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Isomorphism and stable isomorphism in “real” and “quaternionic” K-theory

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ABSTRACT. We find lower bounds on the rank of a “real” vector bundle over an involutive space, such that “real” vector bundles of higher rank have a trivial summand and such that a stable isomorphism for such bundles implies ordinary isomorphism. We prove similar lower bounds also for “quaternionic” bundles. These estimates have consequences for the classification of topological insulators with time-reversal symmetry.

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1. Introduction

Topological insulators are materials that are insulators such that some special topology enforces the existence of conducting states on their boundaries. These conducting boundary states tend to be very robust under disorder. See, for instance, [6] for a survey on this subject from the perspective of non-commutative geometry and index theory. It is common to model such materials in the one-particle approximation. The physical system is then described through a C^*-algebra A of observables and an invertible, self-adjoint element $H \in A$, the Hamiltonian. Two such systems with the same C^*-algebra A are considered in the same topological phase if there is a homotopy of invertible self-adjoint elements in A between their Hamiltonians. Up to homotopy, we may replace

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self-adjoint invertible elements by self-adjoint unitaries. By a linear transformation, these may be replaced by projections in A. So the set of possible topological phases of the system described by A is the set of homotopy classes of projections in A.

A projection in A has a K-theory class in $\mathrm{K}_0(A)$. However, if two projections have the same class in $\mathrm{K}_0(A)$, then they are only stably homotopic, that is, they become homotopic after adding the same projection to them both. Since K-theory is easier to compute than sets of homotopy classes of projections, it is important to know situations where stable homotopy implies homotopy.

If disorder in the system is neglegcted, then the observable algebra becomes isomorphic to a matrix algebra over the algebra of continuous functions on the d-torus $\mathbb{T}^d$, where d is the dimension of the material. Many interesting topological materials only exhibit a nontrivial topological phase when we require extra symmetries that are anti-unitary or that anticommute with the Hamiltonian. A particularly important case is a time-reversal symmetry. This is actually two cases because the symmetry may have square $+1$ or -1. It is explained in [4, 5] how topological phases in these two symmetry types are classified using “real” and “quaternionic” vector bundles over tori with involution. We briefly sketch this here.

The torus $\mathbb{T}^d$ above appears through the Fourier transform. The relevant observable algebra is the group C*-algebra of $\mathbb{Z}^d$, tensored by a matrix algebra. Here the time-reversal symmetry acts by entrywise complex conjugation combined with conjugation by a suitable scalar matrix Θ. Under Fourier transform, this becomes the real C*-algebra

$$\left\{f\colon \mathbb{T}^d \to \mathbb{M}_n(\mathbb{C}) \;\middle|\; f(\overline{z}) = \Theta\overline{f(z)}\Theta^{-1} \text{ for all } z \in \mathbb{T}^d\right\}.$$

The conjugation map $z \mapsto \overline{z}$ on $\mathbb{T}^d$ is an involution and generates an action of the group $\mathbb{Z}/2$. Projections in matrix algebras over the real group C*-algebra of $\mathbb{Z}^d$ correspond by the Serre–Swan Theorem to “real” vector bundles over $\mathbb{T}^d$ as defined by Atiyah [1]. That is, they carry a map on their total space that lifts the involution on the base space, is fibrewise conjugate-linear, and squares to the identity map. For a time reversal symmetry with square -1, we instead need the map on the total space to square to the map of multiplication by -1, and complex bundles with this kind of extra structure are called “quaternionic”.

Thus it is physically interesting to know sufficient criteria for two “real” or “quaternionic” vector bundles over $\mathbb{T}^d$ that are stably isomorphic to be isomorphic. Without the extra “real” or “quaternionic” structure map, such criteria are well known: for each d there is an explicit $k(d) \in \mathbb{N}$ such that two complex vector bundles of rank at least $k(d)$ over a space of covering dimension d are isomorphic once they are stably isomorphic (see [2, Theorem 1.5 in Chapter 9]). Similar results are known for real and quaternionic vector bundles, but the more general cases of “real” and “quaternionic” vector bundles have not yet been treated. This will be done here.

A related result says that any "real" vector bundle of sufficiently high rank is a direct sum of a trivial "real" vector bundle of rank 1 and another "real" vector bundle. In fact, a relative version of this result, saying that a trivial direct summand on a subspace may be extended to one on the whole space, implies that stable isomorphism and isomorphism are equivalent for bundles of sufficiently high rank. However, to prove that stable isomorphism and isomorphism are equivalent for certain bundles over a space X, we need the statement for the space $X \times [0,1]$, relative to the subspace $X \times \{0,1\}$.

Some results about trivial direct summands in vector bundles of sufficiently high rank are already proven in [4, Theorem 4.25] for "real" vector bundles and [5, Theorem 2.5] for "quaternionic" vector bundles. However, these statements assume that the set of fixed points of the "real" involution on the underlying space is discrete, so that they never apply to a space of the form $X \times [0,1]$. Thus they cannot help to relate stable isomorphism to isomorphism. Therefore, our main task is to remove this restriction from the results in [4, 5].

As in [4, 5], we work with $\mathbb{Z}/2$-CW-complexes. The key proof technique is induction over cells, extending certain equivariant maps already defined on the boundary of a $\mathbb{Z}/2$-cell to the interior. Any smooth manifold with a smooth $\mathbb{Z}/2$-action carries an equivariant triangulation by [3], and this implies immediately that it may be turned into a $\mathbb{Z}/2$-CW-complex. We now formulate our main results.

Theorem 1.1. *Let $d_1, d_0, k \in \mathbb{N}$. Let X be a $\mathbb{Z}/2$-CW-complex. Assume that the free cells in (X, A) have at most dimension d_1 and that the trivial cells have at most dimension d_0. Let*

$$k_0 := \max\left\{d_0, \left\lceil \frac{d_1 - 1}{2} \right\rceil\right\}, \qquad k_1 := \max\left\{d_0 + 1, \left\lceil \frac{d_1}{2} \right\rceil\right\}.$$

(1) *Let E be a "real" vector bundle over X of rank $k \geq k_0$. There is an isomorphism $E \cong E_0 \oplus (X \times \mathbb{C}^{k-k_0})$ for some "real" vector bundle E_0 over X and the trivial "real" vector bundle $X \times \mathbb{C}^{k-k_0}$ of rank $k - k_0$.*
(2) *Let E_1 and E_2 be two "real" vector bundles over X of rank $k \geq k_1$. If E_1 and E_2 are stably isomorphic, that is, $E_1 \oplus E_3 \cong E_2 \oplus E_3$ for some "real" vector bundle E_3, then they are isomorphic.*

Theorem 1.2. *Let $d_1, d_0, k \in \mathbb{N}$. Let X be a $\mathbb{Z}/2$-CW-complex. Assume that the free cells in (X, A) have at most dimension d_1 and that the trivial cells have at most dimension d_0. Let*

$$k_0 := \max\left\{\left\lceil \frac{d_0 - 3}{2} \right\rceil, \left\lceil \frac{d_1 - 1}{2} \right\rceil\right\}, \qquad k_1 := \max\left\{\left\lceil \frac{d_0 - 2}{2} \right\rceil, \left\lceil \frac{d_1}{2} \right\rceil\right\}.$$

(1) *Let E be a "quaternionic" vector bundle over X of rank $k \geq k_0$. There is an isomorphism $E \cong E_0 \oplus \theta_X^{2\lfloor (k-k_0)/2 \rfloor}$ for some "quaternionic" vector bundle E_0 over X and the trivial "quaternionic" vector bundle $\theta_X^{2\lfloor (k-k_0)/2 \rfloor}$ of rank $2\lfloor (k - k_0)/2 \rfloor$.*

(2) *Let E_1 and E_2 be two “quaternionic” vector bundles over X of rank $k \geq k_1$. If E_1 and E_2 are stably isomorphic, that is, $E_1 \oplus E_3 \cong E_2 \oplus E_3$ for some “quaternionic” vector bundle E_3, then they are isomorphic.*

Proposition 1.3. *Two “real” or “quaternionic” vector bundles over $\mathbb{T}^d$ with $d \leq 4$ are already isomorphic once they are stably isomorphic.*

Proof. Here we are dealing with the $\mathbb{Z}/2$-CW-complex $\mathbb{T}^d$, which has $d_0 = 0$ and $d_1 = d \leq 4$. This gives $k_1 \leq 2$ both in Theorem 1.1 and in Theorem 1.2. So the assertion holds for “real” and “quaternionic” bundles of rank at least 2. All rank-zero bundles are trivial. So it only remains to prove the statement for bundles of rank one. All rank-one “real” bundles over $\mathbb{T}^d$ for all $d \in \mathbb{N}$ are trivial by [4, Proposition 5.3]. Since the torus has τ-fixed points, any “quaternionic” bundle over $\mathbb{T}^d$ has even rank (see [5]), so there are no “quaternionic” vector bundles of rank one. □

Remark 1.4. Consider the stabilisation map $[E] \mapsto [E \oplus \theta_X^{m-k}]$ between the sets of “real” or “quaternionic” vector bundles of rank k and m for $m \geq k$; in the “quaternionic” case, this only works if $m - k$ is even. Theorems 1.1 and 1.2 say that this map is injective for $k \geq k_1$ and surjective for $k \geq k_0$. So it is bijective for $k \geq \max\{k_0, k_1\} = k_1$. For instance, if $d_0, d_1 \leq 4$, this happens if $k \geq 2$. So in these low dimensions, it is no loss of generality to restrict attention to “real” and “complex” vector bundles of rank at most 2. This is claimed in [5, Corollary 2.1] for $d_0 = 0$ and $d_1 = 5$ as well. We can only confirm the surjectivity of the map in this case, however, and the injectivity of the map is not addressed in [5]. If $d_0 = 0$, then the threshold k_0 for the map above to be surjective is the same as the threshold $\lfloor d/2 \rfloor$ in the “real” case in [5, Theorem 4.25] and the threshold $2\lfloor (d+2)/4 \rfloor$ in the “quaternionic” even rank case in [5, Theorem 2.5].

In the body of the paper, we state and prove generalisations of Theorems 1.1 and 1.2 for relative $\mathbb{Z}/2$-CW-complexes, which allow to extend a given direct sum decomposition on a subspace. We need the relative versions of the first statements in Theorems 1.1 and 1.2 to prove the second statements. Section 2 contains our results and some notation in the “real” case, and Section 3 treats the “quaternionic” case. Finally, in Section 4, we explain how our results imply statements about stable conjugacy and conjugacy of projections in the physical observable C*-algebra.

An *involutive space* (X, τ) is a topological space X with a continuous involution $\tau : X \to X$, that is, $\tau^2 = \mathrm{id}_X$. Throughout this article, let (X, A) be a relative $\mathbb{Z}/2$-CW-complex, that is, $A \subseteq X$ is a closed τ-invariant subspace and X is gotten from A by attaching $\mathbb{Z}/2$-cells of increasing dimensions. There are two different types of $\mathbb{Z}/2$-cells, namely, the *free cells* $\mathbb{D}^j \times \mathbb{Z}/2$ with the generator of $\mathbb{Z}/2$ acting by $\tau(x, j) := (x, j+1)$ and the *fixed cells* $\mathbb{D}^j$ with the trivial $\mathbb{Z}/2$-action. Let d_0 be the supremum of the dimensions of the fixed cells, which is the dimension of $X^\tau \setminus A$. Let d_1 be the supremum of the dimensions of the free cells, which is the dimension of $X \setminus (X^\tau \cup A)$. Our results only work if $d_0, d_1 < \infty$.

2. “Real” vector bundles

This section proves our main result for “real” vector bundles. Even more, we state and prove a relative version over relative $\mathbb{Z}/2$-CW-complexes.

Definition 2.1 ([1,4]). A *“real” vector bundle* over an involutive space (X, τ) is a complex vector bundle $\pi\colon E \to X$ together with a homeomorphism $\Theta\colon E \to E$ such that

(1) $\pi\circ\Theta = \tau\circ\pi$;
(2) Θ is fibrewise additive and $\Theta(\lambda p) = \overline{\lambda} p$ for all $\lambda \in \mathbb{C}$ and $p \in E$, where $\overline{\lambda}$ is the complex conjugate of λ;
(3) $\Theta^2 = \mathrm{id}_E$.

The bundle has *rank* k if all its fibres are isomorphic to $\mathbb{C}^k$.

The *trivial “real” vector bundle* of rank k over X is $X \times \mathbb{C}^k$ with the obvious, trivial $\mathbb{C}$-vector bundle structure and $\Theta(x, v) := (\tau(x), \overline{v})$. It is denoted by θ^k_X.

Proposition 2.2 ([4]). *Let X be a space and let $k \in \mathbb{N}$. Then any “real” vector bundle over (X, id_X) is the complexification of an ordinary real vector bundle.*

Therefore, for any involutive space (X, τ), the restriction E on the subset $X^\tau \subseteq X$ of τ-fixed points is a complexification of a real vector bundle, namely,

$$E|_{X^\tau} \cong E^\Theta \otimes_{\mathbb{R}} \mathbb{C},$$

where E^Θ is the set of fixed points of Θ, which is an $\mathbb{R}$-vector bundle over X^τ.

Let $\mathbb{F}$ denote $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and let $c = \dim_{\mathbb{R}} \mathbb{F}$. Recall a classical result:

Proposition 2.3 ([2, Chapter 9, Proposition 1.1]). *Let ξ^k be a k-dimensional $\mathbb{F}$-vector bundle over a CW-complex X with $d \le ck - 1$. Then ξ is isomorphic to $\eta \oplus (X \times \mathbb{F})$ for some $\mathbb{F}$-vector bundle η over X.*

The key ingredient in the proof is [2, Theorem 7.1 in Chapter 2], which allows to extend sections of fibre bundles under a higher connectedness assumption. This proof technique provides a relative version of the proposition for a relative CW-complex (X, A), which shows that a given direct sum decomposition on A extends to X.

We now formulate and prove a relative version of Theorem 1.1.(1), which generalises the relative version of Proposition 2.3 for $\mathbb{F} = \mathbb{R}$ to “real” bundles. Our proof follows the proof of [2, Theorem 1.2] and [4, Proposition 4.23]. Our main task is to remove the extra assumption in the latter result that fixed point cells are only of dimension 0.

Theorem 2.4. *Let $d_1, d_0, k \in \mathbb{N}$. Let (X, A) be a relative $\mathbb{Z}/2$-CW-complex. Assume that the free cells in (X, A) have at most dimension d_1 and that the trivial cells have at most dimension d_0. Let*

$$k_0 := \max\left\{\left\lceil \frac{d_1 - 1}{2} \right\rceil, d_0\right\}.$$

Let E be a “real” vector bundle over X of rank $k \geq k_0$. Let an isomorphism $E|_A \cong E_0^A \oplus \theta_A^{k-k_0}$ for some “real” vector bundle E_0^A over A be given. This extends to an isomorphism $E \cong E_0 \oplus \theta_X^{k-k_0}$ for some “real” vector bundle E_0 over X.

Proof. We are going to extend an isomorphism $E|_A \cong E_0^A \oplus \theta_A^1$ to an isomorphism $E \cong E_0 \oplus \theta_X^1$, assuming $k > k_0$. Repeating this step $k - k_0$ times then gives the result that is stated. An isomorphism $E \cong E_0 \oplus \theta_X^1$ contains an injective “real” vector bundle map $\theta_X^1 \hookrightarrow E$. Conversely, such an embedding implies an isomorphism $E \cong E_0 \oplus \theta_X^1$ because any “real” vector subbundle has an orthogonal complement, which is again a “real” vector subbundle, and then the direct sum is isomorphic to the whole bundle (see [4]). An injective “real” vector bundle map $\theta_X^1 \hookrightarrow E$ is equivalent to a section $s\colon X \to E$ that satisfies $s(x) \neq 0$ and $\Theta(s(x)) = s(\tau(x))$ for all $x \in X$: then we map $\theta_X^1 = X \times \mathbb{C}$ to E by $(x, \lambda) \mapsto \lambda \cdot s(x)$. We call the section $\mathbb{Z}/2$-equivariant if $\Theta(s(x)) = s(\tau(x))$ for all $x \in X$. Let $E^\times \subset E$ be the subbundle of nonzero vectors. Our task is to extend a $\mathbb{Z}/2$-equivariant section $A \to E^\times|_A$ to a $\mathbb{Z}/2$-equivariant section $X \to E^\times$. The fibres of $E^\times$ are $(\mathbb{C}^k)^\times = \mathbb{C}^k \setminus \{0\}$. This is homotopy equivalent to the sphere $\mathbb{S}^{2k-1}$, which is $2k-2$-connected.

First, we construct our section on $A \cup X^\tau$. This is equivalent to extending a given section on A^τ to a section $X^\tau \to (E^\Theta)^\times$ (compare Proposition 2.2). The $\mathbb{R}$-vector bundle $E^\Theta \twoheadrightarrow X^\tau$ has dimension k. The proof of Proposition 2.3 also allows to extend a section that is given on a closed subspace. The assumption $d_0 \leq k_0 \leq k-1$ ensures that the section that is given on A^τ may be extended to a section $X^\tau \to (E^\Theta)^\times$. Together with the given section on A, we get the desired $\mathbb{Z}/2$-equivariant section of $E^\times$ on $A \cup X^\tau$.

We prolong this section to all of X by induction over skeleta. Since the cells of the same dimension are disjoint, we may work one $\mathbb{Z}/2$-cell at a time. Since we have already found the section on X^τ, we only encounter free cells of the form $\mathbb{Z}/2 \times \mathbb{D}^j$, and $j \leq d_1$ by assumption. We are given a $\mathbb{Z}/2$-equivariant section on the boundary $\mathbb{Z}/2 \times \partial\mathbb{D}^j$, which we have to extend to $\mathbb{Z}/2 \times \mathbb{D}^j$. The involution τ flips the two copies of $\mathbb{D}^j$ in $\mathbb{Z}/2 \times \partial\mathbb{D}^j$. So it suffices to construct the section $s\colon \{+1\} \times \mathbb{D}^j \to E^\times$ and then define $s(-1, x) := \Theta(s(1, x))$. This is automatically $\mathbb{Z}/2$-equivariant. The restriction of the bundle E to $\{+1\} \times \mathbb{D}^j$ is trivial because $\mathbb{D}^j$ is contractible. So our task becomes equivalent to extending a map $\partial\mathbb{D}^j \to (\mathbb{C}^k)^\times$ to a map $\mathbb{D}^j \to (\mathbb{C}^k)^\times$. This is possible because $j \leq d_1 \leq 2k_0 + 1 \leq 2k - 1$. □

Theorem 2.5. *Let $d_1, d_0, k \in \mathbb{N}$. Let (X, A) be a relative $\mathbb{Z}/2$-CW-complex. Assume that the free cells in (X, A) have at most dimension d_1 and that the trivial cells have at most dimension d_0. Let*

$$k_1 := \max\left\{\left\lceil \frac{d_1}{2} \right\rceil, d_0 + 1\right\}.$$

Let E_1 and E_2 be two “real” vector bundles over X of rank $k \geq k_1$. An isomorphism $E_1|_A \cong E_2|_A$ that extends to a stable isomorphism between E_1 and E_2 on X extends to an isomorphism $E_1 \cong E_2$.

Proof. Let $\varphi_A : E_1|_A \cong E_2|_A$ be the given isomorphism on A. Any “real” vector bundle over a finite-dimensional $\mathbb{Z}/2$-CW-complex is a direct summand in a trivial “real” bundle. Therefore, our stable isomorphism assumption implies that there is an isomorphism $\psi : E_1 \oplus \theta_X^\ell \cong E_2 \oplus \theta_X^\ell$ for some $\ell \geq 0$ such that $\psi|_A$ is $\varphi_A \oplus \mathrm{id}_{\theta_X^\ell}$. There is nothing to do if $\ell = 0$. We are going to prove that there is an isomorphism $E_1 \oplus \theta_X^{\ell-1} \cong E_2 \oplus \theta_X^{\ell-1}$ that extends $\varphi_A \oplus \mathrm{id}_{\theta_X^{\ell-1}}$. Repeating this step ℓ times gives the result we need. Replacing E_j by $E_j \oplus \theta_X^{\ell-1}$, we reduce to the case where $\ell = 1$. Thus we may assume an isomorphism $\psi : E_1 \oplus \theta_X^1 \cong E_2 \oplus \theta_X^1$ in the following.

We want to reduce the proof to Theorem 2.4, as in the proof of [2, Theorem 1.5 in Chapter 9]. We work on $Y = X \times I$ with $I = [0, 1]$, equipped with the involution $\tau(x, t) := (\tau(x), t)$. We let E be the pullback of $E_1 \oplus \theta_X^1$ to Y. The relevant dimensions and ranks are now

$$d_1(Y) = d_1(X) + 1, \qquad d_0(Y) = d_0(X) + 1, \qquad \mathrm{rank}(E) = k + 1.$$

We identify the restriction of E to $B := X \times \partial I \cup A \times I \subseteq Y$ with a “real” vector bundle of the form $E_0 \oplus \theta_B^1$. Here we glue the identity isomorphism from E to the pull back of $E_1 \oplus \theta_X^1$ on $X \times \{0\} \cup A \times [0, 1]$ and the isomorphism $E|_{X\times\{1\}} = E_1 \oplus \theta_X^1 \cong E_2 \oplus \theta_X^1$ on $X \times \{1\}$; we may glue this on $A \times \{1\}$ because the isomorphism ψ is of the form $\varphi_A \oplus \mathrm{id}_{\theta_A^1}$ on A. Now Theorem 2.4 provides an isomorphism $E \cong E_0 \oplus \theta_X^1$ on all of Y extending the given isomorphism on $B \subseteq Y$. By construction, the bundle E_0 on Y restricts to E_1 on $X \times \{0\} \cup A \times [0, 1]$ and to E_2 on $X \times \{1\}$, glued together using the given isomorphism φ_A. Now, as in the proof in [2], the existence of such a “real” vector bundle over Y implies that there is an isomorphism of “real” vector bundles $E_1 \cong E_2$ that restricts to the given isomorphism on A. □

3. “Quaternionic” vector bundles

The goal of this section is to prove a relative version of Theorem 1.2.

Definition 3.1 ([5]). A *“quaternionic” bundle* over the involutive space (X, τ) is a complex vector bundle $\pi : E \to X$ together with a homeomorphism $\Theta : E \to E$ such that

- $\pi \circ \Theta = \tau \circ \pi$;
- Θ is fibrewise additive and $\Theta(\lambda p) = \overline{\lambda} p$ for all $\lambda \in \mathbb{C}$ and $p \in E$;
- $\Theta^2(x, v) = (x, -v)$ is fibrewise multiplication by -1 for all $x \in X$, $v \in E_x$.

The bundle has *rank k* if all its fibres are isomorphic to $\mathbb{C}^k$.

The name “quaternionic” vector bundles is justified by the following:

Proposition 3.2 ([5]). *A "quaternionic" vector bundle over (X, id_X) is equivalent to an $\mathbb{H}$-vector bundle, where $a + bi + cj + dk$ acts by $(a + bi) + (c + di)\Theta$ on each fibre. The rank as a "quaternionic" vector bundle is twice the rank as an $\mathbb{H}$-vector bundle because $\mathbb{H}^k = \mathbb{C}^{2k}$.*

In particular, the restriction of a "quaternionic" vector bundle to X^τ must have even rank. If X is connected and $X^\tau \neq \emptyset$, then this implies that the rank is even on all of X. Nevertheless, "quaternionic" vector bundles of odd rank are possible if $X^\tau = \emptyset$. All trivial "quaternionic" bundles have even rank. Namely, the trivial "quaternionic" vector bundle θ_X^{2k} over (X, τ) of rank $2k$ is the space $X \times \mathbb{C}^{2k}$ with

$$\Theta(x, \lambda_1, \lambda_2, \dots, \lambda_{2k-1}, \lambda_{2k}) := (\tau(x), \overline{\lambda_2}, -\overline{\lambda_1}, \dots, \overline{\lambda_{2k}}, -\overline{\lambda_{2k-1}}).$$

A vector bundle map $f : \theta_X^1 \to E$ is of the form $f(x, \lambda_1, \lambda_2) = \lambda_1 s_1(x) + \lambda_2 s_2(x)$ for two sections s_1, s_2 of E. This map is $\mathbb{Z}/2$-equivariant if and only if $s_2(x) = -\Theta(s_1(\tau(x)))$ for all $x \in X$. Thus the section s_1 already determines f if it is $\mathbb{Z}/2$-equivariant. Of course, f is injective if and only if $s_1(x)$ and $s_2(x)$ are linearly independent for all $x \in X$. If $\tau(x) = x$, this is true once $s_1(x) \neq 0$ because then the restriction of f to the fibre at x is an $\mathbb{H}$-linear map $\mathbb{H} \to E_x$. If, however, $x \neq \tau(x)$, then $s_1(x) \neq 0$ is not sufficient. We must ensure that $s_2(x) = -\Theta(s_1(\tau(x)))$ is linearly independent of $s_1(x)$ as well. Here a mistake is made in [5]: their argument above Definition 2.1 why $s_1(x) \neq 0$ should suffice for $s_1(x)$ and $s_2(x)$ to be linearly independent is wrong because it only implies that the functions s_1 and s_2 are linearly independent, which is much weaker; so their proofs of Propositions 2.4 and 2.7 are incomplete. It is easy to fix the proof of their Proposition 2.4, and our theorem above may replace their Proposition 2.7.

Theorem 3.3. *Let $d_1, d_0, k \in \mathbb{N}$. Let (X, A) be a relative $\mathbb{Z}/2$-CW-complex. Assume that the free cells in (X, A) have at most dimension d_1 and that the trivial cells have at most dimension d_0. Let*

$$k_0 := \max\left\{\left\lceil \frac{d_0 - 3}{2} \right\rceil, \left\lceil \frac{d_1 - 1}{2} \right\rceil\right\}.$$

Let E be a "quaternionic" vector bundle over X of rank $k \geq k_0$. Assume an isomorphism $E|_A \cong E_0^A \oplus \theta_A^{2\lfloor (k-k_0)/2 \rfloor}$ for some "quaternionic" vector bundle E_0^A over A is given. This isomorphism extends to an isomorphism $E \cong E_0 \oplus \theta_X^{2\lfloor (k-k_0)/2 \rfloor}$ for some "quaternionic" vector bundle E_0 over X.

Proof. We are going to prove that any injective $\mathbb{Z}/2$-equivariant vector bundle map $f_A : \theta_A^2 \to E|_A$ extends to an injective $\mathbb{Z}/2$-equivariant vector bundle map $f : \theta_X^2 \to E$ if $k \geq k_0 + 2$. This implies the statement as in the proof of Theorem 2.4.

We first construct f on the subset $X^\tau \cup A$. This is equivalent to extending $f_A|_{A^\tau}$ from A^τ to X^τ. In this part of the proof, we may assume without loss of generality that $X^\tau \neq \emptyset$. This forces k to be even. Since the involution acts trivially on X^τ, the "quaternionic" bundle E of rank k becomes an $\mathbb{H}$-vector bundle of rank $k/2$. Our assumptions imply that all cells in the relative CW-complex (X^τ, A^τ) have dimension $j \le d_0 \le 2k_0+3 \le 2k-1 = 4(k/2)-1$. Now the relative version of Proposition 2.3 allows us to extend $f_A|_{A^\tau}$ to an injective $\mathbb{Z}/2$-equivariant vector bundle map $\theta^1_{X^\tau} \to E|_{X^\tau}$.

Next, we extend our section further from $X^\tau \cup A$ to X. It suffices to extend an injective $\mathbb{Z}/2$-equivariant vector bundle map from the boundary of any cell in $(X, A \cup X^\tau)$ to the whole cell: if we can do this, we may build the required section by induction over the skeleta. Since we work relative to X^τ, only free cells $\mathbb{D}^j \times \mathbb{Z}/2$ occur. An injective $\mathbb{Z}/2$-*equivariant* vector bundle map on $\mathbb{D}^j \times \mathbb{Z}/2$ is equivalent to an injective vector bundle map on one of the pieces $\mathbb{D}^j$. Here our problem becomes equivalent to extending a $\mathbb{C}$-vector bundle isomorphism $E|_{\partial\mathbb{D}^j} \cong E_0 \oplus (\partial\mathbb{D}^j \times \mathbb{C}^2)$ for some $\mathbb{C}$-vector bundle E_0 over $\partial\mathbb{D}^j$ to a $\mathbb{C}$-vector bundle isomorphism $E|_{\mathbb{D}^j} \cong \tilde{E}_0 \oplus (\mathbb{D}^j \times \mathbb{C}^2)$ for some $\mathbb{C}$-vector bundle $\tilde{E}_0$ over $\mathbb{D}^j$. This amounts to applying Proposition 2.3 for $\mathbb{F} = \mathbb{C}$ twice and is possible if $j \le 2(k-1)-1 = 2k-3$. This is indeed the case because $j \le d_1 \le 2k_0+1 \le 2k-3$. □

Theorem 3.4. *Let $d_1, d_0, k \in \mathbb{N}$. Let (X, A) be a relative $\mathbb{Z}/2$-CW-complex. Assume that the free cells in (X, A) have at most dimension d_1 and that the trivial cells have at most dimension d_0. Let*

$$k_1 := \max\left\{\left\lceil \frac{d_0-2}{2} \right\rceil, \left\lceil \frac{d_1}{2} \right\rceil\right\}.$$

Let E_1 and E_2 be two "quaternionic" vector bundles over X of rank $k \ge k_1$. An isomorphism $E_1|_A \cong E_2|_A$ that extends to a stable isomorphism between E_1 and E_2 on X extends to an isomorphism $E_1 \cong E_2$.

Proof. This follows from Theorem 3.3 in exactly the same way as Theorem 2.5 follows from Theorem 2.4. First, we use that any "quaternionic" vector bundle over a finite-dimensional $\mathbb{Z}/2$-CW-complex is a direct summand in a trivial "quaternionic" bundle. The relevant quantities for d_1, d_0, k on $X \times I$ are now $d_1 + 1$, $d_0 + 1$ and $k + 2$ because the smallest trivial "quaternionic" bundle has rank 2. So the estimate about the rank in Theorem 3.3 for the extension problem on $X \times I$ is equivalent to the assumption made in this theorem. □

4. Conjugacy of projections

Our physical motivation was about projections in the observable algebra being homotopic. In this very short section, we briefly comment on the link between this original problem and our results on vector bundles.

Let E be a "real" or "quaternionic" vector bundle over (X, τ). Then E is a direct summand in a trivial bundle. Equivalently, there is another "real" or "quaternionic" vector bundle $E^\perp$ over X so that $E \oplus E^\perp \cong \theta^k_X$ for some $k \in \mathbb{N}$, with $k \in 2\mathbb{N}$ in the "quaternionic" case. The projection onto E is an endomorphism of the trivial bundle θ^k_X. In the "real" case, the endomorphism ring of θ_X^k is the ring of functions $X \to \mathbb{M}_k(\mathbb{C})$ that satisfy $\overline{f(x)} = f(\tau(x))$. In the "quaternionic" case, let $\Theta_0 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathbb{M}_2(\mathbb{R})$ and let $\Theta_0^{(k)} \in \mathbb{M}_{2k}(\mathbb{R})$ be the block diagonal sum of k copies of Θ_0. Then the endomorphism ring of θ_X^k is the ring of functions $X \to \mathbb{M}_k(\mathbb{C})$ that satisfy $\Theta_0^{(k)}\overline{f(x)}\Theta_0^{(k)} = f(\tau(x))$. Taking the pointwise adjoint of a matrix-valued function makes this endomorphism ring into a unital C*-algebra A. For $X = \mathbb{T}^d$, this is the observable C*-algebra for a translation-invariant physical system in dimension d with a time-reversal symmetry of square $+1$ in the "real" case and of square -1 in the "quaternionic" case. Homotopy classes of projections in A are in bijection with homotopy classes of invertible, self-adjoint elements of A, which are the possible Hamiltonians for insulators when the observable algebra is A. So the physical question is to classify the projections in A up to homotopy.

Each projection in A generates a direct sum decomposition of the trivial bundle θ_X^k as $E \oplus E^\perp$, where E and $E^\perp$ are the image bundles of p and $1-p$, respectively. The following result is well known.

Lemma 4.1. *Let p and q be two such projections and let $\theta_X^k = E \oplus E^\perp$ and $\theta_X^k = F \oplus F^\perp$ be the resulting direct sum decompositions. There is an invertible element $v \in A$ with $vpv^{-1} = q$ if and only if $E \cong F$ and $E^\perp \cong F^\perp$ as "real" or "quaternionic" vector bundles.*

Proof. If $E \cong F$ and $E^\perp \cong F^\perp$, then the two isomorphisms together produce an automorphism $\theta_X^k = E \oplus E^\perp \cong F \oplus F^\perp = \theta_X^k$. This is simply an invertible element $v \in A$, and it satisfies $vpv^{-1} = q$. Conversely, such an invertible element defines an automorphism of θ_X^k that restricts to isomorphisms $E \cong F$ and $E^\perp \cong F^\perp$. $\square$

We call p and q *conjugate* if there is an invertible element $v \in A$ with $vpv^{-1} = q$. This is well known to be equivalent to the existence of a unitary $v \in A$ with $vpv^{-1} = q$. It is also well known that homotopic projections are conjugate. The converse is only known up to stabilisation, however. The issue is whether the invertible element implementing the conjugacy is homotopic to the unit in A.

The projections p and q are stably conjugate, meaning that there is a projection r in another matrix algebra so that $p \oplus r$ and $q \oplus r$ are conjugate, if and only if E and F are stably isomorphic and $E^\perp$ and $F^\perp$ are stably isomorphic. So Proposition 1.3 says that if $X = \mathbb{T}^d$ with $d \leq 4$, then stable conjugacy and conjugacy are equivalent for projections in A. It is impossible, however, to prove a result that relates stable homotopy and homotopy by working only with vector bundles.

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