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On noncommutative frame bundles

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Abstract. The question of whether a right Hilbert bimodule admits a noncommutative frame bundle - i. e., a C*-algebraic noncommutative principal bundle with which the right Hilbert bimodule is associated via some fundamental representation - is both pivotal and difficult. In this paper, we contribute to this topic by providing an axiomatic characterization of a right Hilbert bimodule, let's say M, that ensures the existence of a unique (up to isomorphism) free C*-dynamical system $(\mathcal{A}_M, \mathrm{SO}(n), \alpha_M)$ with the property that its associated noncommutative vector bundle, with respect to the standard representation of $\mathrm{SO}(n)$, is isomorphic to M. Our approach is inspired by potential applications in noncommutative Riemannian spin geometry.

Contents

1. Introduction

Vector bundles in classical geometry typically arise as objects associated with something more profound, a principal bundle. In particular, each vector bundle E with fibre V is naturally associated with a principal $\mathrm{GL}(V)$-bundle, called the *frame bundle* of E and denoted by $\mathrm{Fr}(E)$. Notably, if the base space of E comes equipped with additional structure, then it is often natural to consider a reduction of $\mathrm{Fr}(E)$ which is adapted to the given structure. Frame bundles thus constitute a key tool for studying vector bundles. Indeed, given a vector bundle E, its frame bundle can be utilized in order to attach to E several new

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vector bundles in a functorial manner. Furthermore, a connection on $\mathrm{Fr}(E)$ induces covariant derivatives on all associated vector bundles in a coherent way, leading to many important geometric constructions. This is the situation in Riemannian geometry where, for a Riemannian manifold X, the Levi-Civita connection on $\mathrm{Fr}(TX)$ - a principal $\mathrm{O}(V)$-bundle - induces a covariant derivative on the tensor fields, leading, for instance, to the Riemannian curvature of X. Another instance is Riemannian spin geometry where, for a spin manifold X, a "spin connection" on $\mathrm{Fr}(TX)$ - a principal $\mathrm{SO}(V)$-bundle - induces a covariant derivative on the spinor bundle, leading to the Dirac and Laplace operator on the spinor bundle.

The noncommutative geometry of frame bundles, however, has not been studied conclusively, although the notion of a noncommutative principal bundle is certainly available, accompanied by a natural procedure of associating noncommutative vector bundles (see, e. g., [BCH17, Ell00, SW17b] and references therein). Noteworthily, the purpose of [Ell00] was to provide a general setting in order to address the problem of finding "a suitable notion of frame bundle over an abstract non-commutative manifold". To the best of our knowledge, the first and only systematic treatment of a quantized notion of frame bundle seems to be due to S. Majid [Maj99, Maj02, Maj05] (see also [BM20, Sec. 5.6] for a cohesive presentation). For the sake of expedience, we briefly recall his main idea. Let B be an algebra equipped with a differential structure (Ω^1, d) which plays the role of the differential 1-forms of an ordinary manifold. A *framing* of B is then essentially a Hopf-Galois-algebraic noncommutative principal bundle over B that recovers Ω^1 in the sense that there exists a suitable associated noncommutative vector bundle isomorphic to Ω^1. Note that the structure quantum group need not be fixed in this approach as one might have several different candidates.

In this paper, we sort of complement Majid's approach within the C^*-algebraic framework of noncommutative principal bundles. More precisely, for a correspondence M over a unital C^*-algebra $\mathcal{B}$, we provide axiomatic conditions that enable the construction of a unique (up to isomorphism) free C^*-dynamical system $(\mathcal{A}_M, \mathrm{SO}(n), \alpha_M)$ with fixed point algebra $\mathcal{B}$ and the property that its associated noncommutative vector bundle with respect to the standard representation of $\mathrm{SO}(n)$ is isomorphic to M. This will be our *noncommutative frame bundle associated with* M. We emphasize that such a construction already exists for the special case of the compact Abelian group $\mathrm{SO}(2)$ (see [SW17a, Sec. 4]); therefore we assume from now on that $n \geq 3$.

Although our approach is primarily of topological nature, we hope that it can be extended to incorporate additional geometric information. In particular, we hope to be able to construct new and interesting characteristic classes in future work. Furthermore, we wish to mention that this paper is part of a broader research program aimed at offering a novel perspective on noncommutative Riemannian spin geometry by systematically developing and studying the key constructions and ideas of Riemannian spin geometry within the C^*-algebraic framework of noncommutative principal bundles. This also explains our focus on the structure group $\mathrm{SO}(n)$. However, we would like to point out that with little effort our arguments and results extend to semisimple Lie groups admitting a faithful irreducible representation.

Organization of the paper. In Section 2, we set out the necessary preliminaries and notation. In Section 3, the main body of this paper, we provide a detailed exposition of our approach and its significant outcomes. More precisely, for a unital C*-algebra $\mathcal{B}$ and the standard representation π of $\mathrm{SO}(n)$, $n \geq 3$, we introduce the central notion of this work which is concerned with correspondences over $\mathcal{B}$ being "*tensorial of type* π" (Definition 3.1 and Definition 3.5). For such a correspondence, let's say M, we are able to provide a construction procedure for a unitary tensor functor from a small tensor subcategory $\mathcal{T} \subseteq \mathrm{Rep}(\mathrm{SO}(n))$ containing representatives of $\mathrm{Irr}(\mathrm{SO}(n))$ to the tensor category $\mathrm{Corr}(\mathcal{B})$ of correspondences over $\mathcal{B}$. The construction procedure relies on the representation theory of the semisimple compact Lie group $\mathrm{SO}(n)$, and it naturally splits into the following two main steps:

(1) We construct a small tensor subcategory $\mathcal{T} \subseteq \mathrm{Rep}(\mathrm{SO}(n))$ containing representatives of $\mathrm{Irr}(\mathrm{SO}(n))$ together with a linear functor $\Gamma_M : \mathcal{T} \to \mathrm{Corr}(\mathcal{B})$ satisfying $\Gamma_M(\pi^{\otimes k}) = M^{\otimes k}$ for all integers $k \geq 0$ and $\Gamma_M(T)^* = \Gamma_M(T^*)$ for all morphisms T in $\mathcal{T}$ (Corollary 3.4).
(2) We construct natural, $\mathcal{B}$-bilinear, and unitary maps
$$m_M(\sigma, \tau) : \Gamma_M(\sigma) \otimes_{\mathcal{B}} \Gamma_M(\tau) \to \Gamma_M(\sigma \otimes \tau)$$
for all objects $\sigma, \tau \in \mathcal{T}$ (Corollary 3.10), and show that they satisfy a certain associativity condition (Lemma 3.11).

Taken together, these results yield a unitary tensor functor $\mathcal{T} \to \mathrm{Corr}(\mathcal{B})$ (Theorem 3.12) which, in turn, gives rise to a free C*-dynamical system $(\mathcal{A}_M, \mathrm{SO}(n), \alpha_M)$ with fixed point algebra $\mathcal{B}$ and the property $\Gamma_{\mathcal{A}_M}(\pi) \cong M$, the *noncommutative frame bundle associated with* M (Corollary 3.13; see Section 4.2 for a justification of this designation). For expediency's sake, we discuss the main steps of the construction of $(\mathcal{A}_M, \mathrm{SO}(n), \alpha_M)$ in Appendix A. Finally, we present a classification result that extends the classical correspondence between frame bundles and their associated vector bundles (Corollary 3.16). Section 4 is devoted to providing examples. Indeed, let V be the representation space of π. In Section 4.1, we show that $\mathcal{B} \otimes V$ is tensorial of type π. In Section 4.2, we demonstrate that each locally trivial hermitian vector bundle over a compact space with typical fibre V and structure group $\mathrm{SO}(n)$ is tensorial of type π, thereby recovering the classical setting of frame bundles. In Section 4.3 and Section 4.4, we introduce, as of yet unknown, free C*-dynamical systems with structure group $\mathrm{SO}(3)$, the *quantum projective 7-space* and the (even part of the) Cuntz algebra $\mathcal{O}_2$, and hence we get two more examples of tensorial correspondences of type π by looking at their associated noncommutative vector bundles with respect to π. Last but not least, in Appendix B, we briefly treat the special case of $\mathrm{SO}(2)$ for the sake of completeness.

2. Preliminaries and notation

Our study deals with noncommutative frame bundles. This preliminary section exhibits the most fundamental definitions and notations in use.

At first, we provide some standard references. For a thorough treatment of Lie theory and representation theory, we refer to the remarkable work [GW09] by Goodman and Wallach (see also [BtD85]). For a comprehensive introduction to the theory of fibre bundles, especially principal bundles and (their associated) vector bundles, we refer to Husemöller's book [Hus94] and the influential exposition [KN96] by Nomizu and Kobayashi. For a recent account of the theory of Hilbert module structures, we refer to the excellent volume [RW98] by Raeburn and Williams and the memoirs [EKQR06] by Echterhoff et al. Our standard references for the theory of operator algebras are the opuses [Bla06, Ped18] by Blackadar and Pedersen, respectively. We also use a categorical description of noncommutative principal bundles, and for the necessary background on category theory we refer to the monographs [EGNO15, Mit65, NT13].

About groups. Let G be a compact group. All representations of G are assumed to be finite-dimensional and unitary unless mentioned otherwise. We denote a representation $\sigma : G \to \mathcal{U}(V_\sigma)$ by the pair (σ, V_σ) or simply by σ. In particular, we let 1 stand for the trivial representation when no ambiguity is possible. We write $\mathrm{Rep}(G)$ for its C^*-tensor category of representations and $\mathrm{Irr}(G)$ for the set of equivalence classes of irreducible representations. By abuse of notation, we also use the symbol σ to denote elements of $\mathrm{Irr}(G)$ and choose a representative representation (σ, V_σ) for $\sigma \in \mathrm{Irr}(G)$ when needed.

One of the key ingredients for our construction procedure in Section 3 is the following result on irreducible representations of the semisimple compact Lie group $\mathrm{SO}(n)$, $n \geq 3$.

Corollary 2.1 (See, e. g., [GW09, Thm. 5.5.21])**.** *Let π be the standard representation of $\mathrm{SO}(n)$, $n \geq 3$. Each irreducible representation of $\mathrm{SO}(n)$ occurs as a subrepresentation of some tensor product representation $\pi^{\otimes k}$, $k \geq 0$ (with $\pi^{\otimes 0} = 1$).*

About frames bundles. Let X be a locally compact space and let $q : E \to X$ be a locally trivial (real or complex) vector bundle over X with typical fibre V. The *frame bundle*

$$\mathrm{Fr}(E) := \bigcup_{x \in X} \mathrm{Iso}\,(V, E_x)\,, \qquad\qquad E_x := q^{-1}(\{x\}),$$

where $\mathrm{Iso}\,(V, E_x)$ denotes the set of linear isomorphisms from V to E_x, carries the structure of a principal $\mathrm{GL}(V)$-bundle over X with respect to the canonical right action of $\mathrm{GL}(V)$ on $\mathrm{Fr}(E)$. The associated vector bundle $\mathrm{Fr}(E) \times_{\mathrm{GL}(V)} V$ with respect to the standard representation (π, V) of $\mathrm{GL}(V)$ recovers E, i. e., $\mathrm{Fr}(E) \times_{\mathrm{GL}(V)} V \cong E$ as vector bundles over X. For simplicity of notation, we use the same symbol $\mathrm{Fr}(E)$ to denote any reduction of the frame bundle.

About Hilbert modules. Let $\mathcal{B}$ be a unital C*-algebra. A *correspondence* over $\mathcal{B}$ is a $\mathcal{B}$-bimodule M equipped with an inner product $\langle \cdot, \cdot \rangle_{\mathcal{B}} : M \times M \to \mathcal{B}$ turning it into a right Hilbert $\mathcal{B}$-module such that the left action of $\mathcal{B}$ on M is through adjointable operators. Given two correspondences M and N over $\mathcal{B}$, we write $M \otimes_{\mathcal{B}} N$ for their tensor product on which the inner product is determined by $\langle x_1 \otimes y_1,\ x_2 \otimes y_2 \rangle_{\mathcal{B}} = \langle y_1, \langle x_1, x_2 \rangle_{\mathcal{B}} . y_2 \rangle_{\mathcal{B}}$ for all $x_1, x_2 \in M$ and $y_1, y_2 \in N$. We are also concerned with multiple tensor products. For a correspondence M over $\mathcal{B}$ and a non-negative integer k, we let $M^{\otimes k}$ stand for the k-fold tensor product of M with itself (with $M^{\otimes 0} = \mathcal{B}$). We use the symbol $\mathrm{Corr}(\mathcal{B})$ to denote the C*-tensor category of correspondences over $\mathcal{B}$.

About C*-dynamical systems. Let $\mathcal{A}$ be a unital C*-algebra, let G be a compact group, and let $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ be a strongly continuous group homomorphism. We refer to such a triple $(\mathcal{A}, G, \alpha)$ as a *C*dynamical system* and adopt the standard shorthand $\alpha_g := \alpha(g)$ for $g \in G$. The corresponding fixed point algebra is typically denoted by $\mathcal{B}$.

Remark 2.2. Like every, possibly infinite, continuous representation of a compact group, the algebra $\mathcal{A}$ can be decomposed into its isotypic components which amounts to saying that their algebraic direct sum forms a dense *-subalgebra of $\mathcal{A}$ (see, e. g., [HM13, Thm. 4.22]).

We also deal to a large extend with the associated spaces

$$\Gamma_{\mathcal{A}}(\sigma) := \{x \in \mathcal{A} \otimes V_\sigma : (\forall g \in G)\, (\alpha_g \otimes \sigma_g)(x) = x\} \tag{1}$$

for all objects σ in $\mathrm{Rep}(G)$, each of which is naturally a correspondence over $\mathcal{B}$ with respect to the canonical left and right actions and the restriction of the right $\mathcal{A}$-valued inner product on $\mathcal{A} \otimes V_\sigma$ determined by $\langle a \otimes v, b \otimes w \rangle_{\mathcal{A}} := \langle v, w \rangle a^* b$ for all $a, b \in \mathcal{A}$ and $v, w \in V_\sigma$. Most notably, the linear functor $\Gamma_{\mathcal{A}} : \mathrm{Rep}(G) \to \mathrm{Corr}(\mathcal{B})$, defined for objects by $\Gamma_{\mathcal{A}}(\sigma)$ and for morphisms by $\Gamma_{\mathcal{A}}(T) := 1_{\mathcal{A}} \otimes T$, together with the natural $\mathcal{B}$-bilinear isometries

$$m_{\mathcal{A}}(\sigma, \tau) : \Gamma_{\mathcal{A}}(\sigma) \otimes_{\mathcal{B}} \Gamma_{\mathcal{A}}(\tau) \to \Gamma_{\mathcal{A}}(\sigma \otimes \tau), \qquad x \otimes y \mapsto x_{12} y_{13}$$

for all objects σ, τ in $\mathrm{Rep}(G)$ constitute a weak unitary tensor functor $\mathrm{Rep}(G) \to \mathrm{Corr}(\mathcal{B})$ which allows to reconstruct the C*-dynamical system $(\mathcal{A}, G, \alpha)$ up to isomorphism (see [Nes13, Sec. 2]).

About freeness. A C*-dynamical system $(\mathcal{A}, G, \alpha)$ is called *free* if the *Ellwood map*

$$\Phi : \mathcal{A} \otimes_{\mathrm{alg}} \mathcal{A} \to C(G, \mathcal{A}), \qquad \Phi(x \otimes y)(g) := x \alpha_g(y)$$

has dense range with respect to the canonical C*-norm on $C(G, \mathcal{A})$. This requirement was originally introduced for actions of quantum groups on C*-algebras by Ellwood [Ell00] and is known to be equivalent to Rieffel's saturatedness [Rie91] and the Peter-Weyl-Galois condition [BCH17]. By [Phi87, Prop. 7.1.12 & Thm. 7.2.6], a continuous action $r : P \times G \to P$ of a compact group G on a compact space P is free in the classical sense if and only if the induced C*-dynamical system $(C(P), G, \alpha)$, where $\alpha_g(f)(p) := f(r(p,g))$ for all $p \in P$ and $g \in G$, is free in the sense of Ellwood. Free C*-dynamical systems thus provide a natural framework for noncommutative principal bundles, and in this context the spaces in (1) play the role of the associated vector bundles. In particular, they are finitely generated and projective as right $\mathcal{B}$-modules (see, e. g., [CY13, Thm. 1.2]).

Yet another crucial result, which plays a significant role in the construction of noncommutative frame bundles, is the bijective correspondence between free C*-dynamical systems and unitary tensor functors [SW17b, Sec. 5] (see also [Đur96]). For the convenience of the reader, we now recall the definition of these functors.

Definition 2.3. Let $\mathcal{B}$ be a unital C*-algebra and let G be a compact group. A *unitary tensor functor* $\operatorname{Rep}(G) \to \operatorname{Corr}(\mathcal{B})$ is a linear functor $\Gamma : \operatorname{Rep}(G) \to \operatorname{Corr}(\mathcal{B})$ together with natural $\mathcal{B}$-bilinear unitary maps

$$m(\sigma,\tau) : \Gamma(\sigma) \otimes_{\mathcal{B}} \Gamma(\tau) \to \Gamma(\sigma \otimes \tau)$$

for all objects $\sigma, \tau \in \operatorname{Rep}(G)$ such that the following conditions are satisfied:

(i) $\Gamma(1) = \mathcal{B}$, and for each object σ in $\operatorname{Rep}(G)$ the map $m(1,\sigma)$ maps $b \otimes x$ to $b \,.\, x$ and, similarly, $m(\sigma, 1)$ maps $x \otimes b$ to $x \,.\, b$ for all $b \in \mathcal{B}$ and $x \in \Gamma(\sigma)$.

(ii) $\Gamma(T)^* = \Gamma(T^*)$ for all morphisms T in $\operatorname{Rep}(G)$.

(iii) $m(\sigma, \tau \otimes \rho)(\mathrm{id} \otimes_{\mathcal{B}} m(\tau,\rho)) = m(\sigma \otimes \tau, \rho)(m(\sigma,\tau) \otimes_{\mathcal{B}} \mathrm{id})$ for all objects σ, τ, ρ in $\operatorname{Rep}(G)$.

Remark 2.4. For the construction of a free C*-dynamical system from a unitary tensor functor $\operatorname{Rep}(G) \to \operatorname{Corr}(\mathcal{B})$, one only needs a small C*-tensor subcategory $\mathcal{T} \subseteq \operatorname{Rep}(G)$ containing representatives of $\operatorname{Irr}(G)$ (see [Nes13, Thm. 2.3]).

3. Noncommutative SO(n)-frame bundles

Let $\mathcal{B}$ be a unital C*-algebra, let M be a correspondence over $\mathcal{B}$, and let π be the standard representation of $\mathrm{SO}(n)$, $n \geq 3$. In this section, we provide axiomatic conditions on M that enable the construction of a unique (up to isomorphism) free C*-dynamical system $(\mathcal{A}_M, \mathrm{SO}(n), \alpha_M)$ with fixed point algebra $\mathcal{B}$ and the property $\Gamma_{\mathcal{A}_M}(\pi) \cong M$. This will be our *noncommutative frame bundle associated with M* (see Remark 3.14). The main idea is to put together a unitary tensor functor from a small tensor subcategory $\mathcal{T} \subseteq \operatorname{Rep}(\mathrm{SO}(n))$ containing representatives of $\operatorname{Irr}(\mathrm{SO}(n))$ to $\operatorname{Corr}(\mathcal{B})$ (see Remark 2.4).

We begin by introducing the main notion of this paper:

Definition 3.1. Let $\mathcal{B}$ be a unital C*-algebra and let π be the standard representation of $\mathrm{SO}(n)$, $n \geq 3$. We say that *a correspondence M over $\mathcal{B}$ is of type π* if there exist injective linear maps

$$\varphi_{k,l} : C_{k,l} =: \operatorname{Hom}_{\mathrm{SO}(n)}\big(V_\pi^{\otimes k}, V_\pi^{\otimes l}\big) \to \mathcal{L}\big(M^{\otimes k}, M^{\otimes l}\big)$$

for all integers $k, l \geq 0$ such that the following compatibility conditions are satisfied:

(C) $\varphi_{l,m}(T')\varphi_{k,l}(T) = \varphi_{k,m}(T'T)$ for all $k, l, m \geq 0$, $T \in C_{k,l}$, and $T' \in C_{l,m}$.
(A) $\varphi_{k,l}(T)^* = \varphi_{l,k}(T^*)$ for all $k, l \geq 0$ and $T \in C_{k,l}$.
(U) $\varphi_{k,k}(\mathrm{id}) = \mathrm{id}$ for all $k \geq 0$.

From now on, let $\mathcal{B}$ be a unital C*-algebra, let π be the standard representation of $\mathrm{SO}(n)$, $n \geq 3$, and let M be a correspondence over $\mathcal{B}$ of type π. Also, let $V := V_\pi$ for brevity. As a first step, we construct a small tensor subcategory $\mathcal{T} \subseteq \mathrm{Rep}(G)$ containing representatives of $\mathrm{Irr}(\mathrm{SO}(n))$ together with a linear functor $\Gamma_M : \mathcal{T} \to \mathrm{Corr}(\mathcal{B})$. For this purpose, we choose for each $\sigma \in \mathrm{Irr}(\mathrm{SO}(n))$ a representative (σ, V_σ) that is a subrepresentation of some tensor product representation $\left(\pi^{\otimes k}, V^{\otimes k}\right)$, $k \geq 0$ (see Corollary 2.1). In particular, for $1 \in \mathrm{Irr}(\mathrm{SO}(n))$ we choose the trivial representation $(1, \mathbb{C})$. Furthermore, we consider the full subcategory $\mathcal{S} \subseteq \mathrm{Rep}(G)$ whose objects consists of all finite tensor product representations generated by the family (σ, V_σ), $\sigma \in \mathrm{Irr}(\mathrm{SO}(n))$.

Let (σ, V_σ) be an object in $\mathcal{S}$. By construction, there exists an integer $k \geq 0$ such that (σ, V_σ) is a subrepresentation of $\left(\pi^{\otimes k}, V^{\otimes k}\right)$. Let P_σ be the orthogonal projection of $V^{\otimes k}$ onto V_σ. Clearly, $P_\sigma \in C_{k,k}$, and hence $\varphi_{k,k}(P_\sigma)$ acts as an adjointable operator on $M^{\otimes k}$. Moreover, Conditions (C) and (A) combined imply that $\varphi_{k,k}(P_\sigma)$ is a projection, and from this it may be concluded that

$$\Gamma_M(\sigma) := \varphi_{k,k}(P_\sigma)\left(M^{\otimes k}\right) \tag{2}$$

is a correspondence over $\mathcal{B}$. We thus have a correspondence over $\mathcal{B}$ available for each object (σ, V_σ) in $\mathcal{S}$. Note that $\Gamma_M\left(\pi^{\otimes k}\right) = M^{\otimes k}$ for all integers $k \geq 0$ by Condition (U).

Next, let (σ, V_σ) and (τ, V_τ) be objects in $\mathcal{S}$ and let $T : V_\sigma \to V_\tau$ be a morphism. Our objective is to relate the correspondences $\Gamma_M(\sigma)$ and $\Gamma_M(\tau)$ by means of a morphism $\Gamma_M(T)$. To this end, let $k, l \geq 0$ be integers such that (σ, V_σ) and (τ, V_τ) are subrepresentations of $\left(\pi^{\otimes k}, V^{\otimes k}\right)$ and $\left(\pi^{\otimes l}, V^{\otimes l}\right)$, respectively, and let P_σ and P_τ be the orthogonal projections of $V^{\otimes k}$ onto V_σ and $V^{\otimes l}$ onto V_τ, respectively. We define a map $W_T : V^{\otimes k} \to V^{\otimes l}$ by

$$W_T(x) := \begin{cases} T(x) & \text{for } x \in V_\sigma, \\ 0 & \text{for } x \in V_\sigma^\perp \subseteq V^{\otimes k}. \end{cases}$$

Note that $W_T^* = W_{T^*}$. Moreover, it is easily seen that $W_T \in C_{k,l}$, and so we may look at its image under the map $\varphi_{k,l}$. In fact, we have

$$\varphi_{l,l}(P_\tau)\varphi_{k,l}(W_T) \overset{(C)}{=} \varphi_{k,l}(P_\tau W_T) = \varphi_{k,l}(W_T)$$

which implies that $\mathrm{ran}(\varphi_{k,l}(W_T)) \subseteq \Gamma_M(\tau)$. With this at hand, we put

$$\Gamma_M(T) := \varphi_{k,l}(W_T) \restriction_{\Gamma_M(\sigma)}^{\Gamma_M(\tau)} : \Gamma_M(\sigma) \to \Gamma_M(\tau).$$

It is worth noting that $W_{P_\sigma} = P_\sigma$, therefore that $\Gamma_M(P_\sigma) = \varphi_{k,k}(P_\sigma)$, and finally that $\operatorname{ran}(\Gamma_M(P_\sigma)) = \Gamma_M(\sigma)$. Furthermore, we see at once that $\Gamma_M(\mathrm{id}_{V_\sigma}) = \mathrm{id}_{\Gamma_M(\sigma)}$.

By attentively following the above construction and applying the assumptions in Definition 3.1, we immediately get:

Lemma 3.2. *For objects (σ, V_σ), (τ, V_τ), and (ρ, V_ρ) in $\mathcal{S}$ the following assertions hold:*

(1) $\Gamma_M(T + cT') = \Gamma_M(T) + c\Gamma_M(T')$ for all morphisms $T, T' : V_\sigma \to V_\tau$ and $c \in \mathbb{C}$.

(2) $\Gamma_M(T'T) = \Gamma_M(T')\Gamma_M(T)$ for all morphisms $T : V_\sigma \to V_\tau$ and $T' : V_\tau \to V_\rho$.

Lemma 3.3. *Let (σ, V_σ) and (τ, V_τ) be objects in $\mathcal{S}$ and let $T : V_\sigma \to V_\tau$ be a morphism. Then $\Gamma_M(T)^* = \Gamma_M(T^*)$.*

Proof. Let $x \in \Gamma_M(\sigma)$ and let $y \in \Gamma_M(\tau)$. Since $\varphi_{k,l}(W_T)^* \overset{\text{(A)}}{=} \varphi_{l,k}(W_T^*) = \varphi_{l,k}(W_{T^*})$, we conclude that

$$\begin{aligned}\langle \Gamma_M(T)(x), y\rangle_{\mathcal{B}} &= \langle \varphi_{k,l}(W_T)(x), y\rangle_{\mathcal{B}} = \langle x, \varphi_{k,l}(W_T)^*(y)\rangle_{\mathcal{B}} \\ &= \langle x, \varphi_{l,k}(W_{T^*})(y)\rangle_{\mathcal{B}} = \langle x, \Gamma_M(T^*)(y)\rangle_{\mathcal{B}}. \qquad \square\end{aligned}$$

We proceed by looking at the full subcategory $\mathcal{T} \subseteq \operatorname{Rep}(\mathrm{SO}(n))$ whose objects are finite direct sums of objects in $\mathcal{S}$. It is clear that $\mathcal{T}$ is a small tensor subcategory of $\operatorname{Rep}(\mathrm{SO}(n))$. Furthermore, for an object $\sigma = \sigma_1 \oplus \cdots \oplus \sigma_m$ in $\mathcal{T}$ we define

$$\Gamma_M(\sigma) := \Gamma_M(\sigma_1) \oplus \cdots \oplus \Gamma_M(\sigma_m).$$

Also, for two objects $\sigma = \sigma_1 \oplus \cdots \oplus \sigma_r$ and $\tau = \tau_1 \oplus \cdots \oplus \tau_s$ in $\mathcal{T}$ we note that

$$\operatorname{Hom}_G(V_\sigma, V_\tau) = \bigoplus_{i,j} \operatorname{Hom}_G(V_{\sigma_i}, V_{\tau_j}),$$

and hence each morphism $T : V_\sigma \to V_\tau$ can be uniquely written as a matrix with entries in $\operatorname{Hom}_G(V_{\sigma_i}, V_{\tau_j})$ for all eligible pairs i, j. Given such a morphism $T = (T_{ij})$, we put

$$\Gamma_M(T) := \big(\Gamma_M(T_{ij})\big).$$

With these definitions, it is easily checked that Lemma 3.2 and Lemma 3.3 extend to objects in $\mathcal{T}$ and morphisms between them. Summarizing, we have thus shown:

Corollary 3.4. *$\mathcal{T}$ is a small tensor subcategory of $\operatorname{Rep}(SO(n))$ containing representatives of $\operatorname{Irr}(\mathrm{SO}(n))$. Furthermore, the map $\Gamma_M : \mathcal{T} \to \operatorname{Corr}(\mathcal{B})$, defined for objects by $\Gamma_M(\sigma)$ and for morphisms by $\Gamma_M(T)$, is a linear functor such that $\Gamma_M(\pi^{\otimes k}) = M^{\otimes k}$ for all integers $k \geq 0$ and $\Gamma_M(T)^* = \Gamma_M(T^*)$ for all morphisms T in $\mathcal{T}$.*

Having completed the first task, we now turn to the construction of natural, $\mathcal{B}$-bilinear, and unitary maps $m_M(\sigma,\tau) : \Gamma_M(\sigma) \otimes_{\mathcal{B}} \Gamma_M(\tau) \to \Gamma_M(\sigma \otimes \tau)$ for all $\sigma, \tau \in \mathcal{T}$. To this end, we consider the canonical multiplication maps $m_M(k,l) : M^{\otimes k} \otimes_{\mathcal{B}} M^{\otimes l} \to M^{\otimes (k+l)}$ for all integers $k, l \geq 0$, which are obviously $\mathcal{B}$-bilinear and unitary, and note that

$$m_M(k, l+m)\,(\mathrm{id} \otimes_{\mathcal{B}} m_M(l,m)) = m_M(k+l, m)\,(m_M(k,l) \otimes_{\mathcal{B}} \mathrm{id}) \tag{3}$$

for all integers $k, l, m \geq 0$. Furthermore, we impose another condition ensuring that the maps $\varphi_{k,l}$ for all integers $k, l \geq 0$ are compatible with respect to taking tensor products:

Definition 3.5. Let $\mathcal{B}$ be a unital C*-algebra and let π be the standard representation of $\mathrm{SO}(n)$, $n \geq 3$. We say that a *correspondence M over $\mathcal{B}$ is tensorial of type π* if it is of type π (see Definition 3.1) and the following compatibility condition is satisfied:

(T) $m_M(k,l)\big(\varphi_{k,k}(T) \otimes_{\mathcal{B}} \varphi_{l,l}(T')\big) = \varphi_{k+l}(T \otimes T')m_M(k,l)$ for all $k, l \geq 0$, $T \in C_{k,k}$, and $T' \in C_{l,l}$.

Remark 3.6.

(1) It is straightforward that each free C*-dynamical system $(\mathcal{A}, \mathrm{SO}(n), \alpha)$ with fixed point algebra $\mathcal{B}$ naturally gives rise to a correspondence over $\mathcal{B}$ that is tensorial of type π, namely the associated module $\Gamma_{\mathcal{A}}(\pi)$; see Definition 2.3.

(2) Conversely, given a correspondence M over $\mathcal{B}$, the task is simply to verify explicitly that it satisfies the axiomatic conditions defining tensoriality of type π.

In the remainder of this section, we assume that M is tensorial of type π. Let (σ, V_σ) and (τ, V_τ) be objects in $\mathcal{S}$, let $k, l \geq 0$ be integers such that (σ, V_σ) and (τ, V_τ) are subrepresentations of $\big(\pi^{\otimes k}, V^{\otimes k}\big)$ and $\big(\pi^{\otimes l}, V^{\otimes l}\big)$, respectively, and let P_σ and P_τ be the orthogonal projections of $V^{\otimes k}$ onto V_σ and $V^{\otimes l}$ onto V_τ, respectively. Then $(\sigma \otimes \tau, V_\sigma \otimes V_\tau)$ is a subrepresentation of $\big(\pi^{\otimes (k+l)}, V^{\otimes (k+l)}\big)$ and $P_\sigma \otimes P_\tau$ is the orthogonal projection of $V^{\otimes (k+l)}$ onto $V_\sigma \otimes V_\tau$. We put

$$\begin{aligned} m_M(\sigma,\tau) &: \Gamma_M(\sigma) \otimes_{\mathcal{B}} \Gamma_M(\tau) \to \Gamma_M(\sigma \otimes \tau) \\ m_M(\sigma,\tau) &:= \underbrace{\varphi_{k+l}(P_\sigma \otimes P_\tau)}_{=\Gamma_M(P_\sigma \otimes P_\tau)} m_M(k,l) \restriction^{\Gamma_M(\sigma\otimes\tau)}_{\Gamma_M(\sigma)\otimes_{\mathcal{B}}\Gamma_M(\tau)} \end{aligned} \tag{4}$$

It is readily seen that the map $m_M(\sigma,\tau)$ is well-defined and $\mathcal{B}$-bilinear. Furthermore, Condition (T) shows that

$$m_M(\sigma,\tau) = m_M(k,l)\big(\underbrace{\varphi_{k,k}(P_\sigma) \otimes_{\mathcal{B}} \varphi_{l,l}(P_\tau)}_{=\Gamma_M(P_\sigma)\otimes_{\mathcal{B}}\Gamma_M(P_\tau)}\big) \restriction^{\Gamma_M(\sigma\otimes\tau)}_{\Gamma_M(\sigma)\otimes_{\mathcal{B}}\Gamma_M(\tau)} \tag{5}$$

We proceed with a series of lemmas.

Lemma 3.7. *We have* $m_M(\sigma,\tau)\left(\Gamma_M(T)\otimes_{\mathcal{B}}\Gamma_M(T')\right)=\Gamma_M(T\otimes T')m_M(\sigma,\tau)$ *for all morphisms* $T : V_\sigma \to V_\sigma$ *and* $T' : V_\tau \to V_\tau$.

Proof. Let $T : V_\sigma \to V_\sigma$ and $T' : V_\tau \to V_\tau$ be morphisms. Using Conditions (T) and (C), on the domain $\Gamma_M(\sigma)\otimes_{\mathcal{B}}\Gamma_M(\tau)$ we deduce that

$$\begin{aligned} m_M(\sigma,\tau)\left(\Gamma_M(T)\otimes_{\mathcal{B}}\Gamma_M(T')\right) &= \varphi_{k+l}(P_\sigma\otimes P_\tau)m_M(k,l)\left(\varphi_{k,k}(W_T)\otimes_{\mathcal{B}}\varphi_{l,l}(W_{T'})\right)\\ &= \varphi_{k+l}(P_\sigma\otimes P_\tau)\varphi_{k+l}(W_{T\otimes T'})m_M(k,l)\\ &= \varphi_{k+l}((P_\sigma\otimes P_\tau)W_{T\otimes T'})m_M(k,l)\\ &= \varphi_{k+l}(W_{T\otimes T'}(P_\sigma\otimes P_\tau))m_M(k,l)\\ &= \varphi_{k+l}(W_{T\otimes T'})\varphi_{k+l}(P_\sigma\otimes P_\tau)m_M(k,l)\\ &= \varphi_{k+l}(W_{T\otimes T'})m_M(\sigma,\tau)=\Gamma_M(T\otimes T')m_M(\sigma,\tau).\square \end{aligned}$$

Lemma 3.8. *The map* $m_M(\sigma,\tau)$ *is isometric.*

Proof. Let $x\in\Gamma_M(\sigma)$ and let $y\in\Gamma_M(\tau)$. By (5), we have

$$\begin{aligned} &\langle m_M(\sigma,\tau)(x\otimes_{\mathcal{B}} y), m_M(\sigma,\tau)(x\otimes_{\mathcal{B}} y)\rangle_{\mathcal{B}}\\ &\qquad\qquad = \langle m_M(k,l)(x\otimes_{\mathcal{B}} y), m_M(k,l)(x\otimes_{\mathcal{B}} y)\rangle_{\mathcal{B}}. \end{aligned}$$

The claim therefore follows from the fact that the map $m_M(k,l)$ is unitary. $\square$

Lemma 3.9. *The map* $m_M(\sigma,\tau)$ *is surjective.*

Proof. Our proof starts with the observation that

$$M^{\otimes k}\otimes_{\mathcal{B}} M^{\otimes l}=\Gamma_M(\sigma)\otimes_{\mathcal{B}}\Gamma_M(\tau)\oplus\ker\left(\varphi_{k,k}(P_\sigma)\otimes_{\mathcal{B}}\varphi_{l,l}(P_\tau)\right).$$

Furthermore, we have

$$\ker\left(\varphi_{k+l}(P_\sigma\otimes P_\tau)m_M(k,l)\right)=\ker\left(\varphi_{k,k}(P_\sigma)\otimes_{\mathcal{B}}\varphi_{l,l}(P_\tau)\right)$$

which is clear from Condition (T). Since

$$\operatorname{ran}\left(\varphi_{k+l}(P_\sigma\otimes P_\tau)m_M(k,l)\right)=\Gamma_M(\sigma\otimes\tau),$$

it follows that $\operatorname{ran}(m_M(\sigma,\tau))=\Gamma_M(\sigma\otimes\tau)$ as required. $\square$

To summarize:

Corollary 3.10. *The map* $m_M(\sigma,\tau) : \Gamma_M(\sigma)\otimes_{\mathcal{B}}\Gamma_M(\tau)\to\Gamma_M(\sigma\otimes\tau)$ *given by* (4) *is natural,* $\mathcal{B}$*-bilinear, and unitary for all objects* σ,τ *in* $\mathcal{S}$.

Note that $m_M(\pi^{\otimes k},\pi^{\otimes l})=m_M(k,l)$ for all integers $k,l\geq 0$. Our next claim is that the maps $m_M(\sigma,\tau)$ for all objects σ,τ in $\mathcal{S}$ satisfy the following associativity condition:

Lemma 3.11. *We have*

$$m_M(\sigma,\tau\otimes\rho)\left(\mathrm{id}\otimes_{\mathcal{B}} m_M(\tau,\rho)\right)=m_M(\sigma\otimes\tau,\rho)\left(m_M(\sigma,\tau)\otimes_{\mathcal{B}}\mathrm{id}\right)$$

for all objects σ,τ,ρ *in* $\mathcal{S}$.

Proof. Let (σ, V_σ), (τ, V_τ), and (ρ, V_ρ) be objects in $\mathcal{S}$. Furthermore, let k, l and m be non-negative integers such that (σ, V_σ), (τ, V_τ), and (ρ, V_ρ) are subrepresentations of $(\pi^{\otimes k}, V^{\otimes k})$, $(\pi^{\otimes l}, V^{\otimes l})$, and $(\pi^{\otimes m}, V^{\otimes m})$, respectively, and let P_σ, P_τ, and P_ρ be the orthogonal projections of $V^{\otimes k}$ onto V_σ, $V^{\otimes l}$ onto V_τ, and $V^{\otimes m}$ onto V_ρ, respectively. On the domain $\Gamma_M(\sigma) \otimes_{\mathcal{B}} \Gamma_M(\tau) \otimes_{\mathcal{B}} \Gamma_M(\rho)$ we find that

$$
\begin{aligned}
& m_M(\sigma, \tau \otimes \rho)\,(\mathrm{id} \otimes_{\mathcal{B}} m_M(\tau, \rho)) \\
& \overset{(4)}{=} \varphi_{k+l+m}(P_\sigma \otimes (P_\tau \otimes P_\rho)) m_M(k, l+m)\,\big(\varphi_{k,k}(P_\sigma) \otimes_{\mathcal{B}} \big(\varphi_{l+m}(P_\tau \otimes P_\rho) m_M(l,m)\big)\big) \\
& \overset{(T)}{=} \varphi_{k+l+m}(P_\sigma \otimes P_\tau \otimes P_\rho) \varphi_{k+l+m}(P_\sigma \otimes P_\tau \otimes P_\rho) m_M(k, l+m)\,(\mathrm{id} \otimes_{\mathcal{B}} m_M(l,m)) \\
& \overset{(3)}{=} \varphi_{k+l+m}(P_\sigma \otimes P_\tau \otimes P_\rho) \varphi_{k+l+m}(P_\sigma \otimes P_\tau \otimes P_\rho) m_M(k+l, m)\,(m_M(k,l) \otimes_{\mathcal{B}} \mathrm{id}) \\
& \overset{(T)}{=} \varphi_{k+l+m}((P_\sigma \otimes P_\tau) \otimes P_\rho) m_M(k+l, m)\,\big(\varphi_{k+l}(P_\sigma \otimes P_\tau) m_M(k,l) \otimes_{\mathcal{B}} \varphi_{m,m}(P_\rho)\big)\,. \\
& \overset{(4)}{=} m_M(\sigma \otimes \tau, \rho)\,(m_M(\sigma, \tau) \otimes_{\mathcal{B}} \mathrm{id})\,.
\end{aligned}
$$

□

Finally, we look once more at the small tensor category $\mathcal{T} \subseteq \mathrm{Rep}(\mathrm{SO}(n))$. For two objects $\sigma = \sigma_1 \oplus \cdots \oplus \sigma_r$ and $\tau = \tau_1 \oplus \cdots \oplus \tau_s$ in $\mathcal{T}$ we put

$$m_M(\sigma, \tau) := \bigoplus_{i,j} m_M(\sigma_i, \tau_j).$$

A straightforward verification shows that this formula yields a natural $\mathcal{B}$-bilinear unitary map $m_M(\sigma, \tau) : \Gamma_M(\sigma) \otimes_{\mathcal{B}} \Gamma_M(\tau) \to \Gamma_M(\sigma \otimes \tau)$ for all objects σ, τ in $\mathcal{T}$ satisfying

$$m_M(\sigma, \tau \otimes \rho)\,(\mathrm{id} \otimes_{\mathcal{B}} m_M(\tau, \rho)) = m_M(\sigma \otimes \tau, \rho)\,(m_M(\sigma, \tau) \otimes_{\mathcal{B}} \mathrm{id})$$

for all objects σ, τ, ρ in $\mathcal{T}$. Thus we have proven the following statement, which constitutes the main result of this paper:

Theorem 3.12. *The linear functor* $\Gamma_M : \mathcal{T} \to \mathrm{Corr}(\mathcal{B})$ *together with the natural* $\mathcal{B}$*-bilinear unitary maps* $m_M(\sigma, \tau) : \Gamma_M(\sigma) \otimes_{\mathcal{B}} \Gamma_M(\tau) \to \Gamma_M(\sigma \otimes \tau)$ *for all objects* σ, τ *in* $\mathcal{T}$ *constitute a unitary tensor functor such that* $\Gamma_M(\pi^{\otimes k}) = M^{\otimes k}$ *for all integers* $k \geq 0$.

Having the unitary tensor functor $\mathcal{T} \to \mathrm{Corr}(\mathcal{B})$ at our disposal, we can now apply the construction procedure presented in [Nes13, Sec. 2] (see also [DR89, SW17b, Đur96]) to obtain a free C*-dynamical system $(\mathcal{A}_M, \mathrm{SO}(n), \alpha_M)$ with fixed point algebra $\mathcal{B}$ and $\Gamma_{\mathcal{A}_M}(\pi) \cong M$. For the sake of expediency we briefly sketch the main steps of the construction of $(\mathcal{A}_M, \mathrm{SO}(n), \alpha_M)$ in Appendix A.

Corollary 3.13. *Let* $\mathcal{B}$ *be a unital* C^**-algebra and let* π *be the standard representation of* $\mathrm{SO}(n)$, $n \geq 3$*. Each correspondence* M *over* $\mathcal{B}$ *that is tensorial of type* π *yields a unitary tensor functor* $\mathcal{T} \to \mathrm{Corr}(\mathcal{B})$ *for some small tensor subcategory* $\mathcal{T}$ *of* $\mathrm{Rep}(SO(n))$ *containing representatives of* $\mathrm{Irr}(\mathrm{SO}(n))$ *such that* $\Gamma_M(\pi^{\otimes k}) = M^{\otimes k}$ *for all integers* $k \geq 0$*, and hence a free* C^**-dynamical system* $(\mathcal{A}_M, \mathrm{SO}(n), \alpha_M)$ *with fixed point algebra* $\mathcal{B}$ *and* $\Gamma_{\mathcal{A}_M}(\pi) \cong \Gamma_M(\pi) = M$*.*

Remark 3.14. In Section 4.2, we show that each locally trivial hermitian vector bundle over a compact space with typical fibre V and structure group $\mathrm{SO}(n)$ is tensorial of type π, thereby recovering the classical setting of frame bundles. This justifies referring to the free C*-dynamical system $(\mathcal{A}_M, \mathrm{SO}(n), \alpha_M)$ described in Corollary 3.13 as the *noncommutative frame bundle associated with M*.

Remark 3.15. By the general theory of free C*-dynamical systems, it follows that M, and hence $\Gamma_M(\sigma)$ for all objects σ in $\mathcal{T}$, is necessarily finitely generated and projective as a right $\mathcal{B}$-module (see, e. g., [CY13, Thm. 1.2]).

We conclude this section with a classification result that extends the classical correspondence between frame bundles and their associated vector bundles. For this, we consider equivalence classes of free C*-dynamical systems with structure group $\mathrm{SO}(n)$ and fixed point algebra $\mathcal{B}$ with respect to equivariant *-isomorphisms that preserve $\mathcal{B}$ and, further, equivalence classes of correspondences over $\mathcal{B}$ with respect to $\mathcal{B}$-bilinear isomorphisms. Combining Corollary 3.13 with [Nes13, Thm. 2.3], we get:

Corollary 3.16. *Let $\mathcal{B}$ be a unital C*-algebra and let π be the standard representation of $\mathrm{SO}(n)$, $n \geq 3$. The map $[(\mathcal{A}, \mathrm{SO}(n), \alpha)] \mapsto [\Gamma_{\mathcal{A}}(\pi)]$ yields a bijective correspondence between equivalence classes of free C*-dynamical systems with structure group $\mathrm{SO}(n)$ and fixed point algebra $\mathcal{B}$ and equivalence classes of correspondences over $\mathcal{B}$ that are tensorial of type π with inverse given by $[M] \mapsto [(\mathcal{A}_M, \mathrm{SO}(n), \alpha_M)]$.*

4. Examples

This section is devoted to discussing examples. For expediency, we continue to use (π, V) to denote the standard representation of $\mathrm{SO}(n)$, $n \geq 3$.

4.1. Example: the free module of rank *n*. Let $\mathcal{B}$ be a unital C*-algebra. In this example we apply ourselves to the tensor product $M := \mathcal{B} \otimes V$ which is naturally a correspondence over $\mathcal{B}$ with respect to the canonical left and right actions and the right $\mathcal{B}$-valued inner product determined by $\langle b \otimes v, b' \otimes v' \rangle_{\mathcal{B}} := \langle v, v' \rangle b^* b'$ for all $b, b' \in \mathcal{B}$ and $v, v' \in V$. Note that, up to the canonical isomorphism $\mathcal{B}^{\otimes k} \cong \mathcal{B}$, we have $M^{\otimes k} = \mathcal{B} \otimes V^{\otimes k}$ for all integers $k \geq 0$. For integers $k, l \geq 0$, $T \in C_{k,l}$, and $x \in M^{\otimes k}$, we obtain an element in $M^{\otimes l}$ by putting $\varphi_{k,l}(x) := \mathrm{id}_{\mathcal{B}} \otimes T(x)$. This yields injective linear maps $\varphi_{k,l} : C_{k,l} \to \mathcal{L}\big(M^{\otimes k}, M^{\otimes l}\big)$ for all integers $k, l \geq 0$ which make M tensorial of type π, as is easy to check. From the construction procedure presented in Appendix A, we infer that the corresponding free C*-dynamical system $(\mathcal{A}_M, \mathrm{SO}(n), \alpha_M)$ is equivalent to $(\mathcal{B} \otimes_{\min} C(\mathrm{SO}(n)), \mathrm{SO}(n), \alpha)$, where $\otimes_{\min}$ denotes the minimal tensor product of C*-algebras and the action α is given by right translation in the argument of the second tensor factor.

4.2. Example: classical vector bundles. Let X be a compact space and let $q : E \to X$ be a locally trivial hermitian vector bundle with typical fibre V, structure group $\mathrm{SO}(n)$, and Hermitian metric $x \mapsto \langle \cdot, \cdot \rangle_x$. In this example, we consider the space $M := \Gamma(E)$ of continuous sections of $q : E \to X$ which carries the structure of a correspondence over $C(X)$ with respect to the obvious (bi-)module structure given by pointwise multiplication and the inner product $\langle \cdot, \cdot \rangle_{C(X)}$ given for $s, t \in \Gamma(E)$ by $\langle s, t \rangle_{C(X)}(x) := \langle s(x), t(x) \rangle_x$, $x \in X$. In case E is trivial, $M \cong C(X) \otimes V$, and therefore we are, up to isomorphism, in the situation of Example 4.1. In particular, we have $\mathcal{A}_M \cong C(X) \otimes_{\min} C(\mathrm{SO}(n))$ which shows that $\mathcal{A}_M$ is commutative with character space given by the trivial frame bundle $\mathrm{Fr}(E) \cong X \times \mathrm{SO}(n)$. In case E is non-trivial, we use bundle charts to conclude similarly that M is tensorial of type π. Let $(\mathcal{A}_M, \mathrm{SO}(n), \alpha_M)$ be the corresponding free C*-dynamical system. A moment's thought shows that $m(\sigma, \tau) = \mathit{flip}(m(\tau, \sigma))$ for all $\sigma, \tau \in \mathcal{T}$, where *flip* denotes the fibre-wise tensor flip. From this it follows that $\mathcal{A}_M$ is commutative, and hence that $\mathcal{A}_M \cong C(\mathrm{Fr}(E))$ by the uniqueness (up to isomorphism) of the geometric construction.

Remark 4.1. If X is a closed orientable manifold, then its tangent space is a locally trivial hermitian vector bundle with typical fibre V and structure group $\mathrm{SO}(n)$.

4.3. Example: the quantum projective 7-space. In this example we introduce a new free C*-dynamical system with structure group $\mathrm{SO}(3)$. In particular, its associated noncommutative vector bundle with respect to the standard representation π of $\mathrm{SO}(3)$ yields another instance of a tensorial correspondence of type π (see Remark 3.6). To the best of our knowledge, this noncommutative principal bundle has, as of yet, not been considered in the literature.

For a start, we recall a noncommutative C*-algebraic version of the classical $\mathrm{SU}(2)$-Hopf fibration over the four sphere (see [LS05] for a generalization in the context of Hopf-Galois extensions). Let $\theta \in \mathbb{R}$ and let θ' be the skewsymmetric 4×4-matrix with $\theta'_{1,2} = \theta'_{3,4} = 0$ and $\theta'_{1,3} = \theta'_{1,4} = \theta'_{2,3} = \theta'_{2,4} = \theta/2$. The Connes-Landi sphere $\mathcal{A}(\mathbb{S}^7_{\theta'})$ is the universal unital C*-algebra generated by normal elements $z_1, \dots, z_4$ subject to the relations

$$z_i z_j = e^{2\pi\imath\theta'_{i,j}}\, z_j z_i, \qquad z_j^* z_i = e^{2\pi\imath\theta'_{i,j}}\, z_i z_j^*, \qquad \sum_{k=1}^{4} z_k^* z_k = 1$$

for all $1 \leq i, j \leq 4$. By [SW17b, Expl. 3.5], it comes equipped with a free action α of $\mathrm{SU}(2)$ given for each $U \in \mathrm{SU}(2)$ on generators by

$$\alpha_U : (z_1, \dots, z_4) \mapsto (z_1, \dots, z_4) \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}.$$

The corresponding fixed point algebra is the universal unital C*-algebra $\mathcal{A}(\mathbb{S}^4_\theta)$ generated by normal elements w_1, w_2 and a self-adjoint element x satisfying

$$w_1 w_2 = e^{2\pi\imath\theta}\, w_2 w_1, \quad w_2^* w_1 = e^{2\pi\imath\theta}\, w_1 w_2^*, \quad w_1^* w_1 + w_2^* w_2 + x^* x = 1.$$

To proceed, we consider the normal subgroup $N := \{\pm 1\} \subseteq \mathrm{SU}(2)$. [SW17a, Prop. 3.18] implies that the induced C*-dynamical system

$$(\mathcal{A}(\mathbb{S}^7_{\theta'})^N, \mathrm{SU}(2)/N, \alpha\restriction_{\mathrm{SU}(2)/N})$$

is free, too. Hence, we arrive at the announced free C*-dynamical system with structure group SO(3) by (simply) putting $\mathcal{A}(\mathbb{P}^7_{\theta'}) := \mathcal{A}(\mathbb{S}^7_{\theta'})^N$ and by identifying SO(3) with $\mathrm{SU}(2)/N$ via the universal covering map $p : \mathrm{SU}(2) \to \mathrm{SO}(3)$, i. e.,

$$(\mathcal{A}(\mathbb{P}^7_{\theta'}), \mathrm{SO}(3), \alpha\restriction_{\mathrm{SU}(2)/N} \circ \bar{p}^{-1}),$$

$\bar{p}$ being the induced isomorphism from $\mathrm{SU}(2)/N$ to SO(3). Note that

$$\mathcal{A}(\mathbb{P}^7_{\theta'})^{\mathrm{SO}(3)} = \mathcal{A}(\mathbb{S}^4_{\theta}).$$

Finally, we conclude from Remark 3.6 that $\Gamma_{\mathcal{A}(\mathbb{P}^7_{\theta'})}(\pi)$ is tensorial of type π.

4.4. Example: the even part of the Cuntz algebra $\mathcal{O}_2$. In this example, we present yet another instance of a free C*-dynamical system with structure group SO(3), and hence of a correspondence that is tensorial of type π. Apparently, this free C*-dynamical system has neither been considered elsewhere in the literature.

To begin with, we bring to mind that the Cuntz algebra $\mathcal{O}_2$ is the universal unital C*-algebra generated by two elements S_1 and S_2 satisfying $S_i^* S_j = \delta_{ij}$ and $S_1 S_1^* + S_2 S_2^* = 1$ (see [Cun77]). On account of [Gab14, Prop. 8.4], it comes equipped with a free action α of SU(2) given for each

$$U := \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \in \mathrm{SU}(2)$$

on generators by

$$\alpha_U : (S_1, S_2) \mapsto (aS_1 + bS_2, -\bar{b}S_1 + \bar{a}S_2).$$

Now, the exact same line of arguments as in Example 4.3 above shows that the induced C*-dynamical system $(\mathcal{O}_2^N, \mathrm{SO}(3), \alpha\restriction_{\mathrm{SU}(2)/N} \circ \bar{p}^{-1})$ is free and, further, that $\Gamma_{\mathcal{O}_2^N}(\pi)$ is tensorial of type π.

Appendix A. The construction of the free C*-dynamical systems from the unitary tensor functor

This section contains a brief summary of the construction of the free C*-dynamical system $(\mathcal{A}_M, \mathrm{SO}(n), \alpha_M)$ from the unitary tensor functor $\mathcal{T} \to \mathrm{Corr}(\mathcal{B})$ put together in Section 3. Some parts of the construction require us to deal with conjugates. Indeed, given an irreducible representation (σ, V_σ) of $\mathrm{SO}(n)$, we identify its conjugate representation $(\bar{\sigma}, \bar{V}_\sigma)$ with the equivalent irreducible representation from our initial choice of representatives and denote the latter, by abuse of notation, also by $(\bar{\sigma}, \bar{V}_\sigma)$.

First, we form an algebra A_M. To do this, we consider the algebraic direct sum

$$A_M := \bigoplus_{\sigma \in \mathrm{Irr}(\mathrm{SO}(n))} \Gamma_M(\bar{\sigma}) \otimes V_\sigma.$$

Moreover, for $\sigma, \tau \in \mathrm{Irr}(\mathrm{SO}(n))$, $x \otimes v \in \Gamma_M(\bar{\sigma}) \otimes V_\sigma$, and $y \otimes w \in \Gamma_M(\bar{\tau}) \otimes V_\tau$ we define a product by the recipe

$$(x \otimes v) \bullet (y \otimes w) =: \sum_{k=1}^{N} \left(\Gamma_M\left(\bar{S}_k\right)^* \otimes S_k^*\right)(m(\bar{\sigma}, \bar{\tau})(x \otimes y) \otimes v \otimes w)$$
$$\in \sum_{k=1}^{N} \Gamma_M(\bar{\sigma}_k) \otimes V_{\sigma_k},$$

where $\{S_1, \dots, S_N\}$ is a complete set of isometric intertwiners $S_k : V_{\sigma_k} \to V_\sigma \otimes V_\tau$, $\sigma_k \in \mathrm{Irr}(\mathrm{SO}(n))$, with respective conjugates $\bar{S}_k : \bar{V}_{\sigma_k} \to \bar{V}_\sigma \otimes \bar{V}_\tau$. Extending this product bilinearly yields a multiplication on A_M which is associative due to condition (iii) in the definition of a unitary tensor functor. Note that $\mathcal{B}$ can be regarded as the subalgebra of A_M corresponding to the equivalence class of the trivial representation. Second, we turn A_M into a *-algebra. For this purpose, let $\sigma \in \mathrm{Irr}(\mathrm{SO}(n))$. We define an involutive map $^+ : \Gamma_M(\sigma) \to \Gamma_M(\bar{\sigma})$ by setting $x^+ := m_x^*(\Gamma_M(R)(1_\mathcal{B}))$, where $R : \mathbb{C} \to V_\sigma \otimes \bar{V}_\sigma$ is any intertwiner and $m_x : \Gamma_M(\bar{\sigma}) \to \Gamma_M(\sigma \otimes \bar{\sigma})$ denotes the map $m_x(y) := m(\sigma, \bar{\sigma})(x \otimes y)$. Now, for $x \otimes v \in \Gamma_M(\bar{\sigma}) \otimes V_\sigma$ we put $(x \otimes v)^+ := x^+ \otimes \bar{v}$ and extend this anilinearly to an involutive map on A_M. Third, we equip A_M with the $\mathrm{SO}(n)$-action by *-automorphisms, let's say a_M, given on each summand $\Gamma_M(\bar{\sigma}) \otimes V_\sigma$, $\sigma \in \mathrm{Irr}(\mathrm{SO}(n))$, by the respective unitary representation of $\mathrm{SO}(n)$ on the second tensor factor. In summary, we have built a *-algebra A_M together with an action of $\mathrm{SO}(n)$ on A_M by *-automorphisms.

We proceed by noting that each summand $\Gamma_M(\bar{\sigma}) \otimes V_\sigma$, $\sigma \in \mathrm{Irr}(\mathrm{SO}(n))$, is naturally a correspondence over $\mathcal{B}$ with respect to the canonical $\mathcal{B}$-bimodule structure and the $\mathcal{B}$-valued inner product determined by $\langle x \otimes v, y \otimes w\rangle_\mathcal{B} := \langle x, y\rangle_\mathcal{B}\langle v, w\rangle$ for all $x, y \in \Gamma_M(\bar{\sigma})$ and $v, w \in V_\sigma$. From this it follows that A_M carries the structure of a right pre-Hilbert $\mathcal{B}$-module. We write $\mathfrak{H}_M$ for its completion. It is easy to check that the left multiplication on A_M yields a faithful *-representation $\lambda : A_M \to \mathcal{L}(\mathfrak{H}_M)$. Furthermore, it is immediate that a_M extends to a unitary representation $U_M : \mathrm{SO}(n) \to \mathcal{U}(\mathfrak{H}_M)$.

Now, we are in a position to introduce the free C*-dynamical system

$$(\mathcal{A}_M, \mathrm{SO}(n), \alpha_M).$$

Indeed, we let $\mathcal{A}_M$ be the closure of $\lambda(A_M)$ with respect to the operator norm on $\mathcal{L}(\mathfrak{H}_M)$. Furthermore, α_M is implemented by the unitary representation U_M in the sense that $(\alpha_M)_g(x) = (U_M)_g x (U_M)_g^*$ for all $g \in \mathrm{SO}(n)$ and $x \in \mathcal{A}_M$. Finally, $(\mathcal{A}_M, \mathrm{SO}(n), \alpha_M)$ is free as asserted, because we initially started with a unitary tensor functor. For a more detailed account of the construction, we refer to [Nes13, Sec. 2] (see also [DR89, SW17b, Đur96]).

Appendix B. The special case of SO(2)

In this section, we briefly deal with the special case of SO(2). More precisely, for a unital C*-algebra $\mathcal{B}$ we show that there is a bijective correspondence between free C*-dynamical systems with structure group SO(2) and fixed point algebra $\mathcal{B}$ and Morita equivalence $\mathcal{B}$-bimodules.

We begin by recalling that $\operatorname{Irr}(\mathrm{SO}(2)) \cong \mathbb{Z}$. Now, let $(\mathcal{A}, \mathrm{SO}(2), \alpha)$ be a free C*-dynamical system with fixed point algebra $\mathcal{B}$. Each isotypic component $A(k)$, $k \in \mathbb{Z}$, is a Morita equivalence $\mathcal{B}$-bimodule with inner products given by ${}_{\mathcal{B}}\langle x, y\rangle := xy^*$ and $\langle x, y\rangle_{\mathcal{B}} := x^*y$ for all $x, y \in A(k)$. Furthermore, the canonical multiplication maps

$$\Psi_{k_1,k_2} : A(k_1) \otimes_{\mathcal{B}} A(k_2) \to A(k_1 + k_2), \qquad x \otimes_{\mathcal{B}} y \mapsto xy$$

are isomorphisms of Morita equivalence $\mathcal{B}$-bimodules for all $k_1, k_2 \in \mathbb{Z}$ and the following associativity condition holds for all $k_1, k_2, k_3 \in \mathbb{Z}$:

$$\Psi_{k_1+k_2,k_3}(\Psi_{k_1,k_2} \otimes_{\mathcal{B}} \mathrm{id}_{k_3}) = \Psi_{k_1,k_2+k_3}(\mathrm{id}_{k_1} \otimes_{\mathcal{B}} \Psi_{k_2,k_3}). \tag{6}$$

Note that $A(0) = \mathcal{B}$ and that, up to the canonical isomorphism $\mathcal{B} \otimes_{\mathcal{B}} \mathcal{B} \cong \mathcal{B}$, $\Psi_{0,0} = \mathrm{id}_{\mathcal{B}}$. The isotypic components $A(k)$, $k \in \mathbb{Z}$, along with the maps Ψ_{k_1,k_2}, $k_1, k_2 \in \mathbb{Z}$, constitute a so-called factor system and allows to reconstruct the C*-dynamical system $(\mathcal{A}, \mathrm{SO}(2), \alpha)$ up to isomorphism (see [SW17a, Def. 4.5]).

Conversely, let $\mathcal{B}$ be a unital C*-algebra and let N be a Morita equivalence $\mathcal{B}$-bimodule with dual module $\bar{N}$ and isomorphisms

$$\Psi_{1,-1} : N \otimes_{\mathcal{B}} \bar{N} \to \mathcal{B}, \qquad x \otimes \tilde{y} \mapsto {}_{\mathcal{B}}\langle x, y\rangle,$$

and

$$\Psi_{-1,1} : \bar{N} \otimes_{\mathcal{B}} N \to \mathcal{B}, \qquad \tilde{x} \otimes y \mapsto \langle x, y\rangle_{\mathcal{B}}.$$

Note that the Morita equivalence condition

$${}_{\mathcal{B}}\langle x, y\rangle . z = x . \langle y, z\rangle_{\mathcal{B}} \quad \text{for all} \quad x, y, z \in N$$

implies that $\Psi_{1,-1} \otimes_{\mathcal{B}} \mathrm{id}_N = \mathrm{id}_N \otimes_{\mathcal{B}} \Psi_{-1,1}$ and $\Psi_{-1,1} \otimes_{\mathcal{B}} \mathrm{id}_{\bar{N}} = \mathrm{id}_{\bar{N}} \otimes_{\mathcal{B}} \Psi_{1,-1}$. The task is now to construct a free C*-dynamical system $(\mathcal{A}_N, \mathrm{SO}(2), \alpha_N)$ with fixed point algebra $\mathcal{B}$ and $A_N(1) = N$. For this, we apply [SW17a, Thm. 4.21] which amounts to the construction of a factor system associated with N: First, for each $k \in \mathbb{Z}$ we form a Morita equivalence $\mathcal{B}$-bimodule $N(k)$ by setting

$$N(k) := \begin{cases} \mathcal{B} & k = 0 \\ N^{\otimes k} & k > 0 \\ \bar{N}^{\otimes -k} & k < 0. \end{cases}$$

Second, we define Morita equivalence $\mathcal{B}$-bimodule isomorphisms

$$\Psi_{k_1,k_2} : N(k_1) \otimes_{\mathcal{B}} N(k_2) \to N(k_1 + k_2)$$

for all $k_1, k_2 \in \mathbb{Z}$ in the following way: For non-negative integers we simply take the tensor product $\otimes_{\mathcal{B}}$, and similarly for negative integers. Note that $N(0) = \mathcal{B}$ and that, up to the canonical isomorphism $\mathcal{B} \otimes_{\mathcal{B}} \mathcal{B} \cong \mathcal{B}$, $\Psi_{0,0} = \mathrm{id}_{\mathcal{B}}$. To deal with integers of mixed parity, we repeatedly make use of the maps $\Psi_{1,-1}$ and $\Psi_{-1,1}$. From $\Psi_{1,-1} \otimes_{\mathcal{B}} \mathrm{id}_1 = \mathrm{id}_1 \otimes_{\mathcal{B}} \Psi_{-1,1}$ and $\Psi_{-1,1} \otimes_{\mathcal{B}} \mathrm{id}_{-1} = \mathrm{id}_{-1} \otimes_{\mathcal{B}} \Psi_{1,-1}$, it may be concluded that Equation (6) holds for all $k_1, k_2, k_3 \in \mathbb{Z}$. Hence the modules $N(k)$, $k \in \mathbb{Z}$, together with the maps Ψ_{k_1,k_2}, $k_1, k_2 \in \mathbb{Z}$, form a factor system which gives a free C*-dynamical system $(\mathcal{A}_N, \mathrm{SO}(2), \alpha_N)$ with fixed point algebra $\mathcal{B}$ and $A(1) = N$ as required. We proceed by looking at the maps

$$(\mathcal{A}, \mathrm{SO}(2), \alpha) \mapsto A(1), \qquad \text{and} \qquad N \mapsto (\mathcal{A}_N, \mathrm{SO}(2), \alpha_N).$$

A few moments thought show that these are, in fact, inverse to each other as claimed in the beginning of this section. Passing over to the set $\mathrm{Ext}(\mathcal{B}, \mathrm{SO}(2))$ of equivalence classes of free C*-dynamical systems with structure group SO(2) and fixed point algebra $\mathcal{B}$ (with respect to SO(2)-equivariant isomorphisms over $\mathcal{B}$) and the set $\mathrm{Pic}(\mathcal{B})$ of equivalence classes of Morita equivalence $\mathcal{B}$-bimodules, the Picard group of $\mathcal{B}$, we can assert that

$$\mathrm{Ext}(\mathcal{B}, \mathrm{SO}(2)) \to \mathrm{Pic}(\mathcal{B}), \qquad [(\mathcal{A}, \mathrm{SO}(2), \alpha)] \mapsto [A(1)].$$

is a bijection.

Remark B.1. The above correspondence can also be obtained from a more abstract result involving group cohomology. In fact, since the group cohomology $\mathrm{H}^n_{\mathrm{gr}}(\mathbb{Z}, \mathcal{U}Z(\mathcal{B}))$ vanishes for each $n > 1$, the correspondence follows from [SW17a, Cor. 5.9 & Thm. 5.14].

Remark B.2. If $\mathcal{B} = C(X)$ for some compact space X and N is the $C(X)$-module of sections of some line bundle L over X, then $\mathcal{A}_N \cong C(\mathrm{Fr}(L))$ (see, e. g., [SW17a, Sec. 6]).

Remark B.3. For a similar discussion in the purely algebraic setting of strongly graded rings we refer to the opus [BM20, Sec. 5.2.3].

Remark B.4. Woronowicz's quantum $\mathrm{SU}_q(2)$, equipped with its natural gauge action by U(1) (identified with SO(2)), provides a prominent example of a free C*-dynamical system with structure group U(1). It gives rise to the so-called quantum Hopf fibration. Moreover, the module associated with the trivial representation corresponds to the quantum Hopf line bundle.

We conclude by establishing a relation between free C*-dynamical systems with structure group SO(2) and their associated noncommutative vector bundles with respect to the standard representation of SO(2). For this, we recall that the standard representation of SO(2) is not irreducible. In fact, we have $\mathbb{C}^2 = \mathbb{C}(1, \imath)^\intercal \bigoplus \mathbb{C}(1, -\imath)^\intercal$ as SO(2)-modules.

Corollary B.5. *Let $(\mathcal{A}, \mathrm{SO}(2), \alpha)$ be a free C*-dynamical system. Furthermore, let π be the standard representation of* SO(2)*. Then $\Gamma_{\mathcal{A}}(\pi) = A(1) \oplus A(-1)$. In particular, for a unital C*-algebra $\mathcal{B}$ and a Morita equivalence $\mathcal{B}$-bimodule N with dual module $\bar{N}$ we have $\Gamma_{\mathcal{A}_N}(\pi) = N \bigoplus \bar{N}$.*

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References

[BCH17] BAUM, PAUL; DE COMMER, KENNY; HAJAC, PIOTR M. Free actions of compact quantum groups on unital C^*-algebras. *Doc. Math.* **22** (2017), 825–849. MR3665403, Zbl 1386.46055, arXiv:1304.2812, doi: 10.4171/DM/579. 838, 841

[BM20] BEGGS, EDWIN J.; MAJID, SHAHN. Quantum Riemannian geometry. Grundlehren der mathematischen Wissenschaften, 355. *Springer, Cham*, 2020. xvi+809 pp. ISBN: 978-3-030-30293-1; 978-3-030-30294-8. MR4292536, Zbl 1436.53001, doi: 10.1007/978-3-030-30294-8. 838, 853

[Bla06] BLACKADAR, BRUCE E. Operator algebras. Theory of C^*-algebras and von Neumann algebras. Encyclopaedia of Mathematical Sciences, 122. Operator Algebras and Non-commutative Geometry, III. *Springer-Verlag, Berlin*, 2006. xx+517 pp. ISBN: 978-3-540-28486-4; 3-540-28486-9. MRMR2188261, Zbl 1092.46003,doi: 10.1007/3-540-28517-2. 840

[BtD85] BRÖCKER, THEODOR; TOM DIECK, TAMMO. Representations of compact Lie groups. Graduate Texts in Mathematics, 98. *Springer-Verlag, New York*, New York, 1995.x+313 pp. ISBN:0-387-13678-9. MR1410059, Zbl 0874.22001, doi: 10.1007/978-3-662-12918-0. 840

[Cun77] CUNTZ, JOACHIM. Simple C^*-algebras generated by isometries. *Comm. Math. Phys.* **57** (1977), no. 2, 173–185. MR0467330, Zbl 0399.46045, doi: 10.1007/BF01625776. 850

[CY13] DE COMMER, KENNY; YAMASHITA, MAKOTO. A construction of finite index C^*-algebra inclusions from free actions of compact quantum groups. *Publ. Res. Inst. Math. Sci.* **49** (2013), no. 4, 709–735. MR3141721, Zbl 1338.46077, arXiv:1201.4022,doi: 10.4171/PRIMS/117. 842, 848

[DR89] DOPLICHER, SERGIO; ROBERTS, JOHN E. Endomorphisms of C^*-algebras, cross products and duality for compact groups. *Ann. of Math. (2)* **130** (1989), no. 1, 75–119. MR1005608, Zbl 0702.46044, doi: 10.2307/1971477. 847, 851

[Đur96] ĐURĐEVIĆ, MIĆO. Quantum principal bundles and Tannaka—Krein duality theory. *Rep. Math. Phys.* **38** (1996), no. 3, 313–324. MR1437631, Zbl 0886.58006, arXiv:q-alg/9507018, doi: 10.1016/S0034-4877(97)84884-7. 842, 847, 851

[EKQR06] ECHTERHOFF, SIEGFRIED; KALISZEWSKI, STEVEN P.; QUIGG, JOHN; RAEBURN, IAIN. A categorical approach to imprimitivity theorems for C^*-dynamical systems. *Mem. Amer. Math. Soc.* **180** (2006), no. 850, viii+169 pp. MR2203930, Zbl 1097.46042, arXiv:math/0205322, doi: 10.1090/memo/0850. 840

[Ell00] ELLWOOD, DAVID A. A new characterisation of principal actions. *J. Funct. Anal.* **173** (2000), no. 1, 49–60. MR1760277, Zbl 0960.46049, doi: 10.1006/jfan.2000.3561. 838, 841

[EGNO15] ETINGOF, PAVEL; GELAKI, SHLOMO; NIKSHYCH, DMITRI; OSTRIK, VICTOR. Tensor categories. Mathematical Surveys and Monographs, 205. *American Mathematical Society, Providence, RI,* 2015. xvi+343 pp. ISBN:978-1-4704-2024-6. MR3242743, Zbl 1365.18001, doi: 10.1090/surv/205. 840

[Gab14] GABRIEL, OLIVIER. Fixed points of compact quantum groups actions on Cuntz algebras. *Ann. Henri Poincaré* **15** (2014), no. 5, 1013–1036. MR3192657, Zbl 1288.81058, arXiv:1210.5630, doi: 10.1007/s00023-013-0265-5. 850

[GW09] GOODMAN, ROE; WALLACH, NOLAN R. Symmetry, representations, and invariants. Graduate Texts in Mathematics, 255. *Springer, Dordrecht*, 2009. xx+716 pp. ISBN:978-0-387-79851-6. MR2522486, Zbl 1173.22001, doi: 10.1007/978-0-387-79852-3. 840

[HM13] HOFMANN, KARL H.; MORRIS, SIDNEY A. The structure of compact groups – a primer for the student – a handbook for the expert. Fifth edition. De Gruyter Studies in Mathematics, 25. *De Gruyter, Berlin*, 2023. xli+1032 pp. ISBN: 978-3-11-117260-6; 978-3-11-117163-0; 978-3-11-117405-1. MR4687187, Zbl 1277.22001, doi: 10.1515/9783110695991. 841

[Hus94] HUSEMÖLLER, DALE. Fibre bundles. Third edition. Graduate Texts in Mathematics, 20. *Springer-Verlag, New York*, 1994. xx+353 pp. ISBN: 0-387-94087-1. MR1249482, Zbl 0794.55001, doi: 10.1007/978-1-4757-2261-1. 840

[KN96] KOBAYASHI, SHOSHICHI; NOMIZU, KATSUMI. Foundations of differential geometry. I. Wiley Classics Library. A Wiley-Interscience Publication. *John Wiley & Sons, Inc., New York*, 1996. xii+329 pp. ISBN: 0-471-15733-3. MR1393940, Zbl 0119.37502. 840

[LS05] LANDI, GIOVANNI; VAN SUIJLEKOM, WALTER. Principal fibrations from noncommutative spheres. *Comm. Math. Phys.* **260** (2005), no. 1, 203–225. MR2175995, Zbl 1093.58004, arXiv:math/0410077, doi: 10.1007/s00220-005-1377-7. 849

[Maj99] MAJID, SHAHN. Quantum and braided group Riemannian geometry. *J. Geom. Phys.* **30** (1999), no. 2, 113–146. MR1690065, Zbl 0940.58004, arXiv:q-alg/9709025, doi: 10.1016/S0393-0440(98)00051-5. 838

[Maj02] MAJID, SHAHN. Riemannian geometry of quantum groups and finite groups with nonuniversal differentials. *Comm. Math. Phys.* **225** (2002), no. 1, 131–170. MR1877313, Zbl 0999.58004, arXiv:math/0006150, doi: 10.1007/s002201000564. 838

[Maj05] MAJID, SHAHN. Noncommutative Riemannian and spin geometry of the standard q-sphere. *Comm. Math. Phys.* **256** (2005), no. 2, 255–285. MR2160795, Zbl 1075.58004, arXiv:math/0307351, doi: 10.1007/s00220-005-1295-8. 838

[Mit65] MITCHELL, BARRY. Theory of categories. Pure and Applied Mathematics, vol. XVII. *Academic Press, New York-London*, 1965. xi+273 pp. MR0202787, Zbl 0136.00604. 840

[Nes13] NESHVEYEV, SERGEY. Duality theory for nonergodic actions. *Münster J. Math.* **7** (2014), no. 2, 413–437. MR3426224, Zbl 1336.46057, arXiv:1303.6207, doi: 10.17879/58269764511. 841, 842, 847, 848, 851

[NT13] NESHVEYEV, SERGEY; TUSET, LARS. Compact quantum groups and their representation categories. Cours Spécialisés, 20. *Société Mathématique de France, Paris*, 2013. vi+169 pp. ISBN: 978-2-85629-777-3. MR3204665, Zbl 1316.46003. 840

[Ped18] PEDERSEN, GERT K. C^*-algebras and their automorphism groups. Second edition. Pure and Applied Mathematics (Amsterdam). *Academic Press, London*, 2018. xviii+520 pp. ISBN: 978-0-12-814122-9. MR3839621, Zbl 1460.46001, doi: 10.1016/C2016-0-03431-9. 840

[Phi87] PHILLIPS, N. CHRISTOPHER. Equivariant K-theory and freeness of group actions on C^*-algebras. Lecture Notes in Mathematics, 1274. *Springer-Verlag, Berlin*, 1987. viii+371 pp. ISBN:3-540-18277-2. MR0911880, Zbl 0632.46062, doi: 10.1007/BFb0078657. 841

[Rie91] RIEFFEL, MARC A. Proper actions of groups on C^*-algebras. *Mappings of operator algebras* (Philadelphia, PA, 1988), 141–182. Progr. Math., 284. *Birkhäuser Boston, Boston, MA*, 1990. MR1103376, Zbl 0749.22003. 841

[RW98] RAEBURN, IAIN; WILLIAMS, DANA P. Morita equivalence and continuous-trace C^*-algebras. Mathematical Surveys and Monographs, 60. *American Mathematical*

Society, Providence, RI, 1998. xiv+327 pp. ISBN: 0-8218-0860-5. MR1634408, Zbl 0922.46050, doi: 10.1090/surv/060. 840

[SW17a] SCHWIEGER, KAY; WAGNER, STEFAN. Part I, free actions of compact abelian groups on C^*-algebras. *Adv. Math.* **317** (2017), 224–266. MR3682668, Zbl 1382.46055, arXiv:1505.00688, doi: 10.1016/j.aim.2017.06.036. 838, 850, 852, 853

[SW17b] SCHWIEGER, KAY; WAGNER, STEFAN. Part III, free actions of compact quantum groups on C^*-algebras. *SIGMA* Symmetry Integrability Geom. Methods Appl. **13** (2017), Paper 062, 19 pp. MR3683928, Zbl 1390.46065, arXiv:1701.05895, doi: 10.3842/SIGMA.2017.062. 838, 842, 847, 849, 851

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