l^1(\mathbb{Z})$, each with no negative coefficients and zero constant term satisfying $h_m(1) = 1$, and each of finite second moment. Let $r$ be a positive real number, and set $P_{m,r}(x) = \operatorname{Exp}\left(rh_m(x^{T(m)}\right)$; this is $\exp\left(r(h_m(x^{T^m}) - 1)\right)$, owing to the normalization of the $h_m$. We form the AT system, $\mathcal{M}(r)$, given by the sequence $(P_{m,r})_m$, yielding a one-parameter family, $r \mapsto \mathcal{M}(r)$. The current aim is to determine sufficient conditions so that $\mathcal{M}(r) \cong \mathcal{M}(r')$ implies $r = r'$, and for this, it is sufficient to show $\mathcal{S}((w_k), \mathcal{M}) \notin \{0, 1\}$, by Corollary 2.2.

For example, this property fails if for some $r$, we have that $\mathcal{M}(r)$ is an odometer, for then $\mathcal{M}(2r) \cong \mathcal{M}(r) \otimes \mathcal{M}(r) \cong \mathcal{M}(r)$. The simplest example of this occurs if $n(m) = n > 1$ for all $m$ and $h_m(x) = g(m)x$ where $g$ is a positive-valued function such that $g(m)/\ln m \to \infty$. Then $P_{m,r} = \operatorname{Exp}(rg(m)x^{n^m})$, and by [H, Theorem 4.4], $\mathcal{M}(r)$ is the $n$-odometer for all choices of $r > 0$; the corresponding supernatural number is $n^\infty$, that is, infinite at the prime divisors of $n$ and zero at other primes.

If we weaken the hypothesis on the growth of $g$, it is known [H] that sufficient for the system to be an odometer is that $\liminf g(m)/\ln m$ be sufficiently large (how large depending on $n$), but it is not known whether $\mathcal{M}(r)$ is an odometer if $g$ grows more slowly, e.g., $g(m) \sim \ln\ln m$. Unfortunately, our new invariant doesn't help with this. However, it will help if, for example, $h_m = h$ for all $m$, and some variations on this, e.g., a bound on second moments of $h_m$.

We can now deal with the relatively simple case of $h_k = h$ for all $k$. First, the case of $T(k) = n^k$, that is, $n(k) = n$ is constant.

**Proposition 2.6.** *Let $n \geq 2$ be an integer, and $h \in l^1(\mathbb{Z})$ have only nonnegative coefficients, finite second moment, and $h(1) = 1$. Set $\mathcal{M}(r)$ to be the system associated to $\left(P_m := \operatorname{Exp}(rh(x^{n^m}))\right)$. Let $w_k = \exp(2\pi i/n^k)$. Then*

$$S := \mathcal{S}((w(k)), \mathcal{M}(1)) = \exp\left(-\sum_{t=1}^{\infty} \operatorname{Re}(1 - h(e^{2\pi i/n^t}))\right),$$

*and this lies strictly between 0 and 1. In particular, $\{\mathcal{M}(r)\}_{r>0}$ are mutually nonisomorphic.*

*Remark.* We can get fairly tight estimates for $S$ if the first few terms of the sum are known.

*Remark.* So in our standard example, $\left(\operatorname{Exp}(rx^{n^i})\right)$, we have $h(x) = x$ and the $\mathcal{S}$-value (at $r = 1$) is $\exp\left(-\sum 2\sin^2(\pi/n^i)\right)$ (which clearly converges).

**Proof.** That $S$ is as given in the display is an immediate consequence of the definitions. To check that $S$ is neither zero nor one, we note that

$$\operatorname{Re}(1 - h(e^{2\pi i/n^t})) \leq 2\mu_2(h)\frac{\pi^2}{n^{2t}},$$

hence the sum converges, and thus $S > 0$; but $S < e^{-\operatorname{Re}(1-h(\exp 2\pi i)/n)}$ (the first term), and so $S < 1$. The rest follows from Corollary 2.2. □

The first remark follows from the inequality $1-\mu_2(h)\theta^2/2 < \operatorname{Re}(1-h(e^{i\theta}) < 1 - \mu_2(h)\theta^2/2 + \mu_4(h)\theta^4/24$, which for $\theta = 2\pi/n^t$ gives a tiny error for sufficiently large $t$.

If $n(k) \to \infty$ (but $h_k = h$ for all $k$), then the invariant does nothing; the value will be 1. This will follow from more general results, where $h_k$ are allowed to vary.

Returning to the broader situation (with $n(m)$, $T(m)$, $h_m$ being more general), let $w_k = \exp(2\pi i/T(k))$, a primitive $T(k)$th root of unity. We will estimate (under relatively modest conditions) the value of $S := \mathcal{S}((w_k), (P_{m,1}))$, at least well enough so that we can say it is not zero or one.

Abbreviate $P_{m,1}$ to $P_m$. First, we note that $|P_{m,r}(z)| = |P_m(z)|^r$ for $z$ on the unit circle. Next, we see that $P_m(w_k) = 1$ if $k \leq m$. For $k > m$, set $\theta_{m,k} =$

$\exp(2\pi i/(T(k)/T(m)))$. We have

$$\begin{aligned} |P_m(w_k)| &= \left|\exp\left(h\left(\exp\frac{2\pi i}{T(k)/T(m)}\right)-1\right)\right| \\ &= \exp\left(-\operatorname{Re}(1-h(\exp(i\theta_{m,k})))\right). \end{aligned}$$

Thus, for $l$, $d$ positive integers, we have

$$\begin{aligned} \prod_{j=l}^{l+d}\left|P_j(w_k)\right| &= \prod_{j=l}^{(l+d)\wedge(k-1)}\left|P_j(w_k)\right|; \quad \text{thus, as } d\to\infty, \\ &= \exp\left(-\sum_{j=l}^{k-1}\operatorname{Re}(1-h_j(e^{i\theta_{j,k}}))\right); \quad \text{substituting } t=k-j, \\ &= \exp\left(-\sum_{t=1}^{k-l}\operatorname{Re}(1-h_{k-t}(e^{i\theta_{j,k}}))\right) \\ &= \exp(-\operatorname{Re}(1-h_{k-1}(\exp(2\pi i/n(k))) \\ &\quad -\operatorname{Re}(1-h_{k-2}(\exp(2\pi i/n(k)n(k-1)))-\ldots)) \end{aligned} \tag{2}$$

The last line is purely expository.

The following yields conditions under which the value of the invariant is not zero.

**Lemma 2.7.** *Suppose that $\mu_2(h_k)<\infty$ for all $k$, and in addition,*

(a)

$$\limsup\frac{\mu_2(h_{k-1})}{\mu_2(h_k))n(k)^2} := C<1;$$

(b)

$$\limsup\frac{\mu_2(h_{k-1})}{n(k)^2 n(k-1)^2} := \rho<\infty.$$

*Then, with $\mathcal{M}$ given by $\left(P_m(x)=\operatorname{Exp}(h_m(x^{T(m)}))\right)$, we have*

$$\mathcal{S}((w_k),\mathcal{M}) \geq \exp\left(-\limsup_{k\to\infty}\operatorname{Re}(1-h_k(e^{2\pi i/n(k+1)}))\right)\exp(-M\rho),$$

*where $M=2\pi^2/(1-C)$, with equality if $\rho=0$.*

*Remark.* In particular, the value of the invariant is not zero here.

*Remark.* Hypothesis (a) is quite weak. Hypothesis (b),

$$\mu_2(h_{k-1}) = \boldsymbol{O}\left(n(k)^2 n(k-1)^2\right),$$

is reasonable.

*Remark.* If $\rho=0$, that is, $\mu_2(h_{k-1})=\boldsymbol{o}\left(n(k)^2 n(k+1)^2\right)$ (which is fairly modest), then we obtain right off the bat that the $\mathcal{S}$-value is not 1 (hence neither zero nor one) if $\limsup\operatorname{Re} h_k(e^{2\pi i/n(k+1)})<1$; unfortunately, there is a marked tendency for a reasonable sequence $(h_k)$ of such functions to have the lim sup equalling 1 when $\limsup n(k) = \infty$.

**Proof.** Expression (2) yields a value for the product, and thus for $S_{k,l}$. For the purposes of simplicity of the terms, set $n(0) = 1$. In the penultimate line thereof, take the sum beginning with $t = 2$; define

$$\begin{aligned} A_k &:= \mathrm{Re}\left(1 - h_{k-2}(e^{2\pi i/n(k)n(k-1)})\right) + \cdots + \mathrm{Re}\left(1 - h_l(e^{2\pi i/n(k)\cdots n(l+1)})\right) \\ &\leq \sum_{t=2}^{k-l} \left(\frac{2\pi}{n(k)n(k-1)\cdots n(k-t+1)}\right)^2 \mu_2(h_{k-t})/2. \\ &= \frac{2\pi^2}{n(k)^2 n(k-1)^2} \sum_{t=2} \frac{\mu_2(h_{k-t})}{(n(k-2)\cdots n(k-t+1))^2} \end{aligned}$$

Now let $C = \limsup \mu_2(h_{k-1})/(\mu_2(h_k)\cdot n(k)^2)$, and pick $C'$ such that $1 > C' > C$. There exists $k_0$ such that $j \geq k_0$ entails $\mu_2(h_{j-1}) \leq C'\mu_2(h_j)n(j)^2$. Iterating this when $l > k_0$, we obtain

$$\begin{aligned} A_k &\leq \frac{2\pi^2}{n(k)^2 n(k-1)^2} \sum_{t=2} \mu_2(h_{k-1})(C')^{t-2} \\ &\leq 2\pi^2 \frac{\mu_2(h_{k-1})}{n(k)^2 n(k-1)^2} \frac{1}{1-C'}. \end{aligned}$$

Now $S_{k,l} = \exp(-\mathrm{Re}(1 - h_k(e^{2\pi i/n(k+1)})) - A_k)$, hence for sufficiently large $l$,

$$S_{k,l} \geq \exp(-\mathrm{Re}(1 - h_k(e^{2\pi i/n(k+1)})) \cdot e^{-M\rho}$$

where $M = 2\pi^2/(1 - C')$.

If $\rho = 0$, we obtain $S_l \geq \exp\left(-\limsup \mathrm{Re}(1 - h_k(e^{2\pi i/n(k+1)}))\right)$ (for sufficiently large $l$), and the reverse inequality is trivial. □

Now we can (almost) finish the $h_m = h$ case.

**Corollary 2.8.** *Let $n(k) \to \infty$ and $h \in l^1(\mathbb{Z})$ have only nonnegative coefficients, finite second moment, and $h(1) = 1$. Set $\mathcal{M}(r)$ to be the system associated to $\left(P_m := \mathrm{Exp}(rh(x^{T(m)})\right)$. Let $w_k = \exp(2\pi i/n(k+1))$. Then*

$$\mathcal{S}\left((w_k), \mathcal{M}(r)\right) = 1.$$

**Proof.** Without loss of generality, we can assume $r = 1$. Conditions (a) and (b) are satisfied with $C = \rho = 0$, yielding

$$\mathcal{S}((w_k), \mathcal{M}(r)) = \exp\left(-\limsup \mathrm{Re}(1 - h(e^{2\pi i/n(k+1)})\right),$$

but this is clearly 1. □

Now we obtain estimates for $\mathrm{Re}(1 - h_{k-1}(e^{2\pi i/n(k)})$; it is equivalent, and slightly more convenient, to work with $\mathrm{Re}(1 - h_k(e^{2\pi i/n(k+1)}))$.

So let $h = \sum a_j x^j$ with $a_j \geq 0$ for all $j$, and $\sum a_j = 1$. Let $n$ be a positive integer exceeding 1, and let $R > 1$ be a real number. Let $\operatorname{supp} h$ denote the set of $j$ such that $a_j \neq 0$.

Define

$$S(h,n,R) = \operatorname{supp} h \cap \left( \bigcup_{t \in \mathbb{Z}} \left( tn + \left[ \frac{n}{R}, n \cdot (1 - 1/R) \right] \right) \right),$$
$$U(h,n,R) = \sum_{j \in S(h,n,r)} a_j.$$

For example, if $\theta = \pi/n$, then $j \in tn + [n/R, n(1-1/R)$ for some integer $t$ entails that $j\theta \in t\pi + [\pi/R, \pi(1-1/R)]$, and thus $\sin^2 j\theta \geq \sin^2 \pi/R$. On an interval of the form $[0,K]$, the proportion of coefficients not in the union is about $2n/R$, so for large $R$, $S(h,n,R)$ is typically most of $\operatorname{supp} h$. If the distribution of $h$ is not concentrated off $S(h,n,R)$, then $U(h,n,R)$ will be close to 1, or at any rate, more than one-half. If we can arrange that this occurs uniformly in $k$ for $h_k$ and $n(k+1)$ (playing the roles of $h, n$ respectively) for some $R$, then we obtain a lower bound for values of the invariants.

**Lemma 2.9.** *Suppose there exists* $R > 1$ *such that* $\liminf_k U(h_k, n(k+1), R) := \eta > 0$. *Then for all sufficiently large* $k$, $\operatorname{Re}(1 - h_k(e^{2\pi i/n(k+1)})) \geq 2\eta \sin^2(\pi/R)$.

**Proof.** Write $h_k = \sum_j a_{j,k}$, so that

$$\operatorname{Re}(1 - h_k(e^{i\theta}) = \sum_j a_{j,k}(1 - \cos j\theta) = 2 \sum_j a_{j,k} \sin^2(j\theta/2).$$

Set $\theta = 2\pi/n(k+1)$. We see that for $j \in S(h_k, n(k+1), R)$, we have $\sin^2(j\theta/2) \geq \sin^2 \pi/R$. Hence for all sufficiently large $k$,

$$\operatorname{Re}(1 - h_k(e^{2\pi i/n(k+1)})) \geq 2\eta \sin^2(\pi/R).$$

□

There follows immediately:

**Corollary 2.10.** *Suppose that there exists* $R > 1$ *such that for all sufficiently large* $k$, *there exists* $\eta > 0$ *such that* $U(h_k, n(k+1), R) \geq \eta$. *Then* $\mathcal{S}((w_k), \mathcal{M}) \leq e^{-2\eta \sin^2(\pi/R)}$.

In particular, this yields a fairly weak sufficient condition (on the sequence $(h_m)$) so that the value of the invariant is strictly less than 1.

We also have a converse to this.

**Proposition 2.11.** *Suppose that for all* $R > 1$, $\liminf_k U(h_k, n(k+1), R) = 0$. *Then* $\mathcal{S}((w_k), \mathcal{M}) = 1$.

**Proof.** Let $h = \sum a_j x^j$, and $\theta$ a small positive real number. For $j \notin S(h,n,R)$, $|\sin j\theta| < \pi/R$. Thus

$$\sum_{j \notin S(h,n,R)} a_j(1-\cos 2j\theta) = 2 \sum_{j \notin S(h,n,R)} a_j \sin^2(j\theta); < \left(\frac{\pi}{R}\right)^2.$$

Therefore

$$\operatorname{Re}(1-h(e^{2i\theta}) = \operatorname{Re}\sum a_j(1-\cos 2j\theta) < U(h,n,R) + \left(\frac{\pi}{R}\right)^2,$$

and so sufficient for the left side to be small is that both summands be small.

Suppose we have $U(h_k, n(k+1), R) < \epsilon$ for infinitely many $k$, and let $\theta = \pi/n(k+1)$. Since $\mu_2(h) = \boldsymbol{o}\left(n(k+1)^2\right)$, $\operatorname{Re}(1-h_k(e^{2\pi i/n(k+1)})$, for infinitely many $k$, $\operatorname{Re}(1-h_k(e^{2\pi/n(k+1)})) < \epsilon + (\pi/R)^2$. Allowing $R \to \infty$ and $\epsilon \to 0$, we deduce $S_l \to 1$ along infinitely many $l$, and thus $\mathcal{S}((w_k), \mathcal{M}) = 1$. □

**Theorem 2.12.** *Let $\mathcal{M}(r)$ be given by $P_{m,r} = \operatorname{Exp}\left(rh_m(x^{T(m)})\right)$, subject to the following conditions.*

(a) $\mu_2(h_k) < \infty$ *for all but finitely many $k$;*
(b) $\limsup \frac{\mu_2(h_{k-1})}{\mu_2(h_k))n(k)^2} < 1$*;*
(c) $\mu_2(h_{k-1}) = \boldsymbol{O}\left(n(k)^2 n(k-1)^2\right)$.
(d) *There exists $R > 1$ such that* $\liminf_k U(h_k, n(k+1), R) > 0$.

*Then $\mathcal{M}(r) \cong \mathcal{M}(r')$ implies $r = r'$.*

**Proof.** Let $\mathcal{M}$ denote $\mathcal{M}(1)$ and set

$$w_k = \exp(2\pi i/T(k).$$

By Lemma 2.7, $\mathcal{S}((w_k), \mathcal{M}) > 0$, and by Corollary 2.10, $\mathcal{S}((w_k), \mathcal{M}) < 1$. Corollary 2.2 allows us to conclude. □

Hypothesis (a) obviously holds if the $h_m$ are Laurent polynomials; (b) is a very weak condition; and (c) is somewhat restrictive (and implies (a)), but it is difficult to see how it could be weakened. Hypothesis (d) is not very strong, but is superficially complicated.

## 3. A different type of one-parameter family

In the cases discussed earlier, the mapping (for appropriate choices of $(w_k)$) $r \mapsto \mathcal{S}((w_k), \mathcal{M}(r))$ is multiplicative, that is,

$$\mathcal{S}((w_k), \mathcal{M}(r) \otimes \mathcal{M}(r'))) = \mathcal{S}((w_k), \mathcal{M}(r)) \cdot \mathcal{S}((w_k), \mathcal{M}(r')).$$

There is another, fairly natural type of one-parameter family, for which similar properties do not apply, but nonetheless, we can obtain similar isomorphism results.

Suppose $h_m$ belong to $l^1(\mathbb{Z})$ and have only nonnegative coefficients. For each positive real $r$, define $P_{m,r}(x) = h_m(rx^{T(m)})/h_m(r)$. The system now need not be divisible (as it was in the earlier case, owing to the definition of Exp).

To distinguish this construction from the earlier ones, we use the notation $\mathcal{N}$ (or $\mathcal{N}(r)$) for the system arising from $(h_m(rx^{T(m)})/h_m(r))$.

A more natural definition might seem to be that arising from

$$P'_{m,r} = h_m((rx)^{T(m)})/h_m(r),$$

but this often results in atoms, for example, if $h_m = (1+x)/2$ and $r \neq 1$, for the resulting sequence $(P'_{m,r})$, the products actually converge, resulting in an atomic dynamical system.

**Lemma 3.1.** *Let $h \in l^1(\mathbb{Z})$ have only nonnegative coefficients, finite second moment, and $h(1) = 1$. Let $n \geq 2$ be a positive integer, and $r$ a positive real number. Let $\mathcal{N}$ be of the form $(P_m)$, where $P_m = h(rx^{n^m})/h(r)$. Set $w_k = \exp 2\pi i/n^k$. Then for all $l$, $\mathcal{S}((w_k), \mathcal{N}) = \lim_{k\to\infty} S_{k,l}$, and this equals*

$$\prod_{t=1}^{\infty} \frac{\left|h(r\exp(2\pi i/n^t))\right|}{h(r)}.$$

*Moreover, this is nonzero unless for some $t$, $h(r\exp(2\pi i/n^t)) = 0$.*

**Proof.** Let $h_r(x) = h(rx)/h(r)$. First, from Corollary 2.5(b), we have that $|h(re^{i\theta})|^2 \geq 1 - V(h_r)\theta^2$. With $\theta$ equalling successively $2\pi/n^m$, we see that $1 \geq |h(re^{2\pi i/n(k+1)\cdots n(l+1))}|^2/h(r)^2 \geq 1 - V(h_r)\pi^2/(n(k+1)\cdots n(l+1))^2$. Thus $\prod_{t=1}^{\infty} \left(|h(r\exp(2\pi i/n^t))|/h(r)\right)$ converges in the sense of infinite products, and the only way the limit can be zero is if one of the factors is.

We have $S_{k,l} = \prod_{t=1}^{k-l} \left(|h(re^{2\pi i/n^t})|/h(r)\right)$; fixing $l$ and taking the infimum over $k$, noting that $k - l \to \infty$, we simply obtain

$$S_l = \prod_{t=1}^{\infty} \left(|h(r\exp(2\pi i/n^t))|/h(r)\right).$$

As this is independent of $l$, we obtain $\mathcal{S}((w_k), \mathcal{N}) = S_l$. □

It is not true that $\mathcal{N}(r+r') \cong \mathcal{N}(r) \otimes \mathcal{N}(r')$ (except under degenerate circumstances), so that multiplicativity is not as interesting as in the previous class of examples.

Asking the same question, can the class of evaluation invariants distinguish members of $\{\mathcal{N}(r)\}$, the answer is somewhat different—it requires the aid of another invariant. The following simple-looking example illustrates what can happen.

**Example 3.2.** *A one-parameter family $\mathcal{N}(r)$ such that $\mathcal{S}((w_k), \cdot)$ distinguishes $\mathcal{N}(r)$ from $\mathcal{N}(r')$ if $r' \neq r, r^{-1}$. An additional invariant distinguishes $\mathcal{N}(r)$ from $\mathcal{N}(r^{-1})$ if $r \neq 1$.*

**Proof.** Set $h_m = 1 + x$, so that $P_{m,r} = (1 + rx^{T(m)})/(1 + r)$. If $n(k) = 2$ for all $k$, when $r = 1$, the corresponding system is the dyadic odometer—but for all other values of $r$, it isn't an odometer (the former statement is elementary, the latter is not difficult, and will follow from the computation of the invariant anyway).

Taking our usual $w_k = \exp(2\pi i/T(k))$, we compute enough of the invariant to obtain a slightly limited classification result, which will later be supplemented by another invariant.

An elementary computation reveals that for $k > m$,

$$\left|P_{m,r}(w_k)\right|^2 = 1 - \frac{4r}{(1+r)^2}\sin^2\frac{\pi}{T(k)/T(m)}.$$

Thus

$$\left|\prod_{t=0}^{k+l-1} P_{l+t,r}(w_k)\right|^2 = \prod_{t=0}^{k+l-1}\left(1 - \frac{4r}{(1+r)^2}\sin^2\frac{\pi}{T(k)/T(l+t)}\right).$$

The smallest term in this product is $1 - \sin^2(\pi/n(l+1))4r/(1+r)^2$, and it is easy to check that the product converges (as we let $k \to \infty$) in the usual sense of infinite products—however, some of the initial terms might turn out to be zero (this occurs with the odometer example), so that the product could be zero. The product is invariant under $r \mapsto r^{-1}$ and here, $\mathcal{N}(r^{-1})$ corresponds to the inverse transformation to $\mathcal{N}(r)$. Thus $\mathcal{S}((w_k), \mathcal{N}(r)) = \mathcal{S}((w_k), \mathcal{N}(r^{-1}))$. In particular, the invariant does not distinguish some pairs of members of the family. We will deal with this shortly.

We will show that $r \mapsto \mathcal{S}((w_k), \mathcal{N}(r))$ is monotone decreasing on $(0, 1]$, and strictly decreasing under mild assumptions. The latter entails members of this part of the family are mutually nonisomorphic. Then we will show that if $r \neq 1$, $\mathcal{N}(r) \not\cong \mathcal{N}(r^{-1})$ by an easy application of the invariants introduced in [GH].

A minor problem arises when a few factors in the product,

$$\alpha(k,r) := \prod_{t=1}^{\infty}\left(1 - \frac{4r}{(1+r)^2}\sin^2\left(\frac{\pi}{n(k+1)n(k)\cdots n(l-+2-t)}\right)\right),$$

might be zero. First, we observe that each term is nonnegative (since $4r/(1+r)^2 \leq 1$). Thus the value zero can only occur if $r = 1$ and $n(k+1) = 2$. Since $n(k) = 2$ for all $k$ and $r = 1$ entails the system corresponds to the dyadic odometer, we can set this case aside. In particular, $r \neq 1$ entails each term is positive. Moreover, since $r \mapsto 4r/(1+r)^2$ is strictly increasing, we see that for $r < r' < 1$, we have $S_{k,l}(r) > S_{k.l}(r')$. A consequence is that $\mathcal{S}((w_k), \mathcal{N}(r)) \geq \mathcal{S}((w_k), \mathcal{N}(r'))$, but we want strict inequality.

This does not always hold (as we will see, when we discuss the condition $n(k) \to \infty$). However, if $n(k) = n$ (for all but finitely many $k$), then we easily see that

$$\mathcal{S}((w_k), \mathcal{N}(r)) = \prod_{j=1}^{\infty} \left(1 - \frac{4r}{(1+r)^2} \sin^2 \left(\pi/n^j\right)\right).$$

The infinite product converges to a nonzero positive number, and it is strictly increasing as $r \to 1$ from below, as the function $r \mapsto 4r/(1+r)^2$ is strictly increasing on $(0,1]$ with maximum at $r = 1$.

Hence if $n(k) = n$, and for $r, r' \in \mathbb{R}^{++}$ with $r, r^{-1} \neq r'$, then $\mathcal{N}(r) \ncong \mathcal{N}(r')$.

Now we show that $\mathcal{N}(r) \ncong \mathcal{N}(r^{-1})$ if $r \neq 1$. We use the other class of invariants, $s((p_k), (P_n))$, defined (in [GH]) as follows. The $p_k$ are elements of $l^1(\mathbb{Z})$ of norm one, and we define for each $k \geq l$, $s_{l,k} = \lim_{d\to\infty} \|p_k P_l \cdot P_{l+1} \cdots P_{l+d}\|$ (the limit exists since the sequence is monotone decreasing; all the $P_k$ have norm 1). Then define $s_l = \inf_k s_{l.k}$, and as with the evaluation invariant, note that the sequence $(s_l)$ is increasing, and we define $s((p_k), (P_n)) = \lim_l s_l$ (the latter limit exists, as $s_l$ is increasing). This is the invariant associated to the sequence $(p_k)$, and to distinguish this type from the other one, we refer to the former as *mass-loss invariants.*

We sometimes abbreviate $P_{k,r} = (1 + rx^{T(k)})/(1+r)$ to $P_k$, if $r$ is understood. □

**Example 3.3.** *Let $\mathcal{N}(r)$ be given by $(P_{k,r})$. If either of the following hold, then $\mathcal{N}(r) \ncong \mathcal{N}(r^{-1})$ when $r \neq 1$.*

(a) *$n(k) > 2$ for all but finitely many $k$*
(b) *$n(k) = 2$ for all but finitely many $k$.*

*Remark.* The mixed case, that $\{k \in \mathbb{N} \mid n(k) = 2\}$ is both infinite and co-infinite in $\mathbb{N}$, is difficult to deal with.

*Remark.* In this example, $\mathcal{N}(r^{-1})$ corresponds the AT action that is the inverse of the original one (corresponding to $\mathcal{N}(r)$). So we obtain a continuum of AT systems that are not conjugate to their inverse.

**Proof.** If $h$ is in $l^1(\mathbb{Z})$, we denote by $h^{\mathrm{op}}$ the element of $l^1$ given by $x \mapsto x^{-1}$, that is, all exponents are replaced by their negatives. When $h$ is a polynomial (that is, has support in $\mathbb{Z}^+$ and this is finite), we can replace $h^{\mathrm{op}}$ by $x^d h^{\mathrm{op}}$, where $d$ is the degree of $h$, and so continue to work with polynomials (rather than Laurent polynomials). Since $h \mapsto h^{\mathrm{op}}$ is an isometry of $l^1(\mathbb{Z})$ preserving all the coefficients (just reflecting them), we see immediately that

$$s((p_k)), ({P_m}^{\mathrm{op}})) = s(({p_k}^{\mathrm{op}})), (P_m)).$$

Hence to show that $s((p_k)), ({P_m}^{\mathrm{op}})) \neq s((p_k)), (P_m))$ for suitable $(p_k)$, it is sufficient to show that $s((p_k)), (P_m)) \neq s(({p_k}^{\mathrm{op}})), (P_m))$.

(a) *$n(k) \geq 3$ for all but finitely many $k$.* Set $p_k = (1 - rx^{T(k)})/(1+r)$. We notice that $p_k \cdot P_k = (1 - r^2 x^{2T(k)})/(1+r)^2$, which has norm $(1-r^2)/(1+r)^2 = (1-r)/(1+r)$. Assuming $k > l$, multiply this by $P_l \cdots P_{k-1}$. This has total degree $T(l) + T(l+1) + \ldots T(k-1) < T(k)$, so that the largest difference between exponents is less that $T(k)$. It follows that there is no further mass cancellation in the product $P_l \cdot P_{k-1}(p_k \cdot P_k)$, that is, $\|p_k P_l \cdots P_k\| = (1-r)/(1+r)$.

Moreover, the product polynomial $p_k P_l \cdots P_k$ has degree $T(l) + T(l+1) + \cdots + T(k-1) + 2T(k) < T(k+1)$ (this uses $n(k) \geq 3$). Any product of the form $P_{k+1} \cdot \cdots \cdot P_{l+d}$ (with $d > k - l$) is supported on $T(k+1)\mathbb{Z}$, so that

$$\begin{aligned} s_{k.l} &= \lim_{d\to\infty} \|p_k P_l \cdot P_{l+1} \cdots P_{l+d}\| \\ &= \lim_{d\to\infty} \|(p_k P_l \cdot P_{l+1} \cdots P_k)(P_{k+1} \cdots P_{l+d})\| \\ &= \lim_{d\to\infty} \|(p_k P_l \cdot P_{l+1} \cdots P_k\| \\ &= \frac{1-r}{1+r} \end{aligned}$$

Thus $s((p_k),(P_m)) = (1-r)/(1+r)$.

We are therefore reduced to showing $s((q_k),(P_m)) \neq s((p_k),(P_m)$ where $q_k = (x^{T(k)} - r)/(1+r)$. Now $q_k P_k = (x^{2T(k)} + (1-r)x^{T(k)} - r)/(1+r)^2$. This has norm $2(1-r)/(1+r)^2$. Now let $Q$ be any polynomial with only nonnegative coefficients. In order for $\|q_k P_k Q\| < \|Q\| \cdot \|q_k P_k\|$, there must exist two points in the support of $Q$ whose difference is either $T(k)$ or $2T(k)$. But no such exists in a polynomial $Q = P_l \cdots P_{k+1} \cdot P_{k+1} \cdot P_{l+d}$. Hence, as in the previous case, $s((q_k),(P_m)) = 2(1-r)/(1+r)^2$. This is not equal to $(1-r)/(1+r)$, unless $r = 1$.

(b) $n(k) = 2$ *for all but finitely many* $k$. Special techniques are needed to deal with non-noninteractivity. We require some preliminary results.

For a nonnegative integer $j$, let $\delta(j)$ denote the number of 1s in its binary expansion, and if $j \neq 0$, let $e(j)$ be the maximum power of 2 that divides $j$. Thus $e(j) = 0$ iff $j$ is odd, $e(j) = 1$ iff $j \equiv 2 \pmod 4$, and so on.

Let $r$ be a positive real number, and form the product of polynomials in the variable $X$,

$$Q(X) := \prod_{i=0}^{d-1}(1 + rX^{2^i}) = \sum_{j=0}^{2^d-1} r^{\delta(j)} X^j.$$

The last line follows easily from uniqueness of binary expansions. Evaluating at $X = 1$, we obtain $\sum r^{\delta(j)} = (1+r)^d$. Let $a$ be a positive real number, and consider the product,

$$(1 - aX) \cdot Q = 1 + \sum_{j=1}^{2^d-1} \left(r^{\delta(j)} - ar^{\delta(j-1)}\right) X^j - ar^d X^{2^d}.$$

In order the compute the $l^1$-norm of this, we observe that for $1 \leq j < 2^d$, we have $\delta(j-1) = \delta(j) - 1 + e(j)$. □

**Lemma 3.4.** *For* $u = 0, 1, \dots, d$, *the following holds:*

$$\sum_{\{1\leq j\leq 2^d-1 | e(j)=u\}} r^{\delta(j)} = r(1+r)^{d-1-u}.$$

*Remark.* Of course, this is consistent with the earlier expansion, since $r\sum_{u=0}^{d-1}(1+r)^u = (1+r)^d - 1$.

**Proof.** Fix $u$; then $e(j) = u$ means $j \equiv 2^u \pmod{2^{u+1}}$ (even when $u = 0$). For $u = 0$, we have $\delta(j) = \delta(j-1)+1$, so the sum on the left becomes $r\sum r^{\delta(j)}$ where now $j$ varies over all the even integers less than or equal to $2^d - 1$. But the latter sum is the same as the sum over all terms up to $2^{d-1}$, hence is just $(1+r)^{d-1}$.

For $u \geq 1$, all $j$ are divisible by $2^u$, and the quotient is odd. Thus applying the result of the previous paragraph (to odd integers less than $2^{d-1-u}$), we obtain the result. □

**Resumption of the proof for Example 3.3(b).** We have

$$\begin{aligned}(1-aX)\cdot Q &= 1 + \sum_{j=1}^{2^d-1}\left(r^{\delta(j)} - ar^{\delta(j-1)}\right)X^j - ar^dX^{2^d}\\ &= 1 - ar^dX^{2^d} + \sum_{j=1}^{2^d-1} r^{\delta(j)}X^j(1-ar^{e(j)-1})\\ &= 1 - ar^dX^{2^d} + \sum_{u=0}^{d-1}\sum_{\{1\leq j\geq 2^d-1 \mid e(j)=u\}} X^jr^{\delta(j)}(1-ar^{e(j)-1});\end{aligned}$$

so

$$\begin{aligned}\|(1-aX)\cdot Q\| &= 1 + ar^d + \sum_{u=0}^{d-1}\sum_{\{1\leq j\geq 2^d-1 \mid e(j)=u\}} r^{\delta(j)}|1-ar^{e(j)-1}|\\ &= 1 + ar^d + \sum_{u=0}^{d-1} r(1+r)^{d-u-1}|1-ar^{e(j)-1}|.\end{aligned}$$

In the special case that $r < 1$ and $a = r$, then $1 - ar^{u-1} = 1 - r^u$; this is nonnegative, so that

$$\begin{aligned}\|(1-rX)\cdot Q\| &= 1 + r\sum_{u=0}^{d-1}(1+r)^{d-u-1}(1-r^u) + r^{d+1}\\ &= 1 + r\sum_{u=0}^{d-1}(1+r)^{d-u-1} - r\sum_{u=0}^{d-1}(1+r)^{d-u-1}r^u + r^{d+1}\\ &= (1+r)^d - r(1+r)^{d-1}\sum_{u=0}^{d-1}\left(\frac{r}{1+r}\right)^u + r^{d+1}\\ &= (1+r)^d - r(1+r)^{d-1}\frac{1-\left(\frac{r}{r+1}\right)^d}{1-\frac{r}{r+1}} + r^{d+1}\end{aligned}$$

$$= (1+r)^d(1-r)\left(1-\left(\frac{r}{r+1}\right)^d\right)+r^{d+1}.$$

Normalizing,

$$\left\|\frac{1-rX}{1+r}\cdot\frac{Q}{(1+r)^d}\right\| = \frac{1-r}{1+r}+\boldsymbol{O}\left(\left(\frac{r}{1+r}\right)^d\right)$$

Now let $a = 1/r$, so that $1-ar^{u-1} = 1-r^{u-2}$. This is negative for for $u = 0, 1$, zero for $u = 2$, and positive for $u \geq 3$. In that case, we obtain

$$\begin{aligned}\left\|(1-r^{-1}X)\cdot Q\right\| &= 1+r(1+r)^{d-1}(r^{-2}-1)+r(1+r)^{d-2}(r^{-1}-1)\\ &\quad + r\sum_{u\geq 3}(1+r)^{d-u-1}(1-r^{u-2})+r^{d-1}\\ &= 1+r^{d-1}+r(1+r)^d\cdot\\ &\quad\times\left(r^{-2}(1-r)+r^{-1}\frac{1-r}{(1+r)^2}+\frac{1}{1+r}\sum_{u\geq 3}\frac{1}{(1+r)^u}\right.\\ &\qquad\left.-\frac{1}{(1+r)r^2}\sum_{u\geq 3}\left(\frac{r}{1+r}\right)^u\right)\\ &= 1+r^{d-1}+r(1+r)^d\cdot\\ &\quad\times\left(r^{-2}(1-r)+r^{-1}\frac{1-r}{(1+r)^2}\right.\\ &\qquad\left.+\frac{1}{(1+r)^4}\frac{1-\left(\frac{1}{1+r}\right)^{d-3}}{1-\frac{1}{1+r}}-\frac{r}{(1+r)^3}\left(1-\left(\frac{r}{1+r}\right)^{d-3}\right)\right).\end{aligned}$$

Normalizing,

$$\begin{aligned}\left\|\frac{1-r^{-1}X}{1+r^{-1}}\cdot\frac{Q}{(1+r)^d}\right\| &= \frac{r^2}{1+r}\left(r^{-2}(1-r)+r^{-1}\frac{1-r}{(1+r)^2}\right.\\ &\qquad\left.+\frac{1}{r(1+r)^3}-\frac{r}{(1+r)^4}\right)+\boldsymbol{O}\left(\left(\frac{r}{1+r}\right)^d\right)\\ &= \frac{1-r}{1+r}+\frac{r(1-r)}{(1+r)^3}+\frac{r}{(1+r)^4}-\frac{r^3}{(1+r)^5}\\ &\qquad+\boldsymbol{O}\left(\left(\frac{r}{1+r}\right)^d\right)\\ &= \frac{1-r}{1+r}\left(1+\frac{1}{1+r}-\frac{1}{(1+r)^3}\right)+\boldsymbol{O}\left(\left(\frac{r}{1+r}\right)^d\right)\end{aligned}$$

Now we note that if $P$ is a polynomial with degree $m$, and $Q$ is a polynomial of the form $q(x^M)$ where $M > m$, then $\|PQ\| = \|P\| \cdot \|Q\|$ (no mass cancellation can take place; in fact, if $x^a$ appears $PQ$ with nonzero coefficient, then there exists a unique pair $(c,d)$ such that $a = c + c'$, $c \in \operatorname{Log} P$, and $c' \in \operatorname{Log} Q$.

We apply this with $P = \prod_{j<k} P_j$ and $Q = p \prod_{k \le j \le d}$, where $p = (1 + rx^{2^k})/(1+r)$ or $(1+r^{-1}x^{2^k})/(1+r)^{-1}$. Therefore $s_{k,l} = (1-r)/(1+r)$ in the former case, and thus $\S((p_k), \mathcal{N}(r)) = (1-r)/(1+r)$, whereas $\S(({p_k}^{\mathrm{op}}), \mathcal{N}(r))) = ((1-r)/(1+r) \cdot (1 + 1/(1+r) - 1/(1+r)^3) \neq \S((p_k), \mathcal{N}(r))$. □

This, together with the earlier results, yields $\mathcal{N}(r) \cong \mathcal{N}(r')$ implies $r = r'$ provided that $n(k) = n$ for all $k$. The case that $n(k) = 2$ yields a continuum ($r > 1$) of mutually non-isomorphic AT systems not isomorphic to their inverses.

If $n(k) \to \infty$ (the condition that merely $\sup n(k) = \infty$ appears to be much more complicated, and we do not deal with it), we run into a difficulty (although the mass-cancellation invariants can probably be used).

**Proposition 3.5.** *If $n(k) \to \infty$, then $\mathcal{S}((w_k), \mathcal{N}(r)) = 1$ for all $r > 0$.*

**Proof.** The condition $n(k) \to \infty$ implies that for each $j$, the set

$$\{l \in \mathbb{N} \mid n(l) = j\}$$

is finite. It follows immediately that there exist infinitely many $l$ with the property that for all $k' > l$, we have $n(l+1) < n(k'+1)$.

It suffices, from the definition of $S_l$ to show that for every $k' > l$ (where $n(l+1) < n(k'+1)$ for all $k' > l$), that $S_{l+1,l} < S_{k',l}$; sufficient for this is,

$$1 - \frac{4r \sin^2 \pi/(n(l+1))}{(1+r)^2} \le \prod_{t=0}^{k'-l} \left(1 - \frac{4r}{(1+r)^2} \sin^2 \frac{\pi}{n(k'+1)n(k')\cdots n(l+1+t)}\right).$$

To this end, we may assume that $l$ is so large that for all $k > l$, we have $n(k+1) \ge 10$, and we observe that the left side is bounded below by

$$\begin{aligned}
&\prod_{t=0}^{k'+l-1} \left(1 - \frac{4r}{(1+r)^2} \left(\frac{\pi}{n(k'+1)n(k')\cdots n(l+1+t)}\right)^2\right) \\
&\ge \left(1 - \frac{4r\pi^2}{n(k'+1)^2(1+r)^2}\right)\left(1 + \sum_{t=0}^{k'-l-1} \frac{1}{(n(k') \cdot n(k'-t))^2}\right) \\
&\ge \left(1 - \frac{4r\pi^2}{n(k'+1)^2(1+r)^2}\right)\left(1 + \frac{1}{(n(l+1)+1)^2 - 1}\right).
\end{aligned}$$

(The last line comes from $n(k'+1) > n(l+1)$.)

Finally $1 - 4r\sin^2(\pi/n(l+1))/(1+r)^2 \le 1 - 4r\pi^2(1+\eta)/n(k'+1)^2(1+r)^2$ is equivalent to $\sin^2(\pi/n(l+1)) \ge (1+\eta)\pi^2/n(k'+1)^2$. The latter is at least as large as $\pi^2/((n(l+1)+1)^2 - 1)$. For $n(l+1) \ge 4$, we have $\sin^2 \pi/n(l+1) =$

$(1 - \cos 2\pi/n(l+1))/2 \geq \pi^2/n(l+1)^2 - 1\pi^4/3n(l+1)^4$. So sufficient is that

$$\frac{\pi^2}{n(l+1)^2} - \frac{\pi^4}{3n(l+1)^4} \geq \frac{\pi^2}{(n(l+1)+1)^2 - 1}.$$

But this is a straightforward consequence of $6n(l+1) \geq 10 > \pi^2$.

This finishes Examples 3.2 and 3.3. □

Difficulties arise when we try to extend this to more general $h$. Suppose that $h$ is a polynomial of degree $d > 1$, and as usual, $n(k) = n$ (constant). The behaviour of the product $S_l = \prod_{t=1}^{\infty} |h(r\exp(2\pi i/n^t))/h(r)|$ is more complicated, when viewed as a function of $r$. Instead of having just one minimum value (as $r$ varies), it can have several critical points (up to $d$ of them). This can be somewhat compensated for.

To give an example, suppose that $h = (1 + x + 2x^2)/4$ and $n = 3$ (so we still have a non-interactive situation). Instead of taking $w_k = \exp(2\pi i/3^k)$, we may make another choice, $w_{k,2} = \exp(2\pi i/2 \cdot 3^k)$. This yields another invariant, and the value will be not zero.

For a general polynomial $h$ and $n(k) = n$ (easier to deal with if $n > \deg h$), for $j = 1, 2, \ldots d$, we can use each of the sequences $(w_{k,j} = \exp(2\pi i/j \cdot n^k))_k$. This yields $d$ invariants, and a corresponding map $\mathbb{R}^{++} \to [0,1]^d$, given by $r \mapsto (\mathcal{S}((w_{k,j}), \mathcal{N}(r)))_{j=1}^d$. It is plausible that when $d > 1$ and $h$ is suitably nondegenerate, this is one to one—which would yield the non-isomorphism result.

However, we develop a different approach.

**Proposition 3.6.** *Let $a$ be a real number, and let $h = x^2 + ax + 1$ be a real irreducible polynomial. Let $0 < \theta \leq \pi/4$ be a positive real number such that $h(re^{i\theta}) \neq 0$ for all real $r > 0$. Set*

$$G(r) \equiv G(r,\theta) = \frac{|h(re^{i\theta}|^2}{h(r)^2}.$$

*If either $a \geq 0$ or $-2 < a < 1 - \sqrt{3}$, then $G$ has a unique minimum at $r_0 = 1$, $G$ is decreasing on $(0,1)$, increasing on $(1,\infty)$, and $\lim_{t\downarrow 0} G(t) = \lim_{t\to\infty} G(t) = 1$.*

**Proof.** Irreducibility and the nonzero hypothesis ensures that $h(r) > 0$ for all positive real $r$. Irreducibility is equivalent to $a^2 < 4$, that is, $|a| < 2$. Elementary calculus yields

$$\begin{aligned}\frac{dG}{dr} &= \frac{2(1-\cos\theta)\left(a(r^4-1) + (r^3 - r)(2-a^2)\right)}{h(r)^3} \\ &= 2(1-\cos\theta)(r^2-1)\frac{(a(r^2+1) + r(2-a^2))}{h(r)^3}\end{aligned}$$

Hence the zeros of $G'$ occur at $\pm 1$ and the roots of the quadratic (in $r$) factor. If $a = 0$, the additional root is just zero. Suppose $a \neq 0$. The quadratic is (up to a scalar multiple)

$$r^2 + \frac{2 - a^2}{a} r + 1,$$

and we only have to give conditions under which this has no positive real roots (so that the only positive real zero of $G'$ is at $r_0 = 1$).

If $(2 - a^2)/a \geq 0$, then there are clearly no positive roots, so we obtain sufficient conditions:

(i) $a \leq \sqrt{2}$ and $a > 0$, so this yields sufficiency of $a \in (0, \sqrt{2}]$.

(ii) $-2 < a < -\sqrt{2}$, yielding $a \in (-2, -\sqrt{2})$.

If $(2 - a^2)/a < 0$, then irreducibility of the quadratic is equivalent to $|(2 - a^2)|/|a| < 2$.

(iii) $a > 0$, $a^2 > 2$ and $|(2 - a^2)|/|a| < 2$ boils down to $a^2 - 2 < 2a$, that is $a^2 - 2a - 2 < 0$; this occurs precisely when $0 < a < \sqrt{3} + 1$; since the latter exceeds 2, we obtain a sufficient condition, $a \in (\sqrt{2}, 2)$.

(iv) $a < 0$, $a^2 < 2$, and $|(2 - a^2)|/|a| < 2$: set $b = -a > 0$, and we obtain $b^2 + 2b - 2 > 0$, which yields $\sqrt{3} - 1 < b < \sqrt{2}$, which amounts to $-\sqrt{2} < a < 1 - \sqrt{3}$.

The union of the four sets described in (i–iv) is $(0, 2) \cup (-2, 1 - \sqrt{3})$.

Clearly $\lim_{t\to\infty} G(t) = 1 = \lim_{t\downarrow 0} G(t)$, $G'(0) < 0$ (from the formula above), and $G(r) \leq 1$ for all $r \geq 0$. It follows easily that whenever $G'(r) = 0$ has only one positive solution, then $G$ has unique local minimum and no maxima on $(0, \infty)$. □

Irreducibility is essential in Proposition 3.6.

**Example 3.7.** *Let $a > \sqrt{3}+1$ and define $h = x^2+ax+1$. Then $h$ has only positive coefficients, is self-reciprocal, factors as $(x + \alpha)(x + \alpha^{-1})$ for some $\alpha > 0$, and the function $r \mapsto |h(re^{\theta})|^2/h(r)^2$ has two local minima and one local maximum (at $r = 1$) on $(0, \infty)$.*

**Proof.** The calculation in Proposition 2.6 shows that the positive zeros of $G'$ appear at 1, and at the positive roots of $r^2 + (2 - a^2)r/a + 1$. But the latter has positive roots when $a > \sqrt{3} + 1$, so that $G'$ has three positive real roots, one less than 1, one at 1, and one exceeding 1. It is easy to check that $G$ is decreasing at 0 (and asymptotic to 1 as $r \to \infty$), so the three roots correspond respectively to minimum, maximum, and minimum.

Since $a > 2$ and the constant term is 1, $h$ is a product of two linear real polynomials of the form indicated. □

**Proposition 3.8.** *Suppose $h$ is a monic real polynomial with no negative coefficients and nonzero constant term, and all roots of $h$ lie in the union of two cones (one closed, one open), $\operatorname{Re} z \leq 0$ and $0 < |\arg z| < \arctan(3 + \sqrt{3})$. Let $h = \prod g_i \prod h_j$ be the factorization of $h$ into monic irreducibles over $\mathbb{R}[x]$ where $g_i$ are linear and $h_j$ are quadratic. Let $0 < \theta < \pi/2$ be such that $h(re^{i\theta}) \neq 0$ for* *all $r > 0$. Then the function $\mathbb{R}^{++} \to (0,1]$ given by*

$$G_h: \ r \mapsto \frac{|h(re^{i\theta}|}{|h(r)|}$$

*is strictly increasing on the interval $\left(\max\{\sqrt{h_j(0)}, g_i(0)\}, \infty\right)$ and is strictly decreasing on the interval $\left(0, \min\{\sqrt{h_j(0)}, g_i(0)\}\right)$.*

*Remark.* $\arctan(3+\sqrt{3})$ is approximately 78.0675373 degrees or 1.36253556 radians.
*Remark.* The qualitative statement is independent of $\theta$ (subject to the constraints given therein).

**Proof.** First, we observe that for each $i$, $j$, $h_i(0)$ (the constant term of $h_i$), $g_i(0)$ must be positive (else being monic, $h_i$ would have a positive root, and thus so would $h$). If $g_i$ is linear, that is $g_i = x + s$ where $s > 0$, we reparameterize it, that is, $H_i(x) = g_i(\lambda x) = \lambda x + s = \lambda(x + s/\lambda)$ (for $\lambda > 0$), and setting $\lambda = s$, so $G(H) := |H(re^{i\theta}|/H(r)$ has unique minimum at $r = 1$, and is increasing for $r > 1$. Undoing the parameterization, we see that $|g_i(re^{i\theta})|/g_i(r)$ is increasing for $r \geq s = g_i(0)$.

For $h_j$ an irreducible quadratic, $h = x^2 + Ax + B$ (with $A^2 < 4B$), reparameterize to obtain $H_j(x) = h_j(\lambda x) = \lambda^2(x^2 + A/\lambda x + B/\lambda^2)$. Setting $\lambda = \sqrt{B}$, we have $H_j(x) = \lambda^2(x^2 + A/\lambda x + 1)$; reparameterizing does not change the arguments of roots, so we have that $G_{H_j}(r)$ is increasing when $r > 1$. Undoing the reparameterization, we see that $G(h_j)$ is increasing on $(\sqrt{B}, \infty)$.

Hence for each of the factors, the corresponding functions $G_{h_j}$, $G(g_i)$ are increasing on the ray given in the statement. Next, we observe that if $R_i$ are positive differentiable functions, and $R_i'$ are positive on an interval of the form $(M, \infty)$, then so is $\prod R_i$: set $G = \ln \prod R$ and differentiate, and observe that $G$ increasing entails that $\prod R_i$ increasing. So the product of all the $G$'s is increasing on the interval, and this is simply $G(h)$. A similar analysis yields the strictly decreasing part of the result. □

In particular, if $h$ is a product of self-reciprocal irreducible polynomials (that is, $1 + x$ and $x^2 + ax + 1$ with $|a| < 2$) , then $G(h)$ is strictly decreasing on $(0, 1)$ and strictly increasing on $(1, \infty)$. Unfortunately, not every self-reciprocal polynomial is a product of self-reciprocal irreducibles, as in Example 3.7, and moreover, the corresponding $G$ need not have unique local minimum.

For a monic polynomial $h$ with only nonnegative coefficients, let $h = \prod g_i \cdot \prod h_j$ be the irreducible factorization (over $\mathbb{R}[x]$) into monic linear factors $(g_i)$

and monic quadratic factors $(h_j)$. Define

$$M \equiv M(h) = \max\{\sqrt{h_j(0)}, g_i(0)\}$$
$$m \equiv m(h) = \min\{\sqrt{h_j(0)}, g_i(0)\}.$$

**Theorem 3.9.** *Let $n > 1$ be an integer, and let $h$ be a (monic) polynomial with no negative coefficients, and such that all roots lie in $\{\pi/2 \le |\arg z| < \pi\} \cup \{0 < |\arg z| < \arctan(\sqrt{3}+3)\}$, and moreover that $h$ has no roots with argument $\{2\pi/n, 2\pi/n^2, 2\pi/n^3, \dots\}$. For $r$ a positive real number, define the AT system, $\mathcal{N}(r) = (h(rx^{n^k})/h(r))$, and define $w_k = \exp 2\pi i/n^k$. If $r_1 > r_2 > M$, then $\mathcal{S}((w_k), \mathcal{N}(r_1)) > \mathcal{S}((w_k), \mathcal{N}(r_2))$, and in particular, $\mathcal{N}(r_1) \not\cong \mathcal{N}(r_2)$. Similarly, $\mathcal{N}(r_1) \not\cong \mathcal{N}(r_2)$ if $r_1 < r_2 < m$.*

**Proof.** Each of the factors in the infinite product appearing in Lemma 3.1 for $r_1$ is strictly greater than that for $r_2$ by Proposition 3.6, and the result follows. □

If for $h$, $M(h) = m(h)$ (as occurs if all the roots of the irreducible quadratics factoring $h$ have the same absolute value, $m$, and the only linear factor is $x+\sqrt{m}$ with arbitrary multiplicity), then we have a result that is about as far as we can go. The case that $M = m = 1$ means that $h$ is a product of self-reciprocal quadratics with a power of $1+x$. In particular, $h$ would then be self-reciprocal, but not all self-reciprocal polynomials are products of irreducible self-reciprocal ones (for example, $(x^2+ax+b)(x^2+ax/b+1/b)$ if $b \neq 1$, $a, b > 0$, and $a^2 < 4b$).

Because we used only products of quadratics and linear terms, the qualitative result in Proposition 3.8 was independent of $\theta$ (which is obviously important for our application to the inequalities in the infinite factor appearing as the value of the invariant). If we tried to prove the analogous result directly for $h$ (not irreducible), then Example 3.7 shows there can be more than one local minimum on the positive reals, and moreover, it is likely true that even when there is a unique minimum, the location of the unique minimum depends on $\theta$. Then we would not be able to obtain the corresponding inequalities in all the terms of the infinite products. So it is difficult to see how to substantially improve Theorem 3.9.

## 4. Nonisomorphism of powers

In this section, we use our invariants to distinguish systems of the form $\mathcal{K}(a,k) := ((1+x^{a^i})/2)^k)$ and $\mathcal{K}(a',k') := ((1+x^{(a')^i})/2)^{k'})$ where $a, a' > 2$; specifically, if $\{\ln a, \ln a'\}$ is rationally linearly independent, then isomorphism of $\mathcal{K}(a,k)$ with $\mathcal{K}(a',k')$ entails both $a = a'$ and $k = k'$.

The following is elementary. Let $a, b$ be integers exceeding 1. Then $\{\ln a, \ln b\}$ is linearly independent (over the rationals) is just another way of saying there exist no positive integers $u, v$ such that $a^u = b^v$. If we write $a = \prod p^{m(p)}$ and $b = \prod p^{m'(p)}$ are their prime decompositions, failure of linear independence is equivalent to the ratios $m(p)/m'(p)$ being independent of $p$.

**Lemma 4.1.** *Let $c, N$ be positive integers such that $\{\ln c, \ln N\}$ is linearly independent over the rationals. Then there exist strictly increasing sequences $(s(i))$ and $(u(i))$ of positive integers such that*

$$\lim_{i\to\infty} \frac{c^{u(i)}}{N^{s(i)}} \text{ exists and equals } \frac{1}{2}.$$

*Remark.* Of course we could replace $\frac{1}{2}$ by any positive real number.

**Proof.** Since $\{\ln c, \ln N\}$ is rationally linearly independent, it follows that $(\ln c)\,\mathbb{Z} + (\ln N)\,\mathbb{Z}$ is a dense subgroup of $\mathbb{R}$. Hence given $\eta > 0$, there exist integers $a, b$ such that $|a \ln c + b \ln N + \ln 2| < \eta$. We now show that we can arrange this so that the integer coefficient of $\ln c$ is positive.

Assume $a < 0$. Given $\eta' > 0$, there exist nonzero integers $e, f$ such that $|e \ln c + f \ln N| < \eta'$. By shrinking $\eta'$, we can arrange that $|e| > |a|$. Replacing, if necessary, $e, f$ by their negatives, we can assume that $e > 0$, so that $e + a > 0$. Adding the two inequalities, we have $|(e+a)\ln c + (b+f)\ln N + \ln 2| < \eta + \eta'$. Obviously if $\eta + \eta'$ is sufficiently small, then $b + f < 0$.

Thus given $\epsilon > 0$ small enough that $e^{\epsilon} < 1 + 2\epsilon$ and $e^{-\epsilon} > 1 - 2\epsilon$, we can find positive integers $u, s$ such that $|u \ln c - s \ln N + \ln 2| < \epsilon$. Exponentiating, we deduce

$$e^{-\epsilon} < \frac{2c^u}{N^s} < e^{-\epsilon}.$$

From $e^{\epsilon} < 1 + 2\epsilon$ and $e^{-\epsilon} > 1 - 2\epsilon$, we obtain after subtracting 1 from each term and dividing by 2,

$$-\epsilon < \frac{c^u}{N^s} - \frac{1}{2} < \epsilon.$$

□

**Proposition 4.2.** *Let $a, a'$ be positive integers such that $\{\ln a, \ln a'\}$ is linearly independent over the rationals. Let $k, k'$ be positive integers, and define the two AT systems $\mathcal{K}(a,k) = \left(\left((1 + x^{a^n})/2\right)^k\right)_n$ and $\mathcal{K}(a',k') = \left(\left((1 + x^{(a')^n})/2\right)^{k'}\right)_n$. Then for all $k, k'$, $\mathcal{K}(a,k)$ is not isomorphic to $\mathcal{K}(a',k')$.*

*Remark.* The conclusion of is probably true if merely $a \neq a'$, and this can be proved if a number-theoretic conjecture holds. Unfortunately, this conjecture appears to be far more difficult than the original problem.

**Proof.** Assume that $\mathcal{K}(a,k)$ is isomorphic to $\mathcal{K}(a',k')$. The $T$-sets of the respective systems (as subsets of the unit circle) are the roots of unity of order dividing, respectively, some power of $a$, some power of $a'$; equality of these (resulting from isomorphism) entails that $a$ and $a'$ have the same set of prime divisors (an easy argument can also be obtained from the evaluation type invariant). We denote the set of prime divisors of $a$, and of $a'$, $D$.

Let $a = \prod_{p\in D} p^{m(p)}$ and $a' = \prod_{p\in D} p^{m'(p)}$ be their prime factorizations. Define $M(a) = \max\left\{\frac{m(p)}{m'(p)} \mid p \in D\right\}$. If $M(a) \leq 1$, then we can interchange $a$ with $a'$, and so assume that $M(a) \equiv M > 1$. The constraints ensure that $D_0 := \{p \in D \mid M(a) = m(p)/m(p')\}$ is not all of $D$.

Now we will apply the evaluation invariant. Define $w_j = \exp(2\pi i/a^j)$. We have already seen that $\mathcal{S}((w_j), \mathcal{K}(a,k)) = \left(\prod_{n=1}^{\infty} \cos(\pi/a^n)\right)^k$, and this is in the open unit interval. We will show that $\mathcal{S}((w_j),(K(a',1)) = 0$, and since this implies (by Lemma 1.2(c)) that $\mathcal{S}((w_j),(K(a',k')) = 0$ for all $k'$, the result will follow. Specifically, we will show that $\{S_l\}$ (computed for $\mathcal{K}(a',1)$) contains a null sequence.

For a prime $p$, let $v_p$ denote the usual (additive) valuation, so that $v_p(a) = m(p)$. We see that for $j,l$ positive integers,

$$v_p\left(\frac{(a')^j}{a^l}\right) = jm'(p) - lm(p)$$

We observe that if $j \geq Ml$ (the latter need not be an integer), then for all $p$, we have the valuation at $p$ of $(a')^j a^{-l}$ is nonnegative, hence the latter is an integer:

$$\begin{aligned} jm'(p) - lm(p) &\geq Mlm'(p) - lm(p) \\ &\geq lm(p) - lm(p) = 0. \end{aligned}$$

Let $t$ denote the the least common multiple of $\{m'(p)\}_{p\in D}$ and restrict our attention to $l = tl_0$, that is, those choices of $l$ that are divisible by $t$. Then for any $p \in D$, we have that $tm(p)/m'(p)$ is an integer, and in particular, $tM$ is integer; we have restricted the choice of $l$ to those such that $lm(p)/m'(p)$ are all integers. Now let $s$ be a positive integer less than $Ml = (Mt)l_0$ (we will specify $s$ in more detail soon). Define (with $j = Ml - s$)

$$f(s,l) = \frac{(a')^{Ml-s}}{a^l}.$$

For $p \in D_0$, $m(p) = Mm'(p)$ and so

$$\begin{aligned} v_p(f(s,l)) &= m'(p)(Ml - s) - lm(p) \\ &= m'(p)(ML - s - Ml) \\ &= -sm'(p). \end{aligned}$$

For $q \in D \setminus D_0$ (not empty, as observed above), let $M_q = m(q)/m'(q) < M$. Then

$$\begin{aligned} v_p(f(s,l)) &= m'(p)(Ml - s - lM_q) \\ &= l(M - M_q) - s \end{aligned}$$

If we write $l = l_0 t$, and observe that $tM$ and $tM_q$ are both integers, then $l(M - M_q) = l_0(tM - tM_q)$ and the factor $tM - tM_q$ is thus a positive integer. Hence if $s < l_0$ (as we hypothesized above), the evaluation of $f(s,l)$ at all primes in $D \setminus D_0$ is positive.

Thus we have

$$
\begin{aligned}
f(j,l) &= \frac{\prod_{q\in D\setminus D_0} q^{l_0(tM-tM_q)-s}}{\prod_{p\in D_0} p^{sm(p)}} \\
&= \frac{\left(\prod_{q\in D\setminus D_0} q^{(tM-tM_q)}\right)^{l_0}}{\left(\prod_{p\in D_0} p^{m(p)}\cdot\prod_{q\in D\setminus D_0} q\right)^{s}}
\end{aligned}
$$

Set $c=\prod_{q\in D\setminus D_0} q^{(tM-tM_q)}$ and $N=\prod_{p\in D_0} p^{m(p)}\cdot\prod_{q\in D\setminus D_0} q$. Since $N$ has a prime divisor that does not divide $c$, it follows that $\{\ln c, \ln N\}$ is rationally linearly independent.

Hence given $\epsilon>0$, there exist positive integers $u, s$ such that $|c^u/N^s - 1/2| < \epsilon$, and thus $0<\cos(\pi c^u/N^s)<2\epsilon$ (assuming $\epsilon$ is sufficiently small).

Next we show that with this choice of $u, s$, we have that $s < utM$ (so we can take $l_0 = u$). We observe that for $\epsilon$ sufficiently small, $u/s$ is close to $\ln N/\ln c - \ln 2/s \ln c$. As we can make $s$ as large as we like (even shrinking the $\epsilon$s along the way), we can arrange that $u/s \geq \ln N/\ln c - \epsilon$. Hence it suffices to show that $\ln N/\ln c > 1/tM$.

We have

$$
\begin{aligned}
\ln N &= \sum_{D_0} m(p)\ln p + \sum_{D\setminus D_0}\ln q \\
\ln c &= \sum_{D\setminus D_0}(tM-tM_q)\ln q;\ \text{thus} \\
tM\ln N &> \sum_{D\setminus D_0}(tM-tM_q)\ln q.
\end{aligned}
$$

Hence given $\epsilon$, for infinitely many choices of $l$ at least one term in the infinite product of cosines is within $2\epsilon$ of 0. It follows that for those $l$, $S_l<\epsilon$, and thus $\mathcal{S}((w_k),\mathcal{K}(a',1))=0$, and thus for all $k'$, $\mathcal{S}((w_k),\mathcal{K}(a',k'))=0$. □

For a positive integer $a>2$, define $f(a)=\prod_{j=1}^{\infty}\cos(\pi/a^j)$. The infinite product clearly converges to a real number in the open unit interval.

If $\{\ln a, \ln a'\}$ is rationally dependent, that is, $a^e=(a')^f$ for some positive integers $e, f$, it might still be possible to prove the same conclusion, although this currently depends on a (reasonable) number-theoretic conjecture. In this case, we have that $m(p)/m'(p)=f/e$ for all primes $p$ dividing $a$. If $e=1$, that is, $a=(a')^f$, we set $w_l=\exp(2\pi i/a^l)$. Then

$$
\mathcal{S}((w_l),\mathcal{K}(a,1)) = \prod_{j=1}^{\infty}\cos\left(\frac{\pi}{a^j}\right) = f(a),
$$

as we have seen before. On the other hand,

$$
\mathcal{S}((w_l),\mathcal{K}(a',1)) = \prod_{j=1}^{\infty}\cos\left(\frac{\pi}{(a')^j}\right) = f(a').
$$

A similar computation applies for more general pairs $(a, a')$ with $(\ln a, \ln a')$ rationally dependent. We see quickly that for all $k$, $\mathcal{K}(a, k) \ncong \mathcal{K}(a', k)$ by Lemma 1.2(c). However, we want $\mathcal{K}(a, k) \ncong \mathcal{K}(a', k')$ regardless of the choice of $k, k'$.

Sufficient for this is that $\{\ln f(a), \ln f(a')\}$ be rationally linearly independent. This is likely true, and an even stronger condition probably holds, that the field $\mathbb{Q}(f(a), f(a'))$ have transcendence degree two. But this looks difficult to prove.

Another interesting class of systems are those of the form $\mathcal{M} := (((1 + x^{a^i})/2)^{k(i)})$, that is, with variable powers appearing. The computation of each of the $S_l$ with $w_j = \exp(2\pi\sqrt{-1}/a^j)$ includes a term of the form $(\cos \pi/a))^{k(i)}$. Hence if $\sup k(i) = \infty$, we deduce $\mathcal{S}((w_j), \mathcal{M}) = 0$, which is not especially helpful. In fact, if $\sum e^{-k(i)\pi^2/a^2} < \infty$ (sufficient for this is $\liminf(k(i)/\ln i) > a^2/\pi^2$), then $\mathcal{M}$ is isomorphic to an $a$-odometer (this follows from results in [H]). It is not known what happens for slower growth of $k(i)$, e.g., $k(i) \sim \ln \ln i$.

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