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2-Permutation Orbifolds of W-algebras

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ABSTRACT. In this paper we construct several infinite families of W-algebras as S_2-invariant subalgebras of the tensor product of two copies of affine vertex W-algebras of 'rank' two. For every value of the central charge (or level) we completely determine their types in terms of strong generators. We also consider simple quotients in the case when the type is different from in the generic case. We also examine examples of conformal embeddings that come out of the construction.

CONTENTS

1. Introduction

W-algebras are important algebraic structures that arise in the context of vertex algebra theory (for introduction see [3, 8, 14]) and conformal field theory (CFT) in theoretical physics [4, 15]. These algebras play a fundamental role in the study of extended symmetries and have applications in various branches of representation theory and string theory. While in mathematical literature, the term "W-algebras" typically pertains to affine W-algebras, we adopt a slightly broader interpretation, as coined by physicists, where a W-algebra is any vertex algebra endowed with a finite set of strong generators such that their OPEs involve non-linear terms.

In this paper, continuing a series of papers by the authors with Sadowski dealing with the Virasoro algebra [11, 12], we focus on the S_2-invariant subalgebra of the tensor product of two copies of the principal W-algebra of a Lie algebra of rank 2 under the action of the symmetric group on two letters. This case presents a greater degree of complexity as it lacks a diagonal Lie algebra

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that can be embedded within the fixed point subalgebra. We refer to this invariant subalgebra as the S_2-orbifold (sub)algebra or 2-permutation orbifold. Note that this terminology deviates somewhat from the standard usage. In the context of vertex algebra theory, the term *orbifold vertex algebra* typically involves the fixed point subalgebra under the action of a finite group of automorphisms *and* a suitable direct sum of twisted modules for the larger algebra, corresponding to this automorphism.

In the present work, we consider affine W-algebras associated to simple Lie algebras of rank two, $\mathcal{W}^k(\mathfrak{sl}_3, f_{\text{prin}})$, $\mathcal{W}^k(\mathfrak{sp}_4, f_{\text{prin}})$, $\mathcal{W}^k(\mathfrak{g}_2, f_{\text{prin}})$, along with their simple quotients for the first two algebras. Our primary objective is to describe a strong system of generators of their S_2-orbifold algebras.

Here is a brief outline. In Section 2, we revisit the standard notation concerning affine vertex algebras associated with a simple Lie algebra $\mathfrak{g}$ and discuss affine W-algebra $\mathcal{W}^k(\mathfrak{g})$ associated to a principal nilpotent element. We also prove a general result that gives the decomposition of the 2-permutation orbifold of an extension $V \oplus M$ in terms of irreducible modules for the 2-permutation orbifold of V. In Section 3, we begin with the definition of $\mathcal{W}^c(2,3)$, depending on the central charge c, using generators and OPE relations and explain how this relates to $\mathcal{W}^k(\mathfrak{sl}_3)$, parametrized by level k. In Section 3.2 we completely describe the structure of the S_2-fixed point subalgebra using minimal strong generators; see Theorem 3.3. In Section 3.3, we consider simple orbifolds that have type different from the generic case due to appearance of singular vectors. The most prominent example here is at $c = -2$, also studied in [1]. In Section 4, we discuss $\mathcal{W}^c(2,4)$ and how it relates to $\mathcal{W}^k(\mathfrak{sp}_4)$, after we switch to parametrization using level k. We show that they are isomorphic precisely when $c \neq -\frac{22}{5}$; for $c = -\frac{22}{5}$ the affine $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{sp}_4)$ does not have a primary generator of weight 4. Then we prove the main result of this section, Theorem 4.1 giving a complete structure of the S_2-fixed subalgebra for all values of the central charge in terms of minimal system of generators. Furthermore, in parallel with Section 3, we analyze simple orbifolds with low lying singular vectors that in the simple quotient give a different system of generators. At the very end, in Section 5, we discuss the S_2-orbifold algebra for the exceptional affine W-algebra of type $\mathfrak{g}_2$; see Theorem 5.1.

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2. Notation and preliminary results

2.1. Notation. Let $V = (V, Y, \mathbb{1}, \omega)$ be a conformal vertex algebra with the conformal vector ω, vacuum vector $\mathbb{1}$, such that $Y(\omega, x) = \sum_{n\in\mathbb{Z}} L(n)x^{-n-2}$ closes a representation of the Virasoro algebra of central charge c. Let also $V \otimes V$ denotes the tensor product of two copies of V equipped with a VOA structure in the standard way. The symmetric group on two letters S_2 acts on

$V \otimes V$ by permuting the tensor factors. The fixed point subalgebra under this automorphism will be denoted by $(V^{\otimes 2})^{S_2}$.

As some computations in this paper are performed using Mathematica OPE package [13], throughout we are adopting notation as in loc.cit. (see [11, 12] also). Under the field-state correspondence, for $A \in V$, $A \mapsto A(z) := Y(A,z) = \sum_{n\in\mathbb{Z}} A_{(n)}z^{-n-1}$, $A_{(n)} \in End(V)$, so we omit the Y-map and write $A(z)$ instead. We also use normal ordered product $:AB:(z)$ (resp. $:AB:$) to denote the field (resp. vector) $Y(A_{(-1)}B,z)$ (resp. $A_{(-1)}B$). We also use $\partial^k A$ to denote $L(-1)^k A$. In other words $\partial^k A = k!A_{(-k-1)}\mathbf{1}$. We also adopt the Operator Product Expansion (OPE), so we often write $A(z)B(w) \sim \sum_{n\geq 0} \frac{(A_{(n)}B)(w)}{(z-w)^{n+1}}$ to display the singular part in the $(z-w)$ expansion. This type of relation is widely used in the physics literature.

We say that a vertex algebra V is *strongly* generated by the set $X = \{v_i\}_{i\in I}$ if the span of vectors $:\partial^{n_i}v_{i_1}\cdots\partial^{n_k}v_{i_k}:$, is all of V, where $k \geq 0$, $i_j \in I$ and $n_j \in \mathbb{Z}_{\geq 0}$. A *minimal set of strong generators*, is a strong generating set that does not have a proper subset of strong generators. If $\{v_i\}_{1\leq i\leq k}$ is such a set consisting of homogeneous vectors of conformal weights $\{d_i\}_{1\leq i\leq k}$, we say that V is of type $(d_1,...,d_k)$.

We briefly recall the definition of an affine W-algebra associated to a simple Lie algebra $\mathfrak{g}$. As usual (see for instance [3]), one starts from a universal affine vertex algebra $V^k(\mathfrak{g})$, of level $k \neq -h^\vee$. For a nilpotent element $f \in \mathfrak{g}$, we consider the Drinfeld-Sokolov reduction of $V^k(\mathfrak{g})$ to obtain a new vertex algebra $\mathcal{W}^k(\mathfrak{g},f) = H^0_f(V^k(\mathfrak{g}))$. We only consider the principal nilpotent element $f = f_{prin}$ here, so we shall write $\mathcal{W}^k(\mathfrak{g})$ instead. Its simple quotient will be denoted by $\mathcal{W}_k(\mathfrak{g})$. In the special case of $\mathfrak{g} = \mathfrak{sl}_2$, $\mathcal{W}^k(\mathfrak{g})$ is isomorphic to the universal Virasoro vertex algebra of $V_{Vir}(c,0)$, with central charge $c = 1 - \frac{6(k+1)^2}{k+2}$. Its simple quotient is denoted by $L_{Vir}(c,0)$; for more about this case see [14].

Definition 2.1. Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra. Then the level $k = -h^\vee + \frac{p}{q} \in \mathbb{Q}_{>-h^\vee}$ is called *admissible* if $p,q \in \mathbb{Z}_{\geq 1}$, $(p,q) = 1$ and $p \geq h^\vee$ if $(r^\vee,q) = 1$ and $p \geq h$, if $(r^\vee,q) = r^\vee$. Here h and $h^\vee$ are the Coxeter and the dual Coxeter number, and $r^\vee \in \{1,2,3\}$ is the lacity of $\mathfrak{g}$.

It is know that the simple principal affine W-algebra $\mathcal{W}_k(\mathfrak{g})$ is C_2-cofinite and rational, whenever k is admissible [3]. These are known as minimal models and are a generalization of the Virasoro minimal models [14].

2.2. Extensions of 2-permutation orbifolds. In this section we consider the 2-permutation orbifold algebra of a simple extension $W = V \oplus M$, where M is an irreducible V-module. We also assume that V is a simple vertex algebra V and also assume for simplicity that V is positively graded and that the lowest conformal weight of M is wt_M.

Proposition 2.2. *Viewed as a $(V^{\otimes 2})^{S_2}$ module,*

$$(W^{\otimes 2})^{S_2} = (V^{\otimes 2})^{S_2} \oplus W_1 \oplus W_2$$

where W_1 is an irreducible $(V^{\otimes 2})^{S_2}$-module of lowest conformal weight wt_M and W_2 is irreducible of lowest conformal weight $2wt_M$.

Proof. We consider the decomposition into S_2-invariant submodules:

$$(W \otimes W)^{S_2} = (V \otimes V)^{S_2} \oplus (M \otimes V \oplus V \otimes M)^{S_2} \oplus (M \otimes M)^{S_2}.$$

Since $M \otimes V$ is an irreducible $(V \otimes V)$-module, then [6, Theorem 3.2] implies that $M \otimes V$ remains irreducible as $(V \otimes V)^{S_2}$-module. Next we observe that the symmetrization map $s(m \otimes v) = v \otimes m + m \otimes v$, $v \in V$, $m \in M$ defines an isomorphism between $M \otimes V$ and the middle term in the above decomposition. This defines our irreducible module W_1 whose lowest weight is clearly wt_M.

To handle the last summand we apply [6, Theorem 3.1]. Since $V \otimes V$-module $M \otimes M$ is clearly irreducible and $(12) \circ (M \otimes M) = (M \otimes M)$ (see [6] for the notation used) under the automorphism (12) of switching two tensor factors, the aformentioned result implies that $W_2 := (M \otimes M)^{S_2}$ is irreducible as $(V \otimes V)^{S_2}$-module. Its lowest conformal weight $2wt_M$. We proved the claim. □

3. 2-permutation orbifolds of $\mathcal{W}^k(\mathfrak{sl}_3)$

3.1. Definition of $\mathcal{W}^c(2,3)$. We begin by recalling two approaches to a universal $\mathcal{W}$-algebra of type (2, 3), in each we take L to be the Virasoro generator and W the primary weight 3 generator. In the physics literature, this algebra is usually defined by giving the OPE in terms of the central charge c [15]:

$$\begin{aligned} W(z)W(w) \sim & \frac{\frac{c}{3}(5c+22)}{(z-w)^6} + \frac{2(5c+22)L(w)}{(z-w)^4} + \frac{(5c+22)\partial L(w)}{(z-w)^3} \\ & + \frac{32{:}L(w)^2{:} + \frac{3}{2}(c-2)\partial^2 L(w)}{(z-w)^2} \\ & + \frac{32{:}(\partial L(w))L(w){:} + \frac{1}{3}(c-2)\partial^3 L(w)}{z-w}, \\ L(z)W(w) \sim & \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w}, \\ L(z)L(w) \sim & \frac{\frac{c}{2}}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial L(w)}{z-w}. \end{aligned} \tag{3.1}$$

More precisely, this system of OPEs determines a nonlinear Lie conformal algebra and hence a vertex algebra freely generated by $L(z)$ and $W(z)$ [5, Theorem 3.9]. We denote it by $\mathcal{W}^c(2,3)$ and its simple quotient $\mathcal{W}_c(2,3)$.

Alternatively, this algebra may be constructed via the quantum Drinfeld-Sokolov reduction starting from the level k vertex operator algebra associated to the affine vertex algebra $V^k(\mathfrak{sl}_3)$, $k \neq -3$, so we have $\mathcal{W}^k(\mathfrak{sl}_3)$. In this realization, we can express the generators using affine vertex algebra generators and fermions. If we let (inside $V^k(\mathfrak{g}) \otimes \mathcal{F}^{ch}$, where $\mathcal{F}^{ch}$ is a fermionic Fock space [3, 9]):

$$
\begin{aligned}
J^{(\alpha_1)} &= h_{\alpha_1} - :b^{(\alpha_2)}c^{(\alpha_2)}: + 2:b^{(\alpha_1)}c^{(\alpha_1)}: + :b^{(\alpha_1+\alpha_2)}c^{(\alpha_1+\alpha_2)}: \\
J^{(\alpha_2)} &= h_{\alpha_2} + 2:b^{(\alpha_2)}c^{(\alpha_2)}: - :b^{(\alpha_1)}c^{(\alpha_1)}: + :b^{(\alpha_1+\alpha_2)}c^{(\alpha_1+\alpha_2)}: \\
J^{(x_{\alpha_1})} &= x_{\alpha_1} + :b^{(\alpha_1+\alpha_2)}c^{(\alpha_2)}: \\
J^{(x_{\alpha_2})} &= x_{\alpha_1} + :b^{(\alpha_1+\alpha_2)}c^{(\alpha_1)}: \\
J^{(x_{\alpha_1+\alpha_2})} &= x_{\alpha_1+\alpha_2} \\
J^{(x_{-\alpha_1})} &= x_{-\alpha_1} + :b^{(\alpha_2)}c^{(\alpha_1+\alpha_2)}: \\
J^{(x_{-\alpha_2})} &= x_{-\alpha_2} - :b^{(\alpha_1)}c^{(\alpha_1+\alpha_2)}: \\
J^{(-x_{\alpha_1-\alpha_2})} &= x_{-\alpha_1-\alpha_2},
\end{aligned}
\tag{3.2}
$$

then

$$
\begin{aligned}
L = \frac{1}{k+3}(&-J^{(x_{-\alpha_1})} - J^{(x_{-\alpha_2})} + \frac{1}{3}:(J^{(\alpha_1)})^2: + \frac{1}{3}:(J^{(\alpha_2)})^2: \\
&+ \frac{1}{3}:J^{(\alpha_1)}J^{(\alpha_2)}: + (k+2)\partial^2 J^{(\alpha_1)} + (k+2)\partial^2 J^{(\alpha_2)})
\end{aligned}
\tag{3.3}
$$

and

$$
\begin{aligned}
\widetilde{W} = {} & J^{(x_{-\alpha_1-\alpha_2})} + \frac{2}{3}:J^{(\alpha_1)}J^{(x_{-\alpha_2})}: - \frac{1}{3}:J^{(\alpha_1)}J^{(x_{-\alpha_1})}: + \frac{2}{27}:(J^{(\alpha_1)})^3: \\
&+ \frac{1}{9}:J^{(\alpha_1)}J^{(\alpha_1)}J^{(\alpha_2)}: - \frac{1}{9}:J^{(\alpha_1)}(J^{(\alpha_2)})^2: - \frac{k+2}{3}:J^{(\alpha_1)}(\partial J^{(\alpha_2)}): \\
&+ \frac{1}{3}:J^{(\alpha_2)}J^{(x_{-\alpha_2})}: - \frac{2}{3}:J^{(\alpha_2)}J^{(x_{-\alpha_1})}: - \frac{2}{27}:(J^{(\alpha_2)})^3: \\
&- \frac{2k+4}{3}:(\partial J^{(\alpha_2)})J^{(\alpha_2)}: + (k+2)\partial J^{(x_{-\alpha_2})} - \frac{(k+2)^2}{3}\partial^2 J^{(\alpha_1)} \\
&- \frac{2}{3}(k+2)^2\partial^2 J^{(\alpha_2)} + \frac{(k+2)(k+3)}{2}\partial L
\end{aligned}
\tag{3.4}
$$

strongly generate the zero-th cohomology $H_f^0(V_k(\mathfrak{sl}_3))$, which is by definition $\mathcal{W}^k(\mathfrak{sl}_3)$. Here the OPE is parametrized by the level k, giving the following

OPE for the weight three generator which we name $\widetilde{W}$,

$$\begin{aligned}\widetilde{W}(z)\widetilde{W}(w) \sim & \frac{\frac{1}{9}(k+3)(3k+4)(3k+5)(4k+9)(5k+12)}{(z-w)^6}\\ &+ \frac{-\frac{1}{3}(k+3)^2(3k+4)(5k+12)L(w)}{(z-w)^4}\\ &+ \frac{-\frac{1}{6}(k+3)^2(3k+4)(5k+12)\partial L(w)}{(z-w)^3}\\ &+ \frac{-\frac{3}{4}(k+2)^2(k+3)^2\partial^2 L(w) + \frac{2}{3}^3 {\circ\atop\circ} L(w)^2 {\circ\atop\circ}}{(z-w)^2}\\ &+ \frac{-\frac{1}{6}(k+2)^2(k+3)^2\partial^3 L(w) + \frac{2}{3}(k+3)^3 {\circ\atop\circ}(\partial L(w))L(w){\circ\atop\circ}}{z-w}.\end{aligned} \tag{3.5}$$

The equivalence of these algebras is due to the fact that the central charge of $\mathcal{W}^k(\mathfrak{sl}_3)$ is

$$c_k = -\frac{2(3k+5)(4k+9)}{k+3},$$

and after rescaling $\widetilde{W}(z) \mapsto \frac{1}{4\sqrt{3}}(k+3)^{3/2}\widetilde{W}(z)$ and replacing $c \mapsto c_k$ the OPEs match. For the remainder of this section we will be working with the "version" of this algebra which has OPE parameterized by the central charge, c.

Observe that when $c = -\frac{22}{5}$, equivalently $k = -\frac{4}{3}$ or $k = -\frac{12}{5}$, the only remaining OPE for the weight 3 field with itself in (3.1) is

$$\begin{aligned} W_{(1)}W &= 32\left({\circ\atop\circ} L^2 {\circ\atop\circ} - \frac{3}{10}\partial^2 L\right)\\ W_{(0)}W &= 16\cdot\partial\left({\circ\atop\circ} L^2 {\circ\atop\circ} - \frac{3}{10}\partial^2 L\right).\end{aligned} \tag{3.6}$$

Thus the ideal I generated by W inside of $\mathcal{W}^{-\frac{22}{5}}(2,3)$ is proper and contains ${\circ\atop\circ} L^2 {\circ\atop\circ} - \frac{3}{10}\partial^2 L$ is the well-known singular vector inside of $V_{Vir}(c,0)$. This provides a simple argument that $\mathcal{W}_{-\frac{22}{5}}(2,3)$ and thus $\mathcal{W}_{-\frac{4}{3}}(\mathfrak{sl}_3)$ and $\mathcal{W}_{-\frac{12}{5}}(\mathfrak{sl}_3)$ are isomorphic to $L_{Vir}(-\frac{22}{5},0)$. This result is of course well-known; see for instance [7].

3.2. 2-permutation orbifold of $\mathcal{W}^c(2,3)$. Now we consider the two-fold tensor product ${}^3\mathcal{W}^c := \mathcal{W}^c(2,3) \otimes \mathcal{W}^c(2,3)$ – the simple quotient will be denoted by ${}^3\mathcal{W}_c$. Inside of this algebra, we define

$$\begin{aligned} L &= L\otimes \mathbb{1} + \mathbb{1}\otimes L, & U &= L\otimes\mathbb{1} - \mathbb{1}\otimes L,\\ W^+ &= W\otimes\mathbb{1} + \mathbb{1}\otimes W, & W^- &= W\otimes \mathbb{1} - \mathbb{1}\otimes W,\end{aligned} \tag{3.7}$$

where we have abused notation by using L as the Virasoro field both in $\mathcal{W}^c(2,3)$ and ${}^3\mathcal{W}^c$. These fields strongly and freely generate ${}^3\mathcal{W}^c$. With this set-up, the automorphism that permutes the tensor factors is diagonalized so that L and W^+ are fixed whereas $U \mapsto -U$ and $W^- \mapsto -W^-$. The OPE for these fields is given by:

$$\begin{aligned} W^{\pm}(z)W^{\pm}(w) \sim & \frac{\frac{2}{3}c(5c+22)}{(z-w)^6} + \frac{2(5c+22)L(w)}{(z-w)^4} + \frac{(5c+22)\partial L(w)}{(z-w)^3} \\ & + \frac{16{:}L(w)^2{:} + 16{:}U(w)^2{:} + \frac{3}{2}(c-2)\partial^2 L(w)}{(z-w)^2} \\ & + \frac{16{:}(\partial L(w))L(w){:} + 16{:}(\partial U(w))U(w){:} + \frac{1}{3}(c-2)\partial^3 L(w)}{z-w}, \end{aligned} \tag{3.8}$$

$$\begin{aligned} W^+(z)W^-(w) \sim & \frac{2(5c+22)U(w)}{(z-w)^4} + \frac{(5c+22)\partial U(w)}{(z-w)^3} \\ & + \frac{32{:}L(w)U(w){:} + \frac{3}{2}(c-2)\partial^2 U(w)}{(z-w)^2} \\ & + \frac{16\partial{:}L(w)U(w){:} + \frac{1}{3}(c-10)\partial^3 U(w)}{z-w}, \end{aligned} \tag{3.9}$$

$$U(z)W^{\pm}(w) \sim \frac{3W^{\mp}(w)}{(z-w)^2} + \frac{\partial W^{\mp}(w)}{z-w}, \tag{3.10}$$

and

$$U(z)U(w) \sim \frac{c}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial L(w)(w)}{z-w}, \tag{3.11}$$

where the remaining OPE are understood as U, W^+, W^-, are primary and L is a Virasoro field with central charge $2c$.

Our goal is to describe a minimal set of strong generators for $({}^3\mathcal{W}^c)^{S_2}$ and explore certain simple quotients $({}^3\mathcal{W}_c)^{S_2}$. We first introduce a filtration on ${}^3\mathcal{W}^c$ and on $({}^3\mathcal{W}^c)^{S_2}$. Let ${}^3\mathcal{W}^c_{(k)}$ denote the space spanned by monomials that are degree k in W^+ and W^-. Then ${}^3\mathcal{W}^c_{(0)} = V_{Vir}(c,0)$ is spanned by monomials entirely in L and U. We obtain an increasing filtration ${}^3\mathcal{W}^c_{(0)} \subset {}^3\mathcal{W}^c_{(1)} \subset \cdots$ and corresponding graded vertex algebra $gr({}^3\mathcal{W}^c)$ - where all generators W^+ and W^- commute which other. Then we introduce associated graded algebra $gr^F(gr({}^3\mathcal{W}^c))$, a commutative Poisson algebra, using Li's F-filtration induced by counting all generators L, U, W^+ and W^-. Using classical invariant theory the commutative algebra $gr^F(gr(({}^3\mathcal{W}^c)^{S_2}))$ is generated by

$$\partial^{j_1}L, \partial^{j_2}W^+, (\partial^{j_3}U)(\partial^{j_4}U), (\partial^{j_5}U)(\partial^{j_6}W^-), (\partial^{j_7}W^-)(\partial^{j_8}W^-),$$

where $j_i \geq 0$, $1 \leq i \leq 8$. In fact, using techniques from [11] (and others), involving the differential structure of $gr^F(gr(({}^3\mathcal{W}^c)^{S_2}))$, it is generated by

$$L, W^+, (\partial^{2j_1}U)U, (\partial^{j_2}U)W^-, (\partial^{2j_3}W^-)W^-,$$

for $j_i \geq 0$ for $1 \leq i \leq 3$. This motivates us to define the following fields

$$\begin{aligned} \Lambda_a &:= {}^\circ_\circ(\partial^a U)U{}^\circ_\circ \\ \Psi_a &:= {}^\circ_\circ(\partial^a U)W^-{}^\circ_\circ \\ \Omega_a &:= {}^\circ_\circ(\partial^a W^-)W^-{}^\circ_\circ, \end{aligned} \tag{3.12}$$

for $a \geq 0$, of weight $a+4$, $a+5$, and $a+6$, respectively as well as their double indexed versions

$$\begin{aligned} \Lambda_{a,b} &:= {}^\circ_\circ(\partial^a U)(\partial^b U){}^\circ_\circ \\ \Psi_{a,b} &:= {}^\circ_\circ(\partial^a U)(\partial^b W^-){}^\circ_\circ \\ \Omega_{a,b} &:= {}^\circ_\circ(\partial^a W^-)(\partial^b W^-){}^\circ_\circ, \end{aligned} \tag{3.13}$$

for $a, b \geq 0$.

In light of the generating set for $gr^F(gr(({}^3\mathcal{W}^c)^{S_2}))$, we have

Lemma 3.1. *The vertex algebra $({}^3\mathcal{W}^c)^{S_2}$ is strongly generated by L, W^+ and Λ_{2a}, $a \geq 0$, Ψ_a, $a \geq 0$ and Ω_{2a}, $a \geq 0$.*

Importantly, the normalization of the fields $W^\pm(z)$ was chosen so that all OPEs, (3.8)-(3.11), of the defining fields have structure constants which are polynomials in c. As such, the OPEs for any two normally ordered polynomials in the fields will have structure constants which are polynomials in c. It follows that when any element of $({}^3\mathcal{W}^c)^{S_2}$ is rewritten in terms of the strong generating set, all constants will be polynomials in c.

Observe that the fields L and Λ_{2a} generate a sub-VOA which is a copy of $(V_{\mathrm{Vir}}(c,0) \otimes V_{\mathrm{Vir}}(c,0))^{S_2}$ and thus can be described by the following result from [11].

Theorem 3.2 ([11] Corollary 3.1). *We have*

1. *For $c \notin \{-12, -\frac{23}{3}, -\frac{34}{7}, -\frac{11}{5}, -\frac{3}{10}, \frac{256}{47}\}$ the 2-permutation orbifold algebra $(V_{Vir}(c,0) \otimes V_{Vir}(c,0))^{S_2}$ is strongly generated by primary vectors of weight 2,4,6,8 and is thus of type (2,4,6,8).*
2. *The orbifold $\left(V_{Vir}(\frac{256}{47},0) \otimes V_{Vir}(\frac{256}{47},0)\right)^{S_2}$ is strongly generated by primary vectors of weight 2,4,6,8,10 and is thus of type (2,4,6,8,10).*
3. *In all other cases $(V_{Vir}(c,0) \otimes V_{Vir}(c,0))^{S_2}$ is strongly generated by vectors in weight 2,4,6,8 some of which are not primary.*

In the language of our current setting, in all cases except when $c = \frac{256}{47}$ the generators of the form Λ_{2a} can be minimized to the three fields Λ_0, Λ_2, and Λ_4, with the addition of Λ_6 in the exceptional case.

Next, we move to minimizing the generators of the form Ψ_a. Towards this goal we observe that expressions of the form

$$R^{\Psi}_{m,n} = {:}\Lambda_m \Psi_n{:} - {:}\Lambda_{m,n}\Psi_0{:} \tag{3.14}$$

will allow us to write higher weight generators in terms of fields of lower weight. As a concrete example we have

$$\begin{aligned} R^{\Psi}_{0,1} =& {:}L\Psi_3{:} + 3{:}W^+\Lambda_{2,1}{:} + \frac{5}{2}{:}(\partial L)\Psi_2){:} + 2{:}(\partial W^+)\Lambda_{1,1}{:} - \frac{1}{2}{:}W^+\Lambda_3{:} \\ &+ \frac{5}{2}{:}(\partial L)\Psi_2{:} + 2{:}(\partial^2 L)\Psi_1{:} + \frac{1}{6}{:}(\partial^3 L)\Psi_0{:} - \frac{1}{2}{:}(\partial W^+)\Lambda_2{:} + \frac{5}{24}\Psi_{0,5} \\ &- \frac{19}{12}\Psi_{1,4} - \frac{3}{2}\Psi_{2,3} + \frac{1}{4}\Psi_{3,2} + \frac{7}{24}\Psi_{4,1} + \left(\frac{1}{15} + \frac{7c}{120}\right)\Psi_{5,0} - \frac{11}{840}\partial^7 W^+. \end{aligned}$$

Using the fact that

$$\Psi_{a,b} = (-1)^b \Psi_{a+b} + \sum_{j=1}^{a+b} (-1)^{b+j} \binom{b}{j} \partial^j \Psi_{a+b-j}, \tag{3.15}$$

we can write

$$R^{\Psi}_{0,1} = \left(-\frac{4}{15} + \frac{7c}{120}\right)\Psi_5 + \Upsilon_{0,1},$$

where $\Upsilon_{0,1}$ is a normally ordered polynomial in lower weight generators. This allows us to eliminate the need for the generator Ψ_5 except for the case when $c = \frac{32}{7}$. Using a combination of $R^{\Psi}_{m+1,1}$ and $R^{\Psi}_{m,2}$ for $m \geq 0$ will allow us to eliminate the need for all generators Ψ_a for $a \geq 6$.

In fact, by direct calculation we have

$$p_1(m) R^{\Psi}_{m+1,1} + p_2(m) R^{\Psi}_{m,2} = \Psi_{m+6} + \Upsilon_{m+6}, \tag{3.16}$$

where

$$\begin{aligned} p_1(m) &= -\frac{300(m+1)(m+2)(m+3)(m+5)(m+6)\left(m^2+8m+15\right)}{f(m)} \\ p_2(m) &= \frac{-30m(m+1)(m+2)(m+3)(m+5)(m+6)\left(2m^3+36m^2+243m+524\right)}{f(m)}, \end{aligned}$$

with

$$\begin{aligned} f(m) =& 16m^9 + 492m^8 + 6190m^7 + 38394m^6 + 103639m^5 \\ &- 30702m^4 - 744735m^3 - 956694m^2 + 718680m + 1152000 \end{aligned}$$

and Υ_{m+6} is a normally ordered polynomial in $L, W^+, \Lambda_0, \Lambda_2, \Lambda_4, \Lambda_6$, and Ψ_n for $0 \leq n \leq m+5$ where the coefficients are polynomials in c and rational functions (without poles at non-negative integers) in m.

Now we observe that for all $m \in \mathbb{Z}_{\geq 0}$ $p_1(m)$, $p_2(m)$, and $f(m)$ are never zero, meaning that Ψ_{m+6} can be in the the subalgebra strongly generated by $L, W^+, \Lambda_0, \Lambda_2, \Lambda_4, \Lambda_6$, and Ψ_n for $0 \leq n \leq m+5$. This gives a clear inductive path to write any generator of the form Ψ_a in terms of $L, W^+ \Lambda_0, \Lambda_2, \Lambda_4, \Lambda_6$ and Ψ_n for $0 \leq n \leq 5$.

Finally, we move to minimizing the need for generators of the form Ω_{2a}. In, parallel to (3.14) we introduce

$$R^{\Omega}_{m,n} = {}^{\circ}_{\circ}\Psi_{m,n}\Psi_0{}^{\circ}_{\circ} - {}^{\circ}_{\circ}\Lambda_m\Omega_n{}^{\circ}_{\circ} \tag{3.17}$$

and observe that

$$R^{\Omega}_{0,0} = {}^{\circ}_{\circ}(\partial^2 W^+)\Psi_0{}^{\circ}_{\circ} - \frac{3}{4}\Omega_{2,2} + \frac{1}{2}\Omega_{3,1} + \left(\frac{62+2c}{24}\right)\Omega_4 + \cdots, \tag{3.18}$$

where the $\cdots$ terms are made up from the fields L and U. Applying (3.15), we have

$$R^{\Omega}_{0,0} = \frac{32+c}{24}\Omega_4 + \Phi_{0,0}, \tag{3.19}$$

where $\Phi_{0,0}$ is a normally ordered polynomial with only lower weight generators. This allows us to remove the need for Ω_4 from our strong generating set when $c \neq -32$. Next we observe that for $m \geq 0$ we have

$$\Omega_{2m+6}+\Phi_{2m+6} = \frac{2(m+3)(2m+3)(2m+5)}{40m^2+158m+137}R^{\Omega}_{2m+2,0} + \frac{(2m+3)(2m+5)^2}{40m^2+158m+137}R^{\Omega}_{2m+1,1},$$

where Φ_{2m+6} is a normally ordered polynomial in the fields $L, W^+, \Lambda_0, \Lambda_2, \Lambda_4, \Lambda_6$, Ψ_n for $0 \leq n \leq 5$, and Ω_{2n} for $0 \leq 2n \leq 2m+6$ where the coefficients are polynomials in c and rational functions(without poles at non-negative integers) in m. Thus we have a inductive method of writing any generator of the form Φ_{2a} in terms of $L, W^+, \Lambda_0, \Lambda_2, \Lambda_4, \Lambda_6, \Psi_n$ for $0 \leq n \leq 5$, and Ω_{2n} for $0 \leq 2n \leq 6$.

The above calculations bring us to the main result of this section

Theorem 3.3. *For*

$$c \notin \left\{-32, -12, -\frac{23}{3}, -\frac{34}{7}, -\frac{11}{5}, -\frac{3}{10}, \frac{256}{47}, \frac{32}{7}\right\},$$

the orbifold $\left({}^3\mathcal{W}^c\right)^{S_2}$ *is strongly generated by the conformal field* T_p*, a weight 3 field* W_p*, a weight 4 field* Λ_0*, a weight 5 field* Ψ_0*, three weight 6 fields* Λ_2*,* Ψ_1*,* Ω_0*, one weight 7 field* Ψ_2*, three weight 8 fields* Λ_4*,* Ψ_3*,* Ω_2*, and one weight 9 field* Ψ_4*. Thus it is of type* $(2,3,4,5,6^3,7,8^3,9)$*. We also have the following exceptional cases:*

- *If* $c = \frac{256}{47}$*, we also need* Λ_6 *and the orbifold is of type* $(2,3,4,5,6^3,7,8^3,9,10)$.
- *If* $c = \frac{32}{7}$*, we also need* Ψ_5 *and the orbifold is of type* $(2,3,4,5,6^3,7,8^3,9,10)$.
- *If* $c = -73$*, we also need* Ω_4 *and the orbifold is of type* $(2,3,4,5,6^3,7,8^3,9,10)$.
- *In all other cases,* $\left({}^3\mathcal{W}^c\right)^{S_2}$ *is generated by vectors of weight* $2,3,4,5,6^3,7,8^3,9$*, some of which may not be primary.*

3.3. The simple quotient $({}^3\mathcal{W}_c)^{S_2}$ for certain values of the central charge. In this part we consider the simple quotient of $({}^3\mathcal{W}^c)^{S_2}$. Our approach is similar as in [11] and [12] used for the Virasoro orbifolds. The strong set of generators in Theorem 3.3 will descend to a strong generating set for $({}^3\mathcal{W}_c)^{S_2}$. But if the maximal ideal ${}^3\mathcal{W}^c$ has components of weight ≤ 9, there may be additional coupling relations and the strong generating need not be minimal. There is only a few special values of the central charge there are low weight singular vectors that allow us to further reduce the generating set in the simple quotient.

Our first special case will be when $c = -\frac{22}{5}$. As explained above we know that $\mathcal{W}_{-\frac{22}{5}}(2,3) \cong L_{Vir}(-\frac{22}{5}, 0)$ and thus

$$\left({}^3\mathcal{W}_{-\frac{22}{5}}\right)^{S_2} = \left(L_{Vir}(-\frac{22}{5}, 0) \otimes L_{Vir}(-\frac{22}{5}, 0)\right)^{S_2} \cong L_{Vir}(-\frac{44}{5}, 0),$$

where the final isomorphism is well known [14]. This W-algebra is rational.

Next, we investigate the case when $c = -\frac{114}{7}$. At this value of central charge the W-algebra is rational. There is a weight 5 singular vector inside $\mathcal{W}^{-\frac{114}{7}}(2,3)$ given by

$$:LW: - \frac{3}{14}\partial^2 W, \tag{3.20}$$

which gives rise to a singular vector with $({}^3\mathcal{W}^{-\frac{114}{7}})^{S_2}$ of

$$v_5^{\text{sing}} = \Psi_0 + :LW^+: - \frac{3}{7}\partial^2 W. \tag{3.21}$$

This obviously allows us to remove the weight 5 generator Ψ_0 but further calculations will allow us to remove more. For instance at weight 6 we have

$$\begin{aligned} 36\Omega_0 + \frac{72}{13}\Lambda_2 = \frac{126}{13}:L\Lambda_0: + \frac{42}{13}:L^2: - 36:W^+W^+: - \frac{72}{13}:(\partial L)^2: \\ - \frac{144}{13}:(\partial^2 L)L: - \frac{36}{13}\partial^2\Lambda_0 + \frac{144}{91}\partial^4 L - (\Psi_1)_{(4)}v_5^{\text{sing}}, \end{aligned} \tag{3.22}$$

so we have removed the need for Ω_0, leaving two weight 6 generators Λ_2 and Ψ_1. At weight 7, we have

$$\begin{aligned} \Psi_2 = {} & \frac{7}{10}:Lv_5^{\text{sing}}: - \frac{7}{10}:L^2W: + \frac{13}{10}:L\partial^2 W^+: + \frac{7}{10}:W^+\Psi_0: \\ & + 2:(\partial L)(\partial W^+): + :(\partial^2 L)W: + 2\partial\Psi_1 - \frac{143}{150}\partial^2 v_5^{\text{sing}} \\ & - \frac{671}{1680}\partial^4 W^+ - \frac{7}{300}(\Lambda_2)_{(3)}v_5^{\text{sing}}, \end{aligned} \tag{3.23}$$

removing the need for the only weight 7 generator. We also have three weight 8 equations

$$\begin{aligned}\Psi_3 = &\frac{149}{175}{\circ\atop\circ}(\partial^3 L)W^+{\circ\atop\circ} + \frac{531}{175}{\circ\atop\circ}(\partial^2 L)(\partial W^+){\circ\atop\circ} + \frac{3}{25}{\circ\atop\circ}(\partial L)v_5^{\mathrm{sing}}{\circ\atop\circ} \\ &+ \frac{108}{35}{\circ\atop\circ}(\partial L)(\partial^2 W){\circ\atop\circ} - \frac{6}{25}{\circ\atop\circ}(\partial L)LW^+{\circ\atop\circ} + \frac{3}{25}{\circ\atop\circ}L(\partial v_5^{\mathrm{sing}}){\circ\atop\circ} \\ &+ \frac{186}{175}{\circ\atop\circ}L(\partial^3 W^+){\circ\atop\circ} - \frac{3}{25}{\circ\atop\circ}LL(\partial W^+){\circ\atop\circ} + \frac{42}{25}{\circ\atop\circ}L\Psi_1{\circ\atop\circ} \\ &+ \frac{24}{25}{\circ\atop\circ}W^+(\partial\Lambda_0){\circ\atop\circ} + \frac{3}{25}{\circ\atop\circ}(\partial W^+)\Lambda_0{\circ\atop\circ} - \frac{177}{175}\partial^3 v_5^{\mathrm{sing}} \\ &- \frac{482}{1225}\partial^5 W^+ + \frac{228}{175}\partial^2\Psi_1 + \frac{3}{7}\partial\Psi_2 - \frac{3}{50}(\Lambda_2)_{(2)}v_5^{\mathrm{sing}},\end{aligned} \tag{3.24}$$

and two more involving Λ_4 and Ω_2 eliminating the need for all weight 8 generators. Similarly there is a relation that allows us to eliminate the weight 9 generator Ψ_4. From all of this we see that the simple quotient of $({}^3\mathcal{W}_{-\frac{114}{7}})^{S_2}$ is strongly generated by fields $L, W^+, \Lambda_0, \Psi_1$, and Λ_2 and is thus of type $(2, 3, 4, 6^2)$. Since we started from a rational W-algebra, we obtain a new rational vertex algebra of type $(2, 3, 4, 6^2)$.

Now we look at the case when $c = \frac{4}{5}$, where there is a weight 6 singular vector given by

$$\begin{aligned}v_6^{\mathrm{sing}} = &-\frac{78}{95}\Omega_0 - \frac{162}{95}\Lambda_2 + {\circ\atop\circ}L\Lambda_0{\circ\atop\circ} + \frac{1}{3}{\circ\atop\circ}L^2{\circ\atop\circ} - \frac{78}{95}{\circ\atop\circ}(W^+)^2{\circ\atop\circ} + \frac{17}{19}{\circ\atop\circ}(\partial L)^2{\circ\atop\circ} \\ &- \frac{77}{95}{\circ\atop\circ}(\partial^2 L)L{\circ\atop\circ} + \frac{17}{38}\partial^2\Lambda_0 - \frac{193}{1140}\partial^4 L.\end{aligned} \tag{3.25}$$

Similarly to above, we can use v_6^{sing} to eliminate the need for one of the weight 6 generators and all of the generators of weights 7-9. As such the simple quotient of $\mathcal{W}_{\frac{4}{5}}^{S_2}$ is strongly generated by fields L, W^+, Λ_0, Ψ_0, Ψ_1, and Λ_2 and is thus of type $(2, 3, 4, 5, 6^2)$. Again, we have a new rational vertex algebra of type $(2, 3, 4, 5, 6^2)$.

We also get the following decomposition.

Proposition 3.4.

$$\left({}^3\mathcal{W}_{\frac{4}{5}}\right)^{S_2} = \mathcal{W}_{-22/5}(\mathfrak{sp}_8) \oplus W_1 \oplus W_2$$

where W_1 and W_2 are irreducible $\mathcal{W}_{-22/5}(\mathfrak{sp}_8)$-modules of lowest conformal weights 3 *and* 6*, respectively.*

Proof. This follows from Theorem 2.2 together with [11, Corollary 5.3] where we obtained an isomorphism among the S_2-Virasoro orbifold at $c = \frac{4}{5}$ and the affine W-algebra $\mathcal{W}_k(\mathfrak{sp}_8)$ at level $k = -\frac{22}{5}$.

□

Next, we consider the case when $c = -23$, in which there is a weight 6 singular vector given by

$$w_6^{\mathrm{sing}} = -\frac{93}{8}\Omega_0 - \frac{11}{16}\Lambda_2 + \frac{1}{3}{:}L^3{:} - \frac{93}{8}{:}(W^+)^2{:} - \frac{17}{16}{:}(\partial L)^2{:} \\ - \frac{7}{4}{:}(\partial^2 L)L{:} - \frac{17}{32}\partial^2\Lambda_0 + \frac{89}{192}\partial^4 L. \tag{3.26}$$

Similarly to above, we can use w_6^{sing} to eliminate the need for one of the weight 6 generators, the weight 7 generator, one of the weight 8 generators, and the weight 9 generator. As such the simple quotient $\mathrm{o}({}^3\mathcal{W}_{-23})^{S_2}$ is strongly generated by fields

$$L, \quad W^+, \quad \Lambda_0, \quad \Psi_1, \quad \Psi_2, \quad \Lambda_2, \quad \Psi_3, \quad \Lambda_3,$$

and is thus of type $(2, 3, 4, 5, 6^2, 8^2)$. This vertex algebra is also rational.

Next we finish with a non-rational orbifold. At $c = -2$, there is a weight 6 singular vector,

$$10\Psi_1 - 4{:}L\partial W^+{:} + 6{:}(\partial L)W^+{:} - 4\partial\Psi_1 + \partial^3 W^+,$$

that leads to the elimination of Ψ_1, Ω_0, Ω_2, and Ψ_4, leaving an algebra of type $(2, 3, 4, 5, 6, 7, 8)$. This example was studied in [1] in connection with $\mathcal{W}_{-\frac{5}{2}}(\mathfrak{sp}_4)$. As discussed there, the simple affine W-algebra $\mathcal{W}_{-\frac{5}{2}}(\mathfrak{sp}_4)$ embeds inside the tensor product of two copies of the $(1, 2)$ singlet vertex algebra $\mathcal{M}(2)$ of central charge $c = -2$. This singlet algebra is known to be isomorphic to $\mathcal{W}_{-2}(\mathfrak{sl}_3) = \mathcal{W}_{-2}(2, 3)$ ($k = -2$ leads to $c = -2$ for $\mathfrak{sl}_3$). We have a semisimple decomposition

$$\mathcal{W}_{-2}(\mathfrak{sl}_3)^{\otimes^2} = \bigoplus_{\lambda \in P_+} a_\lambda E_\lambda$$

into irreducible $\mathcal{W}_{-\frac{5}{2}}(\mathfrak{sp}_4)$-modules E_λ [10]. Coefficients $a_\lambda \in \mathbb{N}_0$ is given by the dimension of the zero weight subspaces of the simple $\mathfrak{sp}_4$-modules $V_{\mathfrak{sp}_4}(\lambda)$ of highest weight λ [1]. It is easy to see that the S_2-orbifold $(\mathcal{W}_{-2}(\mathfrak{sl}_3)^{\otimes^2})^{S_2}$ also contains $\mathcal{W}_{-\frac{5}{2}}(\mathfrak{sp}_4)$ as a subalgebra. We also recall that the parafermion algebra $N_{-1}(\mathfrak{sl}_2)$ embeds inside $\mathcal{W}_{-2}(\mathfrak{sl}_3)^{\otimes^2}$ and also inside the S_2-orbifold, and we have a semisimple decomposition (see [1]):

$$(\mathcal{W}_{-2}(\mathfrak{sl}_3)^{\otimes^2})^{S_2} = \bigoplus_{s \geq 0} N_{-1}(4s\omega), \tag{3.27}$$

where $N_{-1}(n\omega)$ are irreducible $N_{-1}(\mathfrak{sl}_2)$-modules coming from the Weyl $\widehat{\mathfrak{sl}}_2$-module whose top component is $V(n\omega)$. Then we have the following result:

Proposition 3.5.

$$\left(\mathcal{W}_{-2}(\mathfrak{sl}_3)^{\otimes^2}\right)^{S_2} = \bigoplus_{\substack{\lambda\in P_+ \\ n\equiv 0 \mod 2}} a_\lambda E_\lambda$$

where $\lambda = n\omega_1 + m\omega_2$ *satisfies*

$$\lambda = \begin{cases} \frac{m(n+1)}{4}, & m \equiv 1 \mod 2 \\ \frac{(m+2)n}{4} + \lfloor \frac{m}{4} \rfloor + 1, & m \equiv 0 \mod 2. \end{cases}$$

Proof. The idea is very similar as in [1] so we only indicate the main steps and omit computational details. Adamović and the first author already demonstrated, that there is an explicit decomposition of $N_{-1}(\mathfrak{sl}_2)$ in terms of irreducible $\mathcal{W}_{-\frac{5}{2}}(\mathfrak{sp}_4)$-modules E_λ. This is done using the semisimplicity of the decomposition and by comparing characters. The same idea applies to each individual module, $N_{-1}(n\omega)$, and we get decompositions into irreducible $\mathcal{W}_{-\frac{5}{2}}(\mathfrak{sp}_4)$-modules by comparing characters. Summing over all s in formula (3.27) then gives the result. □

In the case of $c = -\frac{186}{5}$ there is a singular vector at weight 8 allowing us to remove a weight 8 generator and the weight 9 generator, leaving a simple quotient of type $(2, 3, 4, 5, 6^3, 7, 8^2)$. This vertex algebra is rational [3].

Finally, in the case that $c = -\frac{490}{11}$, there is a singular vector at weight 9 allowing us to remove a weight 8 generator and the weight 9 generator, leaving a simple quotient of type $(2, 3, 4, 5, 6^3, 7, 8^3)$. This vertex algebra is also rational [3].

Remark 3.6. It is interesting to note, as a direct consequence of our construction, that the S_2-orbifold algebra $\left({}^3\mathcal{W}_c\right)^{S_2}$ is (weakly) generated by W and L for all values of c, except when $c = -2$, in which case we require a generator of weight of 6 (see (3.27)).

4. 2-permutation orbifolds of $\mathcal{W}^k(\mathfrak{sp}_4)$

4.1. Definition of $\mathcal{W}^c(2, 4)$. In parallel to section 4.1, we now recall two approaches to a universal $\mathcal{W}$-algebra of type $(2, 4)$, taking L to be the conformal field and W the primary weight 4 generator. In fact, we parallel most of the notation from our previous section here. We will denote by $\mathcal{W}^c(2, 4)$ the universal algebra whose OPE is parameterized with respect to the central charge, c. As above, we take $\mathcal{W}_c(2,4)$ to be the simple quotient. In this setting we have

$$\begin{aligned} W(z)W(w) \sim\ & \frac{\frac{147}{2} c p_0(c)}{(z-w)^8} + \frac{588 p_0(c) L(w)}{(z-w)^6} + \frac{294 p_0(c)\partial L(w)}{(z-w)^5} \\ & + \frac{63 p_1(c)(2W(w) + 7p_2(c)(28 {:}L(w)^2{:} + (c-4)\partial^2 L(w))}{(z-w)^4} \\ & + \frac{7p_1(c)(9\partial W(w) + 14 p_2(c)(126 {:}(\partial L(w))L(w){:} + (c-4)\partial^3 L(w)))}{(z-w)^3} \\ & + \frac{\frac{7}{2} p_3(c)((5c+64)\partial^2 W(w) + 336 {:}L(w)W(w){:} + (c+24)Z_0(w))}{(z-w)^2} \\ & + \frac{\frac{7}{20} p_3(c)(10(c-4)\partial^3 W(w) + 1680 \partial {:}L(w)W(w){:} + (c+24)Z_1(w))}{z-w}, \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} p_0(c) &= (c+24)(2c-1)(5c+22)(7c+68)\left(c^2 - 172c + 196\right) \\ p_1(c) &= (c+24)\left(c^2 - 172c + 196\right) \\ p_2(c) &= (2c-1)(7c+68) \\ p_3(c) &= \left(c^2 - 172c + 196\right) \\ Z_0 &= 2016(13+72c){:}L^3{:} + 84\left(176c^2 + 117c - 2528\right){:}(\partial^2 L)L{:} \\ &\quad + 42\left(295c^2 + 2592c + 2048\right){:}(\partial L)^2{:} + 14\left(5c^3 + 5c^2 - 764c - 116\right)\partial^4 L \\ Z_1 &= 30240(72c+13){:}(\partial L)L^2{:} + 1260(59c^2 + 316){:}(\partial^2 L)(\partial L){:} \\ &\quad + 2520(13c^2 + 13c - 316){:}(\partial^3 L)L{:} + 21(5c^3 + 5c^2 - 764c - 116)\partial^5 L. \end{aligned}$$

As in Section 3, we can alternatively view this algebra as the universal affine $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{sp}_4)$ with central charge

$$c_k = -\frac{2(5k+12)(6k+13)}{k+3}.$$

We omit explicit formulas here for the sake of brevity. The OPE parameterized with respect to the level, k, can be found in [7]. The OPE presented in [7] is written under the assumption that the weight 4 generator, W, is primary. A scaling has been chosen so that the $747 + 674k + 150k^2$ does not appear in the denominator of any structure constants. With this presentation of the OPE if k is a root of $747 + 674k + 150k^2$ the corresponding VOA is not simple – its simple quotient is the simple Virasoro VOA of central charge $c = -\frac{22}{5}$. In the process of scaling some information has been lost and for these values of k, the OPE of $\mathcal{W}^k(\mathfrak{sp}_4)$ is different and not a specialization of the OPEs in [7].

We briefly present the explicit construction of $\mathcal{W}^k(\mathfrak{sp}_4)$ which starts with the vertex operator algebra

$$\mathcal{C}^k(\mathfrak{sp}_4) = V^k(\mathfrak{sp}_4) \otimes \mathcal{F}^{\text{ch}},$$

where $V^k(\mathfrak{sp}_4)$ is the universal level k associated with $\mathfrak{sp}_4$ and $\mathcal{F}^{\mathrm{ch}}$ is a vertex algebra of charge free fermions corresponding with the decomposition of $\mathfrak{sp}_4$ with respect to the principal nilpotent $f = x_{-\alpha_1} + x_{-\alpha_2}$. Also, set

$$\begin{aligned} d^{(1)} =& {:}x_{\alpha_1}c^{(\alpha_1)}{:} + {:}x_{\alpha_2}c^{(\alpha_2)}{:} + {:}x_{\alpha_1+\alpha_2}c^{(\alpha_1+\alpha_2)}{:} + {:}x_{\alpha_1+2\alpha_2}c^{(\alpha_1+2\alpha_2)}{:} \\ &- {:}b^{(\alpha_1+\alpha_2)}c^{(\alpha_1)}c^{(\alpha_2)}{:} + 2{:}b^{(\alpha_1+2\alpha_2)}c^{(\alpha_2)}c^{(\alpha_1+\alpha_2)}{:}, \\ d^{(2)} =& c^{(\alpha_1)} + 2c^{(\alpha_2)}, \end{aligned}$$

and

$$d := d^{(1)} + d^{(2)}.$$

The map $d_{(0)} : \mathcal{C}^k(\mathfrak{sp}_4) \to \mathcal{C}^k(\mathfrak{sp}_4)$, the zeroth mode of d, squares to the zero map leading us to define $\mathcal{W}^k(\mathfrak{sp}_4) := \ker(d_{(0)})/\mathrm{im}(d_{(0)})$, the zeroth homology of the complex $(\mathcal{C}^k(\mathfrak{sp}_4), d_{(0)})$. Next, we set

$$\begin{aligned} J^{(\alpha_1)} &= h_{\alpha_1} + 2{:}b^{(\alpha_1)}c^{(\alpha_1)}{:} - {:}b^{(\alpha_2)}c^{(\alpha_2)}{:} + {:}b^{\alpha_1+\alpha_2}c^{\alpha_1+\alpha_2}{:}, \\ J^{(\alpha_2)} &= h_{\alpha_2} - {:}b^{(\alpha_1)}c^{(\alpha_1)}{:} + 2{:}b^{(\alpha_2)}c^{(\alpha_2)}{:} + 2{:}b^{\alpha_1+2\alpha_2}c^{\alpha_1+2\alpha_2}{:}, \\ J^{(x_{\alpha_1})} &= x_{\alpha_1} + {:}b^{(\alpha_1+\alpha_2)}c^{(\alpha_1+\alpha_2)}{:}, \\ J^{(x_{\alpha_2})} &= x_{\alpha_2} - {:}b^{(\alpha_1+\alpha_2)}c^{(\alpha_1)}{:} - 2{:}b^{(\alpha_1+2\alpha_2)}c^{(\alpha_1+\alpha_2)}{:}, \\ J^{(x_{\alpha_1+\alpha_2})} &= x_{\alpha_1+\alpha_2} + 2{:}b^{(\alpha_2)}c^{(\alpha_1+2\alpha_2)}{:}, \\ J^{(x_{\alpha_1+2\alpha_2})} &= x_{\alpha_1+2\alpha_2}, \\ J^{(x_{-\alpha_1})} &= x_{-\alpha_1} + {:}b^{(\alpha_2)}c^{(\alpha_1+\alpha_2)}{:}, \\ J^{(x_{-\alpha_2})} &= x_{-\alpha_2} - 2{:}b^{(\alpha_1)}c^{(\alpha_1+\alpha_2)}{:} - {:}b^{(\alpha_1+\alpha_2)}c^{(\alpha_1+2\alpha_2)}{:}, \\ J^{(x_{-\alpha_1-\alpha_2})} &= x_{-\alpha_1-\alpha_2} + {:}b^{(\alpha_2)}c^{(\alpha_1+2\alpha_2)}{:} \\ J^{(x_{-\alpha_1-2\alpha_2})} &= x_{-\alpha_1-2\alpha_2} \end{aligned}$$

By Theorem 4.1 of [9] we know that $\mathcal{W}^k(\mathfrak{sp}_4)$ is strongly generated by the homology classes of $J^{(x_{-\alpha_1}+x_{-\alpha_2})}$ and $J^{(x_{-\alpha_1-2\alpha_2})}$ of weight 2 and 4, respectively.

Our strategy to find elements in the kernel of $d_{(0)}$ is for $v \in \mathfrak{sp}_4$ to recursively define $U_n^{(v)}$ by $U_1^{(v)} = J^{(v)}$ and for $n > 1$, by the equation

$$d_{(0)}^{(2)}U_n^{(v)} = d_{(0)}^{(1)}U_{n-1}^{(v)}.$$

and then finally

$$W^{(v)} = \sum_{n\geq 1}(-1)^n U_n^{(v)},$$

where this sum truncates at the weight of the element in $\mathcal{W}^k(\mathfrak{sp}_4)$ associated to $J^{(v)}$ as given in [9]. Observe that with respect to the conformal field inside of $\mathcal{C}^k(\mathfrak{sp}_4)$, we have $\mathrm{wt}\left(U_n^{(v)}\right) = n$ for $v \in \mathfrak{sp}_4$. As such, finding the homology class of $J^{(x_{-\alpha_1}+x_{-\alpha_2})}$ will take two iterations of the procedure, while for $J^{(x_{-\alpha_1-2\alpha_2})}$

we will need four. Explicitly we have

$$\begin{aligned} U_1^{(x_{-\alpha_1}+x_{-\alpha_2})} =& x_{-\alpha_1} + x_{-\alpha_2} - 2{:}b^{(\alpha_1)}c^{(\alpha_1+\alpha_2)}{:} + {:}b^{(\alpha_2)}c^{(\alpha_1+\alpha_2)}{:} \\ & - {:}b^{(\alpha_1+\alpha_2)}c^{(\alpha_1+2\alpha_2)}{:} \\ U_2^{(x_{-\alpha_1}+x_{-\alpha_2})} =& \frac{1}{2}{:}(J^{(\alpha_1)})^2{:} + \frac{1}{2}{:}J^{(\alpha_1)}J^{(\alpha_2)}{:} + \frac{1}{4}{:}(J^{(\alpha_2)})^2{:} \\ & + \frac{4k+9}{2}\partial J^{(\alpha_1)} + \frac{3k+7}{2}\partial J^{(\alpha_2)}, \end{aligned}$$

and thus

$$W(x_{-\alpha_1} + x_{-\alpha_2}) = U_1^{(x_{-\alpha_1}+x_{-\alpha_2})} - U_2^{(x_{-\alpha_1}+x_{-\alpha_2})}.$$

Next we have,

$$\begin{aligned} U_1^{(x_{-\alpha_1-2\alpha_2})} =& J^{(x_{-\alpha_1-2\alpha_2})} \\ U_2^{(x_{-\alpha_1-2\alpha_2})} =& \frac{1}{2}{:}J^{(\alpha_1)}J^{(\alpha_1+\alpha_2)}{:} + \frac{1}{2}{:}J^{(\alpha_2)}J^{(\alpha_1+\alpha_2)}{:} - \frac{2k+5}{4}{:}(J^{(x_{-\alpha_1})})^2{:} \\ & - \frac{2k+10}{2}{:}J^{(x_{-\alpha_1})}J^{(x_{-\alpha_2})}{:} - \frac{k+2}{2}{:}(J^{(x_{-\alpha_2})})^2{:} \\ U_3^{(x_{-\alpha_1-2\alpha_2})} =& -\frac{k+2}{2}{:}(J^{(\alpha_1)})^2J^{(x_{-\alpha_1-\alpha_2})}{:} + \cdots - \frac{2k^2+9k+10}{4}\partial^2 J^{(x_{-\alpha_1-\alpha_2})} \\ U_4^{(x_{-\alpha_1-2\alpha_2})} =& -\frac{k+2}{8}{:}(J^{(\alpha_1)})^4{:} + \cdots - \frac{6k^3+41k^2+93k+70}{16}\partial^2 J^{(\alpha_2)}, \end{aligned}$$

where the $\cdots$ indicates terms involving $J^{(v)}$ where $v \in \mathfrak{sp}_4$. Then we have

$$W^{(x_{-\alpha_1-2\alpha_2})} = U_1^{(x_{-\alpha_1-2\alpha_2})} - U_2^{(x_{-\alpha_1-2\alpha_2})} + U_3^{(x_{-\alpha_1-2\alpha_2})} - U_4^{(x_{-\alpha_1-2\alpha_2})}.$$

So $\mathcal{W}^k(\mathfrak{sp}_4)$ is strongly generated by $W^{(x_{-\alpha_1}+x_{-\alpha_2})}$ and $W^{(x_{-\alpha_1-2\alpha_2})}$. We can correct these fields so that $W^{(x_{-\alpha_1}+x_{-\alpha_2})}$ has the OPE of the Virasoro field and $W^{(x_{-\alpha_1}+x_{-\alpha_2})}$ is primary via

$$\begin{aligned} L =& -\frac{1}{k+3}W^{(x_{-\alpha_1}+x_{-\alpha_2})} \\ \widetilde{W} =& W^{(x_{-\alpha_1}+x_{-\alpha_2})} - \frac{(k+3)^2\left(300k^3+2044k^2+4653k+3540\right)}{4\,(150k^2+674k+747)}{:}L^2{:} \\ & - \frac{(k+3)(5k+11)\left(12k^3+52k^2+41k-36\right)}{8\,(150k^2+674k+747)}\partial^2 L. \end{aligned} \tag{4.2}$$

Observe that if $k = \frac{1}{150}(-337 \pm 7\sqrt{31})$ the denominators of ${:}L^2{:}$ and $\partial^2 L$ are zero in (4.2) and thus $W^{(x_{-\alpha_1-2\alpha_2})}$ cannot be corrected to a primary field, namely, $\mathcal{W}^{\frac{1}{150}(-337\pm7\sqrt{31})}(\mathfrak{sp}_4)$ necessarily has a non-primary weight 4 generator.

Further, when $k = \frac{1}{150}(-337 \pm 7\sqrt{31})$ we have $c = -\frac{22}{5}$. Since the universal algebra $\mathcal{W}^{\frac{1}{150}(-337\pm7\sqrt{31})}(\mathfrak{sp}_4)$ has an essential non-primary generator whereas the algebra $\mathcal{W}^{-\frac{22}{5}}(2,4)$ has a primary weight 4 generator and thus only in this special case these algebras are non-isomorphic. Since $k \notin -3+\mathbb{Q}_{>0}$, this vertex algebra is simple.

Much as we did in the previous section we will now highlight some cases when the simple quotient of this algebra coincides with $L_{Vir}(c,0)$. These cases are essentially known but we recall some of the details here. At $c=-\frac{68}{7}$ there is a singular vector within $V_{Vir}(c,0)$ given by

$$v_6^{Vir} = 14{:}L^3{:} - 2{:}(\partial L)^2{:} - 11{:}(\partial^2 L)L{:} - \frac{19}{42}\partial^4 L.$$

We check that for this central charge

$$\begin{aligned} W_{(n)}W &= 0, \text{ for } n \geq 4 \\ W_{(3)}W &= \frac{172980000}{49}W \\ W_{(2)}W &= \frac{86490000}{49}\partial W \\ W_{(1)}W &= \frac{5189400}{49}\partial^2 W + 2306400{:}LW{:} - \frac{3324675600000}{343}v_6^{Vir} \\ W_{(0)}W &= -\frac{4612800}{49}\partial^3 W + 1153200\partial{:}LW{:} - \frac{1662337800000}{343}\partial v_6^{Vir} \end{aligned} \tag{4.3}$$

So the ideal generated by W is proper and contains v_6^{Vir} providing that $\mathcal{W}_{-\frac{68}{7}}(2,4)$, thus $\mathcal{W}_{-\frac{18}{7}}(\mathfrak{sp}_4)$ and $\mathcal{W}_{-\frac{11}{6}}(\mathfrak{sp}_4)$ are isomorphic to $L_{Vir}(-\frac{68}{7},0)$.

Very similar calculations exist for $c=\frac{1}{2}$ using the weight 6 singular vector inside of $V_{Vir}(\frac{1}{2},0)$, meaning that $\mathcal{W}_{\frac{1}{2}}(2,4)$, thus $\mathcal{W}_{-\frac{19}{8}}(\mathfrak{sp}_4)$ and $\mathcal{W}_{-\frac{11}{5}}(\mathfrak{sp}_4)$ are isomorphic to $L_{Vir}(\frac{1}{2},0)$.

Something a bit different is happening at $c=-24$ where we have

$$\begin{aligned} W_{(n)}W &= 0, \text{ for } n \geq 2 \\ W_{(1)}W &= -960400\partial^2 W + 5762400{:}LW{:} \\ W_{(0)}W &= -480200\partial^3 W + 2881200\partial{:}LW{:}. \end{aligned} \tag{4.4}$$

Taking the ideal, I, generated by W we have

$$\mathcal{W}^{-24}(2,4)/I \cong V_{Vir}(-24,0) \cong L_{Vir}(-24,0), \tag{4.5}$$

as the central charge $c=-24$ is not part of the minimal series. Thus $\mathcal{W}_{-\frac{8}{3}}(\mathfrak{sp}_4)$ and $\mathcal{W}_{-\frac{3}{2}}(\mathfrak{sp}_4)$ collapse to $L_{Vir}(-24,0)$.

4.2. $c=-\frac{22}{5}$. In this case it is not hard to see that the weight 4 generator W and the singular vector for $V_{Vir}(-\frac{22}{5},0)$,

$$2{:}L(w)^2{:} - \frac{3}{5}\partial^2 L(w),$$

generate a maximal ideal inside $\mathcal{W}^{-\frac{22}{5}}(2,4)$. So in this case we also get that $\mathcal{W}_{-\frac{22}{5}}(2,4) = L_{Vir}(-\frac{22}{5},0)$. We again stress that $\mathcal{W}_{\frac{1}{150}\left(7\sqrt{31}\pm 337\right)}(\mathfrak{sp}_4)$ with $c = -\frac{22}{5}$ is simple and not isomorphic to $\mathcal{W}_{-\frac{22}{5}}(2,4)$.

4.3. Automorphism of $\mathcal{W}^c(2,4)$, for $c = 86 \pm 60\sqrt{2}$. Something intriguing occurs with $\mathcal{W}^{86\pm 60\sqrt{2}}(2,4)$. At these specific central charges, we can normalize the weight 4 generator W, in such a way that its operator product expansion with itself becomes non-trivial but excludes any terms involving W. In these exceptional cases, this vertex algebra exhibits a $\mathbb{Z}_2$ symmetry, while for all other values of c, $\mathcal{W}^c(2,4)$ possesses the trivial group of automorphisms. It can be verified, using calculations similar to those presented in this work or in [2], that $\left(\mathcal{W}^{86\pm 60\sqrt{2}}(2,4)\right)^{\mathbb{Z}_2}$ is of type $(2,8,10,12,14,16)$.

4.4. 2-permutation orbifold of $\mathcal{W}^c(2,4)$. In this section, we essentially repeat the outline of the previous section but now working with ${}^4\mathcal{W}^c = \mathcal{W}^c(2,4)\otimes \mathcal{W}^c(2,4)$. This algebra is strongly and freely generated by the conformal field L, a primary weight 2 field U, and two primary weight 4 fields W^+ and W^-, which have been defined in parallel to (3.7) and diagonalize the S_2 action. The OPE of these field follows from (4.1) and will not be given here.

In parallel with the previous section we introduce a starting set of generators for $\left(\mathcal{W}^4_c\right)^{S_2}$,

$$\begin{aligned}\Lambda_a &:= {:}(\partial^a U)U{:}\\ \Psi_a &:= {:}(\partial^a U)W^-{:}\\ \Omega_a &:= {:}(\partial^a W^-)W^-{:},\end{aligned}$$

for $a \geq 0$, as well as their double indexed versions

$$\begin{aligned}\Lambda_{a,b} &:= {:}(\partial^a U)(\partial^b U){:}\\ \Psi_{a,b} &:= {:}(\partial^a U)(\partial^b W^-){:}\\ \Omega_{a,b} &:= {:}(\partial^a W^-)(\partial^b W^-){:},\end{aligned}$$

for $a, b \geq 0$. Which are related via

$$\begin{aligned}\Lambda_{a,b} &= \Lambda_{a+b} + \partial^2(\text{lower weight terms})\\ \Psi_{a,b} &= (-1)^b\Psi_{a+b} + \partial(\text{lower weight terms})\\ \Omega_{a,b} &= \Omega_{a+b} + \partial^2(\text{lower weight terms}).\end{aligned} \tag{4.6}$$

Families of relations similar to those above can be used to minimize this generating set. For instance

$$\begin{aligned}\frac{7c}{120}\Psi_5 = {}& {:}\Lambda_0\Psi_1{:} - {:}\Lambda_1\Psi_0{:} - {:}L\Psi_3{:} - 4{:}W^+\Lambda_{2,1}{:} + \frac{2}{3}{:}W^+\Lambda_3{:} - \frac{5}{2}{:}(\partial L)\Psi_2{:}\\ & - 2{:}(\partial W^+)\Lambda_{1,1}{:} + \frac{1}{2}{:}(\partial W^+)\Lambda_2{:} - 2{:}(\partial^2 L)\Psi_1{:} - \frac{1}{6}{:}(\partial^3 L)\Psi_0{:}\end{aligned}$$

$$-\frac{14}{3}\partial\Psi_4+\frac{83}{6}\partial^2\Psi_3-\frac{43}{3}\partial^3\Psi_2+\frac{131}{24}\partial^4\Psi_1-\frac{17}{40}\partial^5\Psi_0+\frac{5}{112}\partial^7W^+$$

and more generally for $m \geq 1$ we have,

$$a_1^{\Psi}(m)\Psi_{m+5} = {}^\circ_\circ\Lambda_{m,1}\Psi_0{}^\circ_\circ - {}^\circ_\circ\Lambda_m\Psi_1{}^\circ_\circ + \cdots$$
$$a_2^{\Psi}(m)\Psi_{m+5} = {}^\circ_\circ\Lambda_{m,1}\Psi_{0,1}{}^\circ_\circ - {}^\circ_\circ\Lambda_m\Psi_{1,1}{}^\circ_\circ + \cdots$$

where the suppressed terms are normally ordered polynomials in the generators Λ_r and Ψ_s with $0 \leq s \leq m+4$ and the $a_i^{\Psi}(m)$ are rational functions in m. For example,

$$a_1^{\Psi}(m) = -\frac{c\left(2m^4+26m^3+113m^2+194m+105\right)+4m\left(3m^3+59m^2+252m+256\right)}{30(m+1)(m+3)(m+4)(m+5)},$$

and

$$a_2^{\Psi}(m) = \frac{c\left(6m^4+78m^3+369m^2+732m+435\right)-4m^4+452m^3+3496m^2+4048m-9792}{90(m+1)(m+3)(m+4)(m+5)},$$

for even m. It is easy to check that $a_1^{\Psi}(m)$ and $a_2^{\Psi}(m)$ are never simultaneously zero meaning that for $c \neq 0$ the only generators of the form Ψ_a that are required are Ψ_0, Ψ_1, Ψ_2, Ψ_3, and Ψ_4, where if $c = 0$ we also require Ψ_5.

Also

$$\frac{c+64}{24}\Omega_4 = {}^\circ_\circ\Lambda_0\Omega_0{}^\circ_\circ - {}^\circ_\circ\Psi_0^2{}^\circ_\circ - \frac{9}{2}\partial^2\Omega_2 + \frac{13}{12}\partial^4\Omega_0 + \cdots$$

and more generally for $m \geq 1$ we have

$$a_1^{\Omega}(m)\Omega_{m+5} = {}^\circ_\circ\Psi_{m,1}\Psi_0{}^\circ_\circ - {}^\circ_\circ\Psi_m\Psi_{0,1}{}^\circ_\circ + \cdots$$
$$a_2^{\Omega}(m)\Omega_{m+5} = {}^\circ_\circ\Psi_m\Psi_0){}^\circ_\circ - {}^\circ_\circ\Lambda_m\Omega_0{}^\circ_\circ + \cdots$$

where the suppressed terms are normally ordered polynomials in the generators $V(r,0)$, $Z(s,0)$, and $U(t,0)$ with $0 \leq t \leq m+4$ and the $a_i^{U}(m)$ are rational functions in m, for example for odd m, we have

$$a_1^{\Omega}(m) = \frac{-cm-2c-8m-8}{3(m+2)(m+4)}$$

$$a_2^{\Omega}(m) = \frac{cm^2+6cm+8c+20m^2+168m+340}{6(m+2)(m+4)(m+5)},$$

which are never simultaneously zero for appropriate values of m. As such, for $c \neq -64$ the only generators of the form Ω_a are are required are Ω_0 and Ω_2, where if $c = -64$ we also require Ω_4.

The above calculations bring us to the main result of this section

Theorem 4.1. *For*

$$c \notin \left\{-32, -12, -\frac{23}{3}, -\frac{34}{7}, -\frac{11}{5}, -\frac{3}{10}, \frac{256}{47}, \frac{32}{7}\right\},$$

the orbifold $({}^4\mathcal{W}^c)^{S_2}$ *is strongly generated by the conformal field* L*, a two weight 4 fields* W^+, Λ_0*, two weight 6 fields* Λ_2 *and* Ψ_0*, one weight 7 field* Ψ_1*, three weight 8 fields* Λ_4, Ψ_2, Ω_0*, one weight 9 field* Ψ_3*, and two weight 10 fields* Ψ_4, Ω_2*. Thus it is of type* $(2, 4^2, 6^2, 7, 8^3, 9, 10^2)$*. We also have the following exceptional cases:*

- *If* $c = \frac{256}{47}$*, we also need* Λ_6 *and the orbifold is of type*
$$(2, 4^2, 6^2, 7, 8^3, 9, 10^3).$$
- *If* $c = 0$*, we also need* Ψ_5 *and the orbifold is of type*
$$(2, 4^2, 6^2, 7, 8^3, 9, 10^2, 11).$$
- *If* $c = -64$*, we also need* Ω_4 *and the orbifold is of type*
$$(2, 4^2, 6^2, 7, 8^3, 9, 10^2, 12).$$
- *In all other cases* $({}^4\mathcal{W}^c)^{S_2}$ *is generated by vectors of weight*
$$(2, 4^2, 6^2, 7, 8^3, 9, 10^2),$$
some of which are not primary.

4.5. The simple quotient $({}^4\mathcal{W}_c)^{S_2}$ for certain values of the central charge. Similar to the previous section, we can further reduce the generating set in the simple quotient for values of the central charge that produce low weight singular vectors.

First we consider the case when $c = -\frac{68}{7}$, where as described above,
$$\mathcal{W}_{-\frac{68}{7}}(2, 4) \cong L_{Vir}(-\frac{68}{7}, 0)$$
and thus
$$\left({}^4\mathcal{W}_{-\frac{68}{7}}\right)^{S_2} \cong \left(L_{Vir}(-\frac{68}{7}, 0) \otimes L_{Vir}(-\frac{68}{7}, 0)\right)^{S_2} \cong \mathcal{W}_{-\frac{136}{7}}(2, 4).$$
as per the results in [11]. In the language of affine $\mathcal{W}$-algebras, we have
$$\left(\mathcal{W}_{-\frac{18}{7}}(\mathfrak{sp}_4)^{\otimes^2}\right)^{S_2} \cong \mathcal{W}_{-\frac{37}{14}}(\mathfrak{sp}_4), \tag{4.7}$$
and the equivalent statements for other values of the level given by Feigin-Frenkel duality.

Similarly, for $c = \frac{1}{2}$, we have $\mathcal{W}_{\frac{1}{2}}(2, 4) \cong L_{Vir}(\frac{1}{2}, 0)$ and thus
$$\left({}^4\mathcal{W}_{\frac{1}{2}}\right)^{S_2} \cong \left(L_{Vir}(\frac{1}{2}, 0) \otimes L_{Vir}(\frac{1}{2}, 0)\right)^{S_2},$$
which is of type $(2, 4, 8)$ and was described both in terms of a fermionic free field realization as well as the fixed point subalgebra of a lattice VOA in [11].

Finally, when $c = -24$ we also have a collapse
$$\left({}^4\mathcal{W}_{-24}\right)^{S_2} \cong (L_{Vir}(-24, 0) \otimes L_{Vir}(-24, 0))^{S_2},$$
which is of type $(2, 4, 6, 8)$ [11].

The remaining special cases rely on the existence of singular vectors within the orbifold $({}^4\mathcal{W}^c)^{S_2}$ and can be summarized in the following result.

Theorem 4.2. *For the following values of the central charge the simple quotient of* $\mathcal{W}_c^{S_2}$ *has the given type*

value of c	*lowest weight singular vector*	*strong generators*
$-\frac{11}{14}$	8	$2, 4^2, 6^2, 7, 8^2, 9$
$-\frac{444}{11}$	8	$2, 4^2, 6^2, 7, 8^2, 9$
-76	8	$2, 4^2, 6^2, 7, 8^2, 10$
1	8	$2, 4^2, 7, 6^2, 8^2$
-11	8	$2, 4^2, 6^2, 7, 8^2, 9$
$-\frac{136}{7}$	10	$2, 4^2, 6^2, 7, 8^3, 9, 10$
$-\frac{752}{13}$	10	$2, 4^2, 6^2, 7, 8^3, 9, 10$

Finally, we will investigate some additional structure for a selection of the cases listed in Theorem 4.2, starting with $c = -\frac{11}{14}$. In this case we have a conformal embedding of the rational Virasoro vertex algebra $L_{Vir}(-\frac{11}{14}, 0)$, so the $\mathcal{W}$-algebra decomposes as

$$\mathcal{W}_{-\frac{11}{14}}(2,4) = L_{Vir}\left(-\frac{11}{14}, 0\right) \oplus L_{Vir}\left(-\frac{11}{14}, 4\right)$$

We already know from [11] that we have an embedding of a W-algebra of type $(2,4,6,8)$ inside $({}^4\mathcal{W}_{-\frac{11}{14}})^{S_2}$. Combined with Theorem 2.2 we also get

$$\left({}^4\mathcal{W}_{-\frac{11}{14}}\right)^{S_2} = W(2,4,6,8) \oplus W_1 \oplus W_2$$

where W_1 is irreducible of lowest conformal weight 4 and W_2 is irreducible of lowest conformal weight 8.

4.6. $c = 1$. The case when $c = 1$ is of particular interest as it is tied to with the Heisenberg algebra. It is well-known that the $\mathbb{Z}_2$-orbifold of the $c = 1$ Heisenberg algebra $\mathcal{H}(1)$ is isomorphic to $\mathcal{W}_1(2,4)$. This makes $\mathcal{W}_1(2,4)$ unitary and therefore can be decomposed into a direct sum of irreducible (unitary) modules with central charge $c = 1$. Furthermore, when we take the tensor product of two copies of $\mathcal{W}_1(2,4)$ or its fixed point subalgebra $({}^4\mathcal{W}_1)^{S_2}$, we obtain a unitary Virasoro module with $c = 2$, which is also semisimple. The structure theorem for Virasoro highest weight modules at $c = 2$ tells us that for (real) $h > 0$, we have $M_{Vir}(2,h) = L_{Vir}(2,h)$, and for $h = 0$, the irreducible module $L_{Vir}(2,0)$ is the vacuum module $V_{Vir}(2,0)$. By combining this with well-known formulas for the character of $\mathcal{W}_1(2,4)^{S_2}$, we can calculate all the multiplicities a_m of $L_{Vir}(2,m)$ for $m \geq 0$ in terms of certain partitions (we omit this formula for brevity). This decomposition takes the following form:

$$\begin{aligned}\left({}^4\mathcal{W}_1\right)^{S_2} &= L_{Vir}(2,0) \oplus \sum_{m\geq 4} a_m L_{Vir}(2,m)\\ &= L_{Vir}(2,0) \oplus 2L_{Vir}(2,4) \oplus 2L_{Vir}(2,6) \oplus L_{Vir}(2,7) \oplus 5L_{Vir}(2,8) \oplus \cdots.\end{aligned}$$

This final formula clearly reveals the position of the strong generating set that we previously constructed.

We finish with a comment on the D_4-permutation orbifold of the rank 4 Heisenberg vertex operator algebra, $\mathcal{H}(4)$. We let α_1, α_2, α_3, and α_4 be the generators of this algebra with OPE given by

$$\alpha_i(z)\alpha_i(w) \sim \frac{\delta_{i,j}}{(z-w)^2}. \tag{4.8}$$

There is an obvious S_4 action on $\mathcal{H}(4)$ given by $\sigma(\alpha_i) = \alpha_{\sigma(i)}$ for $\sigma \in S_4$. We consider the subgroup $\langle (1\quad 2), (1\quad 3\quad 2\quad 4)\rangle \leq S_4$ which is isomorphic to the dihedral group D_4. Further, since $D_4 \cong V_4 \rtimes S_2$, where V_4 is the Klein-4 group, we have

$$\mathcal{H}(4)^{D_4} \cong \left(\mathcal{H}(4)^{V_4}\right)^{S_2}, \tag{4.9}$$

where $V_4 \cong \langle (1\quad 2), (3\quad 4)\rangle$ and $S_2 \cong \langle (1\quad 3)(2\quad 4)\rangle$. Through the change of basis on the generators of $\mathcal{H}(4)$ given by

$$\begin{aligned} h_1 &= \alpha_1 + \alpha_2 & h_2 &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\\ \alpha_1^- &= \alpha_1 - \alpha_2 & \alpha_2^- &= \alpha_3 - \alpha_4. \end{aligned} \tag{4.10}$$

Then, using the fact that $\mathcal{H}(1)^{\mathbb{Z}_2} \cong \mathcal{W}_1(2,4)$, it is clear that

$$\mathcal{H}(4)^{V_4} \cong \mathcal{H}(2) \otimes \mathcal{W}_1(2,4) \otimes \mathcal{W}_1(2,4) \tag{4.11}$$

which is generated by h_1, h_2, L_1, L_2, W_1, and W_2, where for $j = 1, 2$ we have

$$\begin{aligned} L_j &= \frac{1}{2} {}^\circ_\circ(\alpha_j^-)^2{}^\circ_\circ\\ W_j &= 2{}^\circ_\circ(\alpha_j^-)^4{}^\circ_\circ - 2{}^\circ_\circ(\partial^2\alpha_j^-)\alpha_j^-{}^\circ_\circ + 3{}^\circ_\circ(\partial\alpha_j^-)^2{}^\circ_\circ. \end{aligned} \tag{4.12}$$

Finally, by (4.9), we have

$$\mathcal{H}(4)^{D_4} \cong \mathcal{H}(2) \otimes \left(\mathcal{W}_1(2,4) \otimes \mathcal{W}_1(2,4)\right)^{S_2}, \tag{4.13}$$

which by Theorem 4.2 is of type $(1^2, 2, 4^2, 6^2, 7, 8^2)$.

5. Affine W-algebra $\mathcal{W}^k(\mathfrak{g}_2)$ and its 2-permutation orbifolds

One can undertake a similar analysis for the principal W-algebra associated with the exceptional Lie algebra $\mathfrak{g}_2$, $\mathcal{W}^k(\mathfrak{g}_2)$. The explicit OPEs of the generators of this algebra were given in [12] (see also [7] for affine W-algebras of $\mathfrak{g}_2$ coming from other nilpotent elements), using explicit generators of degree 2 and 6. More precisely, we showed (with Sadowski) that for

$$(336k^2 + 2301k + 3940)(588k^2 + 3991k + 6752) \neq 0,$$

$\mathcal{W}^k(\mathfrak{g}_2)$ is freely generated by the conformal vector of central charge

$$c = -\frac{2(7k+24)(12k+41)}{k+4}$$

and a primary vector of weight 6, denoted by W. We also gave explicit OPE for W with itself, resulting in some complicated formulas. In particular, we had

$$W(z)W(w) \sim \frac{p(k)}{(z-w)^{12}} + \text{lower},$$

where $p(k)$ is a monic degree 19 polynomial and all coefficients of lower terms in the OPE are polynomials in k of degree strictly less than 19. The roots of the two quadratic polynomials above are the following four irrational levels:

$$\frac{1}{1176}\left(-3991 \pm \sqrt{47377}\right), \frac{1}{672}\left(-2301 \pm i\sqrt{759}\right),$$

and the corresponding central charges are $c = -\frac{68}{7}$ and $c = \frac{1}{2}$. For these levels, the weight 6 generator has to be adjusted and is no longer primary, so we have the same situation as in the case of $\mathcal{W}^k(\mathfrak{sp}_4)$ of central charge $-\frac{22}{5}$, discussed in Section 4.1. Clearly, at these four values of the level the affine W-algebra $\mathcal{W}^k(\mathfrak{g}_2)$ is generic and simple [3].

As in previous sections we denote the generators for the two copies of $\mathcal{W}^k(\mathfrak{g}_2)$ by $L_1(z)$, $L_2(z)$, $W_1(z)$, and $W_2(z)$, and then set

$$L(z) = L_1(z) + L_2(z),$$
$$U(z) = L_1(z) - L_2(z),$$

and

$$W^+(z) = W_1(z) + W_2(z),$$
$$W^-(z) = W_1(z) - W_2(z).$$

We also set ${}^6\mathcal{W}^c = (\mathcal{W}^k(\mathfrak{g}_2))^{\otimes 2}$, where the OPEs have been parameterized with respect to the central charge c. Next, introduce a starting set of generators for $\left({}^6\mathcal{W}_c\right)^{S_2}$,

$$\Lambda_a := {:}(\partial^a U)U{:}$$
$$\Psi_a := {:}(\partial^a U)W^-{:}$$
$$\Omega_a := {:}(\partial^a W^-)W^-{:},$$

for $a \geq 0$, as well as their double indexed versions

$$\Lambda_{a,b} := {:}(\partial^a U)(\partial^b U){:}$$
$$\Psi_{a,b} := {:}(\partial^a U)(\partial^b W^-){:}$$
$$\Omega_{a,b} := {:}(\partial^a W^-)(\partial^b W^-){:},$$

for $a, b \geq 0$. Which are related via

$$\begin{aligned} \Lambda_{a,b} &= \Lambda_{a+b} + \partial^2(\text{lower weight terms}) \\ \Psi_{a,b} &= (-1)^b \Psi_{a+b} + \partial(\text{lower weight terms}) \\ \Omega_{a,b} &= \Omega_{a+b} + \partial^2(\text{lower weight terms}). \end{aligned} \tag{5.1}$$

Of course generators of the form Λ_a and Ψ_a are reduced via calculations very similar to those above. In this case we require the generators $\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4$, which are of weight 8,9,10,11,12, for $c \neq -\frac{870}{7}$. In the case that $c = -\frac{870}{7}$ we also require Ψ_5

For the generators of the form Ω_a we employ a different strategy for building the necessary relations that simplifies the calculations. Set

$$R_m^6 = {}^\circ_\circ\Lambda_{2m-1}\Lambda_1\Omega_0{}^\circ_\circ - {}^\circ_\circ\Lambda_{2m-1,1}\Lambda_0\Omega_0{}^\circ_\circ$$

and observe that since the W^- fields stay in the rightmost term of this expression reassociating this object only requires the OPEs $U(z)W^-(w)$, $L(z)W^{\pm}(w)$, and $W^+(z)W^-(w)$, all of which are quite simple. Importantly, we do not require the OPEs of $W^+(z)W^+(w)$ or $W^-(z)W^-(w)$, which are unreasonably complicated for direct calculation. That being said, we do require a few more parts to end at a usable relation. These include

$$\begin{aligned} R_m^5 = &- \frac{2m-1}{2m+1}{}^\circ_\circ L\Lambda_{2m+1,1}\Omega_0{}^\circ_\circ - \frac{4m^2+8m+5}{(2m+1)(m+1)}{}^\circ_\circ L\Lambda_{2m+2}\Omega_0{}^\circ_\circ - 2{}^\circ_\circ W^+\Lambda_{2m-1,3}\Psi_0{}^\circ_\circ \\ &- \frac{10m^2+3m+2}{2m(2m+1)}{}^\circ_\circ(\partial L)\Lambda_0\Omega_0{}^\circ_\circ - \frac{1}{6}{}^\circ_\circ(\partial^3 L)\Lambda_{2m-1}\Omega_0{}^\circ_\circ - \frac{2m-1}{m}{}^\circ_\circ(\partial W^+)\Lambda_{2m,1}\Psi_0{}^\circ_\circ \\ &+ 12{}^\circ_\circ W^+\Lambda_{2m-1,2}\Psi_1{}^\circ_\circ - \frac{2m-1}{2m}{}^\circ_\circ(\partial L)\Lambda_{2m,1}\Omega_0{}^\circ_\circ - \frac{6(2m-1)}{2m+1}{}^\circ_\circ W^+\Lambda_{2m+1,1}\Psi_0{}^\circ_\circ \\ &+ 4{}^\circ_\circ(\partial W^+)\Lambda_{2m-1,1}\Psi_1{}^\circ_\circ \end{aligned}$$

$$\begin{aligned} R_m^4 = &\frac{80m^3+282m^2+237m+4}{2m(m+2)(2m+1)(2m+3)}{}^\circ_\circ\Lambda_0\Omega_{2m+4}{}^\circ_\circ \\ &+ \frac{(2m-1)(60m^2+64m+3)}{2m(m+1)(2m+1)(2m+3)}{}^\circ_\circ\Lambda_1\Omega_{2m+3}{}^\circ_\circ \\ &- \frac{2(5m-1)}{m(m+1)(2m+1)}{}^\circ_\circ\Lambda_{1,1}\Omega_{2m+2}{}^\circ_\circ + \frac{5m-1}{2m(m+1)(2m+1)}{}^\circ_\circ\Lambda_2\Omega_{2m+2}{}^\circ_\circ \\ &- \frac{6(10m-1)}{m(2m+1)}{}^\circ_\circ\Lambda_{2,1}\Omega_{2m+1}{}^\circ_\circ + \frac{10m-1}{m(2m+1)}{}^\circ_\circ\Lambda_3\Omega_{2m+1}{}^\circ_\circ + \frac{127}{60}{}^\circ_\circ\Lambda_{2m-1}\Omega_5{}^\circ_\circ \\ &- \frac{50}{3}{}^\circ_\circ\Lambda_{2m-1,1}\Omega_4{}^\circ_\circ - \frac{35}{2}{}^\circ_\circ\Lambda_{2m-1,2}\Omega_3{}^\circ_\circ + 4{}^\circ_\circ\Lambda_{2m-1,3}\Omega_2{}^\circ_\circ + \frac{2m-1}{3m}{}^\circ_\circ\Lambda_{2m}\Omega_4{}^\circ_\circ \\ &+ \frac{2m-1}{2m}{}^\circ_\circ\Lambda_{2m,1}\Omega_{3,0}{}^\circ_\circ - \frac{2m-1}{2m}{}^\circ_\circ\Lambda_{2m,2}\Omega_2{}^\circ_\circ - \frac{2m-1}{m}{}^\circ_\circ\Lambda_{2m,3}\Omega_1{}^\circ_\circ \\ &+ \frac{20m^2-6m+1}{m(2m+1)}{}^\circ_\circ\Lambda_{2m+1}\Omega_3{}^\circ_\circ + \frac{2(10m+1)(m-1)}{m(2m+1)}{}^\circ_\circ\Lambda_{2m+1}\Omega_3{}^\circ_\circ \\ &+ \frac{2(m-1)(10m+1)}{m(2m+1)}{}^\circ_\circ\Lambda_{2m+1,1}\Omega_2{}^\circ_\circ - \frac{38m^2+29m+12}{2m(2m+1)}{}^\circ_\circ\Lambda_{2m+1,2}\Omega_1{}^\circ_\circ \\ &- \frac{142m^2-67m+1}{12m(2m+1)}{}^\circ_\circ\Lambda_{2m+1,3}\Omega_0{}^\circ_\circ + \frac{32(m+2)}{2m+1}{}^\circ_\circ\Lambda_{2m+2}\Omega_2{}^\circ_\circ \end{aligned}$$

$$- \frac{124m^2+336m+221}{6(m+1)(2m+1)} {}^\circ_\circ \Lambda_{2m+2,2}\Omega_1 {}^\circ_\circ - \frac{52m^2+144m+95}{(m+1)(2m+1)} {}^\circ_\circ \Lambda_{2m+2,2}\Omega_0 {}^\circ_\circ$$
$$- \frac{46m^2+167m+157}{12(m+1)(2m+3)} {}^\circ_\circ \Lambda_{2m+3}\Omega_1 {}^\circ_\circ$$
$$- \frac{c(4m^4+4m^3-m^2-m)+824m^4+3224m^3+4054m^2+1360m-6}{24m(m+1)(2m+1)(2m+3)} {}^\circ_\circ \Lambda_{2m+3,1}\Omega_0 {}^\circ_\circ$$
$$- \frac{c(12m^2+52m+61)+500m^2+2460m+3010}{60(m+2)(2m+3)} {}^\circ_\circ \Lambda_{2m+4}\Omega_0 {}^\circ_\circ + \cdots,$$

where the appended terms, $\cdots$, are of the form ${}^\circ_\circ(\partial^a L)(\partial^b W^+)\Psi_{c,d}{}^\circ_\circ$ and also do not require reassociation to be in terms of the orbifold generators. There is a similar cubic term R_m^3 that is a linear combination of terms of the form ${}^\circ_\circ(\partial^a L)\Omega_{b,c}{}^\circ_\circ$ and ${}^\circ_\circ(\partial^a W^+)\Psi_{b,c}{}^\circ_\circ$, which also do not require reassociation. Finally, we have

$$R_m^2 = - \frac{589(10m-1)}{168m(2m+1)}\Omega_{2m+1,7} + \frac{30m^2-833m+169}{20m(2m+1)(m+1)}\Omega_{2m+2,6}$$
$$+ \frac{520m^3+10648m^2+9568m-739}{80m(m+1)(2m+1)(2m+3)}\Omega_{2m+3,5}$$
$$- \frac{880m^4+3922m^3+5461m^2+3371m+1174}{12m(m+1)(m+2)(2m+1)(2m+3)}\Omega_{2m+4,4}$$
$$- \frac{840m^5+3316m^4+22738m^3+58475m^2+41687m+6066}{24m(m+1)(m+2)(2m+1)(2m+3)(2m+5)}\Omega_{2m+5,3}$$
$$- \frac{26040m^5+270916m^4+997238m^3+1517899m^2+791997m+22410}{60m(m+1)(m+3)(2m+1)(2m+3)(2m+5)}\Omega_{2m+6,2}$$
$$- \frac{p_1(m)}{120m(m+1)(m+2)(2m+1)(2m+3)(2m+5)(2m+7)}\Omega_{2m+7,1}$$
$$+ \frac{p_2(m)}{5040m(m+1)(m+2)(m+3)(m+4)(2m+1)(2m+3)(2m+5)(2m+7)}\Omega_{2m+8}.$$

where

$$p_1(m) = 21040m^6 + 225456m^5 + 838488m^4 + 1374016m^3 + 1036127m^2 + 400733m + 88350$$

and

$$p_2(m) = c(20160m^8 + 352744m^7 + 2774884m^6 + 12140254m^5 + 30256975m^4 + 40851706m^3 + 26368951m^2 + 6076266m) + 5639040m^8 + 77127136m^7 + 385630000m^6 + 773436664m^5 + 85506820m^4 - 1846104176m^3 - 2278390640m^2 - 710866284m + 59623200.$$

All of these expressions are connected via

$$R_m^2 = R_m^5 + R_m^4 + R_m^3,$$

which holds for $m \geq 4$. Now using (5.1), we have

$$R_m^2 = \frac{f_1(m)}{g_1(m)}\Omega_{2m+8} + \partial^2(\text{lower}),$$

where

$$\begin{aligned} f_1(m) = c(&2880m^8 + 50392m^7 + 396412m^6 + 1734322m^5 \\ &+ 4322425m^4 + 5835958m^3 + 3766993m^2 + 868038m) \\ &+ 337920m^8 + 609088m^7 - 39488072m^6 - 344996708m^5 \\ &- 1243829570m^4 - 2247062903m^3 - 1988750273m^2 \\ &- 679886262m + 78120 \end{aligned}$$

and

$$g_1(m) = 720m(m+1)(m+2)(m+3)(m+4)(2m+1)(2m+3)(2m+5)(2m+7).$$

Combining everything together, we have

$$\frac{f_1(m)}{g_1(m)}\Omega_{2m+8} = R^6_m + R^5_m + R^4_m + R^3_m + \partial^2(\text{lower}).$$

We can similarly define

$$\widehat{R}^6_m = {}^\circ_\circ\Lambda_{2m-2}\Lambda_{1,1}\Omega_0{}^\circ_\circ - {}^\circ_\circ\Lambda_{2m-2,1}\Lambda_1\Omega_0{}^\circ_\circ,$$

and companion expressions $\widehat{R}^4_m$, $\widehat{R}^3_m$, and $\widehat{R}^2_m$, which leads to

$$\frac{f_2(m)}{g_2(m)}\Omega_{2m+8} = \widehat{R}^6_m + \widehat{R}^5_m + \widehat{R}^4_m + \widehat{R}^3_m + \partial^2(\text{lower})$$

for $m \geq 4$, where

$$\begin{aligned} f_1(m) = c(&2880m^8 + 50392m^7 + 396412m^6 + 1734322m^5 \\ &+ 4322425m^4 + 5835958m^3 + 3766993m^2 + 868038m) \\ &+ 337920m^8 + 609088m^7 - 39488072m^6 - 344996708m^5 \\ &- 1243829570m^4 - 2247062903m^3 - 1988750273m^2 \\ &- 679886262m + 78120 \end{aligned}$$

and

$$g_2(m) = 720m(m+1)(m+2)(m+3)(m+4)(2m+1)(2m+3)(2m+5)(2m+7).$$

Next, define

$$\begin{aligned} &a_1(m) = \\ &1440m(m+1)(m+2)(m+3)(m+4)(2m-1)(2m+1)(2m+3)(2m+5)(2m+7)\cdot \\ &\cdot\big(5460m^6 + 95524m^5 + 747187m^4 + 3068993m^3 + 6606719m^2 + 6802293m + 2480814\big), \end{aligned}$$

$$\begin{aligned} b_1(m) = {}&23040000000m^{15} + 705894528000m^{14} + 9765464009600m^{13} \\ &+ 79149493837760m^{12} + 407269605851616m^{11} \\ &+ 1337405203436816m^{10} + 2571286617208376m^9 \\ &+ 1673660577799188m^8 - 4340377921054766m^7 \\ &- 11966909453679605m^6 - 11085412237308783m^5 \end{aligned}$$

$$- 1066807248803933m^4 + 4930749881340985m^3 + 2250198386390034m^2 - 528331939883928m - 342669742425360,$$

$$a_2(m) = 5040m(m+1)^2(m+2)(m+3)(m+4)(2m-1)(2m+1)(2m+3)(2m+5)(2m+7)\cdot \cdot \left(1440m^5 + 23036m^4 + 162932m^3 + 611245m^2 + 1162879m + 868038\right),$$

and

$$\begin{aligned} b_2(m) = {} & 23040000000m^{15} + 705894528000m^{14} + 9765464009600m^{13} \\ & + 79149493837760m^{12} + 407269605851616m^{11} \\ & + 1337405203436816m^{10} + 2571286617208376m^9 \\ & + 1673660577799188m^8 - 4340377921054766m^7 \\ & - 11966909453679605m^6 - 11085412237308783m^5 \\ & - 1066807248803933m^4 + 4930749881340985m^3 \\ & + 2250198386390034m^2 - 528331939883928m \\ & - 342669742425360. \end{aligned}$$

It is clear that $a_1(m), a_2(m), b_1(m)$, and $b_2(m)$ are all nonzero for all positive integers m. Further, these polynomials have been constructed so that

$$\begin{aligned} \Omega_{2m+8} = & \frac{a_1(m)}{b_1(m)}(R_m^6 + R_m^5 + R_m^4 + R_m^3 + \partial^2(\text{lower})) \\ & + \frac{a_2(m)}{b_2(m)}(\widehat{R}_m^6 + \widehat{R}_m^5 + \widehat{R}_m^4 + \widehat{R}_m^3 + \partial^2(\text{lower})), \end{aligned}$$

for $m \geq 4$. By repeated uses of these relations we can remove the need for the generators $\Omega_{16}, \Omega_{18}, \Omega_{20}, \cdots$. At this point, we require the generators Ω_0,Ω_2, $\ldots$, Ω_{14}, which are of weight 12,14,16,18,20,22,24, and 26. The generators Ω_{10}, Ω_{12}, and Ω_{14} can be eliminated with R_m^6, R_m^5, R_m^4, and R_m^3 with $\widehat{R}_m^6, \widehat{R}_m^5, \widehat{R}_m^4$, and $\widehat{R}_m^3$ for $m = 1, 2, 3$ where a companion quadratic expression can be constructed for each of these cases. It remains to remove the need for Ω_4, Ω_6, and Ω_8 which will be done each with their own relation. For example, we have

$$\frac{c+260}{24}\Omega_4 = {}^{\circ}_{\circ}\Psi_0\Psi_0{}^{\circ}_{\circ} - {}^{\circ}_{\circ}\Lambda_0\Omega_0{}^{\circ}_{\circ} - 12\partial^2\Omega_2 + 3\partial^4\Omega_0 + \cdots,$$

where the $\cdots$ represents a normally ordered polynomial in L, W^+, and the generators Ψ_a and Λ_b, that is at most linear in Ψ_a generators. As such, if $c \neq -260$, Ω_4 is not needed as a generator. At higher weights there are pairs of expressions of the form ${}^{\circ}_{\circ}\Psi_a\Psi_b{}^{\circ}_{\circ} - {}^{\circ}_{\circ}\Lambda_c\Omega_d{}^{\circ}_{\circ}$ that allow us to remove the remaining generators without a restriction on c. In particular, we have

$$\begin{aligned} \Omega_6 = & \frac{40}{223}({}^{\circ}_{\circ}\Psi_2\Omega_0{}^{\circ}_{\circ} - {}^{\circ}_{\circ}\Lambda_2\Omega_0{}^{\circ}_{\circ}) + \frac{100}{669}({}^{\circ}_{\circ}\Psi_{1,1}\Omega_0{}^{\circ}_{\circ} - {}^{\circ}_{\circ}\Lambda_1\Omega_1{}^{\circ}_{\circ}) - \frac{1565}{4014}\partial^2\Omega_4 \\ & + \frac{1565}{4014}\partial^4\Omega_2 - \frac{313}{4014}\partial^6\Omega_0 + \cdots, \end{aligned}$$

where the $\cdots$ represents the same type of normally ordered polynomial described above. All of this brings us to the following result.

Theorem 5.1. *For*

$$c \notin \left\{\frac{256}{47}, -\frac{870}{7}, -260\right\},$$

the orbifold $\left({}^3\mathcal{W}^c\right)^{S_2}$ *is strongly generated by the conformal field L, a weight 4 field* Λ_0*, two weight 6 fields* Λ_2, W^+*, two weight 8 fields* Λ_4*,* Ψ_0*, a weight 9 field* Ψ_1*, a weight 10 field* Ψ_2*, a weight 11 field* Ψ_3*, two weight 12 fields* Ψ_4*,* Ω_0*, and a weight 14 field* Ω_2*. We also have the following exceptional cases:*

- *If* $c = \frac{256}{47}$*, we also need* Λ_6*.*
- *If* $c = -\frac{870}{47}$*, we also need* Ψ_5*.*
- *If* $c = -260$*, we also need* Ω_4*.*

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