

Amenable actions of discrete quantum groups on von Neumann algebras

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ABSTRACT. We introduce the notion of Zimmer amenability for actions of discrete quantum groups on von Neumann algebras. We prove non-commutative generalizations of several fundamental results from the classical setting. In particular, we characterize Zimmer amenability of such an action in terms of injectivity of the associated von Neumann algebra crossed product as a module over the dual quantum group. As an application, we show that actions of any discrete quantum group on its Poisson boundaries are always Zimmer amenable.

CONTENTS

1. Introduction	1140
2. Preliminaires	1143
3. von Neumann algebra braided tensor products	1145
4. Amenable actions	1147
5. Examples	1154
6. Amenable actions and crossed products: Kac algebra case	1155
7. Amenable actions and crossed products: general case	1159
References	1162

1. Introduction

There are many different equivalent conditions that characterize the amenability of a locally compact group G . One such characterization is in terms of a fixed-point property of affine actions of G . In [24], Zimmer introduced the notion of amenable actions as a natural generalization of this property of fixed points. He also gave a criterion in terms of the associated cross-product. In subsequent work, Adams, Elliott, and Giordano characterized Zimmer amenability in terms of the existence of an equivariant conditional expectation [1]. In [2], Delaroché extended Zimmer’s definition to the setting of group actions on von

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Neumann algebras. In this paper, we introduce the notion of Zimmer amenability for actions of discrete quantum groups on von Neumann algebras.

Definition 4.1. *Let $\alpha : \mathbb{G} \curvearrowright N$ be an action of a discrete quantum group $\mathbb{G}$ on a von Neumann algebra N . Then α is called amenable if there exists a conditional expectation $E_\alpha : \ell^\infty(\mathbb{G}) \overline{\otimes} N \rightarrow \alpha(N)$ such that*

$$(\text{id} \otimes E_\alpha)(\Delta \otimes \text{id}) = (\Delta \otimes \text{id})E_\alpha.$$

This definition coincides with Delaroche's definition [2, Définition. 3.4] when G is a discrete group. Also observe that a discrete quantum group is amenable if and only if its action on the trivial space is amenable in the above sense. We prove, similarly to the classical result, the action of every discrete quantum group on itself by its co-multiplication is amenable (Proposition 5.1). Moreover, we show a connection between amenability of discrete quantum groups and amenability of their actions on von Neumann algebras which is in fact a non-commutative version of [2, Proposition 3.6]:

Theorem 4.7. *Let $\alpha : \mathbb{G} \curvearrowright N$ be an action of a discrete quantum group $\mathbb{G}$ on a von Neumann algebra N . The following are equivalent:*

1. *The quantum group $\mathbb{G}$ is amenable.*
2. *The action α is amenable and there exists an invariant state on N .*

In the case of Kac algebras, this theorem provides a new characterization for amenability of $\mathbb{G}$ in terms of amenability of the canonical action of $\mathbb{G}$ on its dual Kac algebra.

Theorem 5.2. *Let $\mathbb{G}$ be a discrete Kac algebra. Then $\mathbb{G}$ is amenable if and only if the canonical action of $\mathbb{G}$ on $L^\infty(\widehat{\mathbb{G}})$ is amenable.*

One of Zimmer's main motivations to introduce and study the notion of amenable actions was the applications in the theory of random walks on G and their associated Poisson boundaries of G . He proved that for any second countable locally compact group G , the action of G on its Poisson boundaries is always amenable [24, Theorem 5.2]. We establish the non-commutative analogue of this result in the case of discrete quantum group actions.

Theorem 5.3. *Let $\mathbb{G}$ be a discrete quantum group and let $\mu \in \ell^1(\mathbb{G})$ be a state. The canonical action of $\mathbb{G}$ on the Poisson boundary $\mathcal{H}_\mu$ is amenable.*

In [25], Zimmer studied more properties of the amenable action and he characterized amenability of the action in terms of injectivity of the corresponding crossed product [25, Theorem 2.1]. In [2], Delaroche generalized this result to the case of actions on an arbitrary von Neumann algebra. In fact she proved that an action $\alpha : G \curvearrowright N$ is amenable if and only if there exists a conditional expectation from $B(L^2(G)) \overline{\otimes} N$ onto $N \rtimes_\alpha G$ [2, Proposition 4.1]. She used this result to show that amenability of the action on an injective von Neumann algebra is equivalent to injectivity of the corresponding crossed product. For discrete quantum group actions, we will characterize Zimmer amenability in

terms of the existence of a conditional expectation that satisfies an equivariant condition coming from the natural $\widehat{\mathbb{G}}$ action. More precisely, we have

Theorem 7.5. *Let $\alpha : \mathbb{G} \curvearrowright N$ be an action of a discrete quantum group $\mathbb{G}$ on a von Neumann algebra N . The following are equivalent:*

1. *The action α is amenable.*
2. *There is an equivariant conditional expectation*

$$E : (B(\ell^2(\mathbb{G})) \overline{\otimes} N, \hat{\Delta}^{op} \otimes \text{id}) \rightarrow (N \rtimes_{\alpha} \mathbb{G}, \hat{\alpha}).$$

As a direct consequence, we will prove a non-commutative analogue of [2, Corollaire 4.2] for the general discrete quantum group actions.

Corollary 7.7. *Let $\alpha : \mathbb{G} \curvearrowright N$ be an action of a discrete quantum group $\mathbb{G}$ on a von Neumann algebra N . The following are equivalent:*

1. *The von Neumann algebra N is injective and the action α is amenable.*
2. *The crossed product $N \rtimes_{\alpha} \mathbb{G}$ is $\widehat{\mathbb{G}}$ -injective.*

In the case of the trivial action of $\mathbb{G}$ on the trivial space, Theorem 7.5 provides a duality between amenability of $\mathbb{G}$ and injectivity of the dual von Neumann algebra $L^{\infty}(\widehat{\mathbb{G}})$ in the category of $\mathcal{T}(\ell^2(\mathbb{G}))$ -modules where $\mathcal{T}(L_2(\mathbb{G}))$ is the pre-dual of $B(L_2(\mathbb{G}))$. This perfect duality was initially investigated by Crann and Neufang in [8], (see also [5, 6]).

Moreover, in the case of discrete Kac algebra actions, we will show that the equivariant condition in Theorem 7.5 can be eliminated. In fact we have

Theorem 6.3. *Let $\alpha : \mathbb{G} \curvearrowright N$ be an action of a discrete Kac algebra $\mathbb{G}$ on a von Neumann algebra N . The following are equivalent:*

1. *The action α is amenable.*
2. *There is a conditional expectation from $B(\ell^2(\mathbb{G})) \overline{\otimes} N$ onto $N \rtimes_{\alpha} \mathbb{G}$.*

Beside this introduction, this paper includes six other sections. In section 2, we recall some notions about discrete quantum groups and their actions on von Neumann algebras. In section 3, we construct the von Neumann algebra braided tensor product and we use this notion to obtain a version of diagonal action in the setting of quantum groups. In section 4, we introduce the notion of amenable actions and we study some of its properties. In section 5, we give some examples of amenable actions. In particular, we prove that the action of any discrete quantum group on any of its Poisson boundaries is amenable. In section 6, we study actions of discrete Kac algebras. The main result of this section generalize the well-known fact about the equivalence of amenability of discrete Kac algebra $\mathbb{G}$ and injectivity of $L^{\infty}(\widehat{\mathbb{G}})$. In section 7, we consider the latter result in the case of discrete quantum group actions.

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2. Preliminaires

In this section, we review some basic notions about discrete quantum groups and their actions on von Neumann algebras. For more details on discrete quantum groups we refer the reader to [9], [22] and [23]. A *discrete quantum group* $\mathbb{G}$ is a quadruple $(\ell^\infty(\mathbb{G}), \Delta, \varphi, \psi)$, where $\ell^\infty(\mathbb{G}) = \bigoplus_{i \in I} M_{n_i}(\mathbb{C})$ is a von Neumann algebra direct sum of matrix algebras, $\Delta : \ell^\infty(\mathbb{G}) \rightarrow \ell^\infty(\mathbb{G}) \overline{\otimes} \ell^\infty(\mathbb{G})$ is a co-associative co-multiplication, and φ and ψ are normal faithful semi-finite left, respectively right, invariant weights on $\ell^\infty(\mathbb{G})$, that is,

$$\begin{aligned}\varphi((\omega \otimes \text{id})\Delta(x)) &= \omega(1)\varphi(x), & x \in \mathcal{M}_\varphi, \omega \in \ell^1(\mathbb{G}), \\ \psi((\text{id} \otimes \omega)\Delta(x)) &= \omega(1)\psi(x), & x \in \mathcal{M}_\psi, \omega \in \ell^1(\mathbb{G}).\end{aligned}$$

A discrete quantum group $\mathbb{G} = (\ell^\infty(\mathbb{G}), \Delta, \varphi, \psi)$ is a *Kac algebra*, if φ equals ψ and is a trace.

The pre-adjoint of Δ induces an associative completely contractive multiplication

$$* : f \otimes g \in \ell^1(\mathbb{G}) \widehat{\otimes} \ell^1(\mathbb{G}) \rightarrow f * g = (f \otimes g)\Delta \in \ell^1(\mathbb{G})$$

on $\ell^1(\mathbb{G})$. Moreover, this map induces left and right actions of $\ell^1(\mathbb{G})$ on $\ell^\infty(\mathbb{G})$ given by:

$$\mu * x := (\text{id} \otimes \mu)\Delta(x), \quad x * \mu := (\mu \otimes \text{id})\Delta(x). \quad (1)$$

For a fixed $\mu \in \ell^1(\mathbb{G})$, the map $x \mapsto x * \mu$ is normal, completely bounded on $\ell^\infty(\mathbb{G})$. This map is called the *Markov operator*, if μ is, moreover, a state. A discrete quantum group $\mathbb{G}$ is said to be *amenable* if there exists a state $m \in \ell^\infty(\mathbb{G})^*$ satisfying

$$\langle m, x * f \rangle = \langle f, 1 \rangle \langle m, x \rangle, \quad x \in \ell^\infty(\mathbb{G}), f \in \ell^1(\mathbb{G}).$$

The corresponding GNS Hilbert spaces $\ell^2(\mathbb{G}, \varphi)$ and $\ell^2(\mathbb{G}, \psi)$ are isomorphic and are denoted by the same notation $\ell^2(\mathbb{G})$. The (left) fundamental unitary W of $\mathbb{G}$ is a unitary operator on $\ell^2(\mathbb{G}) \otimes \ell^2(\mathbb{G})$, satisfying the pentagonal relation $W_{12}W_{13}W_{23} = W_{23}W_{12}$, in which we used the leg notation $W_{12} = W \otimes 1$, $W_{23} = 1 \otimes W$ and $W_{13} = (1 \otimes \sigma)W_{12}$, where $\sigma(x \otimes y) = y \otimes x$ is the flip map on $B(H \otimes K)$. The right fundamental unitary V with the same properties is defined in a similar way on $\ell^2(\mathbb{G}) \otimes \ell^2(\mathbb{G})$.

Let $\mathcal{T}(\ell^2(\mathbb{G}))$ be the predual of $B(\ell^2(\mathbb{G}))$. Define the von Neumann algebra $L^\infty(\hat{\mathbb{G}})$ to be the weak*-closure of $\{(\rho \otimes \text{id})W : \rho \in \mathcal{T}(\ell^2(\mathbb{G}))\}$. Consider the map $\hat{\Delta} : L^\infty(\hat{\mathbb{G}}) \rightarrow L^\infty(\hat{\mathbb{G}}) \overline{\otimes} L^\infty(\hat{\mathbb{G}})$ given by $\hat{\Delta}(\hat{x}) = \hat{W}^*(1 \otimes \hat{x})\hat{W}$, where $\hat{W} = \sigma W^* \sigma$. There exists a normal state $\hat{\phi}$ on $L^\infty(\hat{\mathbb{G}})$ which is both invariant of left and right such that the triple $\hat{\mathbb{G}} = (L^\infty(\hat{\mathbb{G}}), \hat{\Delta}, \hat{\phi})$ is a *compact quantum group* called the *dual quantum group* of $\mathbb{G}$.

The opposite co-multiplication $\hat{\Delta}^{\text{op}}$ is given by $\hat{\Delta}^{\text{op}} = \sigma \circ \hat{\Delta}$. The fundamental unitary $\hat{W}^{\text{op}}$ associated to $\hat{\Delta}^{\text{op}}$ is defined by $\hat{W}^{\text{op}} = \sigma \hat{V} \sigma$, and therefore $\hat{W}^{\text{op}} \in L^\infty(\hat{\mathbb{G}}) \overline{\otimes} \ell^\infty(\mathbb{G})'$.

The fundamental unitary W of $\mathbb{G}$ induces a co-associative co-multiplication on $B(\ell^2(\mathbb{G}))$ defined by

$$\Delta^\ell : T \in B(\ell^2(\mathbb{G})) \mapsto W^*(1 \otimes T)W \in B(\ell^2(\mathbb{G})) \overline{\otimes} B(\ell^2(\mathbb{G})).$$

It is clear that the restriction of Δ^ℓ to $\ell^\infty(\mathbb{G})$ is the original co-multiplication Δ on $\ell^\infty(\mathbb{G})$. The pre-adjoint of Δ^ℓ induces associative completely contractive multiplication on the predual $\mathcal{T}(\ell^2(\mathbb{G}))$.

$$* : \omega \otimes \tau \in \mathcal{T}(\ell^2(\mathbb{G})) \hat{\otimes} \mathcal{T}(\ell^2(\mathbb{G})) \mapsto \omega * \tau = \Delta_*^\ell(\omega \otimes \tau) \in \mathcal{T}(\ell^2(\mathbb{G})).$$

If $\langle \mathcal{T}(\ell^2(\mathbb{G})) * \mathcal{T}(\ell^2(\mathbb{G})) \rangle$ denotes the linear span of $\omega * \tau$ with $\omega, \tau \in \mathcal{T}(\ell^2(\mathbb{G}))$ we have

$$\langle \mathcal{T}(\ell^2(\mathbb{G})) * \mathcal{T}(\ell^2(\mathbb{G})) \rangle = \mathcal{T}(\ell^2(\mathbb{G})). \quad (2)$$

Similarly to the equations (1), there are left and right actions of $\mathcal{T}(\ell^2(\mathbb{G}))$ on $B(\ell^2(\mathbb{G}))$.

There is also a co-associative co-multiplication on $B(\ell^2(\mathbb{G}))$ induced by the right fundamental unitary V which is defined by

$$\Delta^r : T \in B(\ell^2(\mathbb{G})) \mapsto V(T \otimes 1)V^* \in B(\ell^2(\mathbb{G})) \overline{\otimes} B(\ell^2(\mathbb{G})).$$

In a similar way, the pre-adjoint of Δ^r induces associative completely contractive multiplication on the predual $\mathcal{T}(\ell^2(\mathbb{G}))$ with the property (2).

Let $\mathbb{G}$ be a discrete quantum group. By [5, Proposition 4.2.18], there is a normal conditional expectation E_0 from $B(\ell^2(\mathbb{G}))$ onto $\ell^\infty(\mathbb{G})$ such that for any $x \in B(\ell^2(\mathbb{G}))$ and $f \in \mathcal{T}(\ell^2(\mathbb{G}))$, we have

$$E_0((f \otimes \text{id})\Delta^\ell(x)) = (f \otimes \text{id})\Delta^\ell(E_0(x)). \quad (3)$$

In particular, for any $\hat{x} \in L^\infty(\hat{\mathbb{G}})$, $E_0(\hat{x}) = \hat{\phi}(\hat{x})\mathbf{1}$ where $\hat{\phi}$ is the normal invariant state of the compact quantum group $\hat{\mathbb{G}}$.

A (left) action $\alpha : \mathbb{G} \curvearrowright N$ of a discrete quantum group $\mathbb{G}$ on a von Neumann algebra N is an injective $*$ -homomorphism $\alpha : N \rightarrow \ell^\infty(\mathbb{G}) \overline{\otimes} N$ satisfying

$$(\Delta \otimes \text{id})\alpha = (\text{id} \otimes \alpha)\alpha.$$

The action of dual quantum group $\hat{\mathbb{G}}$ is defined similarly.

Let $\alpha : \mathbb{G} \curvearrowright N$ be an action of the discrete quantum group $\mathbb{G}$ on the von Neumann algebra N . A state ω on N is said to be *invariant* if for any $f \in \ell^1(\mathbb{G})$,

$$(f \otimes \omega)\alpha = \langle f, 1 \rangle \omega.$$

We denote by $N^\alpha = \{x \in N : \alpha(x) = 1 \otimes x\}$ the *fixed point algebra* of the action $\alpha : \mathbb{G} \curvearrowright N$. Let θ be a normal semi-finite faithful weight on N , and let H_θ be the GNS Hilbert space of θ . It is proved in [19, Theorem 4.4] that α is implemented by a unitary $U_\alpha \in \ell^\infty(\mathbb{G}) \overline{\otimes} B(H_\theta)$, that is,

$$\alpha(x) = U_\alpha (1 \otimes x) U_\alpha^* \quad (x \in N). \quad (4)$$

Definition 2.1. Let $\alpha : \mathbb{G} \curvearrowright N$ and $\beta : \mathbb{G} \curvearrowright M$ be two actions of the discrete quantum group $\mathbb{G}$ on von Neumann algebras N and M . Then a map $\Phi : N \rightarrow M$ is equivariant if

$$(\text{id} \otimes \Phi)\alpha = \beta \circ \Phi.$$

To indicate the actions, we say that the map $\Phi : (N, \alpha) \rightarrow (M, \beta)$ is equivariant, or that Φ is (α, β) -equivariant. In the case $\alpha = \beta$, we say that Φ is α -equivariant.

The (von Neumann algebra) crossed product of the action $\alpha : \mathbb{G} \curvearrowright N$ is defined by

$$\mathbb{G} \ltimes_{\alpha} N := \{\alpha(N) \cup (L^{\infty}(\hat{\mathbb{G}}) \otimes \mathbf{1})\}'' \subseteq B(\ell^2(\mathbb{G})) \otimes N.$$

Analogously to the classical setting, there is a characterization of the crossed product $\mathbb{G} \ltimes_{\alpha} N$ as the fixed point algebra of a certain action of $\mathbb{G}$ on $B(\ell^2(\mathbb{G})) \otimes N$ as follows:

Theorem 2.2 ([10], Theorem 11.6). Let $\alpha : \mathbb{G} \curvearrowright N$ be an action of a discrete quantum group $\mathbb{G}$ on a von Neumann algebra N and let σ be the flip map defined by $\sigma(a \otimes b) = b \otimes a$. Then there is a left action β on the von Neumann algebra $B(\ell^2(\mathbb{G})) \otimes N$ defined by

$$\beta : x \in B(\ell^2(\mathbb{G})) \otimes N \mapsto (\sigma V^* \sigma \otimes 1)((\sigma \otimes \text{id})(\text{id} \otimes \alpha)(x))(\sigma V \sigma \otimes 1),$$

such that

$$\mathbb{G} \ltimes_{\alpha} N = (B(\ell^2(\mathbb{G})) \otimes N)^{\beta}.$$

If $\alpha : \mathbb{G} \curvearrowright N$ is an action of a discrete quantum group $\mathbb{G}$ on a von Neumann algebra N , there is also a natural action $\hat{\alpha}$ of $(\hat{\mathbb{G}}, \hat{\Delta}^{\text{op}})$ on $\mathbb{G} \ltimes_{\alpha} N$ which is called the dual action of α and is defined by

$$\begin{aligned} \hat{\alpha}(\alpha(x)) &= 1 \otimes \alpha(x), \quad \text{for all } x \in N \\ \hat{\alpha}(\hat{x} \otimes 1) &= \hat{\Delta}^{\text{op}}(\hat{x}) \otimes 1, \quad \text{for all } \hat{x} \in L^{\infty}(\hat{\mathbb{G}}). \end{aligned}$$

In fact, we have $\alpha(N) = (N \ltimes_{\alpha} \mathbb{G})^{\hat{\alpha}}$ [19, Theorem 2.7].

3. von Neumann algebra braided tensor products

In order to overcome some technical obstacles, we need to use a version of diagonal action for discrete quantum group actions. This section is devoted to a brief introduction to Yetter–Drinfeld actions and braided tensor products in von Neumann algebra setting. For an overview of these notions, we refer to [3] and [16].

Let $\mathbb{G} = (\ell^{\infty}(\mathbb{G}), \Delta, \varphi, \psi)$ be a discrete quantum group and M be a von Neumann algebra. Consider actions $\beta : \mathbb{G} \curvearrowright M$ and $\gamma : \hat{\mathbb{G}} \curvearrowright M$ of the discrete quantum group $\mathbb{G}$ and its dual $\hat{\mathbb{G}}$ on M , respectively. We say M is the $\mathbb{G}$ -YD-algebra if the actions β and γ satisfy the following Yetter–Drinfeld condition:

$$(\text{ad}(W) \otimes \text{id})(\text{id} \otimes \gamma)\beta = (\sigma \otimes \text{id})(\text{id} \otimes \beta)\gamma, \quad (5)$$

where $\text{ad}(W) = W \cdot W^*$.

In this case, if α is any action of $\mathbb{G}$ on a von Neumann algebra N , then similarly to [20, Proposition 8.3], we have

$$\overline{\text{span}\{\gamma(M)_{12}\alpha(N)_{13}\}}^{\text{weak}^*} = \overline{\text{span}\{\alpha(N)_{13}\gamma(M)_{12}\}}^{\text{weak}^*}.$$

Hence, the weak*-closed linear span of $\{\gamma(a)_{12}\alpha(b)_{13} : a \in M, b \in N\}$ is a von Neumann subalgebra of $B(\ell^2(\mathbb{G})) \overline{\otimes} M \overline{\otimes} N$, which is called the *braided tensor product* of von Neumann algebras M and N , and is denoted by $M \overline{\boxtimes} N$. There is a *-homomorphism $\beta \boxtimes \alpha : M \overline{\boxtimes} N \rightarrow \ell^\infty(\mathbb{G}) \overline{\otimes} (M \overline{\boxtimes} N)$ given by

$$\beta \boxtimes \alpha(X) = W_{12}^* U_{\beta_{13}} (1 \otimes X) U_{\beta_{13}}^* W_{12},$$

where the unitary operator U_β implements the action β by (4).

In particular, on the set of generators $\{\gamma(M)_{12}\alpha(N)_{13}\}$ we have

$$\begin{aligned} (\beta \boxtimes \alpha)(\gamma(a)_{12}\alpha(b)_{13}) &= W_{12}^* U_{\beta_{13}} \gamma(a)_{23} \alpha(b)_{24} U_{\beta_{13}}^* W_{12} \\ &= W_{12}^* U_{\beta_{13}} \gamma(a)_{23} U_{\beta_{13}}^* \alpha(b)_{24} W_{12} \\ &= W_{12}^* (\sigma \otimes \text{id})(U_{\beta_{23}} \gamma(a)_{13} U_{\beta_{23}}^*) \alpha(b)_{24} W_{12} \\ &= W_{12}^* ((\sigma \otimes \text{id})(\text{id} \otimes \beta) \gamma(a))_{123} \alpha(b)_{24} W_{12} \\ &= W_{12}^* ((\sigma \otimes \text{id})(\text{id} \otimes \beta) \gamma(a))_{123} W_{12} W_{12}^* \alpha(b)_{24} W_{12} \\ &= W_{12}^* ((\sigma \otimes \text{id})(\text{id} \otimes \beta) \gamma(a))_{123} W_{12} ((\Delta \otimes \text{id}) \alpha(b))_{124} \\ &= W_{12}^* W_{12} ((\text{id} \otimes \gamma) \beta(a))_{123} W_{12}^* W_{12} ((\Delta \otimes \text{id}) \alpha(b))_{124} \\ &= ((\text{id} \otimes \gamma) \beta(a))_{123} ((\Delta \otimes \text{id}) \alpha(b))_{124} \\ &= ((\text{id} \otimes \gamma) \beta(a))_{123} ((\text{id} \otimes \alpha) \alpha(b))_{124}. \end{aligned}$$

Therefore,

$$(\beta \boxtimes \alpha)(\gamma(a)_{12}\alpha(b)_{13}) = ((\text{id} \otimes \gamma) \beta(a))_{123} ((\text{id} \otimes \alpha) \alpha(b))_{124}.$$

Now it is straightforward to check that the normal *-homomorphism $\beta \boxtimes \alpha$ is in fact an action of the discrete quantum group $\mathbb{G}$ on the von Neumann algebra $M \overline{\boxtimes} N$.

For any discrete quantum group $\mathbb{G}$, there is an action $\gamma : \hat{\mathbb{G}} \curvearrowright \ell^\infty(\mathbb{G})$ given by

$$\gamma(x) = \hat{W}^*(1 \otimes x) \hat{W}.$$

Observe that

$$(\text{ad}(W) \otimes \text{id})(\text{id} \otimes \gamma) \Delta(x) = (\sigma \otimes \text{id})(\text{id} \otimes \Delta) \gamma(x).$$

It implies that the pair (Δ, γ) satisfies the compatibility condition (5) and therefore $\ell^\infty(\mathbb{G})$ is a $\mathbb{G}$ -YD-algebra. In this paper, we always consider braided tensor products whose first legs are $\ell^\infty(\mathbb{G})$.

The following is the von Neumann algebraic version of [3, Lemma 1.24]. We included the proof for the convenience of the reader.

Lemma 3.1. *Let $\alpha : \mathbb{G} \curvearrowright N$ be an action of a discrete quantum group $\mathbb{G}$ on a von Neumann algebra N . There exists an equivariant $*$ -isomorphism*

$$T_\alpha : (\ell^\infty(\mathbb{G}) \overline{\boxtimes} N, \Delta \boxtimes \alpha) \rightarrow (\ell^\infty(\mathbb{G}) \overline{\boxtimes} N, \Delta \otimes \text{id})$$

such that $T_\alpha(1 \boxtimes a) = \alpha(a)$ for all $a \in N$ and $T_\alpha(x \boxtimes 1) = x \otimes 1$ for all $x \in \ell^\infty(\mathbb{G})$.

Proof. Given $x \in B(\ell^2(\mathbb{G})) \overline{\boxtimes} \ell^\infty(\mathbb{G}) \overline{\boxtimes} N$, the map T_α is defined spatially by

$$T_\alpha(x) := (\text{id} \otimes \alpha^{-1}) \left((\sigma \otimes \text{id}) ((\sigma W^* \sigma \otimes 1) x (\sigma W \sigma \otimes 1)) \right).$$

The restriction of T_α to $\ell^\infty(\mathbb{G}) \overline{\boxtimes} N$ is a map from $\ell^\infty(\mathbb{G}) \overline{\boxtimes} N$ onto $\ell^\infty(\mathbb{G}) \overline{\boxtimes} N$. Indeed, for all $a \in \ell^\infty(\mathbb{G})$ and $b \in N$ we have

$$\begin{aligned} & (\sigma W^* \sigma \otimes 1) (\gamma(a)_{12} \alpha(b)_{13}) (\sigma W \sigma \otimes 1) \\ &= (\sigma W^* \sigma \otimes 1) ((\hat{W}^* \otimes 1) (1 \otimes a \otimes 1) (\hat{W} \otimes 1) \alpha(b)_{13}) (\sigma W \sigma \otimes 1) \\ &= (\hat{W} \otimes 1) ((\hat{W}^* \otimes 1) (1 \otimes a \otimes 1) (\sigma W^* \sigma \otimes 1) \alpha(b)_{13}) (\sigma W \sigma \otimes 1) \\ &= (1 \otimes a \otimes 1) (\sigma W^* \sigma \otimes 1) \alpha(b)_{13} (\sigma W \sigma \otimes 1) \\ &= (1 \otimes a \otimes 1) (\sigma \otimes \text{id}) (W_{12}^* \alpha(b)_{23} W_{12}) \\ &= (\sigma \otimes \text{id}) ((a \otimes 1 \otimes 1) W_{12}^* \alpha(b)_{23} W_{12}) \\ &= (\sigma \otimes \text{id}) ((a \otimes 1 \otimes 1) (\Delta \otimes \text{id}) (\alpha(b))). \end{aligned}$$

Therefore, by definition of T_α , we have

$$T_\alpha(\gamma(a)_{12} \alpha(b)_{13}) = (\text{id} \otimes \alpha^{-1}) ((a \otimes 1 \otimes 1) (\text{id} \otimes \alpha) (\alpha(b))) = (a \otimes 1) \alpha(b).$$

Since the linear span of $\{(\ell^\infty(\mathbb{G}) \otimes 1) \alpha(N)\}$ is weak* dense in $\ell^\infty(\mathbb{G}) \overline{\boxtimes} N$, T_α is a $*$ -isomorphism from $\ell^\infty(\mathbb{G}) \overline{\boxtimes} N$ onto $\ell^\infty(\mathbb{G}) \overline{\boxtimes} N$ and it is clear that for all $a \in N$ and $x \in \ell^\infty(\mathbb{G})$, $T_\alpha(1 \boxtimes a) = \alpha(a)$ and $T_\alpha(x \boxtimes 1) = x \otimes 1$. $\square$

4. Amenable actions

In this section, we introduce the notion of amenable action of discrete quantum groups on von Neumann algebras. This definition is a generalization of the amenable action of discrete groups on von Neumann algebras introduced in [2, Définition 3.4]. If τ denotes the left translation action of G on $\ell^\infty(G)$, then the action $\alpha : G \curvearrowright M$ is called *amenable* if there exists an equivariant conditional expectation $P : (\ell^\infty(G) \overline{\boxtimes} M, \tau \otimes \alpha) \rightarrow (\mathbf{1} \overline{\boxtimes} M, \alpha)$, i.e.,

$$P(\tau_g \otimes \alpha_g) = \alpha_g \circ P, \quad g \in G.$$

There exists an automorphism T_α on $\ell^\infty(G) \overline{\boxtimes} M$ defined by

$$T_\alpha \left(\sum_{g \in G} (\delta_g \otimes x) \right) = \sum_{g \in G} (\delta_g \otimes \alpha_g^{-1}(x)).$$

If $\alpha(x) = \sum_{g \in G} \delta_g \otimes \alpha_g^{-1}(x)$ for all $x \in M$, we have $T_\alpha(\mathbf{1} \overline{\boxtimes} M) = \alpha(M)$. It is straightforward to check that for all $g \in G$,

$$(\tau_g \otimes \text{id}) \circ T_\alpha = T_\alpha \circ (\tau_g \otimes \alpha_g).$$

Therefore, the map T_α is an equivariant isomorphism from $(\ell^\infty(G) \bar{\otimes} M, \tau \otimes \alpha)$ onto $(\ell^\infty(G) \bar{\otimes} M, \tau \otimes \text{id})$.

In summary, we have the following commutative diagram for the amenable action α of a discrete group G on a von Neumann algebra N :

$$\begin{array}{ccc} (\ell^\infty(G) \bar{\otimes} M, \tau \otimes \alpha) & \xrightarrow{T_\alpha} & (\ell^\infty(G) \bar{\otimes} M, \tau \otimes \text{id}) \\ P \downarrow & & \downarrow \bar{P} \\ (\mathbf{1} \bar{\otimes} M, \tau \otimes \alpha) & \xrightarrow{T_\alpha} & (\alpha(M), \tau \otimes \text{id}) \end{array} \quad (6)$$

This diagram allows us to define an equivalent definition for the amenable action of discrete groups on von Neumann algebras. Let $\alpha : G \rightarrow \text{Aut}(M)$ be an action of a discrete group G on a von Neumann algebra M and let τ be the left translation action on $\ell^\infty(G)$. Then the action α is called amenable if there exists an equivariant conditional expectation

$$P : (\ell^\infty(G) \bar{\otimes} M, \tau \otimes \text{id}) \rightarrow (\alpha(M), \tau \otimes \text{id}).$$

Motivated by this definition, we introduce the notion of the amenable action of discrete quantum groups on von Neumann algebras.

Definition 4.1. Let $\alpha : \mathbb{G} \curvearrowright N$ be an action of a discrete quantum group $\mathbb{G}$ on a von Neumann algebra N . Then α is called amenable if there exists a conditional expectation $E_\alpha : \ell^\infty(\mathbb{G}) \bar{\otimes} N \rightarrow \alpha(N)$ such that

$$(\text{id} \otimes E_\alpha)(\Delta \otimes \text{id}) = (\Delta \otimes \text{id})E_\alpha. \quad (7)$$

Remark 4.2. The diagram (6) shows that the Definition 4.1 coincides with the classical definition of amenable actions introduced in [2].

Remark 4.3. The trivial action $\text{tr} : \mathbb{G} \curvearrowright \mathbb{C}$ of a discrete quantum group $\mathbb{G}$ on the trivial space is amenable if and only if $\mathbb{G}$ is amenable. Indeed, if the trivial action tr is amenable, then there is an equivariant conditional expectation $E_{\text{tr}} : (\ell^\infty(\mathbb{G}) \bar{\otimes} \mathbb{C}, \Delta \otimes \text{id}) \rightarrow (\mathbb{C} \otimes 1, \Delta \otimes \text{id})$. Define a state m on $\ell^\infty(\mathbb{G})$ by $E_{\text{tr}}(x \otimes 1) = m(x)1 \otimes 1$. Then

$$\begin{aligned} m(x * f)1 \otimes 1 &= E_{\text{tr}}((x * f) \otimes 1) \\ &= E_{\text{tr}}((f \otimes \text{id} \otimes \text{id})(\Delta \otimes \text{id})(x \otimes 1)) \\ &= (f \otimes \text{id} \otimes \text{id})(\text{id} \otimes E_{\text{tr}})(\Delta \otimes \text{id})(x \otimes 1) \\ &= (f \otimes \text{id} \otimes \text{id})(\Delta \otimes \text{id})E_{\text{tr}}(x \otimes 1) \\ &= m(x)(f \otimes \text{id} \otimes \text{id})(\Delta \otimes \text{id})(1 \otimes 1) \\ &= m(x)f(1)1 \otimes 1. \end{aligned}$$

Therefore m is an invariant mean on $\ell^\infty(\mathbb{G})$. For the converse, if $\mathbb{G}$ is amenable, there exists an invariant state m on $\ell^\infty(\mathbb{G})$. Define the conditional expectation $P : \ell^\infty(\mathbb{G}) \bar{\otimes} \mathbb{C} \rightarrow \mathbb{C} \bar{\otimes} \mathbb{C}$ by $P = m \otimes \text{id}$. It is easy to check that P is $(\Delta \otimes \text{id})$ -equivariant. (See also Theorem 4.7.)

Definition 4.4. Let $\alpha : \mathbb{G} \curvearrowright M$ and $\beta : \mathbb{G} \curvearrowright N$ be actions of a discrete quantum group $\mathbb{G}$ on von Neumann algebras M and N , respectively, where M is a von Neumann subalgebra of N . Then

1. The triple $(N, \mathbb{G}, \beta)$ is an extension of $(M, \mathbb{G}, \alpha)$ if α is the restriction of β to M and there is a conditional expectation from M onto N .
2. For the extension $(N, \mathbb{G}, \beta)$ of $(M, \mathbb{G}, \alpha)$, the pair (N, M) is called amenable, if there is an equivariant conditional expectation P from (N, β) onto (M, α) .

Proposition 4.5. Let $(N, \mathbb{G}, \beta)$ be an extension of $(M, \mathbb{G}, \alpha)$.

1. If the action α is amenable, then the pair (N, M) is amenable.
2. If the action β is amenable and the pair (N, M) is amenable, then the action α is amenable.

Proof. (1): Assume that $(N, \mathbb{G}, \beta)$ is an extension of $(M, \mathbb{G}, \alpha)$ and therefore there is a conditional expectation Q from N onto M . Since α is amenable we have an equivariant conditional expectation E_α from $(\ell^\infty(\mathbb{G}) \overline{\otimes} M, \Delta \otimes \text{id})$ onto $(\alpha(M), \Delta \otimes \text{id})$. Define the conditional expectation $P : N \rightarrow M$ by

$$P = \alpha^{-1} \circ E_\alpha \circ (\text{id} \otimes Q) \circ \beta.$$

Then we have

$$\begin{aligned} (\text{id} \otimes P)\beta &= (\text{id} \otimes \alpha^{-1} \circ E_\alpha)(\text{id} \otimes Q)(\text{id} \otimes \beta)\beta \\ &= (\text{id} \otimes \alpha^{-1})(\text{id} \otimes E_\alpha)(\text{id} \otimes Q)(\Delta \otimes \text{id})\beta \\ &= (\text{id} \otimes \alpha^{-1})(\text{id} \otimes E_\alpha)(\Delta \otimes \text{id})(\text{id} \otimes Q)\beta \\ &= (\text{id} \otimes \alpha^{-1})(\Delta \otimes \text{id})E_\alpha(\text{id} \otimes Q)\beta \\ &= (\text{id} \otimes \alpha^{-1})(\text{id} \otimes \alpha)E_\alpha(\text{id} \otimes Q)\beta \\ &= \alpha \circ P. \end{aligned}$$

It shows that $P : (N, \beta) \rightarrow (M, \alpha)$ is equivariant.

(2): Now suppose that β is amenable, then there is an equivariant conditional expectation E_β from $(\ell^\infty(\mathbb{G}) \overline{\otimes} N, \Delta \otimes \text{id})$ onto $(\beta(N), \Delta \otimes \text{id})$. Since the pair (N, M) is amenable, there is also a conditional expectation P from N onto M such that $(\text{id} \otimes P)\beta = \alpha \circ P$. Hence the composition $(\text{id} \otimes P)E : \ell^\infty(\mathbb{G}) \overline{\otimes} N \rightarrow \alpha(M)$ is a conditional expectation such that

$$\begin{aligned} (\text{id} \otimes \text{id} \otimes P)(\text{id} \otimes E)(\Delta \otimes \text{id}) &= (\text{id} \otimes \text{id} \otimes P)(\Delta \otimes \text{id})E \\ &= (\Delta \otimes \text{id})(\text{id} \otimes P)E. \end{aligned}$$

Since $\alpha(M) \subseteq \ell^\infty(\mathbb{G}) \overline{\otimes} M$, by restricting of $(\text{id} \otimes P)E$ to $\ell^\infty(\mathbb{G}) \overline{\otimes} M$ we obtain an equivariant conditional expectation from $(\ell^\infty(\mathbb{G}) \overline{\otimes} M, \Delta \otimes \text{id})$ onto $(\alpha(M), \Delta \otimes \text{id})$, and therefore the action α is amenable. $\square$

Let $\alpha : \mathbb{G} \curvearrowright N$ be an action of a discrete quantum group $\mathbb{G}$ on a von Neumann algebra N . Consider the normal conditional expectation E_0 as the equation (3) and fix an arbitrary state $f \in \ell^1(\mathbb{G})$. Then $E_0 \otimes f \otimes \text{id}$ is a normal conditional expectation from $B(\ell^2(\mathbb{G})) \overline{\otimes} \ell^\infty(\mathbb{G}) \overline{\otimes} N$ onto $\ell^\infty(\mathbb{G}) \overline{\otimes} \mathbf{1} \overline{\otimes} N$. By

restricting we obtain a conditional expectation from $\ell^\infty(\mathbb{G}) \overline{\otimes} N$ onto $\mathbf{1} \overline{\otimes} N$. So the triple $(\ell^\infty(\mathbb{G}) \overline{\otimes} N, \Delta \overline{\otimes} \alpha, \mathbb{G})$ is an extension of $(\mathbf{1} \overline{\otimes} N, \Delta \overline{\otimes} \alpha, \mathbb{G})$.

Proposition 4.6. *Let $\alpha : \mathbb{G} \curvearrowright N$ be an action of a discrete quantum group $\mathbb{G}$ on a von Neumann algebra N . Then the action α is amenable if and only if for the extension $(\ell^\infty(\mathbb{G}) \overline{\otimes} N, \Delta \overline{\otimes} \alpha, \mathbb{G})$ of $(\mathbf{1} \overline{\otimes} N, \Delta \overline{\otimes} \alpha, \mathbb{G})$, the pair $(\ell^\infty(\mathbb{G}) \overline{\otimes} N, \mathbf{1} \overline{\otimes} N)$ is amenable.*

Proof. By Lemma 3.1, there exists a $*$ -isomorphism between $(\ell^\infty(\mathbb{G}) \overline{\otimes} N, \Delta \overline{\otimes} \alpha)$ and $(\ell^\infty(\mathbb{G}) \overline{\otimes} N, \Delta \otimes \text{id})$ which is equivariant. Since $(\ell^\infty(\mathbb{G}) \overline{\otimes} N, \Delta \overline{\otimes} \alpha, \mathbb{G})$ is an extension of $(\mathbf{1} \overline{\otimes} N, \Delta \overline{\otimes} \alpha, \mathbb{G})$, the action α is amenable if and only if the pair $(\ell^\infty(\mathbb{G}) \overline{\otimes} N, \mathbf{1} \overline{\otimes} N)$ is amenable. $\square$

The following result is a non-commutative version of [2, Proposition 3.6].

Theorem 4.7. *Let $\alpha : \mathbb{G} \curvearrowright N$ be an action of a discrete quantum group $\mathbb{G}$ on a von Neumann algebra N . The following are equivalent:*

1. *The quantum group $\mathbb{G}$ is amenable.*
2. *The action α is amenable and there exists an invariant state on N .*

Proof. (2) $\Rightarrow$ (1): suppose that ω is an invariant state on N and E_α is an equivariant conditional expectation from $\ell^\infty(\mathbb{G}) \overline{\otimes} N$ onto $\alpha(N)$ coming from amenability of the action α . Define a state m on $\ell^\infty(\mathbb{G})$ by

$$\langle m, x \rangle = \langle \omega, \alpha^{-1} \circ E_\alpha(x \otimes 1) \rangle.$$

Then m is a left invariant state on $\ell^\infty(\mathbb{G})$. Indeed, for any $x \in \ell^\infty(\mathbb{G})$ and $f \in \ell^1(\mathbb{G})$ we have

$$\begin{aligned} \langle m, x * f \rangle &= \langle m, (f \otimes \text{id})\Delta(x) \rangle \\ &= \langle \omega, \alpha^{-1} \circ E_\alpha(((f \otimes \text{id})\Delta(x)) \otimes 1) \rangle \\ &= \langle \omega, \alpha^{-1}((f \otimes \text{id} \otimes \text{id})(\text{id} \otimes E_\alpha)(\Delta \otimes \text{id})(x \otimes 1)) \rangle \\ &= \langle \omega, \alpha^{-1}((f \otimes \text{id} \otimes \text{id})(\Delta \otimes \text{id})E_\alpha(x \otimes 1)) \rangle \\ &= \langle \omega, \alpha^{-1}((f \otimes \text{id} \otimes \text{id})(\text{id} \otimes \alpha)E_\alpha(x \otimes 1)) \rangle \\ &= \langle \omega, (f \otimes \text{id})E_\alpha(x \otimes 1) \rangle \\ &= f(1) \langle \omega, \alpha^{-1} \circ E_\alpha(x \otimes 1) \rangle \\ &= f(1) \langle m, x \rangle, \end{aligned}$$

where we use the fact that ω is invariant in the penultimate step.

(1) $\Rightarrow$ (2): suppose that m is a left invariant mean on $\ell^\infty(\mathbb{G})$. Fix $\eta \in N_*$ and define $\omega := (m \otimes \eta)\alpha$. Then for any $f \in \ell^1(\mathbb{G})$ and $x \in N$, we have

$$\begin{aligned} \langle f \otimes \omega, \alpha(x) \rangle &= \langle \omega, (f \otimes \text{id})\alpha(x) \rangle \\ &= \langle m \otimes \eta, (f \otimes \text{id} \otimes \text{id})(\text{id} \otimes \alpha)\alpha(x) \rangle \\ &= \langle m \otimes \eta, (f \otimes \text{id} \otimes \text{id})(\Delta \otimes \text{id})\alpha(x) \rangle \\ &= \langle m, (f \otimes \text{id})\Delta((\text{id} \otimes \eta)\alpha(x)) \rangle \end{aligned}$$

$$\begin{aligned}
&= f(1)\langle m, (\text{id} \otimes \eta)\alpha(x) \rangle \\
&= f(1)\langle m \otimes \eta, \alpha(x) \rangle \\
&= f(1)\langle \omega, x \rangle,
\end{aligned}$$

it shows ω is an invariant state on N .

Now we prove that the action α is amenable. First, we claim that for any $x \in \ell^\infty(\mathbb{G}) \overline{\otimes} N$, we have

$$(\Delta \boxtimes \alpha)(x) = (\text{ad}(W^*) \otimes \text{id} \otimes \text{id})(\sigma \otimes \text{id} \otimes \text{id})(\text{id} \otimes \Delta \otimes \text{id})(x). \quad (8)$$

Since the linear span of $\{\gamma(a_i)_{12}\alpha(b_i)_{13}\}$ is weak* dense in $\ell^\infty(\mathbb{G}) \overline{\otimes} N$, we need to show (8) for the set of generators. For any $a \in \ell^\infty(\mathbb{G})$ and $b \in N$, we have

$$\begin{aligned}
(\Delta \boxtimes \alpha)(\gamma(a)_{12}\alpha(b)_{13}) &= ((\text{id} \otimes \gamma)\Delta(a))_{123}((\text{id} \otimes \alpha)\alpha(b))_{124} \\
&= ((\text{ad}(W^*) \otimes \text{id})(\sigma \otimes \text{id})(\text{id} \otimes \Delta)\gamma(a))_{123}((\text{id} \otimes \alpha)\alpha(b))_{124} \\
&= (W_{12}^*(\sigma \otimes \text{id})(\text{id} \otimes \Delta)\gamma(a))_{123}((\text{id} \otimes \alpha)\alpha(b))_{124} \\
&= (W_{12}^*(\sigma \otimes \text{id})(\text{id} \otimes \Delta)\gamma(a))_{123}((\Delta \otimes \text{id})\alpha(b))_{124} \\
&= (W_{12}^*(\sigma \otimes \text{id})(\text{id} \otimes \Delta)\gamma(a))_{123}W_{12}^*\alpha(b)_{24}W_{12} \\
&= W_{12}^*((\sigma \otimes \text{id})(\text{id} \otimes \Delta)\gamma(a))_{123}\alpha(b)_{24}W_{12} \\
&= (\text{ad}(W^*) \otimes \text{id} \otimes \text{id})(\sigma \otimes \text{id} \otimes \text{id})[(\text{id} \otimes \Delta)\gamma(a)]_{123}\alpha(b)_{14} \\
&= (\text{ad}(W^*) \otimes \text{id} \otimes \text{id})(\sigma \otimes \text{id} \otimes \text{id})(\text{id} \otimes \Delta \otimes \text{id})(\gamma(a)_{12}\alpha(b)_{13}).
\end{aligned}$$

Hence, we conclude the equality (8). Let $E_0 : B(\ell^2(\mathbb{G})) \rightarrow \ell^\infty(\mathbb{G})$ be the normal conditional expectation given by (3). Therefore, for any $\hat{x} \in L^\infty(\hat{\mathbb{G}})$ and $f, \omega \in \ell^1(\mathbb{G})$, we have

$$\begin{aligned}
\langle \omega \otimes f, (\text{id} \otimes E_0)\Delta^\ell(\hat{x}) \rangle &= \langle f, E_0((\omega \otimes \text{id})\Delta^\ell(\hat{x})) \rangle \\
&= \langle f, (\omega \otimes \text{id})\Delta^\ell(E_0(\hat{x})) \rangle \\
&= \langle \omega \otimes f, \hat{\phi}(\hat{x})1 \otimes 1 \rangle.
\end{aligned}$$

Hence,

$$(\text{id} \otimes E_0)\Delta^\ell(\hat{x}) = \hat{\phi}(\hat{x})1 \otimes 1, \quad (9)$$

for all $\hat{x} \in L^\infty(\hat{\mathbb{G}})$. Consider the conditional expectation $E_0 \otimes m \otimes \text{id}$ from the von Neumann algebra $B(\ell^2(\mathbb{G})) \overline{\otimes} \ell^\infty(\mathbb{G}) \overline{\otimes} N$ onto $\ell^\infty(\mathbb{G}) \overline{\otimes} \mathbf{1} \overline{\otimes} N$. Then by restricting, there is a conditional expectation E from $\ell^\infty(\mathbb{G}) \overline{\otimes} N$ onto $\mathbf{1} \overline{\otimes} N$. We show that the conditional expectation E is $(\Delta \boxtimes \alpha)$ -equivariant. For any $a \in \ell^\infty(\mathbb{G})$ and $b \in N$, by the equality (9) we have

$$\begin{aligned}
&(\text{id} \otimes E_0 \otimes \text{id} \otimes \text{id})[W_{12}^*\gamma(a)_{23}\alpha(b)_{24}W_{12}] \\
&= (\text{id} \otimes E_0 \otimes \text{id} \otimes \text{id})[W_{12}^*\gamma(a)_{23}W_{12}W_{12}^*\alpha(b)_{24}W_{12}] \\
&= (\text{id} \otimes E_0 \otimes \text{id} \otimes \text{id})[(\Delta^\ell \otimes \text{id})\gamma(a)]_{123}((\text{id} \otimes \alpha)\alpha(b))_{124}
\end{aligned}$$

$$\begin{aligned}
&= ((\text{id} \otimes E_0 \otimes \text{id})(\Delta^\ell \otimes \text{id})\gamma(a))_{123} ((\text{id} \otimes \alpha)\alpha(b))_{124} \\
&= (\text{id} \otimes \hat{\phi} \otimes \text{id} \otimes \text{id})(\gamma(a)_{23})((\text{id} \otimes \alpha)\alpha(b))_{124}.
\end{aligned}$$

Consider $x \in \ell^\infty(\mathbb{G}) \overline{\boxtimes} N$ as $\sum_{i \in I} \gamma(a_i)_{12} \alpha(b_i)_{13}$. Since m is an invariant mean, the equality (8) and the above calculation yield that

$$\begin{aligned}
&(\text{id} \otimes E)(\Delta \boxtimes \alpha)(x) \\
&= (\text{id} \otimes E) \left[(\text{ad}(W^*) \otimes \text{id} \otimes \text{id})(\sigma \otimes \text{id} \otimes \text{id})(\text{id} \otimes \Delta \otimes \text{id})(x) \right] \\
&= (\text{id} \otimes E) \left[W_{12}^* ((\sigma \otimes \text{id} \otimes \text{id})(\text{id} \otimes \Delta \otimes \text{id})(x)) W_{12} \right] \\
&= (\text{id} \otimes E_0 \otimes m \otimes \text{id}) \left[W_{12}^* ((\sigma \otimes \text{id} \otimes \text{id})(\text{id} \otimes \Delta \otimes \text{id})(x)) W_{12} \right] \\
&= (\text{id} \otimes E_0 \otimes \text{id}) \left[W_{12}^* ((\sigma \otimes \text{id})(\text{id} \otimes \text{id} \otimes m \otimes \text{id})(\text{id} \otimes \Delta \otimes \text{id})(x)) W_{12} \right] \\
&= (\text{id} \otimes E_0 \otimes \text{id}) \left[W_{12}^* ((\sigma \otimes \text{id})(\text{id} \otimes \text{id} \otimes m \otimes \text{id})(x_{134})) W_{12} \right] \\
&= (\text{id} \otimes E_0 \otimes m \otimes \text{id}) \left[W_{12}^* (1 \otimes x) W_{12} \right] \\
&= (\text{id} \otimes E_0 \otimes m \otimes \text{id}) \left[W_{12}^* \sum_{i \in I} \gamma(a_i)_{23} \alpha(b_i)_{24} W_{12} \right] \\
&= (\text{id} \otimes \text{id} \otimes m \otimes \text{id}) \left[\sum_{i \in I} (\text{id} \otimes \hat{\phi} \otimes \text{id} \otimes \text{id})(\gamma(a_i)_{23})((\text{id} \otimes \alpha)\alpha(b_i))_{124} \right],
\end{aligned}$$

where we use the normality of the conditional expectation E_0 in the last equality. On the other hand, repeating the calculation show that

$$\begin{aligned}
&(\Delta \boxtimes \alpha)E(x) = (\Delta \boxtimes \alpha)(E_0 \otimes m \otimes \text{id})(x) \\
&= (\Delta \boxtimes \alpha)(E_0 \otimes m \otimes \text{id}) \left[\sum_{i \in I} \gamma(a_i)_{12} \alpha(b_i)_{13} \right] \\
&= (\Delta \boxtimes \alpha)(\text{id} \otimes m \otimes \text{id}) \left[\sum_{i \in I} (E_0 \otimes \text{id} \otimes \text{id})[\gamma(a_i)_{12} \alpha(b_i)_{13}] \right] \\
&= (\Delta \boxtimes \alpha)(\text{id} \otimes m \otimes \text{id}) \left[\sum_{i \in I} (E_0 \otimes \text{id} \otimes \text{id})[\gamma(a_i)_{12}] \alpha(b_i)_{13} \right] \\
&= (\Delta \boxtimes \alpha)(\text{id} \otimes m \otimes \text{id}) \left[\sum_{i \in I} (\hat{\phi} \otimes \text{id} \otimes \text{id})[\gamma(a_i)_{12}] \alpha(b_i)_{13} \right] \\
&= (\text{id} \otimes \text{id} \otimes m \otimes \text{id}) \left[\sum_{i \in I} (\text{id} \otimes \hat{\phi} \otimes \text{id} \otimes \text{id})(\gamma(a_i)_{23})(\text{id} \otimes \alpha)\alpha(b_i))_{124} \right]
\end{aligned}$$

From these two calculations, it follows that the pair $(\ell^\infty(\mathbb{G}) \overline{\boxtimes} N, \mathbf{1} \overline{\boxtimes} N)$ is amenable and by Proposition 4.6 the action α is amenable. $\square$

Theorem 4.8. *Let $\alpha : \mathbb{G} \curvearrowright N$ be an action of a discrete quantum group $\mathbb{G}$ on a von Neumann algebra N . Then there exists an equivariant isomorphism Φ from*

$((\ell^\infty(\mathbb{G}) \overline{\boxtimes} N) \ltimes_{\Delta \boxtimes \alpha} \mathbb{G}, \widehat{\Delta \boxtimes \alpha})$ onto $(B(\ell^2(\mathbb{G})) \overline{\boxtimes} N, \widehat{\Delta \boxtimes \alpha})$ such that Φ maps $(\mathbf{1} \overline{\boxtimes} N) \ltimes_{\Delta \boxtimes \alpha} \mathbb{G}$ onto $N \ltimes_\alpha \mathbb{G}$.

Proof. Consider the equivariant $*$ -isomorphism T_α , given by Lemma 3.1, from $(\ell^\infty(\mathbb{G}) \overline{\boxtimes} N, \Delta \boxtimes \alpha)$ onto $(\ell^\infty(\mathbb{G}) \overline{\boxtimes} N, \Delta \otimes \text{id})$. Then the isomorphism Φ is obtained from the identification:

$$\begin{aligned} (\ell^\infty(\mathbb{G}) \overline{\boxtimes} N) \ltimes_{\Delta \boxtimes \alpha} \mathbb{G} &= [(\Delta \boxtimes \alpha)(\ell^\infty(\mathbb{G}) \overline{\boxtimes} N) \cup (L^\infty(\hat{\mathbb{G}}) \overline{\boxtimes} \mathbf{1}_{\ell^\infty(\mathbb{G}) \overline{\boxtimes} N})]'' \\ &\cong [(\text{id} \otimes T_\alpha)(\Delta \boxtimes \alpha)(\ell^\infty(\mathbb{G}) \overline{\boxtimes} N) \cup (L^\infty(\hat{\mathbb{G}}) \overline{\boxtimes} \mathbf{1}_{\ell^\infty(\mathbb{G}) \overline{\boxtimes} N})]'' \\ &= [(\Delta \otimes \text{id})T_\alpha(\ell^\infty(\mathbb{G}) \overline{\boxtimes} N) \cup (L^\infty(\hat{\mathbb{G}}) \overline{\boxtimes} \mathbf{1} \overline{\boxtimes} \mathbf{1})V_{12}^*V_{12}]'' \\ &= [(\Delta \otimes \text{id})T_\alpha(\ell^\infty(\mathbb{G}) \overline{\boxtimes} N) \cup V_{12}(L^\infty(\hat{\mathbb{G}}) \overline{\boxtimes} \mathbf{1} \overline{\boxtimes} \mathbf{1})V_{12}^*]'' \\ &= [(\Delta \otimes \text{id})T_\alpha(\ell^\infty(\mathbb{G}) \overline{\boxtimes} N) \cup (\Delta \otimes \text{id})(L^\infty(\hat{\mathbb{G}}) \overline{\boxtimes} \mathbf{1})]'' \\ &\cong [T_\alpha(\ell^\infty(\mathbb{G}) \overline{\boxtimes} N) \cup (L^\infty(\hat{\mathbb{G}}) \overline{\boxtimes} \mathbf{1})]'' \\ &\cong [(\ell^\infty(\mathbb{G}) \overline{\boxtimes} N) \cup (L^\infty(\hat{\mathbb{G}}) \overline{\boxtimes} \mathbf{1})]'' \\ &= B(\ell^2(\mathbb{G})) \overline{\boxtimes} N \end{aligned}$$

where in the fourth equality, we used the fact that $V \in L^\infty(\hat{\mathbb{G}})' \overline{\boxtimes} \ell^\infty(\mathbb{G})$. In particular, for any $x \in \ell^\infty(\mathbb{G}) \overline{\boxtimes} N$ and any $\hat{x} \in L^\infty(\hat{\mathbb{G}})$ we have

$$\Phi((\Delta \boxtimes \alpha)(x)) = T_\alpha(x), \quad \Phi(\hat{x} \otimes \mathbf{1}_{\ell^\infty(\mathbb{G}) \overline{\boxtimes} N}) = \hat{x} \otimes \mathbf{1}_{\ell^\infty(\mathbb{G}) \overline{\boxtimes} N}. \quad (10)$$

From Lemma 3.1, we know that $T_\alpha(\mathbf{1} \overline{\boxtimes} N) = \alpha(N)$, and therefore by the same calculations we have

$$(\mathbf{1} \overline{\boxtimes} N) \ltimes_{\Delta \boxtimes \alpha} \mathbb{G} \cong N \ltimes_\alpha \mathbb{G}.$$

In order to show the equivariant condition, it is sufficient to check the equality on the set of generators of $(\ell^\infty(\mathbb{G}) \overline{\boxtimes} N) \ltimes_{\Delta \boxtimes \alpha} \mathbb{G}$. Suppose that $x \in \ell^\infty(\mathbb{G}) \overline{\boxtimes} N$, then since $\hat{W}^{\text{op}} \in L^\infty(\hat{\mathbb{G}}) \overline{\boxtimes} \ell^\infty(\mathbb{G})'$, by (10) we have

$$\begin{aligned} ((\hat{\Delta}^{\text{op}} \otimes \text{id}) \circ \Phi)(\Delta \boxtimes \alpha)(x) &= (\hat{\Delta}^{\text{op}} \otimes \text{id})(\Phi(\Delta \boxtimes \alpha)(x)) \\ &= (\hat{\Delta}^{\text{op}} \otimes \text{id})T_\alpha(x) \\ &= \hat{W}_{12}^{\text{op}*} (1 \otimes T_\alpha(x)) \hat{W}_{12}^{\text{op}} \\ &= 1 \otimes T_\alpha(x) \\ &= 1 \otimes \Phi((\Delta \boxtimes \alpha)(x)) \\ &= (\text{id} \otimes \Phi)(1 \otimes (\Delta \boxtimes \alpha)(x)) \\ &= (\text{id} \otimes \Phi)(\widehat{(\Delta \boxtimes \alpha)}(\Delta \boxtimes \alpha)(x)). \end{aligned}$$

On the other hand, by (10), for any $\hat{x} \in L^\infty(\hat{\mathbb{G}})$ we have

$$\begin{aligned} ((\hat{\Delta}^{\text{op}} \otimes \text{id}) \circ \Phi)(\hat{x} \otimes \mathbf{1}_{\ell^\infty(\mathbb{G}) \overline{\boxtimes} N}) &= \hat{\Delta}^{\text{op}}(\hat{x}) \otimes \mathbf{1}_{\ell^\infty(\mathbb{G}) \overline{\boxtimes} N} \\ &= (\text{id} \otimes \Phi)(\widehat{(\Delta \boxtimes \alpha)}(\hat{x} \otimes \mathbf{1}_{\ell^\infty(\mathbb{G}) \overline{\boxtimes} N})). \quad \square \end{aligned}$$

5. Examples

In this section, we give some examples of amenable actions of discrete quantum groups on von Neumann algebras. In Theorem 4.7, we showed that the amenable quantum group $\mathbb{G}$ acts amenably on any von Neumann algebras. Also, Proposition 4.5 shows that it is possible to get new amenable actions by appropriate restrictions. Below, we give more concrete examples of amenable actions. As an example of amenable actions, the action of any discrete group G on $\ell^\infty(G)$ is always amenable [2, Remarques 3.7.(b)]. The next result is the non-commutative analogue of that.

Proposition 5.1. *Every discrete quantum group acts amenably on itself.*

Proof. Define the map $\Phi : \ell^\infty(\mathbb{G}) \overline{\otimes} \ell^\infty(\mathbb{G}) \rightarrow \ell^\infty(\mathbb{G})$ by $\Phi(A) = (\text{id} \otimes \varepsilon)(A)$, in which ε is the unit in $\ell^1(\mathbb{G})$. Then

$$\Phi \circ \Delta = (\text{id} \otimes \varepsilon)\Delta = \text{id}.$$

So, the map Φ is a left inverse of the co-multiplication Δ and therefore the map $E_\Delta := \Delta \circ \Phi$ is a conditional expectation from $\ell^\infty(\mathbb{G}) \overline{\otimes} \ell^\infty(\mathbb{G})$ onto $\Delta(\ell^\infty(\mathbb{G}))$. Moreover, for any $A \in \ell^\infty(\mathbb{G}) \overline{\otimes} \ell^\infty(\mathbb{G})$ we have

$$\begin{aligned} (\text{id} \otimes E_\Delta)(\Delta \otimes \text{id})(A) &= (\text{id} \otimes \Delta \circ \Phi)(\Delta \otimes \text{id})(A) \\ &= (\text{id} \otimes \Delta)(\text{id} \otimes \text{id} \otimes \varepsilon)(\Delta \otimes \text{id})(A) \\ &= (\text{id} \otimes \Delta)(\text{id} \otimes \varepsilon)(A) \\ &= (\Delta \otimes \text{id})(\Delta \circ \Phi)(A) \\ &= (\Delta \otimes \text{id})E_\Delta(A). \end{aligned}$$

Hence, $E_\Delta : (\ell^\infty(\mathbb{G}) \overline{\otimes} \ell^\infty(\mathbb{G}), \Delta \otimes \text{id}) \rightarrow (\Delta(\ell^\infty(\mathbb{G})), \Delta \otimes \text{id})$ is an equivariant conditional expectation. $\square$

For a discrete quantum group $\mathbb{G}$, the restriction of the extended comultiplication Δ^ℓ to $L^\infty(\hat{\mathbb{G}})$ provides an action of $\mathbb{G}$ on the von Neumann algebra $L^\infty(\hat{\mathbb{G}})$. We next prove that in the case of Kac algebras, amenability of the latter action is equivalent to amenability of the Kac algebra $\mathbb{G}$.

Theorem 5.2. *Let $\mathbb{G}$ be a discrete Kac algebra. Then $\mathbb{G}$ is amenable if and only if the canonical action $\Delta^\ell|_{L^\infty(\hat{\mathbb{G}})} : \mathbb{G} \curvearrowright L^\infty(\hat{\mathbb{G}})$ is amenable.*

Proof. Since $\mathbb{G}$ is a Kac algebra, by [11, Corollary 3.9], the tracial Haar state $\hat{\phi}$ of the dual quantum group $\hat{\mathbb{G}}$ is invariant with respect to the action $\Delta^\ell|_{L^\infty(\hat{\mathbb{G}})}$. Hence by Theorem 4.7, amenability of the action $\Delta^\ell|_{L^\infty(\hat{\mathbb{G}})}$ is equivalent to amenability of $\mathbb{G}$. $\square$

Remark 5.3. In [7], Crann defined a notion of inner amenability for quantum groups as the existence of an invariant state for the canonical action $\Delta^\ell|_{L^\infty(\hat{\mathbb{G}})} : \mathbb{G} \curvearrowright L^\infty(\hat{\mathbb{G}})$. The same proof shows that Theorem 5.2 holds for the inner amenable quantum group $\mathbb{G}$ in the sense of Crann.

In the next result, we state the non-commutative version of Zimmer's classical result [24, Theorem 5.2] that all Poisson boundaries are amenable G -space. Let us first recall the definition of non-commutative Poisson boundaries in the sense of Izumi [12]. Let $\mu \in \ell^1(\mathbb{G})$ be a state. Recall in this case $\Phi_\mu(x) = (\mu \otimes \text{id})\Delta(x)$ is a unital, normal completely positive map on $\ell^\infty(\mathbb{G})$. The space of fixed point $\mathcal{H}_\mu = \{x \in \ell^\infty(\mathbb{G}) : \Phi_\mu(x) = x\}$ is a weak*-closed operator system in $\ell^\infty(\mathbb{G})$. There is a conditional expectation $\mathcal{E}_\mu$ from $\ell^\infty(\mathbb{G})$ onto $\mathcal{H}_\mu$. Then the corresponding Choi–Effros product induces the von Neumann algebraic structure on $\mathcal{H}_\mu$ [4]. This von Neumann algebra is called *non-commutative Poisson boundary* with respect to μ . For more details on non-commutative Poisson boundaries, we refer the reader to [14] and [15]. By [15, Proposition 2.1], the restriction of Δ to $\mathcal{H}_\mu$ induces a left action Δ_μ of $\mathbb{G}$ on the von Neumann algebra $\mathcal{H}_\mu$. We prove this action is amenable.

Theorem 5.4. *Let $\mathbb{G}$ be a discrete quantum group and let $\mu \in \ell^1(\mathbb{G})$ be a state. The left action Δ_μ of $\mathbb{G}$ on the Poisson boundary $\mathcal{H}_\mu$ is amenable.*

Proof. By Proposition 5.1, the action of $\mathbb{G}$ on itself is amenable. Since the conditional expectation $\mathcal{E}_\mu : \ell^\infty(\mathbb{G}) \rightarrow \mathcal{H}_\mu$ is equivariant (see e.g. the proof of [15, Proposition 2.1]), a similar argument to Proposition 4.5 (2) shows that the left action Δ_μ is amenable. $\square$

6. Amenable actions and crossed products: Kac algebra case

In this section, we characterize amenability of actions in term of von Neumann algebra crossed products. Classically, the action $\alpha : G \curvearrowright X$ of a discrete group G on a standard probability space (X, ν) is amenable if and only if the crossed product $L^\infty(X, \nu) \rtimes_\alpha G$ is injective [25, Theorem 2.1]. Delaroche extended this result to the action of locally compact groups on arbitrary von Neumann algebras [2, Proposition 4.1]. We prove a non-commutative version of this result in the case of discrete Kac algebra actions on von Neumann algebras. This, in particular, generalizes a part of Theorem 4.5 in [17] and also [18, Corollary 3.17] which establish the equivalence between amenability of a discrete Kac algebra $\mathbb{G}$ and injectivity of $L^\infty(\hat{\mathbb{G}})$. The case of actions of general discrete quantum groups on von Neumann algebras is discussed in next section.

Lemma 6.1. *Let $\alpha : \mathbb{G} \curvearrowright N$ be an action of a discrete Kac algebra $\mathbb{G}$ on a von Neumann algebra N and let M be a von Neumann subalgebra of N which is invariant under α . The following are equivalent:*

1. *There is an equivariant conditional expectation $P : (N, \alpha) \rightarrow (M, \alpha)$.*
2. *There is a conditional expectation $E : N \rtimes_\alpha \mathbb{G} \rightarrow M \rtimes_\alpha \mathbb{G}$.*

Proof. (1) $\Rightarrow$ (2): since $P : (N, \alpha) \rightarrow (M, \alpha)$ is an equivariant conditional expectation, it follows $E = \text{id} \otimes P$ is a conditional expectation from $B(\ell^2(\mathbb{G})) \otimes N$ onto $B(\ell^2(\mathbb{G})) \otimes M$ such that

$$(\sigma \otimes \text{id})(\text{id} \otimes \alpha)E = (\text{id} \otimes E)(\sigma \otimes \text{id})(\text{id} \otimes \alpha),$$

where σ is the flip map. therefore the map $\text{id} \otimes E$ is a conditional expectation from $\ell^\infty(\mathbb{G}) \overline{\otimes} B(\ell^2(\mathbb{G})) \overline{\otimes} N$ onto $\ell^\infty(\mathbb{G}) \overline{\otimes} B(\ell^2(\mathbb{G})) \overline{\otimes} M$. Recall the left action β of $\mathbb{G}$ on $B(\ell^2(\mathbb{G})) \overline{\otimes} N$, defined in Theorem 2.2. One can see that for any $y \in B(\ell^2(\mathbb{G})) \overline{\otimes} N$ we have $\beta \circ E(y) = (\text{id} \otimes E)\beta(y)$. In particular, if $y \in (B(\ell^2(\mathbb{G})) \overline{\otimes} N)^\beta$ we have

$$\beta \circ E(y) = (\text{id} \otimes E)\beta(y) = (\text{id} \otimes E)(1 \otimes y) = 1 \otimes E(y),$$

which implies $E(y) \in (B(\ell^2(\mathbb{G})) \overline{\otimes} M)^\beta$. Thus in view of Theorem 2.2, the restriction of E is a conditional expectation from $N \rtimes_\alpha \mathbb{G}$ onto $M \rtimes_\alpha \mathbb{G}$.

(2) $\Rightarrow$ (1): suppose $E : N \rtimes_\alpha \mathbb{G} \rightarrow M \rtimes_\alpha \mathbb{G}$ is the conditional expectation and $\hat{\phi}$ is the tracial Haar state of the dual Kac algebra $\hat{\mathbb{G}}$. There is a canonical conditional expectation $E_{\hat{\phi}}$ from $M \rtimes_\alpha \mathbb{G}$ onto $\alpha(M)$ defined by

$$E_{\hat{\phi}}(x) = (\hat{\phi} \otimes \text{id})\hat{\alpha}(x), \quad \text{for all } x \in M \rtimes_\alpha \mathbb{G}. \quad (11)$$

We claim that $E_{\hat{\phi}} \circ E : (N \rtimes_\alpha \mathbb{G}, \Delta \otimes \text{id}) \rightarrow (\alpha(M), \Delta \otimes \text{id})$ is an equivariant conditional expectation. Since the left fundamental unitary W lies in $\ell^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\hat{\mathbb{G}})$, for all $z \in N \rtimes_\alpha \mathbb{G}$ we have

$$\begin{aligned} (\text{id} \otimes E)(\Delta \otimes \text{id})(z) &= (\text{id} \otimes E)(W_{12}^* z_{23} W_{12}) \\ &= W_{12}^* (1 \otimes E(z)) W_{12} \\ &= (\Delta \otimes \text{id})(E(z)). \end{aligned}$$

Therefore, in order to conclude the claim, it is sufficient to show that the canonical conditional expectation $E_{\hat{\phi}} : (M \rtimes_\alpha \mathbb{G}, \Delta \otimes \text{id}) \rightarrow (\alpha(M), \Delta \otimes \text{id})$ is equivariant. First, consider $\hat{x} \in L^\infty(\hat{\mathbb{G}})$. Then

$$\begin{aligned} (\text{id} \otimes E_{\hat{\phi}})(\Delta \otimes \text{id})(\hat{x} \otimes 1) &= (\text{id} \otimes E_{\hat{\phi}})(W_{12}^* (1 \otimes \hat{x} \otimes 1) W_{12}) \\ &= (\text{id} \otimes \hat{\phi} \otimes \text{id})(\text{id} \otimes \hat{\alpha})(W_{12}^* (1 \otimes \hat{x} \otimes 1) W_{12}) \\ &= (\text{id} \otimes \hat{\phi} \otimes \text{id} \otimes \text{id})(\text{id} \otimes \hat{\Delta}^{\text{op}} \otimes \text{id})(W_{12}^* (1 \otimes \hat{x} \otimes 1) W_{12}) \\ &= (\text{id} \otimes \text{id} \otimes \hat{\phi} \otimes \text{id})(\text{id} \otimes \hat{\Delta} \otimes \text{id})(W_{12}^* (1 \otimes \hat{x} \otimes 1) W_{12}) \\ &= (\text{id} \otimes \text{id} \otimes \hat{\phi} \otimes \text{id})(W_{13}^* (1 \otimes 1 \otimes \hat{x} \otimes 1) W_{13}). \end{aligned}$$

Now, consider a complete orthonormal system $\{e_j\}_{j \in J}$. Then similarly to the proof of [18, Corollary 3.17], for any normal states $f \in \ell^1(\mathbb{G})$ and $\omega \in M_*$, and any vector state $\omega_\xi \in \mathcal{T}(\ell^2(\mathbb{G}))$ we get:

$$\begin{aligned} \langle \omega_\xi \otimes f \otimes \omega, (\text{id} \otimes E_{\hat{\phi}})(\Delta \otimes \text{id})(\hat{x} \otimes 1) \rangle &= f(1)\omega(1)\langle \omega_\xi \otimes \hat{\phi}, W^* (1 \otimes \hat{x}) W \rangle \\ &= \langle \hat{\phi}, (\omega_\xi \otimes \text{id})(W^* (1 \otimes \hat{x}) W) \rangle \\ &= \langle \hat{\phi}, \sum_{j \in J} (\omega_{\xi, e_j} \otimes \text{id})(W)^* \hat{x} (\omega_{\xi, e_j} \otimes \text{id})(W) \rangle \\ &= \sum_{j \in J} \langle \hat{\phi}, (\omega_{\xi, e_j} \otimes \text{id})(W)^* \hat{x} (\omega_{\xi, e_j} \otimes \text{id})(W) \rangle \\ &= \sum_{j \in J} \langle \hat{\phi}, (\omega_{\xi, e_j} \otimes \text{id})(W) (\omega_{\xi, e_j} \otimes \text{id})(W)^* \hat{x} \rangle \end{aligned}$$

$$\begin{aligned}
&= \sum_{j \in J} \langle \hat{\phi}, (\omega_{j\xi, j e_j} \otimes \text{id})(W)^* (\omega_{j\xi, j e_p} \otimes \text{id})(W) \hat{x} \rangle \\
&= \omega_{j\xi}(1) \hat{\phi}(\hat{x}) \\
&= \omega_\xi(1) \hat{\phi}(\hat{x}),
\end{aligned}$$

where $\hat{j}$ is the modular conjugation for the tracial Haar state $\hat{\phi}$. So, for any $\hat{x} \in L^\infty(\hat{\mathbb{G}})$ we have:

$$\begin{aligned}
(\text{id} \otimes E_{\hat{\phi}})(\Delta \otimes \text{id})(\hat{x} \otimes 1) &= \hat{\phi}(\hat{x}) 1 \otimes 1 \otimes 1 \\
&= (\Delta \otimes \text{id})(\hat{\phi} \otimes \text{id} \otimes \text{id})(\hat{\Delta}^{\text{op}}(\hat{x}) \otimes 1) \\
&= (\Delta \otimes \text{id})(\hat{\phi} \otimes \text{id})(\hat{\alpha}(\hat{x} \otimes 1)) \\
&= (\Delta \otimes \text{id})E_{\hat{\phi}}(\hat{x} \otimes 1).
\end{aligned}$$

Hence, for any $\hat{x} \in L^\infty(\hat{\mathbb{G}})$ and $x \in M$ we get

$$\begin{aligned}
(\text{id} \otimes E_{\hat{\phi}})(\Delta \otimes \text{id})((\hat{x} \otimes 1)\alpha(x)) &= (\text{id} \otimes E_{\hat{\phi}})((\Delta \otimes \text{id})(\hat{x} \otimes 1)(\Delta \otimes \text{id})\alpha(x)) \\
&= (\text{id} \otimes E_{\hat{\phi}})((\Delta \otimes \text{id})(\hat{x} \otimes 1)(\text{id} \otimes \alpha)\alpha(x)) \\
&= (\text{id} \otimes E_{\hat{\phi}})((\Delta \otimes \text{id})(\hat{x} \otimes 1))(\text{id} \otimes \alpha)\alpha(x) \\
&= (\Delta \otimes \text{id})(E_{\hat{\phi}}(\hat{x} \otimes 1))(\Delta \otimes \text{id})\alpha(x) \\
&= (\Delta \otimes \text{id})E_{\hat{\phi}}((\hat{x} \otimes 1)\alpha(x)).
\end{aligned}$$

Since the crossed product $M \rtimes_\alpha \mathbb{G}$ is generated by $\{\ell^\infty(\hat{\mathbb{G}}) \overline{\otimes} \mathbf{1}, \alpha(M)\}$, it follows that the conditional expectation $E_{\hat{\phi}}$ is $(\Delta \otimes \text{id})$ -equivariant which completes the proof of claim. Define the conditional expectation $P := \alpha^{-1} \circ E_{\hat{\phi}} \circ E \circ \alpha$ from N onto M . We show that $(\text{id} \otimes P)\alpha = \alpha \circ P$. Since $E_{\hat{\phi}} \circ E$ is equivariant with respect to the action $\Delta \otimes \text{id}$, for all $a \in N$ we have

$$\begin{aligned}
(\text{id} \otimes E_{\hat{\phi}} \circ E)(\text{id} \otimes \alpha)\alpha(a) &= (\text{id} \otimes E_{\hat{\phi}} \circ E)(\Delta \otimes \text{id})\alpha(a) \\
&= (\Delta \otimes \text{id})(E_{\hat{\phi}} \circ E)\alpha(a) \\
&= (\text{id} \otimes \alpha) \circ (E_{\hat{\phi}} \circ E)(\alpha(a)),
\end{aligned}$$

where in the last equality we use that $(E_{\hat{\phi}} \circ E)\alpha(a) \in \alpha(M)$. Now it follows

$$(\text{id} \otimes P)\alpha = (\text{id} \otimes \alpha^{-1})(\text{id} \otimes E_{\hat{\phi}} \circ E)(\text{id} \otimes \alpha)\alpha = E_{\hat{\phi}} \circ E \circ \alpha = \alpha \circ P. \quad \square$$

The following is the non-commutative analogue of the main result of [2] for discrete Kac algebra actions: (See [2, Theorem 4.2].)

Theorem 6.2. *Let $\alpha : \mathbb{G} \curvearrowright N$ be an action of a discrete Kac algebra $\mathbb{G}$ on a von Neumann algebra N . The following are equivalent:*

1. *The action α is amenable.*
2. *There exists a conditional expectation from $(\ell^\infty(\mathbb{G}) \overline{\otimes} N) \rtimes_{\Delta \boxtimes \alpha} \mathbb{G}$ onto $(\mathbf{1} \overline{\otimes} N) \rtimes_{\Delta \boxtimes \alpha} \mathbb{G}$.*
3. *For any extension $(M, \mathbb{G}, \beta)$ of $(\mathbf{1} \overline{\otimes} N, \mathbb{G}, \Delta \boxtimes \alpha)$, the pair $(M, \mathbf{1} \overline{\otimes} N)$ is amenable.*

Proof. (1) $\Rightarrow$ (2): Suppose that α is amenable, then by Proposition 4.6 the pair $(\ell^\infty(\mathbb{G}) \overline{\boxtimes} N, \mathbf{1} \overline{\boxtimes} N)$ is amenable which means there is an equivariant conditional expectation from $(\ell^\infty(\mathbb{G}) \overline{\boxtimes} N, \Delta \boxtimes \alpha)$ onto $(\mathbf{1} \overline{\boxtimes} N, \Delta \boxtimes \alpha)$. Hence, (2) follows by Lemma 6.1.

(2) $\Rightarrow$ (3): Suppose that $(M, \mathbb{G}, \beta)$ is an extension of $(\mathbf{1} \overline{\boxtimes} N, \mathbb{G}, \Delta \boxtimes \alpha)$, let q be a conditional expectation from M onto $\mathbf{1} \overline{\boxtimes} N$. Then $\text{id} \otimes q$ is a conditional expectation from $B(\ell^2(\mathbb{G})) \overline{\otimes} M$ onto $B(\ell^2(\mathbb{G})) \overline{\otimes} (\mathbf{1} \overline{\boxtimes} N)$, and thus, Theorem 4.8 yields a conditional expectation

$$E : (\ell^\infty(\mathbb{G}) \overline{\boxtimes} M) \ltimes_{\Delta \boxtimes \beta} \mathbb{G} \rightarrow (\ell^\infty(\mathbb{G}) \overline{\boxtimes} (\mathbf{1} \overline{\boxtimes} N)) \ltimes_{\Delta \boxtimes \beta} \mathbb{G}.$$

Moreover, by the assumption there exists a conditional expectation from the crossed product $(\ell^\infty(\mathbb{G}) \overline{\boxtimes} N) \ltimes_{\Delta \boxtimes \alpha} \mathbb{G}$ onto $(\mathbf{1} \overline{\boxtimes} N) \ltimes_{\Delta \boxtimes \alpha} \mathbb{G}$. Now by identifying $N \cong \alpha(N) = \mathbf{1} \overline{\boxtimes} N$, it is equivalent to the existence of a conditional expectation E_0 from $(\ell^\infty(\mathbb{G}) \overline{\boxtimes} (\mathbf{1} \overline{\boxtimes} N)) \ltimes_{\Delta \boxtimes \beta} \mathbb{G}$ onto $(\mathbf{1} \overline{\boxtimes} (\mathbf{1} \overline{\boxtimes} N)) \ltimes_{\Delta \boxtimes \beta} \mathbb{G}$. By composing, we obtain the conditional expectation

$$E_0 \circ E : (\ell^\infty(\mathbb{G}) \overline{\boxtimes} M) \ltimes_{\Delta \boxtimes \beta} \mathbb{G} \rightarrow (\mathbf{1} \overline{\boxtimes} (\mathbf{1} \overline{\boxtimes} N)) \ltimes_{\Delta \boxtimes \beta} \mathbb{G}.$$

Hence, from Lemma 6.1, we have an equivariant conditional expectation Q from $(\ell^\infty(\mathbb{G}) \overline{\boxtimes} M, \Delta \boxtimes \beta)$ onto $(\mathbf{1} \overline{\boxtimes} (\mathbf{1} \overline{\boxtimes} N), \Delta \boxtimes \beta)$. Since $(M, \mathbb{G}, \beta)$ is an extension of $(\mathbf{1} \overline{\boxtimes} N, \mathbb{G}, \Delta \boxtimes \alpha)$, the restriction of Q to $\mathbf{1} \overline{\boxtimes} M$ is also a conditional expectation $Q_0 : \beta(M) \cong \mathbf{1} \overline{\boxtimes} M \rightarrow \mathbf{1} \overline{\boxtimes} (\mathbf{1} \overline{\boxtimes} N) \cong \beta(\mathbf{1} \overline{\boxtimes} N)$ such that $(\text{id} \otimes Q_0)(\Delta \boxtimes \beta) = (\Delta \boxtimes \beta) \circ Q_0$. Hence, for any $a \in M$, we have

$$(\text{id} \otimes Q_0)(\text{id} \otimes \beta)\beta(a) = (\text{id} \otimes \beta)Q_0(\beta(a)).$$

Now define a conditional expectation $P := \beta^{-1} \circ Q_0 \circ \beta$ from M onto $\mathbf{1} \overline{\boxtimes} N$. Then for all $a \in M$ we have

$$\begin{aligned} (\text{id} \otimes P)\beta(a) &= (\text{id} \otimes \beta^{-1})(\text{id} \otimes Q_0)(\text{id} \otimes \beta)\beta(a) \\ &= (\text{id} \otimes \beta^{-1})(\text{id} \otimes \beta)Q_0(\beta(a)) \\ &= \beta \circ P(a), \end{aligned}$$

which implies $P : (M, \beta) \rightarrow (\mathbf{1} \overline{\boxtimes} N, \Delta \boxtimes \alpha)$ is an equivariant conditional expectation. Hence, the pair $(M, \mathbf{1} \overline{\boxtimes} N)$ is amenable.

(3) $\Rightarrow$ (1): consider the canonical extension $(\ell^\infty(\mathbb{G}) \overline{\boxtimes} N, \mathbb{G}, \Delta \boxtimes \alpha)$ of the triple $(\mathbf{1} \overline{\boxtimes} N, \mathbb{G}, \Delta \boxtimes \alpha)$. Then by the assumption the pair $(\ell^\infty(\mathbb{G}) \overline{\boxtimes} N, \mathbf{1} \overline{\boxtimes} N)$ must be amenable. Hence, the action α is amenable by Proposition 4.6. $\square$

Theorem 6.3. Let $\alpha : \mathbb{G} \curvearrowright N$ be an action of a discrete Kac algebra $\mathbb{G}$ on a von Neumann algebra N . Then the following are equivalent:

1. The action α is amenable.
2. There is a conditional expectation from $B(\ell^2(\mathbb{G})) \overline{\otimes} N$ onto $N \ltimes_\alpha \mathbb{G}$.

Proof. By Theorem 4.8, there is an isomorphism from $(\ell^\infty(\mathbb{G}) \overline{\boxtimes} N) \ltimes_{\Delta \boxtimes \alpha} \mathbb{G}$ onto $B(\ell^2(\mathbb{G})) \overline{\otimes} N$ which maps $(1 \overline{\boxtimes} N) \ltimes_{\Delta \boxtimes \alpha} \mathbb{G}$ onto $N \ltimes_\alpha \mathbb{G}$. So, the theorem follows by the equivalence of (1) and (2) in Theorem 6.2. $\square$

Corollary 6.4. *Let $\alpha : \mathbb{G} \curvearrowright N$ be an action of a discrete Kac algebra $\mathbb{G}$ on a von Neumann algebra N . Then the following are equivalent:*

1. *The von Neumann algebra N is injective and the action α is amenable.*
2. *The crossed product $N \ltimes_\alpha \mathbb{G}$ is injective.*

Proof. (1) $\Rightarrow$ (2): if N is injective then so is $B(\ell^2(\mathbb{G})) \overline{\otimes} N$. If α is amenable, Theorem 6.3 yields a conditional expectation from $B(\ell^2(\mathbb{G})) \overline{\otimes} N$ onto the crossed product $N \ltimes_\alpha \mathbb{G}$. Since $B(\ell^2(\mathbb{G})) \overline{\otimes} N$ is injective, $N \ltimes_\alpha \mathbb{G}$ is also injective.

(2) $\Rightarrow$ (1): since the crossed product $N \ltimes_\alpha \mathbb{G}$ is injective, there is a conditional expectation from $B(\ell^2(\mathbb{G})) \overline{\otimes} N$ onto $N \ltimes_\alpha \mathbb{G}$. Therefore, by Theorem 6.3, the action α is amenable. Moreover, since there is always the canonical conditional expectation from $N \ltimes_\alpha \mathbb{G}$ on $\alpha(N)$, it follows that $\alpha(N)$ and equivalently N , is injective. $\square$

7. Amenable actions and crossed products: general case

In this section, we generalize the duality of Corollary 6.3 to the setting of general discrete quantum group actions. To this end, we basically need to show Lemma 6.1 for general discrete quantum groups. Recall that in the proof of the implication (2) to (1) of Lemma 6.1, we construct an equivariant conditional expectation from $(\alpha(N), \Delta \otimes \text{id})$ onto $(\alpha(M), \Delta \otimes \text{id})$ by composing the restriction $E|_{\alpha(N)}$ with the canonical conditional expectation $E_{\hat{\phi}}$. In the case of discrete Kac algebras, $E_{\hat{\phi}}$ is automatically equivariant with respect to the action $\Delta \otimes \text{id}$. But this is no longer the case in the general setting of discrete quantum group actions, since the Haar state $\hat{\phi}$ is not a trace. To overcome this issue, we impose an extra assumption on the conditional expectation $E : N \ltimes_\alpha \mathbb{G} \rightarrow M \ltimes_\alpha \mathbb{G}$ to be equivariant with respect to the dual action $\hat{\alpha}$. This would imply that E maps $\alpha(N)$ onto $\alpha(M)$, hence use of the canonical conditional expectation $E_{\hat{\phi}}$ is no longer necessary. This is inspired by the work of Crann and Neufang in [8], where they proved a characterization of amenability of the general locally compact quantum group $\mathbb{G}$ in terms of covariant injectivity of the dual von Neumann algebra $L^\infty(\hat{\mathbb{G}})$.

Lemma 7.1. *Let $\mathbb{G}$ be a discrete quantum group. Then for any $y \in B(\ell^2(\mathbb{G}))$ we have*

$$(\text{id} \otimes \Delta^r) \hat{\Delta}^{\text{op}}(y) = (\hat{\Delta}^{\text{op}} \otimes \text{id}) \Delta^r(y).$$

Proof. Let $x \in \ell^\infty(\mathbb{G})$ and $\hat{x} \in L^\infty(\hat{\mathbb{G}})$. Since the fundamental unitaries $\hat{W}^{\text{op}}$ and V lie in $L^\infty(\hat{\mathbb{G}}) \overline{\otimes} \ell^\infty(\mathbb{G})'$ and $L^\infty(\hat{\mathbb{G}})' \overline{\otimes} \ell^\infty(\mathbb{G})$, respectively, we have

$$\hat{\Delta}^{\text{op}}(x) = 1 \otimes x \quad \text{and} \quad \Delta^r(\hat{x}) = \hat{x} \otimes 1.$$

Therefore,

$$(\text{id} \otimes \Delta^r) \hat{\Delta}^{\text{op}}(x) = (\text{id} \otimes \Delta^r)(1 \otimes x) = 1 \otimes \Delta^r(x) = (\hat{\Delta}^{\text{op}} \otimes \text{id}) \Delta^r(x),$$

and

$$(\text{id} \otimes \Delta^r) \hat{\Delta}^{\text{op}}(\hat{x}) = \hat{\Delta}^{\text{op}}(\hat{x}) \otimes 1 = (\hat{\Delta}^{\text{op}} \otimes \text{id})(\hat{x} \otimes 1) = (\hat{\Delta}^{\text{op}} \otimes \text{id}) \Delta^r(\hat{x}).$$

Since the co-multiplications Δ^r and $\hat{\Delta}^{\text{op}}$ are homomorphisms, and the linear span of $\{x\hat{x} : x \in \ell^\infty(\mathbb{G}), \hat{x} \in L^\infty(\hat{\mathbb{G}})\}$ is weak* dense in $B(\ell^2(\mathbb{G}))$ [21, Proposition 2.5], we obtain the desired equality on $B(\ell^2(\mathbb{G}))$. $\square$

In the following, we use the same idea as [6, Proposition 4.2] to show an automatic equivariant property with respect to the dual action.

Proposition 7.2. *Let $\mathbb{G}$ be a discrete quantum group and let N be a von Neumann algebra. Then any $(\Delta^r \otimes \text{id})$ -equivariant map on $B(\ell^2(\mathbb{G})) \bar{\otimes} N$ is automatically $(\hat{\Delta}^{\text{op}} \otimes \text{id})$ -equivariant.*

Proof. Let Φ be an equivariant map on $(B(\ell^2(\mathbb{G})) \bar{\otimes} N, \Delta^r \otimes \text{id})$. Consider normal states $\tau, \omega \in \mathcal{T}(\ell^2(\mathbb{G}))$, $f \in \ell^1(\mathbb{G})$ and $g \in N_*$. Then for any $x \in B(\ell^2(\mathbb{G})) \bar{\otimes} N$ we have

$$\begin{aligned} & \langle f \otimes \tau \otimes \omega \otimes g, (\text{id} \otimes \Delta^r \otimes \text{id})(\text{id} \otimes \Phi)(\hat{\Delta}^{\text{op}} \otimes \text{id})(x) \rangle \\ &= \langle \tau \otimes \omega \otimes g, (\Delta^r \otimes \text{id})\Phi((f \otimes \text{id} \otimes \text{id})(\hat{\Delta}^{\text{op}} \otimes \text{id})(x)) \rangle \\ &= \langle \tau \otimes \omega \otimes g, (\text{id} \otimes \Phi)[(\Delta^r \otimes \text{id})((f \otimes \text{id} \otimes \text{id})(\hat{\Delta}^{\text{op}} \otimes \text{id})(x))] \rangle \\ &= \langle \omega \otimes g, \Phi[(\tau \otimes \text{id} \otimes \text{id})(\Delta^r \otimes \text{id})((f \otimes \text{id} \otimes \text{id})(\hat{\Delta}^{\text{op}} \otimes \text{id})(x))] \rangle \\ &= \langle \omega \otimes g, \Phi[(f \otimes \tau \otimes \text{id} \otimes \text{id})(\text{id} \otimes \Delta^r \otimes \text{id})(\hat{\Delta}^{\text{op}} \otimes \text{id})(x))] \rangle \\ &= \langle \omega \otimes g, \Phi[(f \otimes \tau \otimes \text{id} \otimes \text{id})(\hat{\Delta}^{\text{op}} \otimes \text{id} \otimes \text{id})(\Delta^r \otimes \text{id})(x))] \rangle \\ &= \langle \omega \otimes g, (f \otimes \tau \otimes \text{id} \otimes \text{id})(\hat{\Delta}^{\text{op}} \otimes \text{id} \otimes \text{id})\Phi((\Delta^r \otimes \text{id})(x)) \rangle \\ &= \langle f \otimes \tau \otimes \omega \otimes g, (\hat{\Delta}^{\text{op}} \otimes \text{id} \otimes \text{id})(\Delta^r \otimes \text{id})(\Phi(x)) \rangle \\ &= \langle f \otimes \tau \otimes \omega \otimes g, (\text{id} \otimes \Delta^r \otimes \text{id})(\hat{\Delta}^{\text{op}} \otimes \text{id})(\Phi(x)) \rangle. \end{aligned}$$

Since $\{(\tau \otimes \omega) \Delta^r : \tau, \omega \in \mathcal{T}(\ell^2(\mathbb{G}))\}$ spans a dense subset of $\mathcal{T}(\ell^2(\mathbb{G}))$, see (2), it follows that

$$(\text{id} \otimes \Phi)((\hat{\Delta}^{\text{op}} \otimes \text{id})(x)) = (\hat{\Delta}^{\text{op}} \otimes \text{id})\Phi(x). \quad \square$$

Corollary 7.3. *Let $\alpha : \mathbb{G} \curvearrowright N$ be an action of a discrete quantum group $\mathbb{G}$ on a von Neumann algebra N and let M be a von Neumann subalgebra of N which is invariant under α . If $E : (N \rtimes_\alpha \mathbb{G}, \Delta^r \otimes \text{id}) \rightarrow (M \rtimes_\alpha \mathbb{G}, \Delta^r \otimes \text{id})$ is an equivariant conditional expectation, then E is equivariant with respect to the dual action $\hat{\alpha}$.*

Proof. Note that the dual action $\hat{\alpha}$ is the restriction of $\hat{\Delta}^{\text{op}} \otimes \text{id}$ to the crossed product $N \rtimes_\alpha \mathbb{G} \subseteq B(\ell^2(\mathbb{G})) \bar{\otimes} N$. Hence, Proposition 7.2 implies that the conditional expectation E is equivariant with respect to $\hat{\alpha}$. $\square$

Lemma 7.4. *Let $\alpha : \mathbb{G} \curvearrowright N$ be an action of a discrete quantum group $\mathbb{G}$ on a von Neumann algebra N and let M be a von Neumann subalgebra of N which is invariant under α . The following are equivalent:*

1. *There is an equivariant conditional expectation $P : (N, \alpha) \rightarrow (M, \alpha)$.*
2. *There is an equivariant conditional expectation*

$$E : (N \rtimes_{\alpha} \mathbb{G}, \hat{\alpha}) \rightarrow (M \rtimes_{\alpha} \mathbb{G}, \hat{\alpha}).$$

Proof. (1) $\Rightarrow$ (2): Similarly as in the proof of Lemma 6.1, we see that the restriction of $\text{id} \otimes P : B(\ell^2(\mathbb{G})) \overline{\otimes} N \rightarrow B(\ell^2(\mathbb{G})) \overline{\otimes} M$ to the crossed product $N \rtimes_{\alpha} \mathbb{G}$ yields a conditional expectation E from $N \rtimes_{\alpha} \mathbb{G}$ onto $M \rtimes_{\alpha} \mathbb{G}$. It is easy to see that E is $(\Delta^r \otimes \text{id})$ -equivariant. Thanks to Corollary 7.3, the conditional expectation E is equivariant with respect to the dual action $\hat{\alpha}$.

(2) $\Rightarrow$ (1): Suppose that $E : (N \rtimes_{\alpha} \mathbb{G}, \hat{\alpha}) \rightarrow (M \rtimes_{\alpha} \mathbb{G}, \hat{\alpha})$ is an equivariant conditional expectation. Then for all $x \in N$, we have

$$\begin{aligned} 1_{L^{\infty}(\mathbb{G})} \otimes E(\alpha(x)) &= (\text{id} \otimes E)(1_{L^{\infty}(\mathbb{G})} \otimes \alpha(x)) \\ &= (\text{id} \otimes E) \circ \hat{\alpha}(\alpha(x)) \\ &= \hat{\alpha} \circ E(\alpha(x)). \end{aligned}$$

It follows that $E(\alpha(x))$ is in the fixed point algebra of the dual action $\hat{\alpha}$ on $M \rtimes_{\alpha} \mathbb{G}$. Hence, $E(\alpha(N)) \subseteq \alpha(M)$. Now define the conditional expectation $P := \alpha^{-1} \circ E \circ \alpha$ from N onto M . Similarly to the proof of Lemma 6.1, we show that $(\text{id} \otimes P)\alpha = \alpha \circ P$. Since the fundamental unitary W lies in $\ell^{\infty}(\mathbb{G}) \overline{\otimes} L^{\infty}(\hat{\mathbb{G}})$, it follows that the conditional expectation E is $(\Delta \otimes \text{id})$ -equivariant. Now for any $x \in N$ we have

$$\begin{aligned} (\text{id} \otimes E)(\text{id} \otimes \alpha)\alpha(x) &= (\text{id} \otimes E)(\Delta \otimes \text{id})\alpha(x) \\ &= (\Delta \otimes \text{id})E(\alpha(x)) \\ &= (\text{id} \otimes \alpha)E(\alpha(x)), \end{aligned}$$

where in the last equality we use that $E(\alpha(x)) \in \alpha(M)$. Now it follows

$$(\text{id} \otimes P)\alpha = (\text{id} \otimes \alpha^{-1})(\text{id} \otimes E)(\text{id} \otimes \alpha)\alpha = E \circ \alpha = \alpha \circ P. \quad \square$$

The equivalence of (1) and (3) in the following result is a non-commutative analogue of Zimmer's classical result [25, Theorem. 2.1].

Theorem 7.5. *Let $\alpha : \mathbb{G} \curvearrowright N$ be an action of a discrete quantum group $\mathbb{G}$ on a von Neumann algebra N . The following are equivalent:*

1. *The action α is amenable.*
2. *There is an equivariant conditional expectation*

$$E : ((\ell^{\infty}(\mathbb{G}) \overline{\otimes} N) \rtimes_{\Delta \overline{\otimes} \alpha} \mathbb{G}, \widehat{\Delta \overline{\otimes} \alpha}) \rightarrow ((1 \overline{\otimes} N) \rtimes_{\Delta \overline{\otimes} \alpha} \mathbb{G}, \widehat{\Delta \overline{\otimes} \alpha}).$$

3. *There is an equivariant conditional expectation*

$$E : (B(\ell^2(\mathbb{G})) \overline{\otimes} N, \hat{\Delta}^{op} \otimes \text{id}) \rightarrow (N \rtimes_{\alpha} \mathbb{G}, \hat{\alpha}).$$

Proof. By Theorem 4.8, the statements (2) and (3) are equivalent. To conclude (1) and (2), thanks to Proposition 4.6 amenability of α is equivalent to amenability of the pair $(\ell^\infty(\mathbb{G}) \overline{\otimes} N, \mathbf{1} \overline{\otimes} N)$ which by Lemma 7.4 is equivalent to the existence of a $\widehat{\Delta} \overline{\otimes} \alpha$ -equivariant conditional expectation E from $(\ell^\infty(\mathbb{G}) \overline{\otimes} N) \ltimes_{\Delta \overline{\otimes} \alpha} \mathbb{G}$ onto $(\mathbf{1} \overline{\otimes} N) \ltimes_{\Delta \overline{\otimes} \alpha} \mathbb{G}$ $\square$

Remark 7.6. Since the trivial action $tr : \mathbb{G} \curvearrowright \mathbb{C}$ is amenable if and only if $\mathbb{G}$ is amenable, and $\mathbb{C} \ltimes_{tr} \mathbb{G} = L^\infty(\widehat{\mathbb{G}})$, the equivalence of (1) and (3) in Theorem 7.5 in fact gives a generalization of the main result of [8].

Suppose that $\beta : \mathbb{G} \curvearrowright K$ is an action of a discrete quantum group $\mathbb{G}$ on a von Neumann algebra K . We say that K is $\mathbb{G}$ -injective if for every unital completely isometric equivariant map $\iota : (M, \alpha_1) \rightarrow (N, \alpha_2)$ and every unital completely positive equivariant map $\Psi : (M, \alpha_1) \rightarrow (K, \beta)$ there is a unital completely positive equivariant map $\overline{\Psi} : (N, \alpha_2) \rightarrow (K, \beta)$ such that $\overline{\Psi} \circ \iota = \Psi$.

Corollary 7.7. Let $\alpha : \mathbb{G} \curvearrowright N$ be an action of a discrete quantum group $\mathbb{G}$ on a von Neumann algebra N . The following are equivalent:

1. The von Neumann algebra N is injective and the action α is amenable.
2. The crossed product $N \ltimes_\alpha \mathbb{G}$ is $\widehat{\mathbb{G}}$ -injective.

Proof. (1) $\Rightarrow$ (2): The proof is similar to the proof of Corollary 6.4, only that we use Theorem 7.5 instead of Theorem 6.3.

(2) $\Rightarrow$ (1): Since the crossed product $N \ltimes_\alpha \mathbb{G}$ is $\widehat{\mathbb{G}}$ -injective, the identity map on the crossed product $N \ltimes_\alpha \mathbb{G}$ can be extended to an equivariant conditional expectation from $(B(\ell^2(\mathbb{G})) \overline{\otimes} N, \widehat{\Delta}^{\text{op}} \otimes \text{id})$ onto $(N \ltimes_\alpha \mathbb{G}, \widehat{\alpha})$. Hence, by Theorem 7.5, the action α is amenable. Moreover, there is always the canonical conditional expectation from $N \ltimes_\alpha \mathbb{G}$ onto $\alpha(N)$, it follows that N is injective. $\square$

Corollary 7.8 ([15], Corollary 2.5). Let $\mathbb{G}$ be a discrete quantum group and let $\mu \in \ell^1(\mathbb{G})$ be a state. The von Neumann algebra crossed product $\mathcal{H}_\mu \ltimes_{\Delta_\mu} \mathbb{G}$ is injective.

Proof. By Theorem 5.4, the action of $\mathbb{G}$ on its Poisson boundaries is always amenable, and therefore the result follows by Corollary 7.7. $\square$

Remark 7.9. Crann and Kalantar informed us in a recent unpublished manuscript that they have independently defined a notion of Zimmer amenability in the setting of actions of locally compact quantum groups on von Neumann algebras, where they used a homological approach. But, their definition is equivalent to Definition 4.1 in the case of discrete quantum groups. They have obtained a similar result as Corollary 7.7 in that general context.

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