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Further results on vanishing coefficients in the series expansion of lacunary eta quotients

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ABSTRACT. For a function $A(q) = \sum_{n\geq 0} a_n q^n$ define $A_{(0)} = \{n \in \mathbb{N} : a_n = 0\}$. If $A(q)$ and $B(q)$ satisfy $A_{(0)} = B_{(0)}$, then we say that $A(q)$ and $B(q)$ have *identically vanishing coefficients*.

In a previous paper the authors proved the existence of various pairs $(A(q), B(q))$ of lacunary eta quotients with identically vanishing coefficients. The work in that previous paper was motivated by a result of Han and Ono, who showed that f_1^8 and f_3^3/f_1 have identically vanishing coefficients (here $f_i = \prod_{n=0}^{\infty}(1-q^{in})$). In each of these pairs, one of the eta quotients was a power of f_1, whose lacunarity was described in a paper by Serre.

Further experiments indicated that the results in this previous paper were just the "tip of the iceberg". In the current work, we demonstrate that there is a much larger list of eta quotients $B(q)$ having coefficients that vanish identically with those of $A(q) = f_1^6$. Similar results hold for f_1^r, $r = 4, 8, 10, 14, 26$, and for $f_1^3 f_2^3$. A natural network structure exists on the set of eta quotients $C(q)$ for which $A_{(0)} \subsetneqq C_{(0)}$. The network may be exhibited by partially ordering the sets of vanishing coefficients by inclusion and constructing a directed graph of collections of eta quotients that have identically vanishing coefficients.

We provide a comprehensive description of what experiment suggests and employ a variety of methods to prove experimentally-derived results. The work is a template and atlas for the subsequent study of lacunary eta quotients. A broad range of proof strategies are applied to confirm the vanishing structure. These comprise a representative sample of techniques that may be used to study the remaining observations and conjectures resulting from the work.

CONTENTS

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1. Introduction

The work in the present paper is a continuation of work begun in [17], and that work in turn was motivated by a result of Han and Ono in [12]. To state this result, first define the sequences $\{a_n\}$ and $\{b_n\}$ by

$$f_1^8 =: \sum_{n=0}^{\infty} a_n q^n, \qquad \frac{f_3^3}{f_1} =: \sum_{n=0}^{\infty} b_n q^n, \qquad f_i := \prod_{n=1}^{\infty}(1-q^{in}), \quad i \in \mathbb{Z}^+. \tag{1.1}$$

The result of Han and Ono may be stated as follows.

Theorem 1.1. (*Han and Ono,* [12, Theorem 1.4, page 307]) *Assuming the notation above, we have that*

$$a_n = 0 \quad \textit{if and only if} \quad b_n = 0. \tag{1.2}$$

Moreover, we have that $a_n = b_n = 0$ precisely for those non-negative n for which $ord_p(3n+1)$ is odd for some prime $p \equiv 2 \pmod 3$.

Their result motivated the work in [17], where experimental investigations were conducted to determine if a similar situation held for other pairs of eta quotients. Recall that an *eta quotient* is a finite product of the form $\prod_j (q^{j/24} f_j)^{n_j}$, for some $j \in \mathbb{N}$ and some $n_j \in \mathbb{Z}$, and that if all $n_j > 0$ the product is termed an *eta product*.

What was eventually proven in [17] may be summarized as follows.

Theorem 1.2. (*Huber, Mc Laughlin, Ye* [17, Theorems 3.1, 4.1, 5.2, 6.1 - 11.1]) *Let $(A(q), B(q))$ be any of the pairs*

$$\left\{\left(f_1^4, \frac{f_1^8}{f_2^2}\right), \left(f_1^4, \frac{f_1^{10}}{f_3^2}\right), \left(f_1^6, \frac{f_2^4}{f_1^2}\right), \left(f_1^6, \frac{f_1^{14}}{f_2^4}\right), \left(f_1^{10}, \frac{f_2^6}{f_1^2}\right), \left(f_1^{14}, \frac{f_3^5}{f_1}\right), \left(f_1^{14}, \frac{f_2^8}{f_1^2}\right)\right\}. \tag{1.3}$$

For any such pair $(A(q), B(q))$, define the sequences $\{a_n\}$ and $\{b_n\}$ by

$$A(q) =: \sum_{n=0}^{\infty} a_n q^n, \qquad B(q) =: \sum_{n=0}^{\infty} b_n q^n. \tag{1.4}$$

Then

$$a_n = 0 \quad \textit{if and only if} \quad b_n = 0,$$

with the criteria determining precisely the n for which $a_n = b_n = 0$ being those of Serre for $a_n = 0$. For the pairs

$$\left\{ \left(f_1^{26}, \frac{f_3^9}{f_1} \right), \left(f_1^{26}, \frac{f_2^{16}}{f_1^6} \right) \right\} \tag{1.5}$$

one has that

$$a_n = b_n = 0$$

if $12n + 13$ satisfies a criteria of Serre for $a_n = 0$.

Remark 1.3. *For a precise statement of the criteria under which $a_n = b_n = 0$, see Serre's paper* [29] *or Theorem* 3.4.

Before discussing this phenomenon further, we introduce some new notation. For a function $A(q) = \sum_{n\geq 0} a_n q^n$ we write

$$A_{(0)} := \{n \in \mathbb{N} : a_n = 0\}.$$

If $A(q)$ and $B(q)$ are two functions for which $A_{(0)} = B_{(0)}$, then for ease of discussion, we say that *the coefficients of $A(q)$ and $B(q)$ vanish identically*, or that $A(q)$ and $B(q)$ have *identically vanishing coefficients*. If $A_{(0)} \subseteq B_{(0)}$, we say that $B(q)$ has *vanishing behaviour similar to $A(q)$*.

A series $\sum_{n=0}^{\infty} c_n q^n$ satisfying

$$\lim_{x\to\infty} \frac{|\{0 \leq n \leq x \mid c_n) = 0\}|}{x} = 1.$$

is said to be *lacunary*, and indeed Serre showed that all the first components of each pair in (1.3) and (1.5) are lacunary.

Theorem 1.2 implies that all three eta quotients in each of the following triples

$$\left\{ f_1^4, \frac{f_1^8}{f_2^2}, \frac{f_1^{10}}{f_3^2} \right\}, \left\{ f_1^6, \frac{f_2^4}{f_1^2}, \frac{f_1^{14}}{f_2^4} \right\}, \left\{ f_1^{14}, \frac{f_3^5}{f_1}, \frac{f_2^8}{f_1^2} \right\}, \left\{ f_1^{26}, \frac{f_3^9}{f_1}, \frac{f_2^{16}}{f_1^6} \right\}$$

have identically vanishing coefficients, and thus all 12 of these eta quotients are lacunary. This observation caused us to wonder if this phenomenon of identically vanishing coefficients might exist more widely, so we conducted further experiments.

What we discovered experimentally appeared to suggest that the results in Theorem 1.2 were, in fact, just "the tip of the iceberg". For example, our limited search (see Section 2 for details about the extent of this search) found 42 eta quotients $B(q)$ for which it appears $f^6_{1(0)} = B_{(0)}$. In addition, this search found 130 additional eta quotients with the property that for each such eta quotient $B(q)$, it seems $f^6_{1(0)} \subsetneqq B_{(0)}$. Moreover, it appears that all 172 eta quotients $B(q)$ may be organized into 28 collections (labelled I - XXVIII in what follows) in a directed graph structure by partially ordering the corresponding $B_{(0)}$ by proper inclusion.

Table 1: Eta quotients with vanishing behavior similar to f_1^6

Collection	# of eta quotients in Collection	Collection	# of eta quotients in Collection
I	42	II	4
III	4	IV	16
V	2	VI	2
VII	4	VIII	4
IX	4	X	10
XI	2	XII	4
XIII	8	XIV	4
XV	8	XVI	10
XVII	2	XVIII	2
XIX	2	XX	4
XXI	6	XXII	2
XXIII	4	XXIV	4
XXV	4	XXVI	2
XXVII	6	XXVIII	6

For example, all 8 eta quotients in the collection labelled XV appear to have identically vanishing coefficients (and likewise for any other pair of eta quotients that both lie in any of the other collections). Note that collection I is the one containing f_1^6. The relationships between eta quotients in different collections is illustrated in Figure 1.

Thus the arrow from XXVII to XVIII indicates that if $A(q)$ is any of the 6 eta quotients in collection XXVII and $B(q)$ is any of the 6 eta quotients in collection XVIII, then $A_{(0)} \subsetneqq B_{(0)}$. A similar meaning for any other arrow, in this figure or any of the other figures in the paper, is to be understood.

These vanishing coefficient phenomena are more easily understood if a dilation $q \to q^c$, where $c > 1$ is a positive integer, is applied to an eta quotient $A(q) = \sum a_n q^n$ and the result is multiplied by q^d, where $d > 1$ is a positive integer to produce a modular form $A'(q) = \sum a_n q^{cn+d}$. In particular, throughout the remainder of this work, we call $A(q)$ an eta quotient of weight k whenever its associated modular form $A'(q)$ is of weight k. The greater structure of the lacunary modular form $A'(q)$ maybe be used to get precise information about when $a_n = 0$ (see Section 3 for more detailed explanation of this).

We stress that at this point we have not proven all of the hundreds of examples of these "$A_{(0)} = B_{(0)}$" and "$A_{(0)} \subsetneqq B_{(0)}$" phenomena. We indicate various procedures whereby a particular example may be proved (with varying degrees

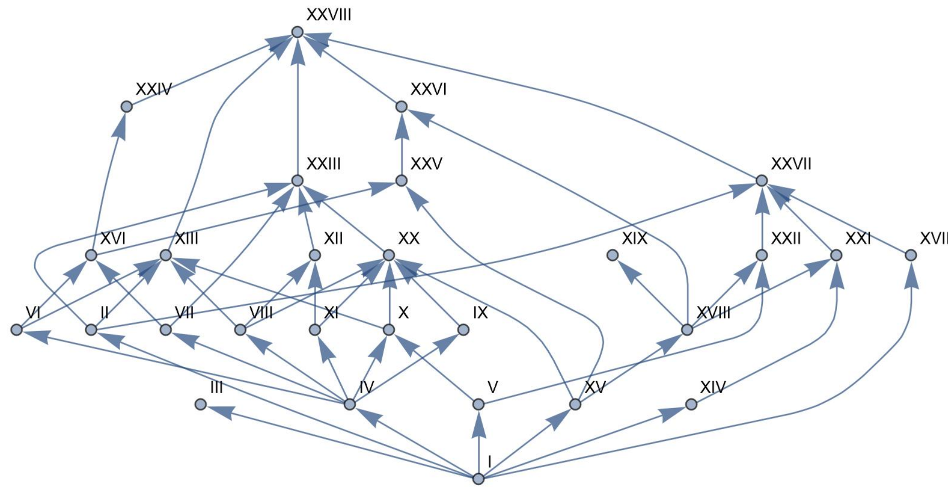

FIGURE 1. The grouping of eta quotients in Table 9, which have vanishing coefficient behaviour similar to f_1^6

of difficulty) in particular cases, if sufficient information (for example, representations of the eta quotient in terms of CM forms and theta series) is available. We prove several examples in later sections by way of illustration.

We briefly preview the method of proof here by way of some examples, but before getting to that, we recall that $\eta(z) = q^{1/24} f_1$, where $q = e^{2\pi i z}$, is the Dedekind eta function. If $A(q) = f_1^6 = \sum a_n q^n$ then by considering $\eta(4z)^6 = \sum a_n q^{4n+1}$, Serre [29] showed that $a_n = 0$ if and only if $4n+1$ has a prime factor $p \equiv -1 \pmod 4$ with odd exponent. If $B(q) = f_1^2 f_2^2 = \sum_{n=0}^{\infty} b_n q^n$, then by considering $\eta(4z)^2\eta(8z)^2 = \sum_{n=0}^{\infty} b_n q^{4n+1}$, it was shown in [18, Theorem 1.1, part (2)] that $b_n = 0$ if and only if $4n+1$ has a prime factor $p \equiv -1 \pmod 4$ with odd exponent, and thus $A_{(0)} = B_{(0)}$. Similarly, if $C(q) = f_1 f_5 = \sum_{n=0}^{\infty} c_n q^n$, by considering $\eta(4z)\eta(20z) = \sum_{n=0}^{\infty} c_n q^{4n+1}$, it was shown in [18, Theorem 1.1, part (15)] that $c_n = 0$ if and only if $4n+1$ has a has a prime factor $p \not\equiv 1, 9 \pmod{20}$ with odd exponent, thus showing that $A_{(0)} \subsetneqq C_{(0)}$.

These examples provide some insight into the directed graph structure exhibited in Figure 1, and the other figures, as follows. Let $A(q) = \sum a_n q^n$ and $B(q) = \sum b_n q^n$ be two eta quotients, and let $A'(q) = \sum a_n q^{cn+d}$ and $B'(q) = \sum b_n q^{cn+d}$ be the corresponding modular forms. Suppose that $a_n = 0$ if and only if the prime factorization of $cn+d$ satisfies some specified condition (such as containing a prime factor $p \equiv 3 \pmod 4$ with odd exponent, or consisting entirely of primes $\equiv 11 \pmod{12}$ with even exponents). If $b_n = 0$ under exactly the same conditions, then $A_{(0)} = B_{(0)}$ and hence $A(q)$ and $B(q)$ lie in the same collection. On the other hand, if $b_n = 0$ under the same conditions but $b_n = 0$ also if some other condition on the prime factorization of

$cn + d$ is satisfied, then $A_{(0)} \subsetneq B_{(0)}$, and in the figure showing the relationships between the various collections of eta quotients, there would be an arrow from the collection containing $A(q)$ to the collection containing $B(q)$.

An obvious question that arises is the following: If $A(q)$ is *any* lacunary eta quotient, does a similar situation hold? In other words, do there exist other lacunary eta quotients $B(q)$ such that $A_{(0)} = B_{(0)}$ or $A_{(0)} \subsetneq B_{(0)}$? To study this question we conducted a similar investigation of a lacunary eta quotient in one of the infinite families of such lacunary eta quotients given by Ono and Robins [25, page 1027], namely the simplest case of their first infinite family, $f_1^3 f_2^3$. It turns out that a similar situation holds (see Table 19 and Figure 8). Moreover, we also find a criterion for the vanishing of the coefficients of $f_1^3 f_2^3$ in a fashion of Serre's (see Theorem 5.44).

In this work, we shall exemplify both relations $A_{(0)}) = B_{(0)}$ and $A_{(0)} \subsetneq B_{(0)}$ for a large number of eta quotients $A(q)$ and $B(q)$ with a directed graph structure rooted in f_1^r for $r \in \{1, 2, 3, 4, 6, 8, 10, 14\}$ or $f_1^3 f_2^3$ in order to provide the reader with some insight into the study of these phenomena. To facilitate the reader with browsing the examples we shall prove, we summarize all of them in the following table as a directory for the reader.

Table	Number (Group) of $A(q)$	Number (Group) of $B(q)$	Relation	Location
3	1 - 6 (I)	1 - 6 (I)	$A_{(0)} = B_{(0)}$	Theorem 4.2
4	1 - 4 (I)	1 - 4 (I)	$A_{(0)} = B_{(0)}$	Theorem 4.6
4	1 - 4 (I)	5 - 10 (II)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 4.7
4	5 - 10 (II)	5 - 10 (II)	$A_{(0)} = B_{(0)}$	Theorem 4.7
5	1 - 6 (I)	1 - 6 (I)	$A_{(0)} = B_{(0)}$	Theorem 4.9
5	1 - 6 (I)	7 - 12 (II)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 4.9
5	7 - 12 (II)	7 - 12 (II)	$A_{(0)} = B_{(0)}$	Theorem 4.9
7	1 (I)	139 - 142 (XVII)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.3
7	139 - 142 (XVII)	145 - 150 (XIX)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.3
7	139 - 142 (XVII)	139 - 142 (XVII)	$A_{(0)} = B_{(0)}$	Theorem 5.3
7	145 - 150 (XIX)	145 - 150 (XIX)	$A_{(0)} = B_{(0)}$	Theorem 5.3
7	1 (I)	120 (XI)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.5
7	1 (I)	91, 93, 95 (VII)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.6
7	1 (I)	131 (XIV)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.6
7	74 (II)	75 (II)	$A_{(0)} = B_{(0)}$	Theorem 5.7
9	1 (I)	22 (I)	$A_{(0)} = B_{(0)}$	Equation (5.13)
9	1 (I)	130 (XVIII)	$A_{(0)} \subsetneq B_{(0)}$	Equation (5.13)
9	1 (I)	141 (XXII)	$A_{(0)} \subsetneq B_{(0)}$	Equation (5.13)
9	1 (I)	153 (XXV)	$A_{(0)} \subsetneq B_{(0)}$	Equation (5.13)
9	1 (I)	161 (XXVIII)	$A_{(0)} \subsetneq B_{(0)}$	Equation (5.14)
9	1 (I)	170 (XXIX)	$A_{(0)} \subsetneq B_{(0)}$	Equation (5.14)
9	1 (I)	65 (IV)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.11
9	1 (I)	151 - 154 (XXV)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.12
9	151 - 154 (XXV)	151 - 154 (XXV)	$A_{(0)} = B_{(0)}$	Theorem 5.12
9	151 - 154 (XXV)	167 - 172 (XXIX)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.12
9	1 (I)	161 - 162 (XXVIII)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.12
9	161 - 162 (XXVIII)	167 - 172 (XXIX)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.12
9	161 - 162 (XXVIII)	161 - 162 (XXVIII)	$A_{(0)} = B_{(0)}$	Theorem 5.12
9	167 - 172 (XXIX)	167 - 172 (XXIX)	$A_{(0)} = B_{(0)}$	Theorem 5.12

9	1 (I)	46 (II)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.13
9	1 (I)	49 (III)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.13
9	52 (IV)	61 (IV)	$A_{(0)} = B_{(0)}$	Theorem 5.15
9	1 (I)	25, 33 (I)	$A_{(0)} \subseteq B_{(0)}$	Theorem 5.17
9	1 (I)	67 (V)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.17
9	1 (I)	129 (XVIII)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.17
9	1 (I)	145 (XXIII)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.17
9	1 (I)	41 (I)	$A_{(0)} \subseteq B_{(0)}$	Theorem 5.18
9	1 (I)	43 (II)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.18
9	1 (I)	107 (XIV)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.18
9	1 (I)	139 (XXII)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.18
11	1 (I)	59 (IV)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.19
11	1 (I)	14 (I)	$A_{(0)} = B_{(0)}$	Equation (5.33)
11	1 (I)	90 (V)	$A_{(0)} \subsetneq B_{(0)}$	Equation (5.33)
11	1 (I)	128 (XVII)	$A_{(0)} \subsetneq B_{(0)}$	Equation (5.33)
11	1 (I)	141 - 144 (XXII)	$A_{(0)} \subsetneq B_{(0)}$	Corollary 5.20
11	141 - 144 (XXII)	151 - 156 (XXV)	$A_{(0)} \subsetneq B_{(0)}$	Corollary 5.20
11	141 - 144 (XXII)	141 - 144 (XXII)	$A_{(0)} = B_{(0)}$	Corollary 5.20
11	151 - 156 (XXV)	151 - 156 (XXV)	$A_{(0)} = B_{(0)}$	Corollary 5.20
11	1 (I)	147 - 150 (XXIV)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.21
11	147 - 150 (XXIV)	151 - 156 (XXV)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.21
11	147 - 150 (XXIV)	147 - 150 (XXIV)	$A_{(0)} = B_{(0)}$	Theorem 5.21
11	103, 104 (IX)	131, 132 (XIX)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.23
11	121, 122 (XIV)	131, 132 (XIX)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.23
11	129, 130 (XVIII)	141, 142 (XXII)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.23
11	137, 138 (XX)	143, 144 (XXII)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.23
11	149, 150 (XXIV)	151, 152 (XXV)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.23
11	7, 8 (I)	75, 76 (IV)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.26
11	9, 10 (I)	73, 74 (IV)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.26
11	15, 16 (I)	73, 74 (IV)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.26
11	17, 18 (I)	75, 76 (IV)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.26
11	19, 20 (I)	71, 72 (IV)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.26
11	21, 22 (I)	73, 74 (IV)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.26
11	25, 26 (II)	83, 84 (IV)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.26
11	1 (I)	15 (I)	$A_{(0)} \subseteq B_{(0)}$	Theorem 5.28
11	1 (I)	127 (XVII)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.28
13	1 (I)	107 - 110 (XXIV)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.29
13	107 - 110 (XXIV)	111 - 116 (XXV)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.29
13	107 - 110 (XXIV)	107 - 110 (XXIV)	$A_{(0)} = B_{(0)}$	Theorem 5.29
13	111 - 116 (XXV)	111 - 116 (XXV)	$A_{(0)} = B_{(0)}$	Theorem 5.29
13	3, 4 (I)	86, 89 (XVII)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.30
13	5, 6 (I)	87, 88 (XVII)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.30
13	21, 22 (I)	87, 88 (XVII)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.30
13	43, 44 (III)	83, 84 (XVII)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.30
13	49, 50 (V)	91, 92 (XVIII)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.30
13	51, 52 (V)	91, 92 (XVIII)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.30
13	61, 62 (VIII)	97, 98 (XX)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.30
13	69, 70 (XI)	99, 100 (XXI)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.30
13	71, 72 (XI)	99, 100 (XXI)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.30
13	1 (I)	71 (XI)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.33
13	1 (I)	81 (XVI)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.34
13	1 (I)	93, 95 (XIX)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.35

13	1 (I)	101 (XXII)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.35
15	1 (I)	81 - 84 (XIV)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.36
15	81 - 84 (XIV)	86 - 90 (XV)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.36
15	81 - 84 (XIV)	81 - 84 (XIV)	$A_{(0)} = B_{(0)}$	Theorem 5.36
15	86 - 90 (XV)	86 - 90 (XV)	$A_{(0)} = B_{(0)}$	Theorem 5.36
15	3, 4 (I)	67, 68 (VIII)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.37
15	37, 38 (III)	65, 66 (VIII)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.37
15	39, 40 (III)	65, 66 (VIII)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.37
15	41, 42 (IV)	73, 74 (X)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.37
15	43, 44 (IV)	73, 74 (X)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.37
15	45, 46 (V)	79, 80 (XIII)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.37
15	1 (I)	41, 43 (IV)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.40
15	1 (I)	45 (V)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.40
19	1 (I)	53 (VI)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.45
19	1 (I)	83 (XI)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.46
19	1 (I)	105 - 110 (XVII)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.47
19	105 - 110 (XVII)	111 - 116 (XVIII)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.47
19	105 - 110 (XVII)	105 - 110 (XVII)	$A_{(0)} = B_{(0)}$	Theorem 5.47
19	111 - 116 (XVIII)	111 - 116 (XVIII)	$A_{(0)} = B_{(0)}$	Theorem 5.47
19	1 (I)	39 (I)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.49
19	1 (I)	45 (II)	$A_{(0)} \subsetneq B_{(0)}$	Theorem 5.49

What's more, in Section 7, we shall justify a general inclusive relation indicating that for any $A(q)$ equal to f_1^r for $r \in \{4, 6, 8, 10, 14\}$ or $f_1^3 f_2^3$,

$$A_{(0)} \subseteq B_{(0)}$$

for any eta quotient $B(q)$ in the table associated with $A(q)$ (see Theorem 7.1), and as an implication, we prove that any such eta quotient $B(q)$ is lacunary (see Theorem 7.3).

The paper is a road-map of proofs and observations laying the groundwork for future work that may reveal the larger structures governing identical vanishing of collections of eta quotients. Moreover, any subsequent analysis of the unproven observations in this paper will likely benefit from the toolbox of proof techniques exhibited here. For instance, we observe that the eta quotients in the tables of the last section of the paper appear to satisfy conditions similar to those Serre derived for the lacunary powers of the eta function in [29]. The conditions for vanishing summarized below in Conjecture 1.4 are expected to follow from the approach used in [17] that is further expanded upon in this paper. Proving the conditions entails writing each eta quotient as a linear combination of Hecke theta series by using the constructive techniques of Lemma 3.5. The next step requires finding an appropriate linear combination of newforms from the database LMFDB that interpolate the eta quotient. Then the multiplicativity of the coefficients s_n of the Hecke theta series reduces the analysis to $s_{p^{e_p}}$ for $p^{e_p}||n$. Finally, by combining the binary quadratic form representation for p and the recursive formula for $s_{p^{e_p}}$ one may obtain a characterization for the vanishing of $s_{p^{e_p}}$.

Conjecture 1.4. *Denote by $a_t(n)$ the coefficient of q^n of any eta quotient in Group I in Table t in Appendix II. Then the following hold.*

(1) $a_7(n) = 0$ *if and only if* $6n + 1$ *has a prime factor* $p \equiv 2 \pmod 3$ *with odd exponent.*
(2) $a_9(n) = 0$ *if and only if* $4n + 1$ *has a prime factor* $p \not\equiv 1 \pmod 4$ *with odd exponent.*
(3) $a_{11}(n) = 0$ *if and only if* $3n + 1$ *has a prime factor* $p \equiv 2 \pmod 3$ *with odd exponent.*
(4) $a_{13}(n) = 0$ *if and only if* $12n + 5$ *has a prime factor* $p \equiv 3 \pmod 4$ *with odd exponent.*
(5) $a_{15}(n) = 0$ *if and only if* $12n + 7$ *has a prime factor* $p \equiv 2 \pmod 3$ *with odd exponent.*
(6) $a_{17}(n) = 0$ *if either of the following holds:*
  (a) $12n + 13$ *has a prime factor* $p_1 \equiv 2 \pmod 3$ *with odd exponent* <u>*and*</u> *a prime* $p_2 \equiv 3 \pmod 4$ *with odd exponent (it may be that* $p_1 = p_2$*),*
  (b) $12n+13$ *is a square and all prime factors p satisfy* $p \equiv -1 \pmod{12}$*.*
(7) $a_{19}(n) = 0$ *if and only if* $8n + 3$ *has a prime factor* $p \equiv 5, 7 \pmod 8$ *with odd exponent.*

The remainder of this work is organized as follows. In the next section, we discuss how we search for those eta quotients with vanishing coefficients similar to f_1^r for $r \in \{1, 2, 3, 4, 6, 8, 10, 14, 26\}$, or $f_1^3 f_2^3$. Following this, in Section 3, we review certain relevant previous work, key notions, and establish some technical lemmas and preliminary results. In Sections 4 and 5, we study a number of eta quotients with vanishing coefficients similar to f_1^r for $r \in \{1, 2, 3, 4, 6, 8, 10, 14\}$, or $f_1^3 f_2^3$ attained in our search and prove all the relations listed in the table above. After this, in Section 6 we make some remarks on the eta quotients associated with f_1^{26} and discuss the subtlety and difficulty in dealing with these cases. In Section 7, besides proving the aforementioned general inclusive relations between the eta quotients and the general lacunarity that follows, we pose a number of conjectures and open problems that are suggested by our computational experiments. Finally, in the last section, we summarize the methods that were used in our discussions and proofs in particular examples. This overview of strategies may provide a foundational blueprint that can contribute to proofs of every observed instance of inclusion and equality of coefficient vanishing in this paper.

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2. Search Methodology

When searching for eta quotients with vanishing coefficient behaviour similar to $A(q) = f_1^r$, $r \in \{1, 2, 3, 4, 6, 8, 10, 14, 26\}$, the search was conducted initially over eta quotients of the form

$$B(q) = f_1^a f_i^b f_j^c f_k^d f_l^e f_m^g, \quad 1 < i < j < k < l < m \leq 12, \quad -12 \leq a, b, c, d, e, g \leq 12, \tag{2.1}$$

with the parameters in the stated range satisfying

$$a + bi + cj + dk + el + gm = r. \tag{2.2}$$

The initial search for eta quotients with vanishing coefficient behaviour similar to $A(q) = f_1^3 f_2^3$ was identical to that just described, except that the r on the right side of Equation (2.2) was replaced with $3(1) + 3(2) = 9$.

We restrict our attention to cases where the associated eta quotients have the same order at infinity. That is, if $B(q) = \prod_j f_j^{n_j}$ has the property that $A_{(0)} = B_{(0)}$, where $A(q) = f_1^r$, then $\sum_j jn_j = r$. For our search space, this is equivalent to requiring that the associated modular forms have Fourier expansions with the identical vanishing property. Although this is a natural condition from a modular forms perspective, there are instances in which this condition is not satisfied (e.g. $A(q) = f_1^6$ and $B(q) = f_1^2/f_2$)). We do not consider these more exotic situations in the present paper.

The comments that follow about searches for eta quotients with vanishing coefficient behaviour similar to f_1^r also apply to the search for eta quotients with vanishing coefficient behaviour similar to $f_1^3 f_2^3$.

The initial search compared coefficients up to q^{50}. For $A(q) = f_1^r = \sum a_n q^n$, an eta quotient $B(q) = \sum b_n q^n$, was added to one of two lists, say L_1 and L_2, depending on whether

$$\{n \leq 50 | a_n = 0\} = \{n \leq 50 | b_n = 0\} \quad \text{or} \quad \{n \leq 50 | a_n = 0\} \subsetneqq \{n \leq 50 | b_n = 0\}. \tag{2.3}$$

Each of these searches took about 10 - 14 days running on a smaller computer cluster. Extending the range of the parameters beyond the bounds indicated at (2.1) can thus be seen to be prohibitive in terms of computing time. As an alternative, new products were added to L_1 and L_2 in the following two ways.

Firstly, if $B_1(q)$, $B_2(q)$ and $B_3(q)$ are any three eta quotients in $L_1 \cup L_2$, then the eta quotient

$$\frac{B_1(q)B_2(q)}{B_3(q)} = \prod_j f_j^{n_j}$$

satisfies a condition similar to (2.2), namely

$$\sum_j jn_j = r.$$

A search was run over all such products $B_1(q)B_2(q)/B_3(q)$ and the product was added to either L_1 or L_2, depending on which (if either) of the conditions in

(2.3) it satisfies. This step was repeated until no new eta quotients are added to either L_1 or L_2.

To describe the second method of adding new eta quotients to L_1 and L_2, we recall that $f_1 = (q;q)_\infty$ and observe that if $q \to -q$, then elementary q-product manipulations give that

$$(-q;-q)_\infty = \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty (q^4;q^4)_\infty} = \frac{f_2^3}{f_1 f_4}. \tag{2.4}$$

Of course q can be replaced with q^m in (2.4), where m is any positive integer. Thus if $B(q) = \prod_j f_j^{n_j}$ and at least one j is odd, then the coefficients of $B(-q)$ vanish identically with those of $B(q)$, and (2.4) can be used to write $B(-q)$ as an eta quotient. A similar situation hold if all the j are even, and their greatest common divisor is, say, the even positive integer $2m$, so that $B(q) = C(q^{2m})$. The coefficients of $C(-q^{2m})$ vanish identically with those of $C(q^{2m})$, and (2.4) can again be used to write $C(-q^{2m})$ as an eta quotient.

It may seem trivial to derive additional "partner" eta quotients with vanishing coefficient behaviour similar to f_1^r in this way. However, we keep each eta quotient and its partner in our tables, since if there are patterns to be found that might assist our understanding of this phenomenon, it is not clear in advance which of the two eta quotients in a "partner pair" might fit the pattern.

The final step was to check that the patterns found held up to q^{3000}. In other words, that for each eta quotient in L_1 that the first "=" statement at (2.3) held with 50 replaced with 3000, and that for each eta quotient in L_2 that the second "$\subsetneq$" statement at (2.3) held with 50 replaced with 3000.

Remark 2.1. *It seemed that in almost all cases, if one of the statements at* (2.3) *held up to q^{50}, then that was sufficient for it to hold up to q^{3000}. This observation leads to an open problem.*

Open Problem 2.2. *Given a weight k and a level ℓ. Is it true that there is always a positive integer $N = N(k,\ell)$ such that for any holomorphic modular forms $\sum_{n=0}^{\infty} a_n q^n$ and $\sum_{n=0}^{\infty} b_n q^n$ of weight k and level $\Gamma_0(\ell)$, whenever*

$$\{n \le N \mid a_n = 0\} = \{n \le N \mid b_n = 0\},$$

then

$$\{n \mid a_n = 0\} = \{n \mid b_n = 0\}?$$

3. Some Technical Lemmas and Preliminary Results

This section is devoted to reviewing and deriving a number of technical lemmas and preliminary results that will be useful for our study of the relations between (and lacunarity of) the coefficients of the eta quotients obtained from the search discussed in Section 2.

3.1. Elementary facts. The following lemmas are elementary but rather useful for our purpose.

Lemma 3.1. *The equation* $x^2 + y^2 = n$, $n > 0$ *has integral solutions if and only if* $\operatorname{ord}_p n$ *is even for every prime* $p \equiv 3 \pmod 4$. *When that is the case, the number of solutions is*

$$\prod_{p\equiv 1 \pmod 4} (1 + \operatorname{ord}_p n).$$

Proof. We omit the proof, as this is well known - see for example [20, Corollary 1, page 279]. □

Lemma 3.2. *Let m and n be positive integers and let p be a prime with* $p \equiv 5$ *or* $7 \pmod 8$.
(1) If $p|m^2 + 2n^2$, *then* $p|m$ *and* $p|n$.
(2) If $m^2/3 + 8n^2/3$ *is an integer such that* $p|(m^2/3 + 8n^2/3)$, *then* $p|m$ *and* $p|n$.
(3) If $p|3m^2 + 24n^2$, *then* $p|m$ *and* $p|n$.
(4) If $3m^2 + 48n^2 = 8t + 3$ *for some integer t and* $p|3m^2 + 48n^2$, *then* $p|m$ *and* $p|m$.

Proof. (1) Suppose $p \nmid m$, so that $p \nmid n$ also. Then $m^2 + 2n^2$ implies that -2 is a quadratic residue modulo p. However this contradicts the fact that -2 is a quadratic residue only for primes $\equiv 1$ or $3 \pmod 8$.
(2) If $p|(m^2/3 + 8n^2/3)$, then $p|m^2 + 2(2n)^2$, and the claim follows from (1).
(3) Similarly, if $p|3m^2 + 24n^2$, then $p|m^2 + 2(2n)^2$, and once again the claim follows from (1).
(4) Suppose $p \nmid m$ so that $p \nmid n$ and then $p|m^2 + (4n^2)$ implies -1 is a quadratic residue modulo p. Thus $p \equiv 1 \pmod 4$ and hence $p \equiv 5 \pmod 8$, from the statement of the lemma. However, since $3m^2 + 48n^2 \equiv 3 \pmod 8$, $3m^2 + 48n^2$ must be divisible by another prime $p' \equiv 7 \pmod 8$ with odd exponent, and hence $p' \equiv 3 \pmod 4$, contradicting the statement in Lemma 3.1. □

Lemma 3.3. *Let p be a prime,* $p \equiv 2 \pmod 3$, *and let a and b be integers such that* $p|3a^2 + b^2$. *Then* $p|a$ *and* $p|b$.

Proof. Suppose $p \nmid a$, so that $p \nmid b$, since $p|3a^2 + b^2$. Then -3 is a square in $(\mathbb{Z}/p\mathbb{Z})^\times$, that is, $\left(\frac{-3}{p}\right) = 1$. By the quadratic reciprocity law, one has that $\left(\frac{p}{3}\right) = 1$, which implies that $p \equiv 1 \pmod 3$, a contradiction. □

3.2. The vanishing of the coefficients of f_1^r. In [29], Serre characterizes the vanishing of the coefficients of f_1^r for $r \in \{2, 4, 6, 8, 10, 14\}$ and establishes sufficient conditions for that of f_1^{26} using the theory of CM newforms. These results in turn imply that these f_1^r are all lacunary by Serre's extension [27] on Landau's density theorem and are summarized in the following theorem.

Theorem 3.4 (Serre)**.** *Let* f_i *be defined as in Section* 1. *Then*

(1) writing

$$f_1^2 = \sum_{n=0}^{\infty} a_n q^n, \tag{3.1}$$

one has that $a_n = 0$ *if and only if* $12n + 1$ *has a prime factor* $p \not\equiv 1 \pmod{12}$ *with odd exponent,*

(2) writing

$$f_1^4 = \sum_{n=0}^{\infty} a_n q^n, \tag{3.2}$$

one has that $a_n) = 0$ *if and only if* $6n + 1$ *has a prime factor* $p \equiv -1 \pmod{3}$ *with odd exponent,*

(3) writing

$$f_1^6 = \sum_{n=0}^{\infty} a_n q^n, \tag{3.3}$$

one has that $a_n = 0$ *if and only if* $4n + 1$ *has a prime factor* $p \equiv -1 \pmod{4}$ *with odd exponent,*

(4) writing

$$f_1^8 = \sum_{n=0}^{\infty} a_n q^n, \tag{3.4}$$

one has that $a_n = 0$ *if and only if* $3n + 1$ *has a prime factor* $p \equiv -1 \pmod{3}$ *with odd exponent,*

(5) writing

$$f_1^{10} = \sum_{n=0}^{\infty} a_n q^n, \tag{3.5}$$

one has that $a_n = 0$ *if and only if* $12n + 5$ *has a prime factor* $p \equiv -1 \pmod{4}$ *with odd exponent,*

(6) writing

$$f_1^{14} = \sum_{n=0}^{\infty} a_n q^n, \tag{3.6}$$

one has that $a_n = 0$ *if and only if* $12n + 7$ *has a prime factor* $p \equiv -1 \pmod{3}$ *with odd exponent,*

(7) writing

$$f_1^{26} = \sum_{n=0}^{\infty} a_n q^n, \tag{3.7}$$

one has that $a_n = 0$ *if either of the following holds:*

(a) $12n + 13$ *has a prime factor* $p_1 \equiv -1 \pmod{3}$ *with odd exponent* *and* *a prime* $p_2 \equiv -1 \pmod{4}$ *with odd exponent (it may be that* $p_1 = p_2$*),*

(b) $12n+13$ *is a square and all prime factors* p *satisfy* $p \equiv -1 \pmod{12}$*.*

In the introduction, we mention that we also take the lacunary eta quotient $f_1^3 f_2^3$ into account as a starting point towards extending our investigations to other lacunary eta quotients. Inspired by Serre's results, it is natural to ask if the vanishing of the coefficients of $f_1^3 f_2^3$ can also be interpreted in terms of prime decomposition. The answer is indeed positive and can be stated as follows: *if*

$$f_1^3 f_2^3 = \sum_{n=0}^{\infty} a_n q^n,$$

then $a_n = 0$ *if and only if* $8n + 3$ *has a prime factor* $p \equiv 5$ *or* $7 \pmod 8$ *with odd exponent.* We shall prove this in Theorem 5.44 in Subsection 5.6.

3.3. Review of CM newforms and their basic properties. As is mentioned in Subsection 3.2, Serre's results are closely related to the notion of CM newforms which are very useful for studying the vanishing of the coefficients of certain eta quotients and shall play a key role in the proof of the aforementioned characterization of the vanishing of the coefficients of $f_1^3 f_2^3$ (see Theorem 5.44). In this subsection, we briefly review relevant materials on CM newforms, such as their basic properties and constructions.

Let $f(z) = \sum_{n=1}^{\infty} a(n) q^n$ be a newform of weight k and level $\Gamma_0(N)$ with some character χ. Then one can first recall that its Fourier coefficients $a(n)$ satisfy the recursive relation

$$a(\ell)a(n) = a(\ell n) + \chi(\ell)\ell^{k-1}a(n/\ell) \tag{3.8}$$

for any positive integer n and any prime ℓ, where $a(x)$ is set to be 0 if x is not an integer. The coefficients also possess the multiplicative property that $a(mn) = a(m)a(n)$ for any positive integers m, n such that $\gcd(m, n) = 1$. Therefore, investigations on $a(n)$ can be boiled down to analysis of $a(p)$ for p prime.

For any Dirichlet character ϕ of conductor m, a newform $f(z)$ is said to have CM by ϕ if $a(p)\phi(p) = a(p)$ for all $p \nmid Nm$. Such an $f(z)$ is also called a CM newform by ϕ. Characterizations of CM newforms of weight $k \geq 2$ have been established by Ribet [26], which are briefly summarized next.

It is known [5, (6.3)] that a CM newform of weight $k \geq 2$ exists only if ϕ is a quadratic character associated to some quadratic field K. In such case, $f(z)$ is also called a CM newform by K. In his groundbreaking work [26], Ribet gives a full characterization of such newforms and justifies that any CM newform of weight $k \geq 2$ by a quadratic field K must come from a Hecke character ψ_K associated to K and be of the form

$$f(z) = \sum_{\substack{\mathfrak{a} \subseteq \mathcal{O}_K \\ \text{integral}}} \psi_K(\mathfrak{a}) \mathcal{N}(\mathfrak{a})^{\frac{k-1}{2}} q^{\mathcal{N}(\mathfrak{a})},$$

where $\mathcal{N}(\cdot)$ denotes the norm of an ideal. In particular, when K is imaginary of discriminant $-d < 0$ and class number 1, one has that (see, e.g., [17, Corollary

2.2]) $f(z)$ must be a linear combination of the generalized theta series

$$\sum_{\alpha\in\beta+\mathfrak{m}} \alpha^{k-1} q^{\mathcal{N}(\alpha)} \quad \text{over } \beta \in (\mathcal{O}_K/\mathfrak{m})^{\times}$$

for some integral ideal $\mathfrak{m}$ with $\mathcal{N}(\mathfrak{m}) = N/d$.

In addition, what is remarkable about CM newforms is their connection with lacunary cusp forms. In [28], Serre proves that a cusp form $\sum_{n=0}^{\infty} a_n q^n$ of weight $k \geq 2$ is lacunary if and only if it lies in some space of CM newforms. By Ribet's characterization of a CM newform, we can tell that such a lacunary cusp form must be a linear combination of generalized theta series. This suggests the following simple implication: a cusp form that is assumed to be lacunary must be a linear combination of generalized theta series. In general, it is hard to verify the lacunarity of a cusp form by computations alone; however, proving that a particular cusp form is a linear combination of generalized theta series is often computationally feasible by Sturm's theorem. If we can find a characterization of the coefficients of a cusp form by generalized theta series, we may use it to go in the other direction and study the vanishing properties of the coefficients of the cusp form. As discussed in Sections 1 and 2, our computational experiments together with Theorem 3.4 are made with the hypothesis that all the eta quotients obtained from our search are lacunary. Moreover, we can validate that the associated modular forms of weight $k \geq 2$ are all cusp forms. The discussion above leads to the following lemma justifying that all these cusp forms are expressible in terms of generalized theta series attached to some integral ideal of $\mathbb{Z}[i]$, $\mathbb{Z}[w]$ with $w = e^{2\pi i/3}$, or $\mathbb{Z}[\sqrt{-2}]$.

Lemma 3.5. *(1) Let $B(q)$ be any of the eta quotients of weight $k \geq 2$ and level N in Table* 7*. Then*

$$qB(q^6) = \sum_{[\alpha]\in\mathbb{Z}[w]/\mathfrak{m}} C_\alpha \sum_{x\in[\alpha]} x^{k-1} q^{N(x)}$$

for some constants C_α, where w is the primitive third root of unity, and $\mathfrak{m}$ is some integral ideal of $\mathbb{Z}[w]$ such that $N(\mathfrak{m}) = N/3$.

(2) Let $B(q)$ be any of the eta quotients of weight $k \geq 2$ and level N in Table 9*. Then*

$$qB(q^4) = \sum_{[\alpha]\in\mathbb{Z}[i]/\mathfrak{m}} C_\alpha \sum_{x\in[\alpha]} x^{k-1} q^{N(x)}$$

for some constants C_α, where $\mathfrak{m}$ is some integral ideal of $\mathbb{Z}[i]$ such that $N(\mathfrak{m}) = N/4$.

(3) Let $B(q)$ be any of the eta quotients of weight $k \geq 2$ and level N in Table 11*. Then*

$$qB(q^3) = \sum_{[\alpha]\in\mathbb{Z}[w]/\mathfrak{m}} C_\alpha \sum_{x\in[\alpha]} x^{k-1} q^{N(x)}$$

for some constants C_α, where w is the primitive third root of unity, and $\mathfrak{m}$ is some integral ideal of $\mathbb{Z}[w]$ such that $N(\mathfrak{m}) = N/3$.

(4) *Let $B(q)$ be any of the eta quotients of weight $k \geq 2$ and level N in Table* 13. *Then*

$$q^5 B(q^{12}) = \sum_{[\alpha]\in\mathbb{Z}[i]/\mathfrak{m}} C_\alpha \sum_{x\in[\alpha]} x^{k-1} q^{N(x)}$$

for some constants C_α, where $\mathfrak{m}$ is some integral ideal of $\mathbb{Z}[i]$ such that $N(\mathfrak{m}) = N/4$.

(5) *Let $B(q)$ be any of the eta quotients of weight $k \geq 2$ and level N in Table* 15. *Then*

$$q^7 B(q^{12}) = \sum_{[\alpha]\in\mathbb{Z}[w]/\mathfrak{m}} C_\alpha \sum_{x\in[\alpha]} x^{k-1} q^{N(x)}$$

for some constants C_α, where w is the primitive third root of unity, and $\mathfrak{m}$ is some integral ideal of $\mathbb{Z}[w]$ such that $N(\mathfrak{m}) = N/3$.

(6) *Let $B(q)$ be any of the eta quotients of weight $k \geq 2$ and level N in Table* 17. *Then*

$$q^{13} B(q^{12}) = \sum_{[\alpha]\in\mathbb{Z}[i]/\mathfrak{m}} C_\alpha \sum_{x\in[\alpha]} x^{k-1} q^{N(x)} + \sum_{[\beta]\in\mathbb{Z}[w]/\mathfrak{n}} C_\beta \sum_{x\in[\beta]} x^{k-1} q^{N(x)} + \sum_{[\gamma]\in\mathbb{Z}[2w]/\mathfrak{l}} C_\gamma \sum_{x\in[\gamma]} x^{k-1} q^{N(x)}$$

for some constants $C_\alpha, C_\beta, C_\gamma$, where $\mathfrak{m}$, $\mathfrak{n}$ and $\mathfrak{l}$ are integral ideals of the orders $\mathbb{Z}[i]$, $\mathbb{Z}[w]$ and $\mathbb{Z}[2w]$, respectively.

(7) *Let $B(q)$ be any of the eta quotients of weight $k \geq 2$ and level N in Table* 19. *Then*

$$q^3 B(q^8) = \sum_{[\alpha]\in\mathbb{Z}[\sqrt{-2}]/\mathfrak{m}} C_\alpha \sum_{x\in[\alpha]} x^{k-1} q^{N(x)},$$

for some constants C_α, where $\mathfrak{m}$ is some integral ideal of $\mathbb{Z}[\sqrt{-2}]$ such that $N(\mathfrak{m}) = N/8$.

In what follows we provide details needed for the construction of the theta series interpolating the eta quotients referenced in Lemma 3.5.

Construction of the ideals $\mathfrak{m}$: Since the ideal norm is multiplicative, it suffices to look at the prime factors $N(\mathfrak{m})$ of the norm of the prospective ideal $\mathfrak{m}$. This norm is determined by the level of the eta quotient as indicated in Lemma 3.5. An ideal for each case can be constructed as follows:

- For $K = \mathbb{Q}[i]$, suppose $N(\mathfrak{m}) = p_1^{e_1} \cdots p_r^{e_r}$. Let

$$\ell(p_j) = \begin{cases} m_j + n_j i, & \text{if } m_j^2 + n_j^2 = p_j, \\ p_j, & \text{else,} \end{cases}, \qquad \mathcal{I}(m+ni) = \begin{cases} m+ni, & n \neq 0, \\ \sqrt{m}, & n = 0. \end{cases}$$

Then

$$\mathfrak{m} = ((\mathcal{I}\circ\ell)(p_1)^{e_1} \cdots (\mathcal{I}\circ\ell)(p_r)^{e_r}).$$

- For $K = \mathbb{Q}[w]$, let $N(\mathfrak{m}) = p_1^{e_1} \cdots p_r^{e_r}$. Define

$$\ell(p_j) = \begin{cases} m_j + n_j w, & \text{if } m_j^2 - m_j n_j + n_j^2 = p_j, \\ p_j, & \text{else,} \end{cases}, \quad \mathcal{I}(p_j) = \begin{cases} \ell(p_j), & \text{if } \left(\frac{-3}{p_j}\right) = 1, \\ \sqrt{p_j} & \text{if } \left(\frac{-3}{p_j}\right) = -1, \\ \sqrt{-3}, & \text{if } \left(\frac{-3}{p_j}\right) = 0. \end{cases}$$

Then

$$\mathfrak{m} = (\mathcal{I}(p_1)^{e_1} \cdots \mathcal{I}(p_r)^{e_r}).$$

- For $K = \mathbb{Q}[\sqrt{-2}]$, let $N(\mathfrak{m}) = p_1^{e_1} \cdots p_r^{e_r}$. Define

$$\ell(p_j) = \begin{cases} m + n\sqrt{-2}, & \text{if } m_j^2 + 2n_j^2 = p_j, \\ p_j, & \text{else,} \end{cases}, \quad \mathcal{I}(p_j) = \begin{cases} \ell(p_j), & \text{if } \left(\frac{-2}{p_j}\right) = 1, \\ \sqrt{p_j} & \text{if } \left(\frac{-2}{p_j}\right) = -1, \\ \sqrt{-2}, & \text{if } \left(\frac{-2}{p_j}\right) = 0. \end{cases}$$

Then

$$\mathfrak{m} = (\mathcal{I}(p_1)^{e_1} \cdots \mathcal{I}(p_r)^{e_r}).$$

Construction of coset representatives for $\mathcal{O}_k/\mathfrak{m}$:

- For $\mathcal{O}_k = \mathbb{Z}[i]$, let $\mathfrak{m} = a + bi$, and define $d = \gcd(a, b)$, and $\ell = N(a+bi)/d$. Find an element $\alpha = k + i$ such that $d\alpha \equiv 0 \pmod{a + bi}$. The element α can be constructed as follows. It suffices to find a k such that $a/g + (b/g)i | k + i$. Since a/g and b/g are coprime, then there are s and t such that

$$s(a/g) + t(b/g) = 1.$$

So, $\alpha = k + i = (a/g + (b/g)i)(t + si)$. Then

$$\mathbb{Z}[i]/(a + bi) = \{[j + h\alpha]\}$$

as $j = 0, \ldots, N - 1$ and $h = 0, \ldots, d - 1$.

- For $O_K = \mathbb{Z}[w]$ with w the primitive third root of unity, and $\mathfrak{m} = a+bw$ with $g = \gcd(a, b)$, write $\ell = N(a + bw)/g$ find s and t such that $(a/g)s - (b/g)t = 1$. Then using the fundamental theorem of finite Abelian groups it can be shown that

$$\mathbb{Z}[w]/(a + bw) = \{[j(s + (b/g)w) + h(t + (a/g)w)]\}$$

as $j = 0, \ldots, g - 1$ and $h = 0, \ldots, \ell - 1$. To find $(Z[w]/(a + bw))^{\times}$.

- For $O_K = \mathbb{Z}[\sqrt{-2}]$, and $\mathfrak{m} = a + b\sqrt{-2}$ with $g = \gcd(a, b)$, write $\ell = N(a + b\sqrt{-2})/g$ and find s and t such that $(a/g)s + (b/g)t = 1$, then

$$\mathbb{Z}[\sqrt{-2}]/(a + b\sqrt{-2}) = \{[j\left((a/g) + (b/g)\sqrt{-2}\right) + h\left(-t + s\sqrt{-2}\right)]\}$$

as $j = 0, \ldots, g - 1$ and $h = 0, \ldots, \ell - 1$.

Construction of the generalized theta series: Although the given algorithm produces a large number of coset representatives, a significant proportion of the representatives can be eliminated by removing associate cosets. Once an appropriate set of non-associate cosets are produced, for each case K above, the theta series for each coset of $\mathcal{O}_K/\mathfrak{m}$ may be constructed as follows:

- For $\mathcal{O}_k = \mathbb{Z}[i]$, define, for each integral ideal $\mathfrak{m} = (\mathcal{M})$ and coset $[c] \in \mathcal{O}_k/\mathfrak{m}$,

$$P_1(c,\mathcal{M}) = \mathrm{Re}\,(c + (m + in)\mathcal{M}), \qquad P_2(c,\mathcal{M}) = \mathrm{Im}\,(c + (m + in)\mathcal{M}).$$

The corresponding theta series of weight $\mathcal{W}$ for the ideal and coset is

$$\sum_{m,n=-\infty}^{\infty} (P_1(c,\mathcal{M}) + P_2(c,\mathcal{M})i)^{\mathcal{W}-1}\, q^{P_1^2(c,\mathcal{M})+P_2^2(c,\mathcal{M})}.$$

- For $\mathcal{O}_k = \mathbb{Z}[w]$, define, for each integral ideal $\mathfrak{m} = (\mathcal{M})$ and coset $[c] \in \mathcal{O}_k/\mathfrak{m}$,

$$P_1(c,\mathcal{M}) = \mathrm{Re}\,(c + (m + nw)\mathcal{M}), \qquad P_2(c,\mathcal{M}) = \frac{\mathrm{Im}\,(c + (m + nw)\mathcal{M})}{\sqrt{3}}.$$

The corresponding theta series of weight $\mathcal{W}$ for the ideal and coset is

$$\sum_{m,n=-\infty}^{\infty} \left(P_1(c,\mathcal{M}) + P_2(c,\mathcal{M})\sqrt{-3}\right)^{\mathcal{W}-1} q^{P_1^2(c,\mathcal{M})+3P_2^2(c,\mathcal{M})}.$$

- For $\mathcal{O}_k = \mathbb{Z}[\sqrt{-2}]$, define, for each integral ideal $\mathfrak{m} = (\mathcal{M})$ and coset $[c] \in \mathcal{O}_k/\mathfrak{m}$,

$$P_1(c,\mathcal{M}) = \mathrm{Re}\left(c + (m + n\sqrt{-2})\mathcal{M}\right), \quad P_2(c,\mathcal{M}) = \frac{\mathrm{Im}\left(c + (m + n\sqrt{-2})\mathcal{M}\right)}{\sqrt{2}}.$$

The corresponding theta series of weight $\mathcal{W}$ for the ideal and coset is

$$\sum_{m,n=-\infty}^{\infty} \left(P_1(c,\mathcal{M}) + P_2(c,\mathcal{M})\sqrt{-2}\right)^{\mathcal{W}-1} q^{P_1^2(c,\mathcal{M})+2P_2^2(c,\mathcal{M})}.$$

The generalized theta series interpolations for the relevant eta quotients in Lemma 3.5 are proven by solving the system obtained by equating the respective Fourier coefficients of the eta quotient and a linear combination of the generalized theta series up to the Sturm bound. In some cases, the construction above does not provide a corresponding linear combination of theta series that represents the eta quotient up to the Sturm bound. In these exceptional cases the requisite linear combination of theta series may be obtained by applying a dilation $q \mapsto q^c$, for small values of c dividing the level, or by possibly changing the construction of the ideal $\mathfrak{m}$, for instance, by switching signs in the cases defining $\mathcal{I}(p_j)$. The explicit constructions described above work without any alteration for the cases in which $\mathcal{O}_K = \mathbb{Z}[w]$. For Table 13, vanishing like f_1^{10}, with $\mathcal{O}_K = \mathbb{Z}[i]$, the only exceptional case is $\eta(12z)^5\eta(60z)$, which is addressed through the dilation $q \mapsto q^5$. Exceptional cases in Table 19, containing quotients vanishing like $f_1^3 f_2^3$, with $\mathcal{O}_K = \mathbb{Z}[\sqrt{-2}]$, include

$$\frac{\eta(16z)^2\eta(48z)^5}{\eta(8z)\eta(96z)^2}, \frac{\eta(24z)^5\eta(32z)^2}{\eta(16z)\eta(48z)^2}, \frac{\eta(8z)^3\eta(16z)^2\eta(96z)}{\eta(32z)\eta(48z)}, \frac{\eta(8z)^3\eta(48z)^5}{\eta(96z)^2},$$

$$\eta(16z)^3\eta(24z), \eta(8z)^3\eta(48z), \frac{\eta(8z)^2\eta(16z)^2\eta(48z)}{\eta(24z)}, \frac{\eta(16z)^3\eta(24z)^5}{\eta(48z)^2},$$

$$\frac{\eta(8z)^2\eta(32z)^9\eta(48z)}{\eta(16z)^4\eta(24z)\eta(64z)^3}, \frac{\eta(16z)^2\eta(24z)\eta(32z)^7\eta(96z)}{\eta(8z)^2\eta(48z)^2\eta(64z)^3}, \frac{\eta(24z)\eta(32z)^9}{\eta(16z)^3\eta(64z)^3},$$

$$\frac{\eta(32z)^9\eta(48z)^3}{\eta(16z)^3\eta(24z)\eta(64z)^3\eta(96z)}, \frac{\eta(16z)\eta(24z)^5\eta(64z)}{\eta(32z)\eta(48z)^2}, \frac{\eta(16z)\eta(48z)^{13}\eta(64z)}{\eta(24z)^5\eta(32z)\eta(96z)^5},$$

$$\frac{\eta(24z)^5\eta(32z)^9}{\eta(16z)^3\eta(48z)^2\eta(64z)^3}, \frac{\eta(32z)^9\eta(48z)^{13}}{\eta(16z)^3\eta(24z)^5\eta(64z)^3\eta(96z)^5}.$$

These may be expressed as a linear combination of generalized theta series from the above algorithm by switching the output in the cases $\left(\frac{-2}{p_j}\right) = \pm 1$ in the definition of $\mathcal{I}(p_j)$. Other exceptional cases in Table 19 include

$$\frac{\eta(8z)\eta(24z)^2\eta(32z)^2}{\eta(48z)}, \quad \frac{\eta(8z)^2\eta(48z)^6}{\eta(16z)\eta(24z)\eta(96z)^2}, \quad \frac{\eta(16z)^2\eta(24z)^5\eta(96z)}{\eta(32z)\eta(48z)^3},$$

$$\frac{\eta(8z)^4\eta(24z)\eta(96z)}{\eta(32z)\eta(48z)}, \quad \frac{\eta(16z)^9\eta(24z)^3\eta(96z)^2}{\eta(8z)^2\eta(32z)^4\eta(48z)^4}, \quad \frac{\eta(16z)^2\eta(24z)^5\eta(96z)}{\eta(32z)\eta(48z)^3},$$

$$\frac{\eta(16z)^9\eta(24z)^3\eta(96z)^2}{\eta(8z)^2\eta(32z)^4\eta(48z)^4}, \quad \frac{\eta(16z)\eta(24z)^7}{\eta(8z)^2\eta(48z)^2}, \quad \frac{\eta(8z)^2\eta(32z)^2\eta(48z)^{19}}{\eta(16z)^5\eta(24z)^7\eta(96z)^7},$$

$$\frac{\eta(8z)^3\eta(32z)^3\eta(48z)^5}{\eta(16z)^3\eta(24z)^2\eta(96z)^2}, \quad \frac{\eta(16z)^5\eta(24z)\eta(48z)^3}{\eta(8z)^2\eta(32z)^2\eta(96z)}, \quad \frac{\eta(8z)^2\eta(16z)^3\eta(48z)^5}{\eta(24z)^3\eta(32z)^2\eta(96z)}.$$

These may be expressed in terms of generalized theta series from the above algorithm by applying the dilation $q \mapsto q^3$ and a corresponding sign alteration the in definition of $\mathcal{I}(p_j)$.

For Table 17, corresponding to f_1^{26}, where integral ideals in both $\mathbb{Z}[i]$ and $\mathbb{Z}[w]$ are required, some exceptional cases occur when the level N has a prime factor that inert in either of $\mathbb{Z}[w]$ and $\mathbb{Z}[i]$ with odd exponent, so that $N/4$ and $N/3$ cannot simultaneously be a norm for $\mathbb{Z}[w]$ and $\mathbb{Z}[i]$. For instance $\eta(12z)\eta(60z)^5$, may be expressed in terms of generalized theta series using only the ideal $\mathfrak{m} = (30)$ in $\mathbb{Z}[i[$ and the dilation $q \mapsto q^5$. This also addresses the expansion for the exceptional quotient $\frac{\eta(24z)^3\eta(120z)^{15}}{\eta(12z)\eta(48z)\eta(60z)^5\eta(240z)^5}$ obtained from

$\eta(12z)\eta(60z)^5$ by mapping q to $-q$. Similarly, a generalized theta series expansion for $\frac{\eta(24z)^6\eta(96z)^4}{\eta(12z)^2\eta(48z)^4}$ addresses that for the exceptional case $\frac{\eta(12z)^2\eta(96z)^4}{\eta(48z)^2}$ since these map to one another under $q \mapsto -q$.

A more computationally intensive strategy involving oldforms is needed to obtain an expansion for the exceptional case $\frac{\eta(24z)^6\eta(36z)\eta(144z)\eta(216z)^9}{\eta(12z)^2\eta(48z)^2\eta(72z)^3\eta(108z)^3\eta(432z)^3}$. We first apply the dilation $q \mapsto q^3$ to its level 432 partner $\frac{\eta(12\ 3z)^2\eta(108\ 3z)^3}{\eta(36\ 3z)}$ under $q \mapsto -q$. We find ideals $\mathfrak{m}_1$ of $\mathbb{Z}[i]$ and $\mathfrak{m}_2$ of $\mathbb{Z}[w]$ such that $N(\mathfrak{m}_1) = 432 \cdot 3/4$ and $N(\mathfrak{m}_2) = 432 \cdot 3/3$. Then for each divisor d of $432 \cdot 3$, we compute the associated Hecke series $H_d(z)$ of level $432 \cdot 3/d$. For each such d, we obtain ideals from the algorithm above of norms $(432 \cdot 3/d)/4$ and $(432 \cdot 3/d)/3$, respectively, and sequentially compute their associated Hecke series. This results in a set of Hecke series $\{H_d(z)\}$ over d dividing $432 \cdot 3$. In particular, when $d > 1$, $H_d(z)$ are of lower level and serve as oldforms. We obtain the quotient $\frac{\eta(12\ 3z)^2\eta(108\ 3z)^3}{\eta(36\ 3z)}$ as a linear combination of $H_d(kz)$ over d dividing $432 \cdot 3$, and k dividing $432 \cdot 3/d$. In fact, an expansion in this case may be obtained from the divisors $d = 1, 2$ and $k = 1, 2, 3$.

3.4. Eta quotients as theta series. Lemke Oliver [22] provided another useful tool for our investigation of eta quotients. Oliver justified that the eta quotients in the following list are the only quotients that can be expressed as unary theta series.

Lemma 3.6 (Lemke Oliver)**.** *Let f_i be defined as in Section* 1. *Then the following identities hold.*

$$\frac{f_2^5}{f_1^2 f_4^2} = \sum_{n=-\infty}^{\infty} q^{n^2}, \quad q\frac{f_8 f_{32}}{f_{16}} = \sum_{n=1}^{\infty} \left(\frac{8}{n}\right) q^{n^2}, \tag{3.9}$$

$$q\frac{f_{16}^2}{f_8} = \sum_{n=1}^{\infty} \left(\frac{n}{2}\right)^2 q^{n^2}, \quad q\frac{f_6^2 f_9 f_{36}}{f_3 f_{12} f_{18}} = \sum_{n=1}^{\infty} \left(\frac{n}{3}\right)^2 q^{n^2}, \tag{3.10}$$

$$q f_{24} = \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) q^{n^2}, \quad q\frac{f_{48}^3}{f_{24} f_{96}} = \sum_{n=1}^{\infty} \left(\frac{24}{n}\right) q^{n^2}, \tag{3.11}$$

$$q\frac{f_{48} f_{72}^2}{f_{24} f_{144}} = \sum_{n=1}^{\infty} \left(\frac{n}{6}\right)^2 q^{n^2}, \quad q\frac{f_{24} f_{96} f_{144}^5}{f_{48}^2 f_{72}^2 f_{288}^2} = \sum_{n=1}^{\infty} \left(\frac{18}{n}\right)^2 q^{n^2}, \tag{3.12}$$

$$q f_8^3 = \sum_{n=1}^{\infty} \left(\frac{-4}{n}\right) n q^{n^2}, \quad q\frac{f_{16}^9}{f_8^3 f_{32}^3} = \sum_{n=1}^{\infty} \left(\frac{-8}{n}\right) n q^{n^2}, \tag{3.13}$$

$$q\frac{f_3^2 f_{12}^2}{f_6} = \sum_{n=1}^{\infty} \left(\frac{n}{3}\right) n q^{n^2}, \quad q\frac{f_{48}^{13}}{f_{24}^5 f_{96}^5} = \sum_{n=1}^{\infty} \left(\frac{-6}{n}\right) n q^{n^2}, \tag{3.14}$$

$$q\frac{f_{24}^5}{f_{48}^2}=\sum_{n=1}^{\infty}\left(\frac{n}{12}\right)nq^{n^2},\quad \frac{f_1^2}{f_2}=\sum_{n=-\infty}^{\infty}(-1)^nq^{n^2}, \tag{3.15}$$

$$\frac{f_1f_4f_6^2}{f_2f_3f_{12}}=\sum_{n=-\infty}^{\infty}\left(1-\frac{3}{2}\left(\frac{n}{3}\right)^2\right)q^{n^2},\quad \frac{f_8^5}{f_4^2f_{16}^2}=\sum_{n=-\infty}^{\infty}\left(1-\left(\frac{n}{2}\right)^2\right)q^{n^2}, \tag{3.16}$$

$$\frac{f_2^2f_3}{f_1f_6}=\sum_{n=-\infty}^{\infty}\left(1-2\left(\frac{n}{2}\right)^2-\frac{3}{2}\left(\frac{n}{2}\right)^2+3\left(\frac{n}{6}\right)^2\right)q^{n^2}, \tag{3.17}$$

$$\frac{f_9^2}{f_{18}}=\sum_{n=-\infty}^{\infty}\left(1-2\left(\frac{n}{2}\right)^2-\left(\frac{n}{3}\right)^2+2\left(\frac{n}{6}\right)^2\right)q^{n^2}, \tag{3.18}$$

$$\frac{f_{18}^5}{f_9^2f_{36}^2}=\sum_{n=-\infty}^{\infty}\left(1-\left(\frac{n}{3}\right)^2\right)q^{n^2},\quad q\frac{f_3f_{18}^2}{f_6f_9}=\sum_{n=1}^{\infty}\left(2\left(\frac{n}{6}\right)^2-\left(\frac{n}{3}\right)^2\right)q^{n^2}, \tag{3.19}$$

$$\frac{f_4f_{16}f_{24}^2}{f_8f_{12}f_{48}}=\sum_{n=-\infty}^{\infty}\left(1-\left(\frac{n}{2}\right)^2-\frac{3}{2}\left(\frac{n}{2}\right)^2+\frac{3}{2}\left(\frac{n}{6}\right)^2\right)q^{n^2}, \tag{3.20}$$

$$\frac{f_{72}^5}{f_{36}^2f_{144}^2}=\sum_{n=-\infty}^{\infty}\left(1-\left(\frac{n}{2}\right)^2-\left(\frac{n}{2}\right)^2+\left(\frac{n}{6}\right)^2\right)q^{n^2}, \tag{3.21}$$

$$q\frac{f_8^2f_{48}}{f_{16}f_{24}}=\sum_{n=1}^{\infty}\left(2\left(\frac{n}{6}\right)^2-\left(\frac{n}{3}\right)^2\right)q^{n^2},\quad q\frac{f_6^5}{f_3^2}=\sum_{n=1}^{\infty}\left(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right)\right)nq^{n^2}. \tag{3.22}$$

We find that a large portion of the eta quotients we consider are indeed in the form of $B(q^m)C(q^n)$ for some eta quotients $B(q)$, $C(q)$ in Lemma 3.6 and some positive integers m, n, and thus, by Lemma 3.6 these eta quotients are all representable by double theta series attached to some binary quadratic form. This will aid us with studying the eta quotients of weight 1 that Serre's theory of CM newforms may not apply to. We summarize our findings for quotients of weight 1 in Lemma 3.7.

Lemma 3.7. *(1) If $B(q)$ is any of the eta quotients of weight 1 in the odd numbered entries of Table 7 other than Entries 91, 97, 99 and 131, then either of $qB(q^6)$ or $qB(-q^6)$ can be expressed as a double theta series of the form either*

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\alpha(m,n)q^{m^2+12n^2}\quad or \quad \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\alpha(m,n)q^{\frac{1}{4}(m^2+3n^2)}.$$

(2) If $B(q)$ is any of the eta quotients of weight 1 in the odd numbered entries of Table 9 other than Entries 25, 33, 41, 43, 67, 107, 129, 139, 145 and 151, then either of $qB(q^4)$ or $qB(-q^4)$ can be expressed as a double theta series

of the form either

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\alpha(m,n)q^{\frac{1}{2}(m^2+n^2)},\qquad \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\alpha(m,n)q^{m^2+4n^2}$$

or

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\alpha(m,n)q^{m^2+16n^2}.$$

(3) *If $B(q)$ is any of the eta quotients of weight* 1 *in the odd numbered entries of Table* 11 *other than Entries* 15 *and* 127*, then either of $qB(q^3)$ or $qB(-q^3)$ can be expressed as a double theta series of the form either*

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\alpha(m,n)q^{\frac{1}{4}(m^2+3n^2)},\qquad \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\alpha(m,n)q^{m^2+3n^2}$$

or

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\alpha(m,n)q^{m^2+12n^2}.$$

(4) *If $B(q)$ is any of the eta quotients of weight* 1 *in the odd numbered entries of Table* 13 *other than Entries* 67*,* 71*,* 81*,* 93*,* 95 *and* 101*, then either of $q^5B(q^{12})$ or $q^5B(-q^{12})$ can be expressed as a double theta series of the form either*

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\alpha(m,n)q^{\frac{1}{2}(m^2+9n^2)},\qquad \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\alpha(m,n)q^{m^2+4n^2}$$

or

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\alpha(m,n)q^{\frac{5}{2}(m^2+n^2)}.$$

(5) *If $B(q)$ is any of the eta quotients of weight* 1 *in the odd numbered entries of Table* 15 *other than Entries* 41*,* 43 *and* 45*, then either of $q^7B(q^{12})$ or $q^7B(-q^{12})$ can be expressed as a double theta series of the form either*

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\alpha(m,n)q^{3m^2+4n^2}\quad\text{or}\quad \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\alpha(m,n)q^{7(m^2+12n^2)}.$$

(6) *If $B(q)$ is any of the eta quotients of weight* 1 *in the odd numbered entries of Table* 17 *other than Entries* 79*, then either of $q^{13}B(q^{12})$ or $q^{13}B(-q^{12})$ can be expressed as a double theta series of the form either*

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\alpha(m,n)q^{4m^2+9n^2}\quad\text{or}\quad \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\alpha(m,n)q^{12m^2+n^2}.$$

(7) *If $B(q)$ is any of the eta quotients of weight 1 in the odd numbered entries of Table 19 other than Entries 39 and 45, then either of $q^3B(q^8)$ or $q^3B(-q^8)$ can be expressed as a double theta series of the form either*

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\alpha(m,n)q^{m^2+2n^2},\qquad \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\alpha(m,n)q^{3(m^2+8n^2)}$$

or

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\alpha(m,n)q^{\frac{1}{3}(m^2+8n^2)}.$$

Proof. These follow from Lemma 3.6; for example,

$$qf_4^6=\left(q^{\frac{1}{2}}f_4^3\right)^2=\left(\sum_{n=1}^{\infty}\left(\frac{-4}{n}\right)nq^{\frac{1}{2}n^2}\right)^2=\sum_{m,n=1}^{\infty}mn\left(\frac{-4}{m}\right)\left(\frac{-4}{n}\right)q^{\frac{1}{2}(m^2+n^2)}$$

by (3.13) of Lemma 3.6. □

4. Eta quotients with vanishing coefficient behaviour similar to f_1, f_1^2 and f_1^3

Since the number of eta quotients involved in each case is small, and since it is possible to prove completely what was discovered experimentally, we treat the cases of f_1, f_1^2 and f_1^3 together in one section.

4.1. Eta quotients with vanishing coefficient behaviour similar to f_1. Our search found just five other eta quotients with vanishing coefficient behaviour similar to f_1.

Table 3: Eta quotients with vanishing behaviour similar to f_1

Number	q-Product	Modular Form	Weight	Level	Group
1	f_1	$\eta(24z)$	$\frac{1}{2}$	576	I
2	$\frac{f_2^3}{f_1f_4}$	$\frac{\eta(48z)^3}{\eta(24z)\eta(96z)}$	$\frac{1}{2}$	2304	I
3	$\frac{f_2f_3^2}{f_1f_6}$	$\frac{\eta(48z)\eta(72z)^2}{\eta(24z)\eta(144z)}$	$\frac{1}{2}$	144	I
4	$\frac{f_1f_4f_6^5}{f_2^2f_3^2f_{12}^2}$	$\frac{\eta(24z)\eta(96z)\eta(144z)^5}{\eta(48z)^2\eta(72z)^2\eta(288z)^2}$	$\frac{1}{2}$	2304	I
5	$\frac{f_1^5}{f_2^2}$	$\frac{\eta(24z)^5}{\eta(48z)^2}$	$\frac{3}{2}$	144	I
6	$\frac{f_2^{13}}{f_1^5f_4^5}$	$\frac{\eta(48z)^{13}}{\eta(24z)^5\eta(96z)^5}$	$\frac{3}{2}$	2304	I

That the series coefficients of all the eta quotients in Table 3 vanish identically is an easy consequence of known q-series identities. We first need a lemma

Lemma 4.1. *Let* $|q| < 1$ *Then*

$$f_1 = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}, \tag{4.1}$$

$$\frac{f_2 f_3^2}{f_1 f_6} = \sum_{n=-\infty}^{\infty} q^{n(3n+1)/2}, \tag{4.2}$$

$$\frac{f_1^5}{f_2^2} = \sum_{n=-\infty}^{\infty} (6n+1) q^{n(3n+1)/2}. \tag{4.3}$$

Proof. The first is the very well known special case ($q \to q^3$ followed by $z \to 1/q$) of the Jacobi triple product identity

$$\sum_{n=-\infty}^{\infty} (-z)^n q^{n(n+1)/2} = (zq, 1/z, q; q)_\infty. \tag{4.4}$$

The second is also a special case of (4.4), this time using $q \to q^3$ followed by $z \to -1/q$. The third is an identity is due to Fine [8, p.83] and is a limiting case ($z \to -1$) of the quintuple product identity:

$$(-z, -q/z, q; q)_\infty (qz^2, q/z^2; q^2)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} z^{3n} (1 + zq^n). \tag{4.5}$$

□

Theorem 4.2. *Let* $A(q)$ *and* $B(q)$ *be any two products in the following list:*

$$\left\{ f_1, \frac{f_2^3}{f_1 f_4}, \frac{f_2 f_3^2}{f_1 f_6}, \frac{f_1 f_4 f_6^5}{f_2^2 f_3^2 f_{12}^2}, \frac{f_1^5}{f_2^2}, \frac{f_2^{13}}{f_1^5 f_4^5} \right\}. \tag{4.6}$$

Then

$$A_{(0)} = B_{(0)}. \tag{4.7}$$

Proof. That the claim holds if $A(q)$ and $B(q)$ are any two of the first, third and fifth eta quotients at (4.6) follows from Lemma 4.1, and the full claim follows since the second, fourth and sixth eta quotients at (4.6) are the $q \to -q$ partners, respectively, of these three, after employing (2.4). □

Remark 4.3. *If one writes*

$$A(q) = f_1 = \sum_{n=0}^{\infty} a_n q^n,$$

then by the well known fact that

$$q f_{24} = \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) q^{n^2},$$

where $\left(\frac{12}{n}\right)$ *denotes the quadratic character of* $\mathbb{Q}[\sqrt{3}]$*, one can also tell that* $a_n = 0$ *whenever* $24n + 1$ *is not a square.*

4.2. Eta quotients with vanishing coefficient behaviour similar to f_1^2. Our search found ten eta quotients with vanishing coefficient behaviour similar to f_1^2.

Table 4: Eta quotients with vanishing behaviour similar to f_1^2

Number	q-Product	Modular Form	Weight	Level	Group
1	f_1^2	$\eta(12z)^2$	1	144	I
2	$\frac{f_2^6}{f_1^2 f_4^2}$	$\frac{\eta(24z)^6}{\eta(12z)^2\eta(48z)^2}$	1	576	I
3	$\frac{f_2^4 f_4}{f_1^2 f_8}$	$\frac{\eta(24z)^4\eta(48z)}{\eta(12z)^2\eta(96z)}$	1	2304	I
4	$\frac{f_1^2 f_4^3}{f_2^2 f_8}$	$\frac{\eta(12z)^2\eta(48z)^3}{\eta(24z)^2\eta(96z)}$	1	2304	I
5	$\frac{f_2 f_8 f_{12}^5}{f_4^2 f_6^2 f_{24}^2}$	$\frac{\eta(24z)\eta(96z)\eta(144z)^5}{\eta(48z)^2\eta(72z)^2\eta(288z)^2}$	$\frac{1}{2}$	2304	II
6	$\frac{f_2^5}{f_4^2}$	$\frac{\eta(24z)^5}{\eta(48z)^2}$	$\frac{3}{2}$	144	II
7	f_2	$\eta(24z)$	$\frac{1}{2}$	576	II
8	$\frac{f_4 f_6^2}{f_2 f_{12}}$	$\frac{\eta(48z)\eta(72z)^2}{\eta(24z)\eta(144z)}$	$\frac{1}{2}$	144	II
9	$\frac{f_4^3}{f_2 f_8}$	$\frac{\eta(48z)^3}{\eta(24z)\eta(96z)}$	$\frac{1}{2}$	2304	II
10	$\frac{f_4^{13}}{f_2^5 f_8^5}$	$\frac{\eta(48z)^{13}}{\eta(24z)^5\eta(96z)^5}$	$\frac{3}{2}$	2304	II

We prove two theorems in this subsection, thereby fully verifying what was suggested by experiment. These proofs also give some insight into how similar results might be proved for pairs of eta quotients in other tables. Note that the eta quotients in Group II of Table 4 arise as a result of a $q \to q^2$ dilation of the eta quotients in Table 3.

The first subsequent theorem shows that all the eta quotients in Group I of Table 3 have series expansions with identically vanishing coefficients. Before getting to that, we need the following lemma, which shows that the series expansions of f_1^2 and $f_2^4 f_4/(f_1^2 f_8)$ have quite similar 4-dissections.

Lemma 4.4. *Define*

$$\begin{aligned}
A_0 = A_0(q) &:= (q^4;q^8)_\infty(-q^{12},-q^{20},q^{32};q^{32})_\infty(-q^4,-q^4,q^8;q^8)_\infty, \qquad (4.8)\\
A_1 = A_1(q) &:= -(q^4;q^8)_\infty(-q^{12},-q^{20},q^{32};q^{32})_\infty(-q^8,-1,q^8;q^8)_\infty,\\
A_2 = A_2(q) &:= -(q^4;q^8)_\infty(-q^4,-q^{28},q^{32};q^{32})_\infty(-q^4,-q^4,q^8;q^8)_\infty,\\
A_3 = A_3(q) &:= (q^4;q^8)_\infty(-q^4,-q^{28},q^{32};q^{32})_\infty(-q^8,-1,q^8;q^8)_\infty.
\end{aligned}$$

Then

$$f_1^2 = A_0 + qA_1 + q^2A_2 + q^3A_3, \qquad (4.9)$$

$$\frac{f_2^4 f_4}{f_1^2 f_8} = A_0 - qA_1 - q^2A_2 + q^3A_3.$$

Remark 4.5. *Note that the functions A_0, A_1, A_2 and A_3 are all functions of q^4, so that the expressions of the right sides of* (4.9) *actually give 4-dissections of the eta quotients on the left sides.*

Proof of Lemma 4.4. We will use the following alternative form of the Jacobi triple product identity,

$$\sum_{n=-\infty}^{\infty} (-z)^n q^{n^2} = (zq, q/z, q^2; q^2)_\infty, \tag{4.10}$$

or, more precisely, the identity arrived at by splitting the sum on the left into two sums (n even and n odd) and recombining into infinite products:

$$(-z^2q^4, -q^4/z^2, q^8; q^8)_\infty - zq(-z^2q^8, -1/z^2, q^8; q^8)_\infty = (zq, q/z, q^2; q^2)_\infty. \tag{4.11}$$

From (4.8) one has that

$$\begin{aligned}
&A_0 + qA_1 + q^2A_q^2 + q^3A_3\\
&\quad= (q^4;q^8)_\infty(-q^{12}, -q^{20}, q^{32}; q^{32})_\infty\left[(-q^4, -q^4, q^8; q^8)_\infty - q(-q^8, -1, q^8; q^8)_\infty\right]\\
&\quad- q^2(q^4;q^8)_\infty(-q^4, -q^{28}, q^{32}; q^{32})_\infty\left[(-q^4, -q^4, q^8; q^8)_\infty - q(-q^8, -1, q^8; q^8)_\infty\right]\\
&\quad= (q, q, q^2; q^2)_\infty(q^4;q^8)_\infty\left[(-q^{12}, -q^{20}, q^{32}; q^{32})_\infty - q^2(-q^4, -q^{28}, q^{32}; q^{32})_\infty\right]\\
&\quad= (q, q, q^2; q^2)_\infty(q^4;q^8)_\infty(q^2, q^6, q^8; q^8)_\infty = (q, q, q^2; q^2)_\infty(q^2;q^2)_\infty = (q,q)_\infty^2.
\end{aligned} \tag{4.12}$$

The second equality above follows from using (4.11) directly with $z = 1$, and the second follows from using (4.11) with q replaced with q^4 and $z = 1/q^2$.

In a quite similar fashion,

$$\begin{aligned}
&A_0 - qA_1 - q^2A_q^2 + q^3A_3\\
&\quad= (q^4;q^8)_\infty(-q^{12}, -q^{20}, q^{32}; q^{32})_\infty\left[(-q^4, -q^4, q^8; q^8)_\infty + q(-q^8, -1, q^8; q^8)_\infty\right]\\
&\quad+ q^2(q^4;q^8)_\infty(-q^4, -q^{28}, q^{32}; q^{32})_\infty\left[(-q^4, -q^4, q^8; q^8)_\infty + q(-q^8, -1, q^8; q^8)_\infty\right]\\
&\quad= (-q, -q, q^2; q^2)_\infty(q^4;q^8)_\infty\left[(-q^{12}, -q^{20}, q^{32}; q^{32})_\infty + q^2(-q^4, -q^{28}, q^{32}; q^{32})_\infty\right]\\
&\quad= (-q, -q, q^2; q^2)_\infty(q^4;q^8)_\infty(-q^2, -q^6, q^8; q^8)_\infty = \frac{(q^2;q^2)_\infty^4(q^4;q^4)_\infty}{(q;q)_\infty^2(q^8;q^8)_\infty}.
\end{aligned} \tag{4.13}$$

This time, the second equality above follows from (4.11) directly with $z = -1$, and the second follows from using (4.11) with q replaced with q^4 and $z = -1/q^2$. The final step follows from simple q-product manipulations and is left to the reader. □

Theorem 4.6. *Let $A(q)$ and $B(q)$ be any two eta quotients in Group I of Table* 4. *Then*

$$A_{(0)} = B_{(0)}. \tag{4.14}$$

Proof. That this holds when $A(q)$ and $B(q)$ are the eta quotients numbered 1 and 3 follows from the 4-dissections for these eta quotients at (4.9), and the full claim at (4.14) follows since the eta quotients numbered 2 and 4 in Group I are $q \to -q$ partners of those numbered 1 and 3. □

Theorem 4.7. *Let*

$$A(q) = \sum_{n=0}^{\infty} a_n q^n, \qquad B(q) = \sum_{n=0}^{\infty} b_n q^n, \tag{4.15}$$

where $A(q)$ is any of the eta quotients in Group I of Table 4, and $B(q)$ is any of those in Group II. Then

$$A_{(0)} \subsetneqq B_{(0)}. \tag{4.16}$$

Proof. We prove the result for $A(q) = f_1^2$ and $B(q) = f_2$ only, since by Theorem 4.6 all the eta quotients in Group I have identically vanishing coefficients, and the same holds for group II by Theorem 4.2. After making the dilation $q \to q^{12}$ and then multiplying by q, one gets

$$\eta(12z)^2 = \sum_{n=0}^{\infty} a_n q^{12n+1}, \quad \eta(24z) = \sum_{n=0}^{\infty} b_n q^{12n+1} = \sum_{m=-\infty}^{\infty} (-1)^m q^{(6m+1)^2},$$

where the last well-known equality arises from the dilation $q \to q^{24}$ in (4.1). Suppose $a_n = 0$, so that by Serre's criterion (3.1), $12n+1$ has a prime factor $p \not\equiv 1 \pmod{12}$ with odd exponent. Clearly $b_n = 0$ under this same circumstance, since $b_n = 0$ if $12n + 1$ is not a square. Since $a_1 = -2$ and $b_1 = 0$, the result follows. □

4.3. Eta quotients with vanishing coefficient behaviour similar to f_1^3.

In the case of f_1^3, our search found eleven other eta quotients with similar vanishing coefficient behaviour.

Table 5: Eta quotients with vanishing behaviour similar to f_1^3

Number	q-Product	Modular Form	Weight	Level	Group
1	f_1^3	$\eta(8z)^3$	$\frac{3}{2}$	64	I
2	$\frac{f_2^9}{f_1^3 f_4^3}$	$\frac{\eta(16z)^9}{\eta(8z)^3\eta(32z)^3}$	$\frac{3}{2}$	256	I
3	$\frac{f_2^5 f_3 f_{12}}{f_1^2 f_4^2 f_6^2}$	$\frac{\eta(16z)^5\eta(24z)\eta(96z)}{\eta(8z)^2\eta(32z)^2\eta(48z)^2}$	$\frac{1}{2}$	2304	I
4	$\frac{f_1^2 f_6}{f_2 f_3}$	$\frac{\eta(8z)^2\eta(48z)}{\eta(16z)\eta(24z)}$	$\frac{1}{2}$	144	I
5	$\frac{f_2^2}{f_1}$	$\frac{\eta(16z)^2}{\eta(8z)}$	$\frac{1}{2}$	16	I
6	$\frac{f_1 f_4}{f_2}$	$\frac{\eta(8z)\eta(32z)}{\eta(16z)}$	$\frac{1}{2}$	256	I
7	f_3	$\eta(24z)$	$\frac{1}{2}$	576	II

8	$\frac{f_6^3}{f_3f_{12}}$	$\frac{\eta(48z)^3}{\eta(24z)\eta(96z)}$	$\frac{1}{2}$	2304	II
9	$\frac{f_3^5}{f_6^2}$	$\frac{\eta(24z)^5}{\eta(48z)^2}$	$\frac{3}{2}$	144	II
10	$\frac{f_6^{13}}{f_3^5f_{12}^5}$	$\frac{\eta(48z)^{13}}{\eta(24z)^5\eta(96z)^5}$	$\frac{3}{2}$	2304	II
11	$\frac{f_6f_9^2}{f_3f_{18}}$	$\frac{\eta(48z)\eta(72z)^2}{\eta(24z)\eta(144z)}$	$\frac{1}{2}$	144	II
12	$\frac{f_3f_{12}f_{18}^5}{f_6^2f_9^2f_{36}^2}$	$\frac{\eta(24z)\eta(96z)\eta(144z)^5}{\eta(48z)^2\eta(72z)^2\eta(288z)^2}$	$\frac{1}{2}$	2304	II

In this subsection we give a characterization of the vanishing coefficient behaviour of the eta quotients in Table 5. We first need a lemma.

Lemma 4.8. *Let* $|q| < 1$. *Then*

$$f_1^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}, \tag{4.17}$$

$$\frac{f_1^2f_6}{f_2f_3} = \sum_{n=0}^{\infty} q^{n(n+1)/2} - 3\sum_{n=0}^{\infty} q^{(3n+1)(3n+2)/2}, \tag{4.18}$$

$$\frac{f_2^2}{f_1} = \sum_{n=0}^{\infty} q^{n(n+1)/2}. \tag{4.19}$$

Proof. The first- and third identities above are well-known implications of the Jacobi triple product identity (4.4). The second identity is a restatement of a result of Cooper [3, Equation (9.3)], who showed that

$$\frac{f_1^2f_6}{f_2f_3} = \sum_{n=0}^{\infty} q^{n(n+1)/2} - 3q\sum_{n=0}^{\infty} q^{9n(n+1)/2}.$$

□

We are now ready to prove the results suggested by Table 5.

Theorem 4.9. *(i) If* $A(q)$ *and* $B(q)$ *are any two eta quotients that are both either in Group I or group II of Table* 5, *then*

$$A_{(0)} = B_{(0)}. \tag{4.20}$$

(ii) If $A(q)$ *and* $B(q)$ *are any two eta quotients such that* $A(q)$ *is in Group I of Table* 5 *and* $B(q)$ *is in Group II of Table* 5, *then*

$$A_{(0)} \subsetneqq B_{(0)}. \tag{4.21}$$

Proof. (i) First observe that the eta quotients in Group II in Table 5 are derived from the group of eta quotients in Table 3 by a $q \to q^3$ dilation, so that $A_{(0)} = B_{(0)}$ holds if $A(q)$ and $B(q)$ are both in Group II follows from Theorem 4.2.

If $A(q)$ and $B(q)$ are both in Group I and are any two of the eta quotients listed in Lemma 4.8, then $A_{(0)} = B_{(0)}$ clearly holds from the form of the series on each of the three right sides at (4.17) –(4.19). Next, $A_{(0)} = B_{(0)}$ holds if $A(q)$ and $B(q)$ are *any* two eta quotients in Group I, since the other three eta quotients in the group are $q \to -q$ partners of the three in Lemma 4.8 after using the transformation from (4.5).

(ii) We will prove the statement for $B(q) = f_3$ only, as all the other eta quotients in Group II have identical vanishing coefficient behaviour. It may be seen from (4.1) that after the dilation $q \to q^3$ is applied to f_1, one gets that

$$f_3 = \sum_{n=0}^{\infty}(-1)^n q^{3n(3n+1)/2} + \sum_{n=1}^{\infty}(-1)^n q^{3n(3n-1)/2} =: \sum_{t=0}^{\infty} b_t q^t.$$

Thus $b_t \neq 0$ only if t has the form $t = m(m+1)/2$ for either $m = 3n$ or $m = 3n-1$, for some integer n, and $A_{(0)} \subsetneqq B_{(0)}$ easily follows. □

5. Relations between $B_{(0)}$'s

In this section, we shall study relations between the vanishing of coefficients of various products in the tables for f_1^r for $r \in \{4, 6, 8, 10, 14\}$ and $f_1^3 f_2^3$ and as a consequence, illustrate a number of branches of the associated directed graphs in the appendix. It is worthy of stressing that in the case of f_1^r where $r \in \{4, 6, 8, 10, 14, 26\}$ and $f_1^3 f_2^3$, there are so many eta quotients in each of the corresponding tables that it is not practical to attempt to prove each equality $A_{(0)} = B_{(0)}$ or inclusion $A_{(0)} \subsetneqq B_{(0)}$ suggested by experiment, for each pair of eta quotients $A(q)$ and $B(q)$. We confine ourselves to providing sample proofs that exhibit the various methods of proof. In some cases, we provide proofs that exhibit equality or inclusion for entire groups of eta quotients in the various tables, or proofs that give completeness of proof for an entire group of eta quotients in some table (by giving the proof for some outstanding member of the group), or proofs that help support the relationships between the various groups in a table that are suggested by experiment.

This section is divided into six subsections, each of which accounts for the eta quotients in the table associated with one of f_1^r for $r \in \{4, 6, 8, 10, 14\}$ and $f_1^3 f_2^3$. Before proceeding, we state and prove a technical lemma that will be frequently adopted throughout this section.

Lemma 5.1. *Let $A(q)$ and $B(q)$ be eta quotients and let $m > 1$ be a positive integer. Let $A'(q) = A(q^m)$ and $B'(q) = B(q^m)$. Then the following hold.*

$$A_{(0)} = B_{(0)} \Longleftrightarrow A'_{(0)} = B'_{(0)}, \tag{5.1}$$

$$A_{(0)} \subsetneqq B_{(0)} \Longleftrightarrow A'_{(0)} \subsetneqq B'_{(0)}. \tag{5.2}$$

Proof. Suppose

$$A(q) =: \sum_{n=0}^{\infty} a_n q^n, \qquad B(q) =: \sum_{n=0}^{\infty} b_n q^n,$$

$$A'(q) =: \sum_{n\geq 0} a'_n q^n = \sum_{n=0}^{\infty} a_n q^{mn}, \qquad B'(q) =: \sum_{n=0}^{\infty} b'_n q^n = \sum_{n\geq 0} b_n q^{mn}.$$

Then clearly $a'_n = b'_n = 0$ if $m \nmid n$, and for $n \geq 0$, $a'_{mn} = a_n$ and $b'_{mn} = b_n$. Both results clearly follow from these facts. □

Remark 5.2. *Some of results proven in the following subsections may seem a bit arbitrary, but in some cases we prove them because they are exceptional cases not covered by Lemmas* 3.5 *and* 3.7*, and are necessary to complete the proof of the statements in Theorems* 7.1 *and* 7.2*.*

5.1. Eta quotients with vanishing coefficient behaviour similar to f_1^4. Theorem 4.7 plays a significant role in the proof of the next theorem.

Theorem 5.3. *Let*

$$A(q) = f_1^4 = \sum_{n=0}^{\infty} a_n q^n, \qquad B(q) = \sum_{n=0}^{\infty} b_n q^n, \qquad C(q) = \sum_{n=0}^{\infty} c_n q^n, \tag{5.3}$$

where $B(q)$ is any of the eta quotients in Group XVII and $C(q)$ is any of the eta quotients in Group XIX of Table 7*. Then*

$$A_{(0)} \subsetneqq B_{(0)} \subsetneqq C_{(0)}. \tag{5.4}$$

Moreover, if $B(q)$ and $B'(q)$ are any two eta quotients in Group XVII and $C(q)$ and $C'(q)$ are any two eta quotients in Group XIX, then

$$B_{(0)} = B'_{(0)}, \qquad\qquad C_{(0)} = C'_{(0)}. \tag{5.5}$$

Proof. That $B_{(0)} \subsetneqq C_{(0)}$ and the equalities in (5.5) hold follows from Theorem 4.7 and Lemma 5.1, since the eta quotients in Group XVII and group XIX of Table 7 are derived from the eta quotients in Groups I and II in Table 4 as a result of a $q \to q^2$ dilation.

We prove that $A_{(0)} \subsetneqq B_{(0)}$ for $B(q) = f_2^2$ only, since, as noted in the preceding paragraph, all the eta quotients in the group have identically vanishing coefficients. Note that this holds by Theorem 7.1, but we give another proof to illustrate an alternative method of showing inclusion. After making the dilation $q \to q^6$ and then multiplying by q, one gets

$$\eta(6z)^4 = \sum_{n=0}^{\infty} a_n q^{6n+1}, \qquad\qquad \eta(12z)^2 = \sum_{n=0}^{\infty} b_n q^{6n+1}.$$

Suppose $a_n = 0$, so that by Serre's criterion (3.2), $6n+1$ has a prime factor $p \equiv -1 \pmod 3$ with odd exponent. If n is odd, then clearly $b_n = 0$ since $B(q) = f_2^2$. If n is even, $n = 2m$, then $6n+1 = 12m+1$ has a prime factor $p \equiv -1 \pmod 3$ with odd exponent, and hence by Serre's criterion (3.1), $b_n = 0$. Since $a_1 = -4$ and $b_1 = 0$, the result follows. □

Before proving the next result related to our study of $A(q) = f_1^4$ in Theorem 5.5, we prove a necessary preliminary result, which is to completely characterize the vanishing behaviour of the coefficients of the series defined by

$$B(q) := \frac{f_1^5}{f_3} =: \sum_{n=0}^{\infty} b_n q^n. \tag{5.6}$$

What makes the proof a little easier is that much of the necessary work was carried out elsewhere in proving an entirely different vanishing coefficient result, as will be described below. We again note that the result in Theorem 5.5 also follows from Theorem 7.1, but here also we choose to illustrate an alternative method of proof.

Lemma 5.4. *Let $B(q)$ and the sequence $\{b_n\}$ be as defined at (5.6). Then $b_n = 0$ if and only if $12n+1$ has a prime factor $p \equiv -1 \pmod 3$ with odd exponent.*

Proof. After applying the dilation $q \to q^{12}$ and multiplying by q, one gets that

$$\frac{\eta(12z)^5}{\eta(36z)} = \sum_{n=0}^{\infty} b_n q^{12n+1} = \frac{S(q) + \bar{S}(q)}{2}, \tag{5.7}$$

where

$$S(q) = q - 3i\sqrt{3}q^7 - 5q^{13} - 3i\sqrt{3}q^{19} + 5q^{25} - 6i\sqrt{3}q^{31} + 11q^{37} + 6i\sqrt{3}q^{43} - 20q^{49} - q^{61} + 9i\sqrt{3}q^{67} + 7q^{73} - 3i\sqrt{3}q^{79} + 15i\sqrt{3}q^{91} + 19q^{97} + \cdots \tag{5.8}$$

is the CM newform of weight 2 for $\Gamma_0(432)$ with $\left(\frac{3}{\cdot}\right)$ by $K = \mathbb{Q}[\sqrt{-3}]$ labelled 432.2.c.a in the LMFDB, and $\bar{S}(q)$ is its $\imath \to -\imath$ conjugate. Define the sequence $\{s_n\}$ by

$$S(q) = \sum_{n=0}^{\infty} s_n q^n. \tag{5.9}$$

In [17, Theorem 8.2], it was shown that if $C(q) = f_1^{14}$ and $D(q) = f_3^5/f_1$, then $C_{(0)} = D_{(0)}$, and in the course of proving that, the following results where shown about the sequence s_n (the interested reader may find details of the computations in the aforementioned paper). Firstly, it was shown that $s_2 = s_3 = 0$, and if $p \equiv 5$ or $11 \pmod{12}$, then $s_p = 0$. It was also shown that if $p = x^2 + 3y^2 \equiv 7 \pmod{12}$ where x and y are positive integers, then up to sign,

$$s_p = \begin{cases} 2y\sqrt{-3}, & \text{if } 3|y, \\ (y+x)\sqrt{-3}, & \text{if } 3|y+x, \\ (y-x)\sqrt{-3}, & \text{if } 3|y-x. \end{cases}$$

Likewise, if $p = x^2 + 3y^2 \equiv 1 \pmod{12}$ for positive integers x and y, then it was shown that, up to sign, $s_p = \pm x + 3y$. The recurrence formula

$$s_{p^k} = s_p s_{p^{k-1}} - \chi(p) p s_{p^{k-2}}$$

gives that if $p \equiv 5$ or $11 \pmod{12}$, then $|s_{p^{2k}}| = p^k$ and $s_{p^{2k+1}} = 0$. Likewise, if $p \equiv 1$ or $7 \pmod{12}$, then the formulae above give that $s_p \neq 0$, and since the recurrence relation gives that $s_{p^k} \equiv s_p^k \not\equiv 0 \pmod{p}$, and thus $s_p^k \neq 0$ for any positive integer k.

Hence, by multiplicity, $s_{12n+1} = 0$ if and only if $12n+1$ has a prime factor $p \equiv 5$ or $11 \pmod{12}$ with odd exponent, and the result follows since by (5.7), $b_n = s_{12n+1}$. □

Theorem 5.5. *Let*

$$A(q) = f_1^4 = \sum_{n=0}^{\infty} a_n q^n, \quad B'(q) := B(q^2) = \frac{f_2^5}{f_6} =: \sum_{n=0}^{\infty} b'_n q^n = \sum_{n=0}^{\infty} b_n q^{2n}, \tag{5.10}$$

where $B(q)$ and the sequence $\{b_n\}$ are as defined at (5.6)*. Then*

$$A_{(0)} \subsetneqq B_{(0)}. \tag{5.11}$$

Proof. Suppose $a_n = 0$, so that by Serre's criterion (3.2), $6n+1$ has a prime factor $p \equiv -1 \pmod{3}$ with odd exponent. If n is odd, it follows from the fact that $B'(q) = B(q^2)$ that $b'_n = 0$. If n is even, $n = 2m$ say, then $6n+1 = 12m+1$ has a prime factor $p \equiv -1 \pmod{3}$ with odd exponent. Hence, by Lemma 5.4, $b_m = 0$, and then $b'_{2m} = b'_n = 0$. The result follows upon observing that $a_1 = -4$, while $b'_1 = 0$. □

Theorem 5.6. *Let*

$$A(q) = f_1^4 = \sum_{n=0}^{\infty} a_n q^n$$

and let $B(q) = \sum_{n=0}^{\infty} b_n q^n$ be any of the eta quotients in Entries 91*,* 97*,* 99 *and* 131 *of Table* 7*. Then*

$$A_{(0)} \subsetneqq B_{(0)}.$$

Proof. By [19, Theorem 3.1, Eq. 3.1.8], Entries 91 and 93 of Table 7 have identically vanishing coefficients, as do entries 97, 99 and 101 by [19, Theorem 3.1, Eq. 3.1.9]. Letting $C(q)$ be a product in Entries 93 or 101, one can tell that $B_{(0)} = C_{(0)}$. On the other hand, By Table 8, one finds that $qC(q^6)$ can be expressed as

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \alpha(m,n) q^{\frac{1}{4}(3m^2+n^2)}.$$

So by Theorem 3.4 (2), one can conclude that $A_{(0)} \subsetneqq C_{(0)} = B_{(0)}$, where is inequality can be verified by comparing the first few terms of the coefficients of the eta quotients.

Moreover, by [19, Theorem 3.2, Eq. 3.1.19], if $C(q)$ is any of the eta quotients in Entry 101, and $B(q)$ is the eta quotient given by Entry 131, then $C_{(0)} \subsetneqq B_{(0)}$.

All told, $A_{(0)} \subsetneqq B_{(0)}$ for any product $B(q) = \sum_{n=0}^{\infty} b_n q^n$ in Entries 91, 97, 99 and 131 of Table 7.

□

Theorem 5.7. *Let*

$$B(q)=\frac{f_1^3f_3f_8^3f_{12}}{f_4^2f_6f_{24}}\quad \textit{and}\quad C(q)=\frac{f_2^2f_6^9f_8^3}{f_1f_3^3f_4^2f_{12}^3f_{24}}$$

be the eta quotients numbered 74 and 75 in Group II of Table 7*, respectively. Then* $B_{(0)}=C_{(0)}$.

Proof. Write $C(q)=\sum_{n=0}^{\infty}c_nq^n$. Then one can verify that

$$B(q)=\sum_{j=0,3,4,7,8,11}\sum_{n=0}^{\infty}c_{12n+j}q^{12n+j}-3\sum_{j=1,2,5,6,9,10}\sum_{n=0}^{\infty}c_{12n+j}q^{12n+j}.$$

Therefore, one can see that

$$\sum_{n=0}^{\infty}b_{12n+j}q^{12n+j}=\sum_{n=0}^{\infty}c_{12n+j}q^{12n+j}\quad \text{for } j=0,3,4,7,8,11,$$

and

$$\sum_{n=0}^{\infty}b_{12n+j}q^{12n+j}=-3\sum_{n=0}^{\infty}c_{12n+j}q^{12n+j}\quad \text{for } j=1,2,5,6,9,10,$$

and deduce that $B_{(0)}=C_{(0)}$. □

Remark 5.8. *The proof of Theorem* 5.7 *is inspired by Lemma* 4.4*.*

5.2. Eta quotients with vanishing coefficient behaviour similar to f_1^6. Throughout this section, we let

$$A(q):=f_1^6=:\sum_{n=0}^{\infty}a_nq^n,$$

and we recall Serre's criterion ([29] or Equation (3.3) above) that $a_n=0$ if and only if $4n+1$ has a prime factor $p\equiv -1 \pmod 4$ with odd exponent. We first contrast some results from [18] about various eta quotients in Table 9 with what Theorem 7.1 gives us about these.

Let

$$B(q)=f_1^2f_2^2=\sum_{n=0}^{\infty}b_nq^n,\quad C(q)=f_1f_5=\sum_{n=0}^{\infty}c_nq^n,\quad D(q)=f_2f_4=\sum_{n=0}^{\infty}d_nq^n, \tag{5.12}$$

$$E(q)=f_3^2=\sum_{n=0}^{\infty}e_nq^n.$$

In [18, Theorem 1.1], exact criteria for the vanishing of the b_n, c_n, d_n and e_n were developed, from which it immediately follows that

$$A_{(0)}=B_{(0)},\qquad A_{(0)}\subsetneq C_{(0)},\qquad A_{(0)}\subsetneq D_{(0)},\qquad A_{(0)}\subsetneq E_{(0)}. \tag{5.13}$$

Theorem 7.1 gives the last three inclusions, but only the weaker result $A_{(0)}\subseteq B_{(0)}$. Indeed this highlights a shortcoming of Theorem 7.1 in comparison with more detailed analyses. While Theorem 7.1 is a very general theorem, it provides no information about exactly when the coefficients in the series expansions of the eta quotients it applies to vanish.

To further reinforce the utility of diverse methods refining conclusions from Theorem 7.1, consider

$$G(q) = f_2^3 = \sum_{n=0}^{\infty} g_n q^n, \qquad H(q) = f_6 = \sum_{n=0}^{\infty} h_n q^n. \tag{5.14}$$

Theorem 7.1 gives $A_{(0)} \subsetneqq G_{(0)}$ and $A_{(0)} \subsetneqq H_{(0)}$, but once again the alternative proof we give provides more insight than simply proving an inclusion of the vanishing sets.

That $A_{(0)} \subsetneqq G_{(0)}$ holds may be seen from the fact that

$$\eta(8z)^3 = \sum_{n=0}^{\infty} g_n q^{4n+1} = \sum_{m=0}^{\infty} (-1)^m (2m+1) q^{(2m+1)^2},$$

and thus that $g_n = 0$ if and only if $4n+1$ is not an odd square.

Likewise, $A_{(0)} \subsetneqq H_{(0)}$ holds since

$$\eta(24z) = \sum_{n=0}^{\infty} h_n q^{4n+1} = \sum_{m=-\infty}^{\infty} (-1)^m q^{(6m-1)^2},$$

and thus $h_n = 0$ if and only if $4n+1$ is not an odd square that is not divisible by 3.

There is one more eta quotient in Table 9, namely that numbered 65, $B(q) := f_1^3 f_3$. Rather than choosing the easy route of using Theorem 7.1 to show $A_{(0)} \subsetneqq B_{(0}$, we choose the more difficult route of determining an exact criterion for the vanishing of the coefficients of $B(q)$. This latter method may be necessary to prove some of the finer structure exhibited in Table 9 and Figure 3.

However, determining the exact conditions under which the coefficients in the series expansion vanish is not so easy in this case. We first need the following lemma. As elsewhere in the paper the results follow upon comparing coefficients up to the Sturm bound.

Lemma 5.9. *The following identity holds.*

$$\begin{aligned}\eta(4z)^3\eta(12z) &= \frac{\left(2 - i\sqrt{2}\right)}{4} f_{144.2.c.a} + \frac{\left(2 + i\sqrt{2}\right)}{4} \bar{f}_{144.2.c.a} \\ &= q - 3q^5 + 4q^{13} + 3q^{17} - 13q^{25} + 3q^{29} + 2q^{37} + 9q^{41} + 7q^{49} \\ &\qquad - 9q^{53} - 10q^{61} - 12q^{65} + 16q^{73} + 18q^{85} - 3q^{89} - 8q^{97},\end{aligned}$$

where $f_{144.2.c.a}$ is the CM form of weight 2 and level 144 labelled 144.2.c.a at the LMFDB and $\bar{f}_{144.2.c.a}$ is the conjugate form. Moreover,

$$f_{144.2.c.a} = q - 3i\sqrt{2}q^5 + 4q^{13} + 3i\sqrt{2}q^{17} - 13q^{25} + 3i\sqrt{2}q^{29} + 2q^{37} + 9i\sqrt{2}q^{41} + 7q^{49} \tag{5.15}$$

$$- 9i\sqrt{2}q^{53} - 10q^{61} - 12i\sqrt{2}q^{65} + 16q^{73} + 18q^{85} - 3i\sqrt{2}q^{89} - 8q^{97} + \cdots$$
$$= g_1 + ig_3 + (1/2 - i(1/2))\sqrt{2}g_5 + (-1/2 - i(1/2))\sqrt{2}g_6,$$

where

$$g_1 = \sum_{m,n}(6m + 1 + 6ni)q^{(6m+1)^2+(6n)^2}, \tag{5.16}$$
$$g_3 = \sum_{m,n}(6m + 3 + (6n - 2)i)q^{(6m+3)^2+(6n-2)^2},$$
$$g_5 = \sum_{m,n}(6m + 1 + (6n - 2)i)q^{(6m+1)^2+(6n-2)^2},$$
$$g_6 = \sum_{m,n}(6m + 1 + (6n + 2)i)q^{(6m+1)^2+(6n+2)^2}.$$

Remark 5.10. *If we write $f_{144.2.c.a} =: S_a + \imath\sqrt{2}S_b$, so that S_a and S_b have integer coefficients and $\bar{f}_{144.2.c.a} =: S_a - \imath\sqrt{2}S_b$, then $\eta(4z)^3\eta(12z) = S_a + S_b$.*

We are now in a position to prove the following theorem.

Theorem 5.11. *Let*

$$B(q) := f_1^3 f_3 =: \sum_{n=0}^{\infty} b_n q^n. \tag{5.17}$$

Then $A_{(0)} \subsetneqq B_{(0)}$.

Proof. Apply the dilation $q \to q^4$ in Equation (5.17) and multiply by q to get

$$\eta^3(4z)\eta(12z) = \sum_{n=0}^{\infty} b_n q^{4n+1}. \tag{5.18}$$

Define the sequence $\{s_t\}$ by

$$f_{144.2.c.a} = \sum_{t=1}^{\infty} s_t q^t. \tag{5.19}$$

From the exponents of the theta series at (5.16) it can be easily deduced that if $t \not\equiv 1 \text{ or } 5 \pmod{12}$, then $s_t = 0$. In particular, if p is a prime such that $p = 2$ or 3 or $p \equiv 7 \text{ or } 11 \pmod{12}$, then $s_p = 0$. The recurrence relation for $f_{144.2.c.a}$ at powers of a prime p may be stated as follows:

$$s_{p^{k+1}} = s_p s_{p^k} - \chi(p)\, p\, s_{p^{k-1}}, \text{ where} \tag{5.20}$$
$$\chi(p) = \begin{cases} 1, & p \equiv 1, 11 \pmod{12} \\ -1, & p \equiv 5, 7 \pmod{12}. \end{cases}$$

Hence if $p \equiv 7 \text{ or } 11 \pmod{12}$ is prime and $k \geq 0$ is an integer, $|s_{p^{2k}}| = p^k$ and $s_{p^{2k+1}} = 0$. Note also that $s_9 = 0$ (3 divides the level of $f_{144.2.c.a}$, namely, 144) and hence $s_{3^k} = 0$ for all integers $k \geq 1$.

Next, if $p \equiv 1 \pmod 4$ is a prime, then $p = x^2 + y^2$ for unique positive integers x and y with x odd and y even. We consider the cases $p \equiv 1 \pmod{12}$ and

$p \equiv 5 \pmod{12}$ separately. If $p \equiv 1 \pmod{12}$, then p occurs as an exponent in exactly one of the series g_1 or g_3 and not as an exponent in either of the series g_5 or g_6. In either case(g_1 or g_3) there are exactly two representations ((m,n) and $(m,-n)$ in the case of g_1, (m,n) and $(-m-1,n)$ in the case of g_3). If p is represented by $(6m+1)^2+(6n)^2$ (so g_1), then by the last equality at (5.15),

$$s_p = (6m+1+6n\imath)+(6m+1-6n\imath) = 2(6m+1),$$

so in this case $s_p = \pm 2x$ (the sign depending on whether $x \equiv 1 \pmod 6$ or 5 (mod 6)). By similar reasoning, if $p = (6m+3)^2+(6n-2)^2$ (so g_3), again by the last equality at (5.15),

$$s_p = \imath\Big(\big(6m+3+(6n-2)\imath\big)+\big(6(-m-1)+3+(6n-2)\imath\big)\Big) = -2(6n-2),$$

so in this case $s_p = \pm 2y$.

Finally, if $p \equiv 5 \pmod{12}$, p occurs in the exponent of only the series g_5 and g_6, and exactly once in each case. In this case, once again using the last equality at (5.15),

$$\begin{aligned} s_p = \left(\frac{1}{2}-\frac{i}{2}\right)\sqrt{2}\Big(6m+1+(6n-2)i\Big)+\left(-\frac{1}{2}-\frac{i}{2}\right)\sqrt{2}\Big(6m+1+(-6n+2)i\Big) \\ = -i\sqrt{2}\Big((6m+1)-(6n-2)\Big), \end{aligned}$$

so that in this case $s_p = \pm i\sqrt{2}(x \pm y)$.

The reason for working out these explicit formulae for s_p is to show that for any prime $p \equiv 1$ or $5 \pmod{12}$ that $s_p \not\equiv 0 \pmod p$. The recurrence relation (5.20) then implies $s_{p^k} \equiv (s_p)^k \not\equiv 0 \pmod p$, and thus $s_{p^k} \neq 0$ for any positive integer k. Upon noting that the primes $\equiv 7, 11 \pmod{12}$ are precisely the primes $\equiv 3 \pmod 4$, and using the multiplicativity of the s_t, we see that, for any non-negative integer n,

$$s_{4n+1} = 0 \iff 3|4n+1 \text{ or } ord_p(4n+1) \text{ is odd for some prime } p \equiv 3 \pmod 4.$$

By the remark following Lemma 5.9, either $s_{4n+1} = b_n$ or $s_{4n+1} = \imath\sqrt{2}b_n$, and hence

$$b_n = 0 \iff 3|4n+1 \text{ or } ord_p(4n+1) \text{ is odd for some prime } p \equiv 3 \pmod 4.$$

The strict inclusion $A_{(0)} \subsetneqq B_{(0)}$ now follows since $b_n = 0$ if $ord_3(4n+1)$ is non-zero and even and $ord_p(4n+1)$ is even for all primes $p \equiv 3 \pmod 4$, while $a_n \neq 0$ under the same conditions. □

Theorem 5.12. *Let*

$$B(q) = \sum_{n=0}^{\infty} b_n q^n, \qquad C(q) = \sum_{n=0}^{\infty} c_n q^n, \qquad D(q) = \sum_{n=0}^{\infty} d_n q^n, \tag{5.21}$$

where $B(q)$ is any of the eta quotients in Group XXV, $C(q)$ is any of the eta quotients in Group XXVIII, and $D(q)$ is any of the eta quotients in Group XXIX of Table 9. Then

$$A_{(0)} \subsetneqq B_{(0)} \subsetneqq D_{(0)},\qquad (5.22)$$
$$A_{(0)} \subsetneqq C_{(0)} \subsetneqq D_{(0)}.$$

Moreover, if $B(q)$ and $B'(q)$ are any two eta quotients in Group XXV, $C(q)$ and $C'(q)$ are any two eta quotients in Group XXVIII and $D(q)$ and $D'(q)$ are any two eta quotients in Group XXIX, then

$$B_{(0)} = B'_{(0)}, \qquad C_{(0)} = C'_{(0)}, \qquad D_{(0)} = D'_{(0)}. \qquad (5.23)$$

Proof. The proof uses Lemma 5.1 and is virtually identical to the proof of Theorem 5.3. Hence details are omitted. We note that the eta quotients in Group XXV and group XXIX of Table 9 are derived from the eta quotients in Groups I and II in Table 4 as a result of a $q \to q^3$ dilation, and likewise the eta quotients in Group XXVIII and group XXIX of Table 9 are derived from the eta quotients in Groups I and II in Table 5 as a result of a $q \to q^2$ dilation.

The inclusion $A_{(0)} \subsetneqq B_{(0)}$ for $B(q) = f_3^2$ and $A_{(0)} \subsetneqq C_{(0)}$ for $C(q) = f_2^3$ were shown above, where these eta quotients were labelled $E(q)$ and $G(q)$). □

The results in the next theorem also follow from Theorem 7.1, but once again we give an alternative proof to illustrate a different method that utilizes other structure.

Theorem 5.13. *Let*

$$B(q) = \frac{f_2^7 f_{12}}{f_1^2 f_4^3 f_6} =: \sum_{n=0}^{\infty} b_n q^n, \qquad C(q) = \frac{f_1 f_4^3 f_6^2}{f_1^2 f_3 f_{12}} =: \sum_{n=0}^{\infty} c_n q^n. \qquad (5.24)$$

Then

$$A_{(0)} \subsetneqq B_{(0)}, \qquad A_{(0)} \subsetneqq C_{(0)}. \qquad (5.25)$$

Proof. Apply the dilation $q \to q^4$ in the eta quotients at (5.24) and multiply by q to get

$$\frac{\eta(8z)^7\eta(48z)}{\eta(4z)^2\eta(16z)^3\eta(24z)} = \sum_{n=1}^{\infty} b_n q^{4n+1}, \qquad \frac{\eta(4z)\eta(16z)^3\eta(24z)^2}{\eta(8z)^2\eta(12z)\eta(48z)} = \sum_{n=1}^{\infty} c_n q^{4n+1}.$$

Define

$$E_1(z) = \sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{-4}{d}\right)\right)q^n, \quad E_2(z) = \sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{-3}{n/d}\right)\left(\frac{12}{d}\right)\right)q^n.$$

Then one can check that

$$\frac{\eta(8z)^7\eta(48z)}{\eta(4z)^2\eta(16z)^3\eta(24z)} =$$
$$-\frac{1}{8}E_1(z) + \frac{1}{8}E_1(2z) - \frac{9}{8}E_1(18z) + \frac{9}{8}E_1(9z) + \frac{3}{8}E_2(z) + \frac{3}{8}E_2(2z) + \frac{3}{4}\eta(12z)^2,$$

$$\frac{\eta(4z)\eta(16z)^3\eta(24z)^2}{\eta(8z)^2\eta(12z)\eta(48z)} =$$
$$\frac{1}{4}E_1(z)-\frac{1}{4}E_1(2z)+\frac{9}{4}E_1(18z)-\frac{9}{4}E_1(9z)-\frac{3}{4}E_2(z)-\frac{3}{4}E_2(2z)+\frac{3}{2}\eta(12z)^2,$$

and thus

$$b_n=-\frac{1}{8}\sum_{d|(4n+1)}\left(\frac{-4}{d}\right)+\frac{9}{8}\sum_{d|(4n+1)/9}\left(\frac{-4}{d}\right)+\frac{3}{8}\sum_{d|(4n+1)}\left(\frac{-3}{n/d}\right)\left(\frac{12}{d}\right)+\frac{3}{4}d_{4n+1},$$

$$c_n=\frac{1}{4}\sum_{d|(4n+1)}\left(\frac{-4}{d}\right)-\frac{9}{4}\sum_{d|(4n+1)/9}\left(\frac{-4}{d}\right)-\frac{3}{4}\sum_{d|(4n+1)}\left(\frac{-3}{n/d}\right)\left(\frac{12}{d}\right)+\frac{3}{2}d_{4n+1},$$

where

$$\sum_{n=1}^{\infty} d_n q^n=\eta(12z)^2.$$

Note that for $m=\prod_{p\equiv 1 \pmod 4} p^{e_p}\prod_{q\equiv 3 \pmod 4} q^{f_q}$,

$$\sum_{d|m}\left(\frac{-4}{d}\right)=\begin{cases}\prod_{p\equiv 1 \pmod 4}(e_p+1) & \text{if } f_q \text{ are all even,}\\ 0 & \text{if at least one of } f_q \text{ is odd,}\end{cases}$$

and for $n=mq^{2f+1}$ with $q\equiv 3 \pmod 4$ and $\gcd(m,q)=1$,

$$\begin{aligned}\sum_{d|n}\left(\frac{-3}{n/d}\right)\left(\frac{12}{d}\right)&=\sum_{i=0}^{2f+1}\sum_{d|m}\left(\frac{-3}{mq^{2f+1}/dq^i}\right)\left(\frac{12}{dq^i}\right)\\&=\sum_{i=0}^{2f+1}\sum_{d|m}\left(\frac{-3}{m/d}\right)\left(\frac{12}{d}\right)\left(\frac{-3}{q^{2f+1-i}}\right)\left(\frac{12}{q^i}\right)\\&=\sum_{i=0}^{2f+1}\sum_{d|m}\left(\frac{-3}{m/d}\right)\left(\frac{12}{d}\right)\left(\frac{-3}{q^{2f+1}}\right)\left(\frac{-4}{q^i}\right)\\&=\left(\sum_{d|m}\left(\frac{-3}{m/d}\right)\left(\frac{12}{d}\right)\left(\frac{-3}{q^{2f+1}}\right)\right)\left(\sum_{i=0}^{2f+1}\left(\frac{-4}{q}\right)^i\right)\\&=0.\end{aligned}$$

Recall that $a_n=0$ if and only if $4n+1$ has a prime factor $q\equiv 3 \pmod 4$ with odd exponent, and by [18, Theorem 1.1, part (14)], $d_n=0$ if and only if $n\not\equiv 1 \pmod{12}$, or $n\equiv 1 \pmod{12}$ and n has a prime factor $p\not\equiv 1 \pmod{12}$ with odd exponent. So clearly, $a_n=0$ implies that $d_n=0$.

These altogether indicate that $a_n=0 \implies b_n=0$, $c_n=0$. Finally, one can check that $a_{81}=110$ and $b_{81}=0$, $a_6=11$ and $c_6=0$, and the claimed results follow. □

Remark 5.14. *There are other ways to prove the inclusion results above.*

(1) *The eta quotients $B(q)$ and $C(q)$ are those numbered 46 and 49 in Table* 9*, and hence the inclusion results for $B(q)$ and $C(q)$ proved in Theorem* 5.13 *also hold for their $q \to -q$ partners, the eta quotients numbered 45 and 50 in Table* 9*.*

(2) *Theorem* 5.13 *can also be proved by Lemma* 3.5 *and has been covered by Theorem* 7.1*.*

Theorem 5.15. *Let $B(q)$ and $C(q)$ be defined by*

$$\frac{f_2^2 f_3^5}{f_1 f_6^2} \quad \text{and} \quad \frac{f_1^2 f_3^4}{f_2 f_6},$$

respectively, the eta quotients numbered 52 and 61 in Table 6. Then $B_{(0)} = C_{(0)}$.

Proof. Setting $q = e^{2\pi i z}$, it is easy to verify that both $qB(q^4)$ and $qC(q^4)$ are holomorphic modular forms, so are their 12-dissections

$$\sum_{n=0}^{\infty} b_{12n+j} q^{12n+j} \quad \text{and} \quad \sum_{n=0}^{\infty} c_{12n+j} q^{12n+j}$$

for $j = 1, 5, 9$. Using Sturm's theorem, one can verify that

$$\sum_{n=0}^{\infty} b_{12n+1} q^{12n+1} = \sum_{n=0}^{\infty} c_{12n+1} q^{12n+1},$$

$$\sum_{n=0}^{\infty} b_{12n+5} q^{12n+5} = -\frac{1}{2} \sum_{n=0}^{\infty} c_{12n+5} q^{12n+5},$$

$$\sum_{n=0}^{\infty} b_{12n+9} q^{12n+9} = \sum_{n=0}^{\infty} c_{12n+9} q^{12n+9}.$$

The assertion follows. □

Remark 5.16. *As in Theorem* 5.7*, the proof of Theorem* 5.15 *is motivated by Lemma* 4.4*.*

We end this subsection with the next two theorems that will be necessary for a full justification for Theorem 7.1.

Theorem 5.17. *Let*

$$A(q) = f_1^6 = \sum_{n=0}^{\infty} a_n q^n,$$

and let $B(q)$ be any of the eta quotients in Entries 25, 33, 67, 129 *and* 145*. Then*

$$A_{(0)} \subseteq B_{(0)}.$$

Proof. Define

$$h_1(q; j, k) = \sum_{m,n=0}^{\infty} q^{(24m+j)^2+(24n+k)^2}, \quad h_2(q; j, k) = \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} q^{(24m+j)^2+4(24n+k)^2},$$

$$g_1(q;j,k) = \sum_{m,n=0}^{\infty} q^{(20m+j)^2+(20n+k)^2}, \quad g_2(q;j,k) = \sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty} q^{(20m+j)^2+4(20n+k)^2}.$$

Then since B is a holomorphic modular form in τ, with $q = e^{2\pi iz}$, $qB(q^4)$ is a linear combination of $h_i(q;j,k)$ for $i \in \{1,2\}$ and $0 \le j,k \le 23$ and $g_i(q;,j,k)$ for $i\{1,2\}$ and $0 \le j,k \le 19$. Any integer $4n+1$ having a prime factor congruent to 3 modulo 4 with odd exponent cannot be represented by either of $(24m+j)^2+(24n+k)^2$, $(24m+j)^2+4(24n+k)^2$, $(20m+j)^2+(20n+k)^2$ or $(20m+j)^2+4(20n+k)^2$, and this implies that $b_n = 0$. □

Theorem 5.18. *Let*

$$A(q) = f_1^6 = \sum_{n=0}^{\infty} a_n q^n,$$

and let $B(q)$ be any of the eta quotients in Entries 41*,* 43*,* 107 *and* 139*. Then*

$$A_{(0)} \subseteq B_{(0)}.$$

Proof. Let $C(q)$ be any of the eta quotients in Entries 19, 45, 109 and 143. By Lemma 3.7, one can tell that $C(q)$ can be expressed as either

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \alpha(m,n) q^{\frac{1}{2}(m^2+n^2)}, \qquad \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \alpha(m,n) q^{4m^2+n^2},$$

or

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \alpha(m,n) q^{16m^2+n^2}$$

and deduce that $A_{(0)} \subseteq C_{(0)}$. On the other hand, by [19, Theorem 3.3, Eq. 3.2.5], Entries 19 and 41 of Table 9 have identically vanishing coefficients, as do entries 43 and 45 by [19, Theorem 3.3, Eq. 3.2.24], entries 107 and 109 by [19, Theorem 3.3, Eq. 3.2.2], and entries 139 and 143 by [19, Theorem 3.3, Eq. 3.2.17]. Therefore, correspondingly, $B_{(0)} = C_{(0)}$, and this together with the inference above yields the desire inclusions.

□

5.3. Eta quotients with vanishing coefficient behaviour similar to f_1^8.

Before considering some examples of eta quotients with vanishing coefficient behaviour similar to f_1^8, we consider a connection with eta quotients with vanishing coefficient behaviour similar to f_1^4. We first prove the following key result. Once again this result is a special case of Theorem 7.1, but we prefer to illustrate another method of proof.

Theorem 5.19. *Let*

$$A(q) = f_1^8 =: \sum_{n=0}^{\infty} a_n q^n, \quad B(q) = f_1^4 =: \sum_{n=0}^{\infty} b_n q^n, \quad B'(q) := B(q^2) =: \sum_{n=0}^{\infty} b'_n q^n, \tag{5.26}$$

Then

$$A_{(0)} \subsetneqq B'_{(0)}. \tag{5.27}$$

Throughout the remainder of this section, $A(q)$ and the sequence $\{a_n\}$ will be as defined at (5.26).

Proof. Suppose a_n=0. From Serre's criterion (3.4), $3n+1$ has a prime factor $p \equiv -1 \pmod 3$ with odd exponent. If n is odd, then $b'_n = 0$. Now suppose n is even, $n = 2m$, so that $b'_n = b'_{2m} = b_m$. Since $a_n = a_{2m} = 0$, $3n+1 = 6m+1$ has a prime factor $p \equiv -1 \pmod 3$ with odd exponent. By Serre's criterion (3.2), $b_m = b'_{2m} = b'_n = 0$. Since $a_1 = -8$ and $b'_1 = 0$, then $A_{(0)} \subsetneqq B'_{(0)}$. □

If $C(q)$ and $D(q)$ are any eta quotients in Table 7 or, alternatively, in any of the groups in Figure 2, such that any of the following hold:

$$B_{(0)} = C_{(0)}, \quad B_{(0)} \subsetneqq C_{(0)}, \quad B_{(0)} \subsetneqq C_{(0)} = D_{(0)}, \quad B_{(0)} \subsetneqq C_{(0)} \subsetneqq D_{(0)}, \tag{5.28}$$

then Lemma 5.1 gives that similar relations hold for the eta quotients derived from these by a $q \to q^2$ dilation, namely,

$$B'(q) := B(q^2), \qquad C'(q) := C(q^2), \qquad D'(q) := D(q^2), \tag{5.29}$$

or more precisely, that

$$\begin{aligned} &A_{(0)} \subsetneqq B'_{(0)} = C'_{(0)}, && A_{(0)} \subsetneqq B'_{(0)} \subsetneqq C'_{(0)}, \\ &A_{(0)} \subsetneqq B'_{(0)} \subsetneqq C'_{(0)} = D'_{(0)}, && A_{(0)} \subsetneqq B'_{(0)} \subsetneqq C'_{(0)} \subsetneqq D'_{(0)}. \end{aligned} \tag{5.30}$$

The last statement is the reason we term the statement in Theorem 5.19 a "key" result.

Some comments are in order.

(1) Most of the equality/inclusion results for sets of vanishing coefficients suggested by experimentation and displayed in Table 7 and in Figure 2 are as yet unproven. However, experimentation strongly suggests they are true, which in particular would mean that the entire graph in Figure 2 corresponds to a subgraph in Figure 4.
(2) The reason that many of the eta quotients corresponding to $q \to q^2$ dilation of eta quotients in Table 7 do not appear in Table 11 is that the $q \to q^2$ dilation puts them outside the search range we used. For a similar reason, some entire groups of eta quotients corresponding to the $q \to q^2$ dilation of groups of eta quotients in Figure 2 are missing in Figure 4.
(3) We do not ignore completely eta quotients $C'(q) = C(q^2)$ that are derived from an eta quotient $C(q)$ in Table 7. While we may not have been able to prove $B_{(0)} = C_{(0)}$ or $B_{(0)} \subsetneqq C_{(0)}$, it may be possible to prove $A_{(0)} \subsetneqq C'_{(0)}$, with $A(q)$ and $B(q)$ as in Theorem 5.19.

The groups of eta quotients in Table 11 that are $q \to q^2$ dilations of groups of eta quotients in Table 7 are listed in the following table.

Table 6: Groups of eta quotients in Table 11 that arise as $q \to q^2$ dilations of groups of eta quotients in Table 7

f_1^4 Group	f_1^8 Group	f_1^4 Group	f_1^8 Group
I	IV	III	V
IV	VI	V	VII
VII	XI	VIII	XII
IX	XIII	XI	XIX
XVI	XXI	XVII	XXII
XVIII	XXIII	XIX	XXV

Corollary 5.20. *Let $B(q)$ and $C(q)$ be any two eta quotients in Group XXII, and $D(q)$ and $E(q)$ be any two eta quotients in Group XXV of Table* 11*. Then*

$$A_{(0)} \subsetneqq B_{(0)} \subsetneqq D_{(0)}, \qquad B_{(0)} = C_{(0)}, \qquad D_{(0)} = E_{(0)}. \tag{5.31}$$

Proof. This follows from the remarks above, the facts that groups XXII and XXV in Table 11 are, respectively, the $q \to q^2$ dilations of groups XVII and XIX in Table 7, and Theorem 5.3. □

The next results are for the remaining eta products in Table 11. Let

$$B(q) = f_1^2 f_3^2 =: \sum_{n=0}^{\infty} b_n q^n, \quad C(q) = f_2 f_6 =: \sum_{n=0}^{\infty} c_n q^n, \quad D(q) = f_1 f_7 =: \sum_{n=0}^{\infty} d_n q^n. \tag{5.32}$$

In [18, Theorem 1.1], exact criteria were also given for the vanishing of the b_n, c_n and d_n, and from these one immediately gets

$$A_{(0)} = B_{(0)}, \qquad A_{(0)} \subsetneqq C_{(0)}, \qquad A_{(0)} \subsetneqq D_{(0)}. \tag{5.33}$$

Once again, Theorem 7.1 gives the last two inclusions, but just the weaker result $A_{(0)} \subseteq B_{(0)}$.

The next results are a consequence of the theta series expansions at (3.10) and (3.22).

Theorem 5.21. *Let $A(q)$ and the sequence $\{a_n\}$ be as defined at* (5.26) *above, let $C(q)$ and $D(q)$ be any two of the eta quotients in Group XXIV and let $E(q)$ be any of the eta quotients in Group XXV of Table* 11*. Then*

$$A_{(0)} \subsetneqq C_{(0)} \subsetneqq E_{(0)}, \qquad C_{(0)} = D_{(0)}. \tag{5.34}$$

Proof. This will be shown for $E(q) = f_8$ only, as all the eta quotients in Group XXV have coefficients that vanish identically, by Lemma 5.1 and the fact that group XXV is the $q \to q^4$ dilation of group II in Table 4. We will also prove the results for $C(q)$ and $D(q)$ being the eta quotients defined in (5.35), since the other two eta quotients in Group XXIV of Table 11 are the $q \to -q$ partners of these two. Define the sequences $\{c_n\}$, $\{d_n\}$ and $\{e_n\}$ by

$$C(q) := \frac{f_2^2 f_3 f_{12}}{f_1 f_4 f_6} =: \sum_{n=0}^{\infty} c_n q^n, \quad D(q) := \frac{f_2^5}{f_1^2} =: \sum_{n=0}^{\infty} d_n q^n, \quad E(q) := f_8 =: \sum_{n=0}^{\infty} e_n q^n. \tag{5.35}$$

As with other proofs in this section, next apply a $q \to q^3$ dilation and multiply by q to get

$$\frac{\eta(6z)^2\eta(9z)\eta(36z)}{\eta(3z)\eta(12z)\eta(18z)} = \sum_{n=0}^{\infty} c_n q^{3n+1} = \sum_{n=1}^{\infty} \left(\frac{n}{3}\right)^2 q^{n^2}, \tag{5.36}$$

$$\frac{\eta(6z)^5}{\eta(3z)^2} = \sum_{n=0}^{\infty} d_n q^{3n+1} = \sum_{n=1}^{\infty} \left(2\left(\frac{n}{12}\right) - \left(\frac{n}{3}\right)\right) n q^{n^2},$$

$$\eta(24z) = \sum_{n=0}^{\infty} e_n q^{3n+1} = \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) q^{n^2}.$$

For each of the resulting modular forms, the second series expansion comes from (3.10), (3.22) and (3.11), respectively.

Serre's criterion (3.4) gives that a_n=0 if and only if $3n+1$ has a prime factor $p \equiv -1 \pmod 3$ with odd exponent. If $a_n = 0$, then $3n+1$ is thus not a square, and (5.36) implies $c_n = d_n = e_n = 0$. The remainder of the statements in the theorem also follow from the theta expansions in (5.36). □

The next results require the 2-dissections of some of the eta quotients listed in Lemma 3.6. Although the method and what it provides regarding vanishing coefficients is entirely elementary, we include it for the sake of completeness.

Lemma 5.22. *The following 2-dissections hold.*

$$\frac{f_2^5}{f_1^2 f_4^2} = \frac{f_8^5}{f_4^2 f_{16}^2} + 2q\frac{f_{16}^2}{f_8}, \tag{5.37}$$

$$\frac{f_1^2}{f_2} = \frac{f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_{16}^2}{f_8}.$$

Proof. Both of these follow from separating the corresponding series expansion at (3.9) and (3.15), respectively, into sums over odd and even n, and then using the Jacobi triple product identity (4.10) to convert the resulting sums back into infinite products. □

Note that the second identity follows from the first by replacing q with $-q$. From the point of view of vanishing coefficients, suppose we take the first equation, for example, and multiply across by any even eta quotient $G(q^2)$ to get

$$B(q) = C(q) + qD(q), \tag{5.38}$$

where

$$B(q) := G(q^2)\frac{f_2^5}{f_1^2 f_4^2}, \quad C(q) := G(q^2)\frac{f_8^5}{f_4^2 f_{16}^2}, \quad D(q) := 2G(q^2)\frac{f_{16}^2}{f_8}, \tag{5.39}$$

(note that $C(q)$ and $D(q)$ are actually functions of q^2) then if $D(q)$ is not identically zero, one has that

$$B_{(0)} \subsetneqq C_{(0)}. \tag{5.40}$$

This observation allows us to prove several inclusion results in Table 11/Figure 4.

Theorem 5.23. *Let $B(q)$ be either the first or second eta quotient in any of the following quadruples, and let $C(q)$ be either the third or the fourth:*

$$\left(\frac{f_2^5 f_4^2}{f_1^2 f_8}, \frac{f_1^2 f_4^4}{f_2 f_8}, \frac{f_4^2 f_8^4}{f_{16}^2}, \frac{f_8^{10}}{f_4^2 f_{16}^4}\right), \left(\frac{f_2^5 f_8^5}{f_1^2 f_4^2 f_{16}^2}, \frac{f_1^2 f_8^5}{f_2 f_{16}^2}, \frac{f_8^{10}}{f_4^2 f_{16}^4}, \frac{f_4^2 f_8^4}{f_{16}^2}\right),$$
$$\left(\frac{f_2^5 f_{16}^3}{f_1^2 f_4^2 f_8 f_{32}}, \frac{f_1^2 f_{16}^3}{f_2 f_8 f_{32}}, \frac{f_8^4 f_{16}}{f_4^2 f_{32}}, \frac{f_4^2 f_{16}^3}{f_8^2 f_{32}}\right), \left(\frac{f_2^5 f_8}{f_1^2 f_4^2}, \frac{f_1^2 f_8}{f_2}, \frac{f_8^6}{f_4^2 f_{16}^2}, f_4^2\right),$$
$$\left(\frac{f_2^5}{f_1^2}, \frac{f_1^2 f_4^2}{f_2}, \frac{f_8^5}{f_{16}^2}, \frac{f_{16}^{13}}{f_8^5 f_{32}^5}\right). \tag{5.41}$$

Then

$$B_{(0)} \subsetneqq C_{(0)}. \tag{5.42}$$

Proof. In each case we prove the result only for $B(q)$ equal to the first eta quotient in each quadruple and $C(q)$ the third eta quotient. The second quotient is the $q \to -q$ partner of the first, and the fourth is the $q^4 \to -q^4$ partner of the third for the first four quadruples. The fourth quotient is also the $q^8 \to -q^8$ partner in the case of the fifth quadruple. In the case of each pair, we just list the function $G(q^2)$ so that the pair $(B(q), C(q))$ at (5.38) and (5.39) are the same as the pair $(B(q), C(q))$ from (5.41). These values for $G(q)$ are, respectively,

$$\frac{f_4^4}{f_8}, \qquad \frac{f_8^5}{f_{16}^2}, \qquad \frac{f_{16}^3}{f_8 f_{13}}, \qquad f_8, \qquad f_4^2.$$

That the resulting $D(q)$ in each case is not identically zero follows from the fact that the resulting $B(q)$ is clearly not an even function. □

We note also that if $B(q)$ and $D(q)$ are as at (5.38) and (5.39) and $E(q) := qD(q)$, then $B_{(0)} \subsetneqq E_{(0)}$. However, none of these functions $E(q)$ fall within our search parameters. As an example, the $E(q)$ corresponding to the first example at (5.41) is $E(q) = 2qf_4^4 f_{16}^2/f_8^2$, so that with $B(q) = f_2^5 f_4^2/(f_1^2 f_8)$, we conclude $B_{(0)} \subsetneqq E_{(0)}$.

Remark 5.24. *The quadruples of eta quotients listed at* (5.41) *are those numbered, respectively,* (104, 103, 132, 131)*,* (121, 122, 131, 132)*,* (129, 130, 141, 142)*,* (138, 137, 143, 144) *and* (150, 149, 152, 151) *in Table* 11.

There are a number of other 2-dissection identities that lead to other inclusion results similar to those in Theorem 5.23. We collect those together in the next lemma.

Lemma 5.25. *The following 2-dissections hold:*

$$\frac{f_1^3}{f_3} = \frac{f_4^3}{f_{12}} - 3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2}, \tag{5.43}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}, \tag{5.44}$$

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}, \tag{5.45}$$

$$\frac{f_1}{f_3^3} = \frac{f_2 f_4^2 f_{12}^2}{f_6^7} - q\frac{f_2^3 f_{12}^6}{f_4^2 f_6^9}, \tag{5.46}$$

$$f_1 f_3 = \frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} - q\frac{f_4^4 f_6 f_{24}^2}{f_2 f_8^2 f_{12}^2}, \tag{5.47}$$

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}, \tag{5.48}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}, \tag{5.49}$$

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}. \tag{5.50}$$

Proof. Identity (5.43) is Equation (1.35) in [15], and (5.44) is derived from (5.43) upon the replacement $q \to -q$. Identities (5.45) and (5.46) are stated in Lemma 2.1 of [16]; identity (5.48) is stated in Lemma 2.4 of [16]; and (5.47) is derived from the latter identity after replacing q with $-q$. Finally, (5.49) is stated in Lemma 2.3 of [16], and (5.50) follows from (5.49), once again, upon replacing q with $-q$. □

We now use some of these dissection results to derive inclusion results for vanishing coefficients similar to those stated in Theorem 5.23.

Theorem 5.26. *Let $B(q)$ be either the first or second eta quotient in any of the following quadruples, and let $C(q)$ be either the third or the fourth:*

$$\left(\frac{f_2^7 f_3}{f_1^3 f_6}, \frac{f_1^3 f_4^3 f_6^2}{f_2^2 f_3 f_{12}}, \frac{f_4^6 f_6^2}{f_2^2 f_{12}^2}, \frac{f_4^9}{f_2^2 f_8^3}\right), \left(\frac{f_2^7 f_3 f_{12}}{f_1^3 f_4^3 f_6}, \frac{f_1^3 f_6^2}{f_2^2 f_3}, \frac{f_4^3 f_6^2}{f_2^2 f_{12}}, \frac{f_2^2 f_8^2 f_{12}^4}{f_6^2 f_{24}^2}\right),$$

$$\left(\frac{f_3^3}{f_1}, \frac{f_1 f_4 f_6^9}{f_2^3 f_3^3 f_{12}^3}, \frac{f_4^3 f_6^2}{f_2^2 f_{12}}, \frac{f_2^2 f_8^2 f_{12}^4}{f_6^2 f_{24}^2}\right), \left(\frac{f_3^3 f_4^3}{f_1 f_{12}}, \frac{f_1 f_4^4 f_6^9}{f_2^3 f_3^3 f_{12}^4}, \frac{f_4^6 f_6^2}{f_2^2 f_{12}^2}, \frac{f_4^9}{f_2^2 f_8^3}\right),$$

$$\left(\frac{f_2^4f_6^2}{f_1f_3f_4^2},\frac{f_1f_2f_3f_{12}}{f_4f_6},\frac{f_2^2f_8^2f_{12}^5}{f_4^3f_6^2f_{24}^2},\frac{f_4f_6^7}{f_2f_{12}^3}\right),\left(\frac{f_2^4f_4f_6^2}{f_1f_3f_{12}},\frac{f_1f_2f_3f_4^2}{f_6},\frac{f_2^2f_8^2f_{12}^4}{f_6^2f_{24}^2},\frac{f_4^3f_6^2}{f_2^2f_{12}}\right),$$
$$\left(\frac{f_2^{10}}{f_1^4f_4^2},\frac{f_1^4f_4^2}{f_2^2},\frac{f_4^{12}}{f_2^4f_8^4},\frac{f_4^{10}f_6^2}{f_2^4f_8^3f_{12}}\right). \quad (5.51)$$

Then

$$B_{(0)} \subsetneq C_{(0)}. \quad (5.52)$$

Proof. As with the previous theorem, we prove the result just for $B(q)$ taken to be the first eta quotient in each quadruple, and $C(q)$ the third eta quotient, as the second is the $q \to -q$ partner of the first, and the fourth is the $q^2 \to -q^2$ partner of the third.

In the case of each pair, we just list the function $G(q^2)$ and a 2-dissection from Lemma 5.25, so that after multiplying the indicated 2-dissection by $G(q^2)$, the resulting eta quotient on the left is equal to $B(q)$, and the first eta quotient resulting on the right is the same as $C(q)$. The pairs ($G(q^2)$, 2-dissection) are, respectively,

$$\left(\frac{f_2^7}{f_6},(5.44)\right),\left(\frac{f_2^7f_{12}}{f_4^3f_6},(5.44)\right),(1,(5.45)),\left(\frac{f_4^3}{f_{12}},(5.45)\right),$$
$$\left(\frac{f_2^4f_6^2}{f_4^2},(5.48)\right),\left(\frac{f_2^4f_4f_6^2}{f_{12}},(5.48)\right),\left(\frac{f_2^{10}}{f_4^2},(5.49)\right). \quad (5.53)$$

□

Remark 5.27. *The quadruples of eta quotients in Theorem* 5.26 *are, respectively,* (7, 8, 75, 76)*,* (9, 10, 73, 74)*,* (15, 16, 73, 74)*,* (17, 18, 75, 76)*,* (19, 20, 72, 71)*,* (21, 22, 74, 73) *and* (26, 25, 84, 83) *in Table* 11.

The next theorem takes care of some exceptional cases that Lemma 3.7 does not cover, and will be necessary for the proof of Theorem 7.1.

Theorem 5.28. *Let*

$$A(q) = f_1^8 = \sum_{n=0}^{\infty} a_n q^n,$$

and let $B(q)$ be any of the eta quotients in Entries 15 *and* 127*. Then*

$$A_{(0)} \subseteq B_{(0)}.$$

Proof. Define

$$g_1(q;j,k) = \sum_{m,n=0}^{\infty} q^{(24m+j)^2+3(24n+k)^2}, \quad g_2(q;j,k) = \sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty} q^{(24m+j)^2+12(24n+k)^2},$$

$$g_3(q;j,k) = \sum_{m,n=0}^{\infty} q^{(84m+j)^2+3(84n+k)^2}, \quad g_4(q;j,k) = \sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty} q^{(84m+j)^2+12(84n+k)^2}.$$

Then $qB(q^3)$ is a linear combination of $g_i(q; j, k)$ for $i = 1, 2$ and $0 \le j, k \le 23$, and $g_i(q; , j, k)$ for $i = 3, 4$ and $0 \le j, k \le 83$. Recall by Item (4) of Theorem 3.4 that $a_n = 0$ if and only if $3n + 1$ has a prime factor congruent to 2 modulo 3. Clearly, such an integer $3n + 1$ cannot written as $(84m + j)^2 + 3(84n + k)^2$ or $(84m + j)^2 + 12(84n + k)^2$, and therefore, $b_n = 0$.

□

5.4. Eta quotients with vanishing coefficient behaviour similar to f_1^{10}.

Similar to the practice adopted in previous sections, throughout the present section $A(q)$ will be defined by

$$A(q) := f_1^{10} =: \sum_{n=0}^{\infty} a_n q^n. \tag{5.54}$$

Since $r = 2$ is the only non-trivial proper divisor of 10 for which f_1^r is lacunary, the only eta quotients $B(q)$ in Table 13 that are derived through dilation of eta products in other tables in the paper arise through the dilation $q \to q^5$ applied to the eta quotients in Table 4. We have the following theorem.

Theorem 5.29. *Let $B(q)$ and $C(q)$ be any two eta quotients in Group XXIV, and $D(q)$ and $E(q)$ be any two eta quotients in Group XXV of Table* 13. *Then*

$$A_{(0)} \subsetneqq B_{(0)} \subsetneqq D_{(0)}, \qquad B_{(0)} = C_{(0)}, \qquad D_{(0)} = E_{(0)}. \tag{5.55}$$

Proof. Will just prove that $A_{(0)} \subsetneqq B_{(0)}$ for $B(q) = f_5^2 = B'(q^5) = \sum_{n=0}^{\infty} b_n q^n$ where $B'(q) = f_1^2 = \sum_{n=0}^{\infty} b'_n q^n$, since the other statements will then follow from Lemma 5.1 and the facts that groups XXIV and XXV in Table 13 are, respectively, the $q \to q^5$ dilations of groups I and II in Table 4.

Apply the dilation $q \to q^{12}$ to both $A(q)$ and $B(q)$ and then multiply by q^5 to

$$\eta(12z)^{10} = \sum_{n=0}^{\infty} a_n q^{12n+5}, \qquad \eta(60z)^2 = \sum_{n=0}^{\infty} b_n q^{12n+5}, \tag{5.56}$$

and suppose $a_n = 0$, so that by Serre's criterion (3.5), $12n+5$ has a prime factor $p \equiv -1 \pmod 4$ with odd exponent. If $5 \nmid n$, then clearly $b_n = 0$, so suppose $5|n$, or $n = 5m$ for some integer m. So $12n + 5 = 5(12m + 1)$ has a prime factor $p \equiv -1 \pmod 4$, and thus that $12m+1$ has a prime factor $p \equiv -1 \pmod 4$. By Serre's criterion (3.1), then $b'_m = 0$, so that $b_n = b_{5m} = b'_m = 0$. Since $a_1 = -10$ and $b_1 = 0$, then $A_{(0)} \subsetneqq B_{(0)}$ follows. □

Theorem 7.1 could have been used to prove the $A_{(0)} \subsetneqq B_{(0)}$ step, but again we prefer to give an alternative proof.

As was the case with the eta quotients in Table 11, there are pairs of eta quotients in Table 13 such that one of the pair is the even part of the other, so that some of the 2-dissections at (5.37) and (5.43) - (5.50) may be used to derive results similar to those in Theorems 5.23 and 5.26.

Theorem 5.30. *Let $B(q)$ be either the first or second eta quotient in any of the quadruples of eta quotients, with let $C(q)$ be either the third or fourth eta quotient in the same quadruple:*

$$\left(\frac{f_2^9}{f_1^4 f_4}, \frac{f_1^4 f_4^3}{f_2^3}, \frac{f_4^{13}}{f_2^5 f_8^4}, \frac{f_2^5 f_8}{f_4^2}\right), \left(\frac{f_2^{10} f_3 f_{12}}{f_1^3 f_4^4 f_6}, \frac{f_1^3 f_2 f_6^2}{f_3 f_4}, \frac{f_2 f_4^2 f_6^2}{f_{12}}, \frac{f_4^5 f_{12}^5}{f_2 f_6^2 f_8 f_{24}^2}\right),$$
$$\left(\frac{f_2^3 f_3^3}{f_1 f_4}, \frac{f_1 f_6^9}{f_3^3 f_{12}^3}, \frac{f_2 f_4^2 f_6^2}{f_{12}}, \frac{f_4^5 f_{12}^5}{f_2 f_6^2 f_8 f_{24}^2}\right), \left(\frac{f_2 f_6^8}{f_1 f_3 f_{12}^3}, \frac{f_1 f_3 f_4 f_6^5}{f_2^2 f_{12}^2}, \frac{f_6^4 f_8^2 f_{12}^2}{f_2 f_4 f_{24}^2}, \frac{f_2 f_8^3 f_{12}^{14}}{f_4^4 f_6^4 f_{24}^6}\right),$$
$$\left(\frac{f_2^8 f_3 f_{12}^2}{f_1^3 f_4^3 f_6^3}, \frac{f_1^3 f_{12}}{f_2 f_3}, \frac{f_4^3}{f_2}, f_2 f_8\right), \left(\frac{f_4^9 f_6^2}{f_1 f_2 f_3 f_8^4}, \frac{f_1 f_3 f_4^{10} f_{12}}{f_2^4 f_6 f_8^4}, \frac{f_4^8 f_{12}^5}{f_2^3 f_6^2 f_8^2 f_{24}^2}, \frac{f_2^3 f_6^2 f_8}{f_4 f_{12}}\right),$$
$$\left(\frac{f_2 f_3^3 f_{12}}{f_1 f_6^2}, \frac{f_1 f_4 f_6^7}{f_2^2 f_3^3 f_{12}^2}, \frac{f_4^3}{f_2}, f_2 f_8\right), \left(\frac{f_2^3 f_6^2}{f_1 f_3 f_4}, \frac{f_1 f_3 f_{12}}{f_6}, \frac{f_2 f_8^2 f_{12}^5}{f_4^2 f_6^2 f_{24}^2}, \frac{f_4 f_6^2 f_8}{f_2 f_{12}}\right),$$
$$\left(\frac{f_1^4 f_8^2 f_{12}^5}{f_2 f_4^2 f_6^2 f_{24}^2}, \frac{f_2^{11} f_8^2 f_{12}^5}{f_1^4 f_4^6 f_6^2 f_{24}^2}, \frac{f_4^8 f_{12}^5}{f_2^3 f_6^2 f_8^2 f_{24}^2}, \frac{f_2^3 f_6^2 f_8}{f_4 f_{12}}\right). \tag{5.57}$$

Then

$$B_{(0)} \subsetneqq C_{(0)}. \tag{5.58}$$

Proof. As with proofs in the previous section, we prove the result just for $B(q)$ being the first eta quotient in each quadruple, and $C(q)$ being the third eta quotient, as the second is the $q \to -q$ partner of the first, and the fourth is the $q^2 \to -q^2$ partner of the third.

As in the proof of Theorem 5.26, we confine the proof, to simply listing, in the case of each pair $B(q)$ and $C(q)$, the function $G(q^2)$ and a 2-dissection from Lemma 5.25, so that after multiplying the indicated 2-dissection by $G(q^2)$, the resulting eta quotient on the left is equal to $B(q)$, and the first eta quotient resulting on the right is the same as $C(q)$. The pairs ($G(q^2)$, 2-dissection) are, respectively,

$$\left(\frac{f_2^9}{f_4}, (5.49)\right), \left(\frac{f_2^{10} f_{12}}{f_4^4 f_6}, (5.44)\right), \left(\frac{f_2^3}{f_4}, (5.45)\right),$$
$$\left(\frac{f_2 f_6^8}{f_{12}^3}, (5.48)\right), \left(\frac{f_2^8 f_{12}^2}{f_4^3 f_6^3}, (5.44)\right), \left(\frac{f_4^9 f_6^2}{f_2 f_8^4}, (5.48)\right),$$
$$\left(\frac{f_2 f_{12}}{f_6^2}, (5.45)\right), \left(\frac{f_2^3 f_6^2}{f_4}, (5.48)\right), \left(\frac{f_8^2 f_{12}^5}{f_2 f_4^2 f_6^2 f_{24}^2}, (5.50)\right). \tag{5.59}$$

□

Remark 5.31. *The quadruples of eta quotients listed at* (5.57) *are those numbered, respectively,*

(3, 4, 89, 86), (5, 6, 88, 87), (21, 22, 88, 87), (44, 43, 84, 83), (72, 71, 100, 99),

$$(52, 51, 91, 92), (70, 69, 100, 99), (62, 61, 97, 98), (50, 49, 91, 92). \quad (5.60)$$

in Table 13.

We next consider an different method than those employed previously, where we showed either that $B_{(0)} = C_{(0)}$ or that $B_{(0)} \subsetneqq C_{(0)}$ for two eta quotients $B(q)$ and $C(q)$. The idea is to look for eta quotients in Table 13 which may be expressed as a product of two eta quotients with single-sum theta series expansions, as in Lemma 3.6. This means that any such eta quotient in Table 13 as a double-sum theta series. We may then employ properties of the theta series. The list of such eta quotients is displayed in Table 14. The numbers and Roman numerals are those assigned to the eta quotients in Table 13.

We briefly explain the purpose of developing the double theta series expansions in Table 14. These explanations also apply to the other similar tables related to f_1^r, $r = 4, 6, 8, 14$ and 26, and $f_1^3 f_2^3$. By Serre's criterion, $a_n = 0$ if and only $12n + 5$ has a prime factor $p \equiv -1 \pmod 4$ with odd exponent. Now consider another eta quotient $B(q) = \sum_{n=0}^{\infty} b_n q^n$ in Table 13. Suppose that the corresponding modular form $q^5 B(q^{12})$ appears in Table 14 and the corresponding double theta series has one of the quadratic forms $m^2 + 4n^2$, $1/2(m^2 + 9n^2)$, $5/2(m^2 + n^2)$. Then by Lemma 3.1, if $a_n = 0$ one also has that $b_n = 0$, implying that $A_{(0)} \subseteq B_{(0)}$. See Lemma 3.7 for more details.

Remark 5.32. *In the case of all the eta quotients from Group I in Table* 14*, ideally, to get* $A_{(0)} = B_{(0)}$*, one would like to show that if* $12t + 5$ *does not have a prime factor* $p \equiv 3 \pmod 4$ *with odd exponent, then* $b_t \neq 0$*. However attempting to use the theta series representations to show this is likely not so straightforward, so we do not consider that here.*

We end this subsection with the following three theorems accounting for the exceptional cases not covered by the item (4) of Lemma 3.7.

Theorem 5.33. *Let* $A(q) = f_1^{10} =: \sum_{n=0}^{\infty} a_n q^n$ *and let* $B(q) =: \sum_{n=0}^{\infty} b_n q^n$ *be the eta quotients in Entry* 71 *of Table* 13*. Then*

$$A_{(0)} \subsetneqq B_{(0)}.$$

Proof. By Item (4) of Lemma 3.7, one can see that $A_{(0)} \subsetneqq C_{(0)}$ where $C(q)$ is the product in Entry 70 of Table 13. On the other hand, by [19, Theorem 3.7, Eq. 3.4.8] one finds that $C_{(0)} = B_{(0)}$. These yield the desired inclusion. □

Theorem 5.34. *Let* $A(q) = f_1^{10} =: \sum_{n=0}^{\infty} a_n q^n$ *and let* $B(q) =: \sum_{n=0}^{\infty} b_n q^n$ *be the eta quotients in Entry* 81 *of Table* 13*. Then*

$$A_{(0)} \subsetneqq B_{(0)}.$$

Proof. Define

$$g(q; j, k) = \sum_{m,n=0}^{\infty} q^{\frac{1}{2}\left(9(24m+j)^2 + (72n+k)^2\right)}.$$

Then viewed as holomorphic modular form for $q = e^{2\pi i z}$,

$$q^5 B(-q^{12}) = \sum_{n=0}^{\infty} b_n' q^{12n+5}$$

is a linear combination of $g(-q; j, k)$ for $0 \leq j \leq 23$ and $0 \leq k \leq 71$. It is clear that any $12n + 5$ having a prime factor congruent to 3 modulo 4 with odd exponent cannot be written as $9(24m + j)^2 + (72n + k)^2$, and this implies that $b_n' = 0$. Clearly, the coefficients of $q^5 B(-q^{12})$ vanish identically with that of $q^5 B(q^{12})$. Therefore, $b_n = 0$, and the desired inclusion follows. □

Theorem 5.35. *Let $A(q) = f_1^{10} =: \sum_{n=0}^{\infty} a_n q^n$ and let $B(q) =: \sum_{n=0}^{\infty} b_n q^n$ be any of the eta quotients in Entries* 93*,* 95 *and* 101 *of Table* 13*. Then*

$$A_{(0)} \subsetneq B_{(0)}.$$

Proof. As the treatments are similar, we illustrate these with the case of Entry 95. By Table 14, one finds that

$$q^5 B(q^{12}) = \frac{1}{2} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^m \left(\frac{12}{n}\right) q^{12m^2+5n^2} = \sum_{N=0}^{\infty} b_N q^{12N+5}.$$

One can assume n to be odd, and by the quadratic reciprocity law, one has that for

$$\left(\frac{12}{n}\right) = (-1)^{\frac{n-1}{2}} \left(\frac{n}{3}\right).$$

Let $12N + 5$ be a positive integer having a prime factor $p \equiv 3 \pmod 4$ with odd exponent e_p, so that $(-1)^m (-1)^{\frac{n-1}{2}} \left(\frac{n}{3}\right)$ is counted into b_N. Suppose that $12N + 5$ is representable by $12X^2 + 5Y^2$, say, $12N + 5 = N\left(\mathfrak{p}\left(2m + \frac{n}{3}\sqrt{-15}\right)\right)$, where $\mathfrak{p}$ is the prime ideal over 3. So by the ideal theory of $\mathbb{Z}[\sqrt{-15}]$ and the assumptions, one can decompose $\mathfrak{p}(2m + \frac{n}{3}\sqrt{-15})$ as

$$\begin{aligned}\mathfrak{p}\left(2m + \frac{n}{3}\sqrt{-15}\right) &= \left(m_1 + n_1\sqrt{-15}\right)\left(m_2 + \frac{n_2}{3}\sqrt{-15}\right)\mathfrak{p} \\ &= \left((m_1 m_2 - 5n_1 n_2) + \frac{m_1 n_2 + 3m_2 n_1}{3}\sqrt{-15}\right)\mathfrak{p},\end{aligned}$$

where exactly one of the ideal factors has norm of p^{e_p}. By assumption, one must have that $N\left(\left(m_2 + \frac{n_2}{3}\sqrt{-15}\right)\mathfrak{p}\right) \equiv 3 \pmod 4$, so $m_2 \equiv 1 \pmod 2$ and $n_2 \equiv 0 \pmod 2$. Also, since $m_1 m_2 - 5n_1 n_2 \equiv 0 \pmod 2$ and $m_1 n_2 + 3m_2 n_1 \equiv 1 \pmod 2$, one deduces that $m_2 n_1 \equiv 1 \pmod 2$. One can see each solution

$$\left(m_1 + n_1\sqrt{-15}\right)\left(m_2 + \frac{n_2}{3}\sqrt{-15}\right)\mathfrak{p}$$

gives rise to exactly one other solution

$$\left(m_1 + n_1\sqrt{-15}\right)\left(m_2 - \frac{n_2}{3}\sqrt{-15}\right)\mathfrak{p} = \left((m_1m_2 + 5n_1n_2) - \frac{m_1n_2 + 3m_2n_1 - 6m_2n_1}{3}\sqrt{-15}\right)\mathfrak{p},$$

so that

$$\begin{aligned}&(-1)^{\frac{m_1m_2+5n_1n_2}{2}}(-1)^{\frac{m_1n_2+3m_2n_1-6m_2n_1-1}{2}}\left(\frac{m_1n_2 + 3m_2n_1 - 6m_2n_1}{3}\right)\\ &= -(-1)^{\frac{m_1m_2-5n_1n_2}{2}}(-1)^{\frac{m_1n_2+3m_2n_1-1}{2}}\left(\frac{m_1n_2 + 3m_2n_1}{3}\right)\\ &= -(-1)^m(-1)^{\frac{n-1}{2}}\left(\frac{n}{3}\right)\end{aligned}$$

is also counted by b_N, where the minus sign follows from the facts that $m_2n_1 \equiv 1 \pmod 2$ and $n_1n_2 \equiv 0 \pmod 2$. Hence, $b_N = 0$. □

5.5. Eta quotients with vanishing coefficient behaviour similar to f_1^{14}. Throughout section 5.5 we follow the practice of previous sections, and define $A(q)$ by

$$A(q) := f_1^{14} =: \sum_{n=0}^{\infty} a_n q^n. \tag{5.61}$$

As in the case of f_1^{10}, the only eta quotients $B(q)$ in Table 15 that are derived through dilation of eta products in other tables in the paper arise through the dilation $q \to q^7$ applied to the eta quotients in Table 4. From this we derive the next theorem.

Theorem 5.36. *Let $B(q)$ and $C(q)$ be any two eta quotients in Group XIV, and $D(q)$ and $E(q)$ be any two eta quotients in Group XV of Table* 15. *Then*

$$A_{(0)} \subsetneqq B_{(0)} \subsetneqq D_{(0)}, \qquad B_{(0)} = C_{(0)}, \qquad D_{(0)} = E_{(0)}. \tag{5.62}$$

Proof. The proof is essentially the same as the proof of Theorem 5.29. Will just prove that $A_{(0)} \subsetneqq B_{(0)}$ for $B(q) = f_7^2 = B'(q^7) = \sum_{n=0}^{\infty} b_n q^n$ where $B'(q) = f_1^2 = \sum_{n=0}^{\infty} b'_n q^n$. As in the proof of Theorem 5.29, the other statements will then follow from Lemma 5.1 and the facts that groups XIV and XV in Table 15 are, respectively, the $q \to q^7$ dilations of groups I and II in Table 4.

Apply the dilation $q \to q^{12}$ to both $A(q)$ and $B(q)$ and then multiply by q^7 to

$$\eta(12z)^{14} = \sum_{n=0}^{\infty} a_n q^{12n+7}, \qquad \eta(84z)^2 = \sum_{n=0}^{\infty} b_n q^{12n+7}, \tag{5.63}$$

and suppose $a_n = 0$, so that by Serre's criterion (3.6), $12n+7$ has a prime factor $p \equiv -1 \pmod 3$ with odd exponent. If $7 \nmid n$, then clearly $b_n = 0$, so suppose $7|n$, or $n = 7m$ for some integer m. So $12n + 7 = 7(12m + 1)$ has a prime factor $p \equiv -1 \pmod 3$, and thus that $12m+1$ has a prime factor $p \equiv -1 \pmod 3$. By Serre's criterion (3.1), then $b'_m = 0$, so that $b_n = b_{7m} = b'_m = 0$. Since $a_1 = -14$ and $b_1 = 0$, then $A_{(0)} \subsetneqq B_{(0)}$ follows. □

As was the case f_1^8 and f_1^{10}, there are pairs of eta quotients in Table 15 such that one of the pair is the even part of the other, so that some of the 2-dissections from (5.37) and (5.43)-(5.50) may be used to derive strict inclusion results for the index sets of vanishing coefficients.

Theorem 5.37. *Let $B(q)$ be either the first or second eta quotient in any of the quadruples of eta quotients, with $C(q)$ either the third or fourth eta quotient in the same quadruple:*

$$\left(\frac{f_2^{11}f_8^2}{f_1^4f_4^5}, \frac{f_1^4f_8^2}{f_2f_4}, \frac{f_4^9}{f_2^3f_8^2}, f_2^3f_8\right), \left(\frac{f_2^2f_4f_6^3f_{24}}{f_1f_3f_8f_{12}^2}, \frac{f_1f_3f_4^2f_{24}}{f_2f_8f_{12}}, \frac{f_8f_{12}^3}{f_6f_{24}}, f_6f_8\right),$$
$$\left(\frac{f_3^3f_4^2f_{24}}{f_1f_6f_8f_{12}}, \frac{f_1f_4^3f_6^8f_{24}}{f_2^3f_3^3f_8f_{12}^4}, \frac{f_4^5f_6f_{24}}{f_2^2f_8f_{12}^2}, \frac{f_2^2f_8f_{12}}{f_4f_6}\right),$$
$$\left(\frac{f_2^7f_3f_{24}}{f_1^3f_4f_6^2f_8}, \frac{f_1^3f_4^2f_6f_{24}}{f_2^2f_3f_8f_{12}}, \frac{f_4^5f_6f_{24}}{f_2^2f_8f_{12}^2}, \frac{f_2^2f_8f_{12}}{f_4f_6}\right),$$
$$\left(\frac{f_1f_4^3f_6^4f_{12}^6}{f_2^3f_3^3f_8f_{24}^3}, \frac{f_3^3f_4^2f_{12}^9}{f_1f_6^5f_8f_{24}^3}, \frac{f_4^5f_{12}^5}{f_2^2f_6f_8f_{24}^2}, \frac{f_2^2f_6f_8f_{12}^2}{f_4f_{24}}\right),$$
$$\left(\frac{f_1^3f_4^2f_{12}^9}{f_2^2f_3f_6^3f_8f_{24}^3}, \frac{f_2^7f_3f_{12}^{10}}{f_1^3f_4f_6^6f_8f_{24}^3}, \frac{f_4^5f_{12}^5}{f_2^2f_6f_8f_{24}^2}, \frac{f_2^2f_6f_8f_{12}^2}{f_4f_{24}}\right). \tag{5.64}$$

Then

$$B_{(0)} \subsetneqq C_{(0)}. \tag{5.65}$$

Proof. Once again, we prove the result just for $B(q)$ being the first eta quotient in each quadruple and $C(q)$ being the third eta quotient, as the second is the $q \to -q$ partner of the first, and the fourth is the $q^2 \to -q^2$ partner of the third.

As with the similar proof of previous theorems, we confine the proof to simply listing, in the case of each pair $B(q)$ and $C(q)$, the function $G(q^2)$ and a 2-dissection from Lemma 5.25, so that after multiplying the indicated 2-dissection by $G(q^2)$, the resulting eta quotient on the left is equal to $B(q)$, and the first eta quotient resulting on the right is the same as $C(q)$. The pairs ($G(q^2)$, 2-dissection) are, respectively,

$$\left(\frac{f_2^{11}f_8^2}{f_4^5}, (5.49)\right), \left(\frac{f_2^2f_4f_6^3f_{24}}{f_8f_{12}^2}, (5.48)\right), \left(\frac{f_4^2f_{24}}{f_6f_8f_{12}}, (5.45)\right),$$
$$\left(\frac{f_2^7f_{24}}{f_4f_6^2f_8}, (5.44)\right), \left(\frac{f_4^3f_6^4f_{12}^6}{f_2^3f_8f_{24}^3}, (5.46)\right), \left(\frac{f_1^3f_4^2f_{12}^9}{f_2^2f_3f_6^3f_8f_{24}^3}, (5.43)\right). \tag{5.66}$$

□

Remark 5.38. *The quadruples of eta quotients listed at* (5.64) *are those numbered, respectively,*

$$(3,4,68,67),(46,45,79,80),(41,42,73,74),(44,43,73,74),\\ (40,39,66,65),(38,37,66,65), \quad (5.67)$$

in Table 15.

As with Table 13, there are a number of eta quotients in Table 15 which may be expressed as a product of two eta quotients with single-sum theta series expansions, as in Lemma 3.6. These are collected in Table 16.

Remark 5.39. *As was the case with Table* 14*, in the case of all the eta quotients from Group I in Table* 16*, ideally one would like to show that if* $12t+5$ *does not have a prime factor* $p \equiv 2 \pmod 3$ *with odd exponent, then* $b_t \neq 0$*, to get* $A_{(0)} = B_{(0)}$*. Once again, this does not seem straightforward, so we do not apply that approach here.*

Theorem 5.40. *Let* $A(q) = f_1^{14} = \sum_{n=0}^{\infty} a_n q^n$ *and let* $B(q) = \sum_{n=0}^{\infty} b_n q^n$ *be any of the eta quotients in Entries* 41*,* 43 *and* 45 *of Table* 15*. Then*

$$A_{(0)} \subsetneqq B_{(0)}.$$

Proof. Define

$$g(q;j,k) = \sum_{m,n=0}^{\infty} q^{4(24m+j)^2+3(24n+k)^2}.$$

Then viewed as holomorphic modular forms for $q = e^{2\pi i z}$, $q^7 B(q^{12})$ is a linear combination of $g(q;j,k)$ for $0 \leq j,k \leq 23$. Recall by Item (6) of Theorem 3.4 that $a_n = 0$ if and only if $12n+7$ has a prime factor congruent to 2 modulo 3 with odd exponent. Clearly, such $12n+7$ cannot be written as $4(24m+j)^2 + 3(24n+k)^2$, and thus, $b_n = 0$. The desired inclusion follows. □

We take this opportunity to give another example of how computer experimentation can generally assist in a proof, and also to exhibit another method for proving inclusion results. We illustrate this by giving an alternative proof of Theorem 5.40 in the case of Entry 45 in Table 15, namely,

$$B(q) = \frac{f_1 f_3 f_4^2 f_{24}}{f_2 f_8 f_{12}}. \quad (5.68)$$

Computer experimentation suggests that the linear combination of the $g(q;j,k)$ that equals $q^7B(q^{12})$, as mentioned in the proof of Theorem 5.40, contains 160 terms. Examination of the $g(q;j,k)$ in this sum and some further experimentation suggests that these 160 terms can be combined into just two sums, as in the following theorem.

Theorem 5.41. *Let* $B(q)$ *be as at* (5.68)*. Then*

$$q^7B(q^{12}) = \\ \sum_{m,n=0}^{\infty} \left(\frac{12}{m}\right)\left(\frac{24}{n}\right) q^{4m^2+3n^2} + \sum_{m,n=0}^{\infty} \left(\frac{m}{3}\right)^2 \left(\frac{n}{2}\right)\left(2-\left(\frac{n}{3}\right)^2\right)(-1)^{\left(\frac{n}{3}\right)^2+m+1} q^{16m^2+3n^2}. \quad (5.69)$$

Remark 5.42. *There are alternative approaches.*

(1) *Note that* $A_{(0)} \subsetneq B_{(0)}$ *follows by the same argument that was used in the proof of Theorem* 5.40*, since the exponents have the same form, namely* $M^2 + 3N^2$.
(2) *One could also prove* (5.69) *by checking that coefficients in the series expansion of each side agree up to the Sturm bound.*

Proof of Theorem 5.41. From (3.11), the first sum at (5.69) may be written as

$$\sum_{m,n=0}^{\infty} \left(\frac{12}{m}\right)\left(\frac{24}{n}\right) q^{4m^2+3n^2} = q^7 \frac{f_{96} f_{144}^3}{f_{72} f_{288}}.$$

By the second summation at (3.10) (first replacing q with $-q$ and then making the dilation $q \to q^{16}$) the sum over m in the second series at (5.69) may be written as

$$\sum_{m=0}^{\infty} \left(\frac{m}{3}\right)^2 (-1)^m q^{16m^2} = -q^{16} \frac{f_{48} f_{288}^2}{f_{96} f_{144}}.$$

Splitting the sum over n in the second series at (5.69) into two sums, one with $3|n$ and the second with $3 \nmid n$, one gets

$$\begin{aligned}\sum_{n=0}^{\infty} \left(\frac{n}{2}\right)\left(2-\left(\frac{n}{3}\right)^2\right)(-1)^{\left(\frac{n}{3}\right)^2+1} q^{3n^2} &= \sum_{n=0}^{\infty} \left(\frac{3n}{2}\right)(2)(-1)q^{27n^2} + \sum_{\substack{n=0\\3\nmid n}}^{\infty} \left(\frac{n}{2}\right)(1)(-1)^2 q^{3n^2}\\ &= \sum_{n=0}^{\infty} \left(\frac{n}{2}\right) q^{3n^2} + 3\sum_{n=0}^{\infty} \left(\frac{n}{2}\right) q^{27n^2}.\end{aligned}$$

Next,

$$\begin{aligned}\sum_{n=0}^{\infty} \left(\frac{n}{2}\right) q^{3n^2} &= \sum_{n=0}^{\infty} \left(q^{3(8n+1)^2} + q^{3(8n+7)^2}\right) - \sum_{n=0}^{\infty} \left(q^{3(8n+3)^2} + q^{3(8n+5)^2}\right)\\ &= \sum_{n=-\infty}^{\infty} q^{3(8n+1)^2} - \sum_{n=-\infty}^{\infty} q^{3(8n+3)^2}\\ &= q^3 \sum_{n=-\infty}^{\infty} q^{192n^2+48n} - q^{27} \sum_{n=-\infty}^{\infty} q^{192n^2+144n}\\ &= q^3(-q^{144}, -q^{240}, q^{384}; q^{384})_\infty - q^{27}(-q^{48}, -q^{336}, q^{384}; q^{384})_\infty\\ &= q^3 \frac{f_{24} f_{96}}{f_{48}}.\end{aligned}$$

The penultimate equality above follows from the Jacobi triple product identity (4.10), and the last equality is a consequence of an identity of Hirschhorn [13, Lemma 1] (after a dilation $q \to q^{24}$),

$$f_1 = \frac{f_2}{f_4}\left(\bar{J}_{6,16} - q\bar{J}_{2,16}\right), \text{ where } \bar{J}_{a,m} := (-q^a, -q^{m-a}, q^m; q^m)_\infty.$$

After treating $\sum_{n=0}^{\infty}\left(\frac{n}{2}\right)q^{27n^2}$ similarly, one gets that (5.69) is true if it can be shown that

$$q^7B(q^{12}) = q^7\frac{f_{96}f_{144}^3}{f_{72}f_{288}} - q^{16}\frac{f_{48}f_{288}^2}{f_{96}f_{144}}\left(q^3\frac{f_{24}f_{96}}{f_{48}} + 3q^{27}\frac{f_{216}f_{864}}{f_{432}}\right),$$

or after dividing through by q^7 and applying the contraction $q^{12} \to q$, if it can be shown that

$$B(q) = \frac{f_1f_3f_4^2f_{24}}{f_2f_8f_{12}} = \frac{f_8f_{12}^3}{f_6f_{24}} - \frac{qf_2f_{24}^2}{f_{12}} - 3q^3\frac{f_4f_{18}f_{24}^2f_{72}}{f_8f_{12}f_{36}}.$$

By (5.47),

$$\frac{f_1f_3f_4^2f_{24}}{f_2f_8f_{12}} = \frac{f_8f_{12}^3}{f_6f_{24}} - q\frac{f_4^6f_6f_{24}^3}{f_2^2f_8^3f_{12}^3},$$

So, upon comparing the two expressions and after a little manipulation, the result will follow if it can be shown that

$$\frac{f_2f_8}{f_4} = \frac{f_4^5f_6f_{24}}{f_2^2f_8^2f_{12}^2} - 3q^2\frac{f_{18}f_{72}}{f_{36}}.$$

However, this is an easy consequence of combining the following two identities (the $q \to -q$ partners of two identities given by Hirschhorn [14, page 132])

$$\frac{f_1f_4}{f_2} = \frac{f_3f_{12}f_{18}^5}{f_6^2f_9^2f_{36}^2} - q\frac{f_9f_{36}}{f_{18}},$$

$$\frac{f_2^5}{f_1^2f_4^2} = \frac{f_{18}^5}{f_9^2f_{36}^2} + \frac{2qf_6^2f_9f_{36}}{f_3f_{12}f_{18}},$$

and then replacing q with q^2. □

5.6. Eta quotients with vanishing coefficient behaviour similar to $f_1^3f_2^3$. The lacunary eta product

$$A(q) := f_1^3f_2^3 =: \sum_{n=0}^{\infty} a_nq^n \tag{5.70}$$

is not one of the eta quotients considered by Serre [29], but as Table 19 and Figure 8 show, there are over one hundred eta quotients $B(q)$ for which evidence suggests that either $A_{(0)} = B_{(0)}$, or $A_{(0)} \subsetneq B_{(0)}$. For this reason we state and prove a criterion for when $a_n = 0$ that appears at the end of Subsection 3.2. Before getting to that, we have the following lemma that arises from first replacing q with q^8 in the expression for $A(q)$ and then multiplying by q^3. The proof is realized by comparing coefficients up to the Sturm bound.

Lemma 5.43. *The following identity holds.*

$$\eta(8z)^3\eta(16z)^3 = \sum_{n=0}^{\infty} a_nq^{8n+3} = \frac{\overline{f}_{128.3.d.b} - f_{128.3.d.b}}{8\sqrt{2}}$$

$$= q^3 - 3q^{11} - 3q^{19} + 14q^{27} - 15q^{43} - 2q^{51} - 15q^{59} + 21q^{67} + 25q^{75} + 9q^{83} - 69q^{99} + \cdots, \tag{5.71}$$

and

$$\begin{aligned} f_{128.3.d.b} &= \sum_{m,n}(4m+1+4n\sqrt{-2})^2 q^{(4m+1)^2+2(4n)^2} \\ &\quad - \sum_{m,n}(4m+1+(4n+2)\sqrt{-2})^2 q^{(4m+1)^2+2(4n+2)^2} \\ &\quad - i\sum_{m,n}(4m+1+(4n+1)\sqrt{-2})^2 q^{(4m+1)^2+2(4n+1)^2} \\ &\quad + i\sum_{m,n}(4m+1+(4n+3)\sqrt{-2})^2 q^{(4m+1)^2+2(4n+3)^2} \\ &= q - 4\sqrt{2}q^3 + 23q^9 + 12\sqrt{2}q^{11} - 2q^{17} + 12\sqrt{2}q^{19} + 25q^{25} \\ &\quad - 56\sqrt{2}q^{27} - 96q^{33} - 46q^{41} + 60\sqrt{2}q^{43} + 49q^{49} + 8\sqrt{2}q^{51} - 96q^{57} \\ &\quad + 60\sqrt{2}q^{59} - 84\sqrt{2}q^{67} + 142q^{73} - 100\sqrt{2}q^{75} + 241q^{81} \\ &\quad - 36\sqrt{2}q^{83} - 146q^{89} + 94q^{97} + 276\sqrt{2}q^{99} + \cdots. \end{aligned} \tag{5.72}$$

Here $f_{128.3.d.b}$ is the CM form of weight 3 and level 128 labelled 128.3.d.b in the LMFDB, and $\overline{f}_{128.3.d.b}$ is its $\sqrt{2} \to -\sqrt{2}$ conjugate. We use this lemma to derive the following criterion.

Theorem 5.44. *Let a_n be the sequence defined at* (5.70) *above. Then*

$$a_n = 0 \iff ord_p(8n+3) \text{ is odd for some prime } p \equiv 5 \text{ or } 7 \pmod 8. \tag{5.73}$$

Proof. Define the sequence $\{s_t\}$ by

$$f_{128.3.d.b} = \sum_{t=1}^{\infty} s_t q^t. \tag{5.74}$$

It is clear from the exponents of the theta series at (5.72) that if $t \not\equiv 1$ or 3 (mod 8), then $s_t = 0$. In particular, if p is a prime such that $p = 2$ or $p \equiv 5$ or 7 (mod 8), then $s_p = 0$. Next consider the recurrence relation for $f_{128.3.d.b}$ at powers of a prime p:

$$s_{p^{k+1}} = s_p s_{p^k} - \chi(p)p^2 s_{p^{k-1}}, \text{ where} \tag{5.75}$$
$$\chi(p) = \begin{cases} 1, & p \equiv 1,3 \pmod 8 \\ -1, & p \equiv 5,7 \pmod 8. \end{cases}$$

Hence if $p \equiv 5$ or 7 (mod 8) is prime and $k \geq 0$ is an integer, $s_{p^{2k}} = p^k$ and $s_{p^{2k+1}} = 0$.

Next, if $p \equiv 1 \pmod 8$ is a prime, then $p = x^2 + 2y^2$ for unique positive integers x and y, with x odd and y even. If $y \equiv 0 \pmod 4$, then p occurs only in the exponent of the first theta series at (5.72), and there are exactly two representations (the second coming from replacing n with $-n$). Hence

$$\begin{aligned} s_p &= (4m+1+4n\sqrt{-2})^2(4m+1-4n\sqrt{-2})^2 \\ &= 2((4m+1)^2-2(4n)^2) = 2(x^2-2y^2). \end{aligned}$$

A similar analysis of the case $y \equiv 2 \pmod 4$ shows that p occurs only in the exponent of the second theta series at (5.72), there again being two representations, and that in this case $s_p = -2(x^2-2y^2)$.

On the other hand, if $p \equiv 3 \pmod 8$ is a prime, then $p = x^2+2y^2$ for unique positive integers x and y, with x and y both odd. In this case p occurs in the exponent of just the third and fourth theta series at (5.72), and occurs exactly once in each case. A similar computation in this case shows that, up to sign, $s_p = 4\sqrt{2}xy$.

A key point is that for any prime $p \equiv 1$ or $3 \pmod 8$, then the explicit formulae given above imply that $s_p \not\equiv 0 \pmod p$. The recurrence relation (5.75) then implies $s_{p^k} \equiv (s_p)^k \not\equiv 0 \pmod p$, and thus $s_{p^k} \neq 0$ for any positive integer k. Upon putting all of these pieces together and using the multiplicativity of the s_t, we see that, for any non-negative integer n,

$$s_{8n+3} = 0 \Longleftrightarrow ord_p(8n+3) \text{ is odd for some prime } p \equiv 5 \text{ or } 7 \pmod 8.$$

The result now follows since from (5.71) and (5.74) one has that

$$a_n = -\frac{s_{8n+3}}{4\sqrt{2}}.$$

□

The next two theorems are inspired by [18, Theorem 1.1, (12)-(18)] in which the eta quotients of weight 1 lie in some space of CM forms of weight 1, so that they can be written as a linear combination of theta series associated with a binary quadratic form (see [21] for details on the notion of CM forms of weight 1).

Theorem 5.45. *Let*

$$B(q) = f_1 f_8 = \sum_{n=0}^{\infty} b_n q^n. \tag{5.76}$$

Then $A_{(0)} \subsetneqq B_{(0)}$.

Proof. It can be checked that

$$q^3 f_8 f_{64} = \frac{1}{2}\left(\sum_{m,n} q^{3x^2+2xy+43y^2} - \sum_{m,n} q^{11x^2+4xy+12y^2}\right).$$

Note that

$$3x^2+2xy+43y^2 = \frac{1}{3}\left((3x+y)^2+2(8y)^2\right)$$

and

$$11x^2+4xy+12y^2 = \frac{1}{11}\left((11x+2y)^2+2(8y)^2\right),$$

and the primes 3 and 11 are split in $\mathbb{Z}[\sqrt{-2}]$, so any integer n such that $8n+3$ has a prime factor $p \equiv 5, 7 \pmod 8$ with odd exponent is not representable by $3x^2+2xy+43y^2$ or $11x^2+4xy+12y^2$. Therefore, for such integers, $b_n = 0$, and the assertion follows. □

Theorem 5.46. *Let*

$$C(q) = \frac{f_1^2 f_6^2}{f_2 f_3} = \sum_{n=0}^{\infty} c_n q^n. \tag{5.77}$$

Then $A_{(0)} \subsetneqq C_{(0)}$.

Proof. One first verifies that

$$q^3 \frac{f_8^2 f_{48}^2}{f_{16} f_{24}} = \frac{S_4 - S_3}{2},$$

where

$$\begin{aligned} S_3 &= \frac{1}{2}\left(\sum_{m,n} q^{x^2+96y^2} - \sum_{m,n} q^{3x^2+32y^2}\right) \\ &\quad - \frac{1}{2}\left(\sum_{m,n} q^{4x^2+4xy+25y^2} - \sum_{m,n} q^{11x^2+10xy+11y^2}\right), \\ S_4 &= \frac{1}{2}\left(\sum_{m,n} q^{x^2+96y^2} + \sum_{m,n} q^{3x^2+32y^2}\right) \\ &\quad - \frac{1}{2}\left(\sum_{m,n} q^{4x^2+4xy+25y^2} + \sum_{m,n} q^{11x^2+10xy+11y^2}\right) \end{aligned}$$

are CM newforms of weight 1. For $i = 3, 4$, writing

$$S_i = \sum_{n=0}^{\infty} s_i(n) q^n,$$

it suffices to prove that $s_4(n) - s_3(n) = 0$ for n having a prime factor $p \equiv 5, 7 \pmod 8$ with odd exponent. Since S_i are CM newforms, for $n = \prod_{p|n} p^{e_p}$,

$$s_i(n) = \prod_{p|n} s_i\left(p^{e_p}\right).$$

Now note that

$$\begin{aligned} x^2 + 96y^2 &\equiv 0, 1 \pmod 8, \\ 3x^2 + 32y^2 &\equiv 0, 3, 4 \pmod 8, \\ 4x^2 + 4xy + 25y^2 &\equiv 0, 1, 4 \pmod 8, \\ 11x^2 + 10xy + 11y^2 &\equiv 0, 3, 4 \pmod 8. \end{aligned}$$

So for $p \equiv 5, 7 \pmod 8$, $s_i(p) = 0$, and thus, $s_i(p^{e_p}) = 0$ for e_p odd by the recursive formula for $s_i(p^m)$. This demonstrates the assertion. □

Theorem 5.47. *Let*

$$B(q) = \sum_{n=0}^{\infty} b_n q^n, \qquad C(q) = \sum_{n=0}^{\infty} c_n q^n, \tag{5.78}$$

where $B(q)$ is any of the eta quotients in Group XVII, and $C(q)$ is any of the eta quotients in Group XVIII of Table 19. *Then*

$$A_{(0)} \subsetneqq B_{(0)} \subsetneqq C_{(0)}. \tag{5.79}$$

Further, all of the eta quotients in Group XVII of Table 19 *have identically vanishing coefficients, as do all of the eta quotients in Group XVIII.*

Proof. It is enough to prove $A_{(0)} \subsetneqq B_{(0)}$ for $B(q) = f_3^3$, since the eta quotients in group XVII and group XVIII of Table 19 are derived, respectively, from the eta quotients in Groups I and II in Table 5 as a result of a $q \to q^3$ dilation, and hence Theorem 5.47 will follow in full from Lemma 5.1.

Apply the dilation $q \to q^8$ to both $A(q)$ and $B(q)$ and then multiply by q^3 to get

$$q^3 A(q^8) = \sum_{n=0}^{\infty} a_n q^{8n+3} = \eta(8z)^3 \eta(16z)^3,$$

$$q^3 B(q^8) = \sum_{n=0}^{\infty} b_n q^{8n+3} = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{3(2n+1)^2},$$

where the last equality follows from (4.17) after the dilation $q \to q^{24}$.

Now suppose $a_t = 0$, so that $8t+3$ has a prime factor $p \equiv 5$ or $7 \pmod 8$ with odd exponent. Clearly such an integer cannot be represented by $3(2n+1)^2$ and hence $b_t = 0$, showing that $A_{(0)} \subseteq B_{(0)}$. Since $a_1 = -3$ and $b_1 = 0$, the result follows. □

As was done in the previous two sections, we examine eta quotients in Table 19 that are expressible as products of eta quotients that have expressions as single sum theta series from Lemma 3.6. These double-sum theta expansions may be seen in Table 20.

Remark 5.48. *Some integers of the form $8t + 3$ with an odd prime factor $p \equiv 5$ or $7 \pmod 8$ with odd exponent are representable by the other two forms that appear as exponents of theta series in Table* 20*, namely $3m^2 + 8n^2$ (for example, $35 = 3(3^2) + 8(1^2) = 3(1^2) + 8(2^2)$) and $3m^2 + 16n^2$ (for example, $91 = 3(3^3) + 16(2^2) = 3(5^2) + 16(1^2)$). In these cases, it is the sum of the coefficients over the various representations of $8t + 3$ that vanishes.*

Finally, the last theorem of this section supplements the cases missed by Item (7) of Lemma 3.7.

Theorem 5.49. *Let* $A(q) = f_1^3 f_2^3 = \sum_{n=0}^{\infty} a_n q^n$ *and let* $B(q) = \sum_{n=0}^{\infty} b_n q^n$ *be any of the eta quotients in Entries* 39 *and* 45 *of Table* 19. *Then* $A_{(0)} \subseteq B_{(0)}$.

Proof. Define

$$g_1(q;j,k) = \sum_{m,n=0}^{\infty} q^{(24m+j)^2+2(24n+k)^2}, \quad g_2(q;j,k) = \sum_{m,n=0}^{\infty} q^{(24m+j)^2+8(24n+k)^2},$$

$$g_3(q;j,k) = \sum_{m,n=0}^{\infty} q^{(54m+j)^2+2(54n+k)^2}, \quad g_4(q;j,k) = \sum_{m,n=0}^{\infty} q^{(54m+j)^2+8(54n+k)^2},$$

Then $q^3 B(q^8)$ is a linear combination of $g_i(q;j,k)$ for $i \in \{1,2\}$ and $0 \le j,k \le 23$ and $g_i(q;,j,k)$ for $i \in \{3,4\}$ and $0 \le j,k \le 53$. By Item (7) of Theorem 5.44, we know that $a_n = 0$ whenever $8n+3$ has a prime factor congruent to 5 or 7 modulo 8. Clearly, such positive integers $8n+3$ cannot be written as either $(24m+j)^2 + 2(24n+k)^2$ or $(24m+j)^2 + 8(24n+k)^2$, and thus, $b_n = 0$. The desired inclusion follows. □

6. Some Remarks on the Eta Quotients associated with f_1^{26}

Define the function $A(q)$ and the sequence $\{a_n\}$ by

$$A(q) := f_1^{26} =: \sum_{n=0}^{\infty} a_n q^n. \tag{6.1}$$

Recall that, from (3.7), one has that $a_n = 0$ if either of two conditions on $12n+13$ hold. Thus we do not have an “if and only if” condition determining precisely when $a_n = 0$ (in contrast to the situation with the other powers of f_1 considered by Serre [29]), so that we do not know exactly the content of $A_{(0)}$.

In previous sections, when the evidence suggests conjectures of the form $A_{(0)} = B_{(0)}$ or $A_{(0)} \subsetneqq B_{(0)}$ for eta quotients $A(q)$ and $B(q)$ where $A_{(0)}$ is known, in theory such conjectures are provable by determining $B_{(0)}$ exactly using one of the methods used elsewhere in the paper, and then comparing $B_{(0)}$ with $A_{(0)}$. For the reason mentioned ($A_{(0)}$ not known precisely), it is not possible to prove similar theorems when $A(q) = f_1^{26}$. In this case, in place of theorems we have conjectures. In support of some conjecture of the form $A_{(0)} = B_{(0)}$ or $A_{(0)} \subsetneqq B_{(0)}$, for some eta quotient $B(q) =: \sum_{n=0}^{\infty} a_n q^n$, we show that $b_n = 0$ if $12n+13$ satisfies the same conditions above that imply $a_n = 0$. Alternatively, the conjectures made may be based on the conjecture that Serre’s criterion (3.7) may be extended to the following “if and only if” statement:

Conjecture 6.1. *Writing*

$$q^{13} f_{12}^{26} = \sum_{n=0}^{\infty} a_n q^{12n+13}, \tag{6.2}$$

one has that $a_n = 0$ *if and only if either of the following holds:*

(1) $12n + 13$ *has a prime factor* $p_1 \equiv -1 \pmod 3$ *with odd exponent* *and* *a prime* $p_2 \equiv -1 \pmod 4$ *with odd exponent (it may be that* $p_1 = p_2$*),*

(2) $12n + 13$ *is a square and all prime factors* p *satisfy* $p \equiv -1 \pmod{12}$.

Conjecture 6.2. *Let* $A(q)$ *be as at* (6.1) *and let* $B(q)$ *be any eta quotient in Group XVII, and* $D(q)$ *be any eta quotient in Group XVIII of Table* 17. *Then*

$$A_{(0)} \subsetneqq B_{(0)} \subsetneqq D_{(0)}. \tag{6.3}$$

Remark 6.3.

(1) We note that if the statement $A_{(0)} \subsetneqq B_{(0)}$ *could be proven for* $B(q) = f_{13}^2 = B'(q^{13}) = \sum_{n=0}^{\infty} b_n q^n$ *where* $B'(q) = f_1^2 = \sum_{n=0}^{\infty} b'_n q^n$*, then, as in the proof of Theorems* 5.29 *and* 5.36*, the other statements would follow from Lemma* 5.1 *and the facts that groups XVII and XVIII in Table* 17 *are, respectively, the* $q \to q^{13}$ *dilations of groups I and II in Table* 4.

(2) The particular case $A_{(0)} \subsetneqq B_{(0)}$ *mentioned in (1) can be shown to hold if the "if and only if" statement following* (6.2) *holds. Suppose that is the case and apply the dilation* $q \to q^{12}$ *to both* $A(q)$ *and* $B(q)$ *and then multiply by* q^{13} *to obtain*

$$\eta(12z)^{26} = \sum_{n=0}^{\infty} a_n q^{12n+13}, \qquad \eta(156z)^2 = \sum_{n=0}^{\infty} b_n q^{12n+13}, \tag{6.4}$$

and suppose $a_n = 0$. *If* $13 \nmid n$*, then clearly* $b_n = 0$*, so suppose* $13|n$*, or* $n = 13m$ *for some integer* m. *So* $12n+13 = 13(12m+1)$ *and since* $a_n = 0$*, and clearly* $12n+13$ *is not a square with all prime factors* $\equiv -1 \pmod{13}$*, one has that* $12n+13$. *Hence* $12m+1$*, has a prime factor* $p \equiv -1 \pmod 3$. *By Serre's criterion* (3.1)*,* $b'_m = 0$*, so that* $b_n = b_{13m} = b'_m = 0$. *Since* $a_1 = -26$ *and* $b_1 = 0$*, then* $A_{(0)} \subsetneqq B_{(0)}$ *would follow.*

As with Table 13 and Table 15, there are also several eta quotients in Table 17 which may be expressed as a product of two eta quotients with the single-sum theta series expansions listed in Lemma 3.6. These are collected in Table 18. To make use of the theta series representations in the table, we need the following elementary result.

Lemma 6.4. *Suppose* $12t + 13$ *satisfies either condition* (1) *or condition* (2) *following* (6.2). *Then* $12t + 13$ *is not represented by either of the quadratic forms* $9m^2 + 4n^2$ *or* $m^2 + 12n^2$.

Proof. Suppose $12t + 13$ satisfies condition (1), so that $12t + 13$ has a prime factor $p_1 \equiv -1 \pmod 3$ with odd exponent and a prime $p_2 \equiv -1 \pmod 4$ with odd exponent (it may be that $p_1 = p_2$). Then $12t + 13$ is not represented by $9m^2 + 4n^2 = (3m)^2 + (2n)^2$ by Lemma 3.1, and $12t + 13$ is not represented by $m^2 + 12n^2 = m^2 + 3(2n)^2$ by Lemma 3.3.

On the other hand, if $12t + 13$ satisfies condition (2), so that $12t + 13$ is a square and all prime factors p satisfy $p \equiv -1 \pmod{12}$, then all prime factors p satisfy $p \equiv 2 \pmod 3$ and $p \equiv 3 \pmod 4$. Hence, if $12t + 13 = 9m^2 + 4n^2 =$

$(3m)^2+(2n)^2$, then all primes p divide both $3m$ and $2n$ by Lemma 3.1. This leads to a contradiction after cancelling all prime factors on each side. Likewise, if $12t+13=m^2+12n^2=m^2+3(2n)^2$, then all primes p divide both m and $2n$ by Lemma 3.3, leading to a contradiction. □

This lemma leads to the result in the following theorem.

Theorem 6.5. *Let $B(q)=\sum_{n=0}^{\infty} b_n q^n$ be the eta quotient in Entries 79 of Table 17. Then $b_n=0$ when $12n+13$ satisfies either condition (1) or condition (2) following (6.2).*

Proof. Define

$$h(q;j,k)=\sum_{m,n=0}^{\infty} q^{4(52m+j)^2+9(52n+k)^2}$$

and

$$g(q;j,k)=\sum_{m,n=0}^{\infty} q^{(52m+j)^2+12(52n+k)^2}.$$

Then $q^{13}B(q^{12})=\sum_{n=0}^{\infty} b_n q^{12n+13}$ is a linear combination of $h(q;j,k)$ and $g(q;,j,k)$ for $0\le j,k\le 51$. By Lemma 6.4, it is clear that any $12n+13$ satisfying either of the conditions cannot be expressed as $4(52m+j)^2+9(52n+k)^2$ or $(52m+j)^2+12(52n+k)^2$. So the conclusion follows. □

Conjecture 6.6. *Let $A(q)=f_1^{26}=:\sum_{n=0}^{\infty} a_n q^n$ and let $B(q)=:\sum_{n=0}^{\infty} a_n q^n$ be any of the eta quotients in the second column of Table 18. Then*

$$A_{(0)}\subsetneqq B_{(0)}. \tag{6.5}$$

Remark 6.7. *Note once again that this conjecture would be true if Serre's criterion (3.7) could be extended to the "if and only if" statement following (6.2), by Lemma 6.4 (for each eta quotient $B(q)$ in Table 18, one can find an integer n such that $b_n=0$ but $a_n\neq 0$).*

We note in passing that it is possible to prove unconditionally some inclusion results that do not involve $A(q)$. As in previous sections, if the eta quotient $B(q)$ is not an even function of q, and the eta quotient $C(q)$ is its even part, then $B_{(0)}\subsetneqq C_{(0)}$. However, this time all the eta quotients which are even parts of eta quotients in Table 17 lie outside the range of our search. We do not compute these even parts explicitly (using the 2-dissections stated in Lemmas 5.22 and 5.25), except as indicated below, as we feel that we have exhibited this phenomenon sufficiently elsewhere. It is possible that computing these even part eta quotients and determining experimentally their vanishing coefficient behaviour may contribute additional structure to the diagram in Figure 7, in terms of additional groups and additional inclusion connections. We do take this opportunity to state some further 2-dissections of eta quotients, which may be applied to produce 2-dissections of eta quotients in Table 17. These may be used to derive further strict inclusion results for index sets of vanishing coefficients.

Lemma 6.8. *The following 2-dissections hold:*

$$\frac{f_1}{f_3} = \frac{f_2 f_{16} f_{24}^2}{f_6^2 f_8 f_{48}} - q\frac{f_2 f_8^2 f_{12} f_{48}}{f_4 f_6^2 f_{16} f_{24}}, \tag{6.6}$$

$$\frac{f_1^2}{f_3^2} = \frac{f_2 f_4^2 f_{12}^4}{f_6^5 f_8 f_{24}} - 2q\frac{f_2^2 f_8 f_{12} f_{24}}{f_4 f_6^4}, \tag{6.7}$$

$$\frac{f_3}{f_1} = \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q\frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}, \tag{6.8}$$

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}. \tag{6.9}$$

Proof. All four of these 2-dissections may be found in [14, Section 30.10]. □

We give a single example as an illustration. If we consider the eta quotient numbered 43 (group XI) in Table 17 in conjunction with (6.6) above, one gets that if

$$B(q) = \frac{f_1 f_6^5}{f_2 f_3}, \qquad C(q) = \frac{f_6^3 f_{16} f_{24}^2}{f_8 f_{48}},$$

then $B_{(0)} \subsetneq C_{(0)}$. Further, computation of the location of the zero coefficients up to 3000 terms suggests that $C(q)$ belongs in Group XV of Table 17 and Figure 7. Note that $C(q)$ is not listed in Table 17, since it lies outside the range of the search parameters.

7. A General Inclusion $A_{(0)} \subseteq B_{(0)}$ and Conjectures

In this section, building upon the preceding results, we deduce a general inclusive relation satisfied by any eta quotients lying in the same table. As an implication, we prove that all the eta quotients obtained from our search are lacunary.

In addition, as is noted in the introduction, a large number of conjectural relations between the coefficients of eta quotients associated with f_1^r or $f_1^3 f_2^3$ have been suggested by our computational experiments. After justifying the aforementioned general inclusive relation, we explicitly state those suggested conjectures for the interested reader and save them for future investigation.

7.1. A general inclusive relation. The general inclusive relation we have been striving for can be enunciated in the following theorem. Each statement relies on Theorem 3.4, Lemma 3.5, Lemma 3.7, and appropriate results from Section 5.

Theorem 7.1. *Let $A(q) = f_1^r$ for $r = 4, 6, 8, 10, 14$, or $f_1^3 f_2^3$. For any $B(q)$ in the table for $A(q)$, one has that $A_{(0)} \subseteq B_{(0)}$.*

Proof. For $r = 4$, by Item (2) of Theorem 3.4, Item (1) of Lemma 3.5 and Item (1) of Lemma 3.7, it follows that $A_{(0)} \subseteq B_{(0)}$ for any eta quotients $B(q)$ in any entries but Entries 91, 97, 99 or 131 of Table 7. Combining this with Theorem 5.6 for those exceptional cases proves the general inclusive relation for f_1^4.

Similarly, the case of $r = 6$ follows from Item (3) of Theorem 3.4, Item (2) of Lemma 3.5, Item (2) of Lemma 3.7 and Theorems 5.17 and 5.18.

The case of $r = 8$ follows from Item (4) of Theorem 3.4, Item (3) of Lemma 3.5, Item (3) of Lemma 3.7 and Theorem 5.28.

The case of $r = 10$ follows from Item (5) of Theorem 3.4, Item (4) of Lemma 3.5, Item (4) of Lemma 3.7 and Theorems 5.33, 5.34 and 5.35.

The case of $r = 14$ follows from Item (6) of Theorem 3.4, Item (5) of Lemma 3.5, Item (5) of Lemma 3.7 and Theorem 5.40.

The case of $f_1^3 f_2^3$ follows from Item (8) of Theorem 3.4, Item (7) of Lemma 3.5, Item (7) of Lemma 3.7 and Theorem 5.49.

□

The following theorem addresses the vanishing of coefficients of the eta quotients in the table for f_1^{26}. It shows that all these eta quotients have coefficients that vanish similar to those of f_1^{26}.

Theorem 7.2. *Let* $B(q) = \sum_{n=0}^{\infty} b_n q^n$ *be any of the eta quotients in Table* 17. *Then if either of*

(1) $12n + 13$ *has a prime factor* $p_1 \equiv -1 \pmod 3$ *with odd exponent* *and* *a prime* $p_2 \equiv -1 \pmod 4$ *with odd exponent (it may be that* $p_1 = p_2$*),*

(2) $12n + 13$ *is a square and all prime factors* p *satisfy* $p \equiv -1 \pmod{12}$

holds, one has that $b_n = 0$.

Proof. This follows from Item (6) of Lemma 3.5, Item (6) of Lemma 3.7 and Theorem 6.5. □

Theorems 7.1 and 7.2 enable us to certify the lacunarity of the eta quotients obtained from our search.

Theorem 7.3. *Any eta quotient in the tables is lacunary.*

Proof. This follows from Theorems 7.1 and 7.2 and an extension of Landau's density theorem on quadratic forms by Serre [27, Section 3.1]. □

Remark 7.4. *It is noteworthy that the cases of eta quotients of weight 1 considered in Theorem* 7.3 *have been covered by a seminal result of Serre* [27, Theorem 4.2] *certifying that any holomorphic modular form of weight 1 is lacunary via Galois representations.*

Looking carefully at the exact double series for the eta quotients of weight 1 considered in Lemma 3.7, one notices that they can be all expressed as a linear combination of the generalized theta series

$$\sum_{m,n=-\infty}^{\infty} q^{(M_1 m+j)^2+(M_2 n+k)^2}$$

for some integers $M_1 > 0$ and $M_2 > 0$ and $0 \le j < M_1$ and $0 \le k < M_2$. This holds for exceptional cases in Theorem 5.34 and others. This observation prompts us to ask the following question.

Open Problem 7.5. *Is it true that any holomorphic modular form of weight 1 can be written as a linear combination of the generalized theta series*

$$\sum_{m,n=-\infty}^{\infty} q^{(M_1 m+j)^2+(M_2 n+k)^2}$$

for some integers $M_1 > 0$ and $M_2 > 0$ and $0 \le j < M_1$ and $0 \le k < M_2$?

7.2. Conjectures. This subsection contains several observations on the coefficients of the eta quotients obtained from our computational experiments in addition to Conjecture 1.4. As one shall see, while many special cases of these conjectures have been proved in the present work or [17, 18, 19], or the references therein, the uncharted cases remain vast. Therefore, we state these conjectures in their general forms but refer the interested reader to those previous work for proven cases.

Conjecture 7.6. *Let $\mathcal{X}$ and $\mathcal{Y}$ be any two groups connected by an arrow in a directed graph in this paper. Then for any eta quotient $A(q)$ in the group $\mathcal{X}$ and any eta quotient $B(q)$ in the group $\mathcal{Y}$, $A_{(0)} \subsetneq B_{(0)}$.*

Conjecture 7.7. *Let $\mathcal{X}$ be any group in any table. Then for any eta quotients $A(q)$ and $B(q)$ in the group $\mathcal{X}$, one has that $A_{(0)} = B_{(0)}$.*

Remark 7.8. *Theorem* 7.1 *is a special case of Conjecture* 7.6 *for $\mathcal{X} = I$ and $A(q) = f_1^r$ for $r \in \{4, 6, 8, 10, 14\}$ or $f_1^3 f_2^3$.*

8. Concluding Remarks

As is readily apparent, we have been just partially successful in proving all the results suggested by our experimental investigations regarding equality/inclusion of the sets of vanishing coefficients in the series expansion of lacunary eta quotients in the various families considered.

Ideally, for any eta quotient in one of the tables one would like to determine an exact criterion for the vanishing of the coefficients in its series expansion, as this would immediately let us group eta quotients with identically vanishing coefficients together. Likewise, this knowledge would immediately allow us to partially order the eta quotients in a table by inclusion of sets of vanishing coefficients, thus allowing us to affirm the structure of a table as indicated by its associated graph. However, as is apparent from the examples considered, it is not a straightforward matter to determine exactly the criterion for the vanishing of the coefficients in the series expansion of any particular lacunary eta quotient.

Instead, the method outlined in the previous paragraph has been supplemented by several other methods, which have afforded us some partial success in attempting to prove what we would like to prove. The methods used may be described as follows.

- In some cases, it is possible to determine an exact criterion for the vanishing of coefficients (as mentioned above), as in Lemma 5.4, which was used in the proof of Theorem 5.5.
- Basic hypergeometric series-product identities may be used, such as those in Lemma 4.1, that were employed in the proof of Theorem 4.2.
- Dissections of eta quotients may demonstrate that two eta quotients have identically vanishing coefficients, such as the dissections in Lemma 4.4 which were used in the proof of Theorem 4.6.
- Elementary dissections of eta quotients into their odd and even parts can be used to show strict inclusion of sets of vanishing coefficients, such as in the proof of Theorem 5.26, for example.
- In some instances, one can use elementary dilation results such as those stated in Lemma 5.1 and applied in the proof of Theorem 5.3, for example.
- One may be able to write the modular form derived from an eta quotient as a single sum theta series of the form
$$\sum_{X} h(X)q^{Q(X)},$$
where $Q(X)$ is quadratic in X such that integers with a certain property are not represented by $Q(X)$, leading to results of the type $A_{(0)} \subseteq B_{(0)}$ (see Theorem 5.21, for example).
- The modular form (of weight $k = 1$) derived from an eta quotient may sometimes be written as a double theta series of the form
$$\sum_{X,Y} h(X,Y)q^{Q(X,Y)},$$
where $Q(X,Y)$ is a binary quadratic form with the property that integers with a certain property are not represented by $Q(X,Y)$, leading to results of the type $A_{(0)} \subseteq B_{(0)}$ (see item (1) in Lemma 3.7 together with Table 7).
- The modular form (of weight $k = 1$) derived from an eta quotient may sometimes be written as a linear combination of double theta series of the form
$$\sum_{X,Y} h(X,Y)q^{Q(X,Y)}$$
described in the previous item, leading to the same type of $A_{(0)} \subseteq B_{(0)}$ result, as in Theorem 5.34.
- General inclusion results of the form $A_{(0)} \subseteq B_{(0)}$ may be applied, where the modular form derived from $B(q)$ has weight $k \geq 2$, as in Lemma 3.5.
- One may use general decomposition theorems for modular forms. For instance, one may use the decomposition of an eta quotient as the sum

of Eisenstein series and CM newforms to obtain an explicit formulation for the coefficients of the eta quotient. From this construction the vanishing of the coefficients can be analyzed through properties of divisor functions and CM newforms, as in Theorem 5.13.
- The use of ideal theory may be used to show inclusion results of the form $A_{(0)} \subseteq B_{(0)}$, as in Theorem 5.35.

In a companion paper [19] we use more sophisticated dissection techniques to prove some of the finer structure of the various tables and graphs, by showing that all of the eta quotients in some of the various groups do indeed have identically vanishing coefficients, and likewise that some of the inclusions between the groups, as indicated by the arrows in Figures 1 - 8 do indeed hold.

9. Appendix I: The Lacunarity of $f_1^3 f_2^3$

It was shown by Ono and Robins [25, page 1023 and Section 4] that $f_1^3 f_2^3$ is lacunary. In this appendix, using Serre's density theorem [27, Theorem 2.8] we give another proof of this fact, showing that the lacunarity of $f_1^3 f_2^3 = \sum_{n=0}^{\infty} a_n q^n$ follows from the criterion for $a_n = 0$ given in Theorem 5.44: $a_n = 0$ if and only if $8n + 3$ has a prime factor p congruent to 5 or 7 modulo 8, i.e., $\left(\frac{-2}{p}\right) = -1$, with odd exponent.

For starters, recall that our aim is to prove that

$$\lim_{X\to\infty} \frac{|\{n \le X \,:\, a_n = 0\}|}{X} = 1,$$

so equivalently, we want

$$\lim_{X\to\infty} \frac{|\{n \le X \,:\, a_n \neq 0\}|}{X} = 0.$$

On the other hand, by Theorem 5.44, one finds that

$$\begin{aligned}
&\lim_{X\to\infty} \frac{|\{n \le X \,:\, a_n \neq 0\}|}{X} \\
&= \lim_{X\to\infty} \frac{\left|\{n \le X \,:\, \text{any prime } p|(8n+3) \text{ and } \left(\frac{-2}{p}\right) = -1, \operatorname{ord}_p(8n+3) \text{ even}\}\right|}{X} \\
&= \lim_{X\to\infty} \frac{\left|\{n \le X \,:\, n \equiv 3 \pmod 8, \text{any prime } p|n \text{ and } \left(\frac{-2}{p}\right) = -1, \operatorname{ord}_p(n) \text{ even}\}\right|}{X},
\end{aligned}$$

and these amount to proving that

$$\lim_{X\to\infty} \frac{\left|\{n \le X \,:\, n \equiv 3 \pmod 8, \text{any prime } p|n \text{ and } \left(\frac{-2}{p}\right) = -1, \operatorname{ord}_p(n) \text{ even}\}\right|}{X} = 0.$$

Note that the set

$$\left\{n \le X \,:\, n \equiv 3 \pmod 8, \text{any prime } p|n \text{ and } \left(\frac{-2}{p}\right) = -1, \operatorname{ord}_p(n) \text{ even}\right\}$$

is a subset of

$$\left\{n \le X : \text{ any prime } p|n \text{ and } \left(\tfrac{-2}{p}\right) = -1, \operatorname{ord}_p(n) \text{ even}\right\},$$

so it suffices to prove that

$$\lim_{X\to\infty} \frac{\left|\{n \le X : \text{ any prime } p|n \text{ and } \left(\tfrac{-2}{p}\right) = -1, \operatorname{ord}_p(n) \text{ even}\}\right|}{X} = 0. \tag{9.1}$$

To this end, we can adopt Serre's density theorem [27, Theorem 2.8] which is to be enunciated as follows. Let P be a subset of primes of Frobenius density $0 < \alpha < 1$ (see [27, (1.4)] for definition), and let E be a subset of positive integers. We say that E is multiplicative if for any coprime positive integers n_1 and n_2,

$$n_1 n_2 \in E \quad \text{if and only if} \quad n_1 \in E \text{ or } n_2 \in E.$$

Then Serre's density theorem tells that

Theorem 9.1 (Serre)**.** *If E is multiplicative containing P, then*

$$\left|\{n \le X : n \notin E\}\right| = O\left(\frac{X}{(\log X)^\alpha}\right).$$

Now specialize Theorem 9.1 to the case of

$$E = \{n : p|n \text{ for some prime } p, \left(\tfrac{-2}{p}\right) = -1, \operatorname{ord}_p(n) \text{ odd}\}$$

and

$$P = \left\{p : \left(\frac{-2}{p}\right) = -1\right\}.$$

It is not hard to see that E is multiplicative and

$$\{n \le X : n \notin E\} = \{n \le X : \text{ any prime } p|n \text{ and } \left(\tfrac{-2}{p}\right) = -1, \operatorname{ord}_p(n) \text{ even}\}.$$

Also, P is contained in E, and by [27, (1), p. 20-02], P is of Frobenius density $\frac{1}{2}$. Therefore, by Theorem 9.1, one finds that

$$\left|\{n \le X : \text{ any prime } p|n \text{ and } \left(\tfrac{-2}{p}\right) = -1, \operatorname{ord}_p(n) \text{ even}\}\right| = \left|\{n \le X : n \notin E\}\right| = O\left(\frac{X}{(\log X)^{1/2}}\right),$$

from which the desired asymptotics (9.1) follows.

10. Appendix II: Tables and Graphs

Table 7: Eta quotients with vanishing behaviour similar to f_1^4

Number	q-Product	Modular Form	Weight	Level	Group
1	f_1^4	$\eta(6z)^4$	2	36	I
2	$\frac{f_2^{12}}{f_1^4f_4^4}$	$\frac{\eta(12z)^{12}}{\eta(6z)^4\eta(24z)^4}$	2	144	I
3	$\frac{f_2^9f_3^7f_{12}}{f_1^5f_4^3f_6^5}$	$\frac{\eta(12z)^9\eta(18z)^7\eta(72z)}{\eta(6z)^5\eta(24z)^3\eta(36z)^5}$	2	432	I
4	$\frac{f_1^5f_4^2f_6^{16}}{f_2^6f_3^7f_{12}^6}$	$\frac{\eta(6z)^5\eta(24z)^2\eta(36z)^{16}}{\eta(12z)^6\eta(18z)^7\eta(72z)^6}$	2	432	I
5	$\frac{f_2^{10}f_3^2}{f_1^4f_4^3f_6}$	$\frac{\eta(12z)^{10}\eta(18z)^2}{\eta(6z)^4\eta(24z)^3\eta(36z)}$	2	576	I
6	$\frac{f_1^4f_4f_6^5}{f_2^2f_3^2f_{12}^2}$	$\frac{\eta(6z)^4\eta(24z)\eta(36z)^5}{\eta(12z)^2\eta(18z)^2\eta(72z)^2}$	2	576	I
7	$\frac{f_2^2f_3^9}{f_1^3f_6^4}$	$\frac{\eta(12z)^2\eta(18z)^9}{\eta(6z)^3\eta(36z)^4}$	2	432	I
8	$\frac{f_1^3f_4^3f_6^{23}}{f_2^7f_3^9f_{12}^9}$	$\frac{\eta(6z)^3\eta(24z)^3\eta(36z)^{23}}{\eta(12z)^7\eta(18z)^9\eta(72z)^9}$	2	432	I
9	$\frac{f_2^6f_3f_6^4}{f_1^3f_4^2f_{12}^2}$	$\frac{\eta(12z)^6\eta(18z)\eta(36z)^4}{\eta(6z)^3\eta(24z)^2\eta(72z)^2}$	2	432	I
10	$\frac{f_1^3f_4f_6^7}{f_2^3f_3f_{12}^3}$	$\frac{\eta(6z)^3\eta(24z)\eta(36z)^7}{\eta(12z)^3\eta(18z)\eta(72z)^3}$	2	432	I
11	$\frac{f_2^8f_6^5}{f_1^3f_3f_4^3f_{12}^2}$	$\frac{\eta(12z)^8\eta(36z)^5}{\eta(6z)^3\eta(18z)\eta(24z)^3\eta(72z)^2}$	2	1728	I
12	$\frac{f_1^3f_3f_6^2}{f_2f_{12}}$	$\frac{\eta(6z)^3\eta(18z)\eta(36z)^2}{\eta(12z)\eta(72z)}$	2	1728	I
13	$\frac{f_2^{12}f_6}{f_1^3f_3f_4^5}$	$\frac{\eta(12z)^{12}\eta(36z)}{\eta(6z)^3\eta(18z)\eta(24z)^5}$	2	1728	I
14	$\frac{f_1^3f_2^3f_3f_{12}}{f_4^2f_6^2}$	$\frac{\eta(6z)^3\eta(12z)^3\eta(18z)\eta(72z)}{\eta(24z)^2\eta(36z)^2}$	2	1728	I
15	$\frac{f_4^{19}}{f_1^2f_2^3f_8^8}$	$\frac{\eta(24z)^{19}}{\eta(6z)^2\eta(12z)^3\eta(48z)^8}$	3	576	I
16	$\frac{f_1^2f_4^{21}}{f_2^9f_8^8}$	$\frac{\eta(6z)^2\eta(24z)^{21}}{\eta(12z)^9\eta(48z)^8}$	3	576	I
17	$\frac{f_2f_4^9}{f_1^2f_8^4}$	$\frac{\eta(12z)\eta(24z)^9}{\eta(6z)^2\eta(48z)^4}$	2	576	I
18	$\frac{f_1^2f_4^{11}}{f_2^5f_8^4}$	$\frac{\eta(6z)^2\eta(24z)^{11}}{\eta(12z)^5\eta(48z)^4}$	2	576	I
19	$\frac{f_2^3f_3^2}{f_1^2f_6}$	$\frac{\eta(12z)^3\eta(18z)^2}{\eta(6z)^2\eta(36z)}$	1	36	I
20	$\frac{f_1^2f_4^2f_6^5}{f_2^3f_3^2f_{12}^2}$	$\frac{\eta(6z)^2\eta(24z)^2\eta(36z)^5}{\eta(12z)^3\eta(18z)^2\eta(72z)^2}$	1	144	I
21	$\frac{f_2^6f_3^2}{f_1^2f_6^2}$	$\frac{\eta(12z)^6\eta(18z)^2}{\eta(6z)^2\eta(36z)^2}$	2	108	I
22	$\frac{f_1^2f_4^2f_6^4}{f_3^2f_{12}^2}$	$\frac{\eta(6z)^2\eta(24z)^2\eta(36z)^4}{\eta(18z)^2\eta(72z)^2}$	2	432	I

23	$\frac{f_2^7 f_6^5}{f_1^2 f_3^2 f_4^2 f_{12}^2}$	$\frac{\eta(12z)^7\eta(36z)^5}{\eta(6z)^2\eta(18z)^2\eta(24z)^2\eta(72z)^2}$	2	144	I
24	$\frac{f_1^2 f_2 f_3^2}{f_6}$	$\frac{\eta(6z)^2\eta(12z)\eta(18z)^2}{\eta(36z)}$	2	144	I
25	$\frac{f_2^9}{f_1^2 f_4^3}$	$\frac{\eta(12z)^9}{\eta(6z)^2\eta(24z)^3}$	2	576	I
26	$\frac{f_1^2 f_2^3}{f_4}$	$\frac{\eta(6z)^2\eta(12z)^3}{\eta(24z)}$	2	576	I
27	$\frac{f_3^3 f_4 f_6^5}{f_1 f_2 f_{12}^3}$	$\frac{\eta(18z)^3\eta(24z)\eta(36z)^5}{\eta(6z)\eta(12z)\eta(72z)^3}$	2	432	I
28	$\frac{f_1 f_4^2 f_6^{14}}{f_2^4 f_3^3 f_{12}^6}$	$\frac{\eta(6z)\eta(24z)^2\eta(36z)^{14}}{\eta(12z)^4\eta(18z)^3\eta(72z)^6}$	2	432	I
29	$\frac{f_2 f_6^{12}}{f_1 f_3^3 f_{12}^5}$	$\frac{\eta(12z)\eta(36z)^{12}}{\eta(6z)\eta(18z)^3\eta(72z)^5}$	2	1728	I
30	$\frac{f_1 f_3^3 f_4 f_6^3}{f_2^2 f_{12}^2}$	$\frac{\eta(6z)\eta(18z)^3\eta(24z)\eta(36z)^3}{\eta(12z)^2\eta(72z)^2}$	2	1728	I
31	$\frac{f_2 f_3^7}{f_1 f_6^3}$	$\frac{\eta(12z)\eta(18z)^7}{\eta(6z)\eta(36z)^3}$	2	108	I
32	$\frac{f_1 f_4 f_6^{18}}{f_2^2 f_3^7 f_{12}^7}$	$\frac{\eta(6z)\eta(24z)\eta(36z)^{18}}{\eta(12z)^2\eta(18z)^7\eta(72z)^7}$	2	432	I
33	$\frac{f_2^3 f_3^5}{f_1 f_4 f_6^2}$	$\frac{\eta(12z)^3\eta(18z)^5}{\eta(6z)\eta(24z)\eta(36z)^2}$	2	1728	I
34	$\frac{f_1 f_6^{13}}{f_3^5 f_{12}^5}$	$\frac{\eta(6z)\eta(36z)^{13}}{\eta(18z)^5\eta(72z)^5}$	2	1728	I
35	$\frac{f_2^5 f_6^8}{f_1 f_3^3 f_4^2 f_{12}^3}$	$\frac{\eta(12z)^5\eta(36z)^8}{\eta(6z)\eta(18z)^3\eta(24z)^2\eta(72z)^3}$	2	1728	I
36	$\frac{f_1 f_2^2 f_3^3}{f_4 f_6}$	$\frac{\eta(6z)\eta(12z)^2\eta(18z)^3}{\eta(24z)\eta(36z)}$	2	1728	I
37	$\frac{f_2^7 f_3^3 f_{12}}{f_1 f_4^3 f_6^3}$	$\frac{\eta(12z)^7\eta(18z)^3\eta(72z)}{\eta(6z)\eta(24z)^3\eta(36z)^3}$	2	432	I
38	$\frac{f_1 f_2^4 f_6^6}{f_3^3 f_4^2 f_{12}^2}$	$\frac{\eta(6z)\eta(12z)^4\eta(36z)^6}{\eta(18z)^3\eta(24z)^2\eta(72z)^2}$	2	432	I
39	$\frac{f_1 f_3^5}{f_6^2}$	$\frac{\eta(6z)\eta(18z)^5}{\eta(36z)^2}$	2	432	I
40	$\frac{f_2^3 f_6^{13}}{f_1 f_3^5 f_4 f_{12}^5}$	$\frac{\eta(12z)^3\eta(36z)^{13}}{\eta(6z)\eta(18z)^5\eta(24z)\eta(72z)^5}$	2	432	I
41	$\frac{f_1^2 f_2^7}{f_4^3}$	$\frac{\eta(6z)^2\eta(12z)^7}{\eta(24z)^3}$	3	576	I
42	$\frac{f_2^{13}}{f_1^2 f_4^5}$	$\frac{\eta(12z)^{13}}{\eta(6z)^2\eta(24z)^5}$	3	576	I
43	$\frac{f_1^3 f_2^5 f_{12}}{f_3 f_4^3 f_6}$	$\frac{\eta(6z)^3\eta(12z)^5\eta(72z)}{\eta(18z)\eta(24z)^3\eta(36z)}$	2	432	I
44	$\frac{f_2^{14} f_3 f_{12}^2}{f_1^3 f_4^6 f_6^4}$	$\frac{\eta(12z)^{14}\eta(18z)\eta(72z)^2}{\eta(6z)^3\eta(24z)^6\eta(36z)^4}$	2	432	I
45	$\frac{f_1^5 f_6^{13}}{f_2^2 f_3^5 f_{12}^5}$	$\frac{\eta(6z)^5\eta(36z)^{13}}{\eta(12z)^2\eta(18z)^5\eta(72z)^5}$	3	1728	I
46	$\frac{f_2^{13} f_3^5}{f_1^5 f_4^5 f_6^2}$	$\frac{\eta(12z)^{13}\eta(18z)^5}{\eta(6z)^5\eta(24z)^5\eta(36z)^2}$	3	1728	I

47	$\frac{f_1^5f_6^3}{f_2^2f_3f_{12}}$	$\frac{\eta(6z)^5\eta(36z)^3}{\eta(12z)^2\eta(18z)\eta(72z)}$	2	1728	I
48	$\frac{f_2^{13}f_3}{f_1^5f_4^5}$	$\frac{\eta(12z)^{13}\eta(18z)}{\eta(6z)^5\eta(24z)^5}$	2	1728	I
49	$\frac{f_1^5f_3}{f_2^2}$	$\frac{\eta(6z)^5\eta(18z)}{\eta(12z)^2}$	2	432	I
50	$\frac{f_2^{13}f_6^3}{f_1^5f_3f_4^5f_{12}}$	$\frac{\eta(12z)^{13}\eta(36z)^3}{\eta(6z)^5\eta(18z)\eta(24z)^5\eta(72z)}$	2	432	I
51	$\frac{f_1^5f_3^5}{f_2^2f_6^2}$	$\frac{\eta(6z)^5\eta(18z)^5}{\eta(12z)^2\eta(36z)^2}$	3	108	I
52	$\frac{f_2^{13}f_6^{13}}{f_1^5f_3^5f_4^5f_{12}^5}$	$\frac{\eta(12z)^{13}\eta(36z)^{13}}{\eta(6z)^5\eta(18z)^5\eta(24z)^5\eta(72z)^5}$	3	432	I
53	$\frac{f_1^6f_4}{f_2^3}$	$\frac{\eta(6z)^6\eta(24z)}{\eta(12z)^3}$	2	576	I
54	$\frac{f_2^{15}}{f_1^6f_4^5}$	$\frac{\eta(12z)^{15}}{\eta(6z)^6\eta(24z)^5}$	2	576	I
55	$\frac{f_1^7f_4f_6^9}{f_2^5f_3^5f_{12}^3}$	$\frac{\eta(6z)^7\eta(24z)\eta(36z)^9}{\eta(12z)^5\eta(18z)^5\eta(72z)^3}$	2	432	I
56	$\frac{f_2^{16}f_3^5f_{12}^2}{f_1^7f_4^6f_6^6}$	$\frac{\eta(12z)^{16}\eta(18z)^5\eta(72z)^2}{\eta(6z)^7\eta(24z)^6\eta(36z)^6}$	2	432	I
57	$\frac{f_1^7f_6}{f_2^3f_3}$	$\frac{\eta(6z)^7\eta(36z)}{\eta(12z)^3\eta(18z)}$	2	108	I
58	$\frac{f_2^{18}f_3f_{12}}{f_1^7f_4^7f_6^2}$	$\frac{\eta(12z)^{18}\eta(18z)\eta(72z)}{\eta(6z)^7\eta(24z)^7\eta(36z)^2}$	2	432	I
59	$\frac{f_1^8}{f_2^2}$	$\frac{\eta(6z)^8}{\eta(12z)^2}$	3	144	I
60	$\frac{f_2^{22}}{f_1^8f_4^8}$	$\frac{\eta(12z)^{22}}{\eta(6z)^8\eta(24z)^8}$	3	144	I
61	$\frac{f_1^9f_6^2}{f_2^4f_3^3}$	$\frac{\eta(6z)^9\eta(36z)^2}{\eta(12z)^4\eta(18z)^3}$	2	432	I
62	$\frac{f_2^{23}f_3^3f_{12}^3}{f_1^9f_4^9f_6^7}$	$\frac{\eta(12z)^{23}\eta(18z)^3\eta(72z)^3}{\eta(6z)^9\eta(24z)^9\eta(36z)^7}$	2	432	I
63	$\frac{f_1^{10}}{f_3^2}$	$\frac{\eta(6z)^{10}}{\eta(18z)^2}$	4	108	I
64	$\frac{f_2^{30}f_3^2f_{12}^2}{f_1^{10}f_4^{10}f_6^6}$	$\frac{\eta(12z)^{30}\eta(18z)^2\eta(72z)^2}{\eta(6z)^{10}\eta(24z)^{10}\eta(36z)^6}$	4	432	I
65	$\frac{f_1^3f_3f_4^7f_{24}}{f_2^3f_8^3f_{12}^2}$	$\frac{\eta(6z)^3\eta(18z)\eta(24z)^7\eta(144z)}{\eta(12z)^3\eta(48z)^3\eta(72z)^2}$	2	1728	I
66	$\frac{f_2^6f_4^4f_6^3f_{24}}{f_1^3f_3f_8^3f_{12}^3}$	$\frac{\eta(12z)^6\eta(24z)^4\eta(36z)^3\eta(144z)}{\eta(6z)^3\eta(18z)\eta(48z)^3\eta(72z)^3}$	2	1728	I
67	$\frac{f_1f_3^3f_4^8f_6f_{24}}{f_2^4f_8^3f_{12}^3}$	$\frac{\eta(6z)\eta(18z)^3\eta(24z)^8\eta(36z)\eta(144z)}{\eta(12z)^4\eta(48z)^3\eta(72z)^3}$	2	1728	I
68	$\frac{f_4^7f_6^{10}f_{24}}{f_1f_2f_3^3f_8^3f_{12}^6}$	$\frac{\eta(24z)^7\eta(36z)^{10}\eta(144z)}{\eta(6z)\eta(12z)\eta(18z)^3\eta(48z)^3\eta(72z)^6}$	2	1728	I
69	$\frac{f_1^3f_2f_3f_8f_{12}^8}{f_4^3f_6^4f_{24}^3}$	$\frac{\eta(6z)^3\eta(12z)\eta(18z)\eta(48z)\eta(72z)^8}{\eta(24z)^3\eta(36z)^4\eta(144z)^3}$	2	1728	I
70	$\frac{f_2^{10}f_8f_{12}^7}{f_1^3f_3f_4^6f_6f_{24}^3}$	$\frac{\eta(12z)^{10}\eta(48z)\eta(72z)^7}{\eta(6z)^3\eta(18z)\eta(24z)^6\eta(36z)\eta(144z)^3}$	2	1728	I

71	$\frac{f_1f_3^3f_8f_{12}^7}{f_4^2f_6^3f_{24}^3}$	$\frac{\eta(6z)\eta(18z)^3\eta(48z)\eta(72z)^7}{\eta(24z)^2\eta(36z)^3\eta(144z)^3}$	2	1728	I
72	$\frac{f_2^3f_6^6f_8f_{12}^4}{f_1f_3^3f_4^3f_{24}^3}$	$\frac{\eta(12z)^3\eta(36z)^6\eta(48z)\eta(72z)^4}{\eta(6z)\eta(18z)^3\eta(24z)^3\eta(144z)^3}$	2	1728	I
73	$\frac{f_2^9f_6^2f_8^3}{f_1^3f_3f_4^5f_{24}}$	$\frac{\eta(12z)^9\eta(36z)^2\eta(48z)^3}{\eta(6z)^3\eta(18z)\eta(24z)^5\eta(144z)}$	2	1728	II
74	$\frac{f_1^3f_3f_8^3f_{12}}{f_4^2f_6f_{24}}$	$\frac{\eta(6z)^3\eta(18z)\eta(48z)^3\eta(72z)}{\eta(24z)^2\eta(36z)\eta(144z)}$	2	1728	II
75	$\frac{f_2^2f_6^9f_8^3}{f_1f_3^3f_4^2f_{12}^3f_{24}}$	$\frac{\eta(12z)^2\eta(36z)^9\eta(48z)^3}{\eta(6z)\eta(18z)^3\eta(24z)^2\eta(72z)^3\eta(144z)}$	2	1728	II
76	$\frac{f_1f_3^3f_8^3}{f_2f_4f_{24}}$	$\frac{\eta(6z)\eta(18z)^3\eta(48z)^3}{\eta(12z)\eta(24z)\eta(144z)}$	2	1728	II
77	f_1f_3	$\eta(6z)\eta(18z)$	1	108	III
78	$\frac{f_2^3f_6^3}{f_1f_3f_4f_{12}}$	$\frac{\eta(12z)^3\eta(36z)^3}{\eta(6z)\eta(18z)\eta(24z)\eta(72z)}$	1	432	III
79	$\frac{f_2^{14}f_3^2f_{12}}{f_1^6f_4^3f_6^4}$	$\frac{\eta(12z)^{14}\eta(18z)^2\eta(72z)}{\eta(6z)^6\eta(24z)^3\eta(36z)^4}$	2	432	IV
80	$\frac{f_1^6f_4^3f_6^2}{f_2^4f_3^2f_{12}}$	$\frac{\eta(6z)^6\eta(24z)^3\eta(36z)^2}{\eta(12z)^4\eta(18z)^2\eta(72z)}$	2	432	IV
81	$\frac{f_1^2f_3^2f_4^3}{f_2^2f_{12}}$	$\frac{\eta(6z)^2\eta(18z)^2\eta(24z)^3}{\eta(12z)^2\eta(72z)}$	2	432	IV
82	$\frac{f_2^4f_4f_6^6}{f_1^2f_3^2f_{12}^3}$	$\frac{\eta(12z)^4\eta(24z)\eta(36z)^6}{\eta(6z)^2\eta(18z)^2\eta(72z)^3}$	2	432	IV
83	$\frac{f_1^4f_4^4f_6^9}{f_2^5f_3^4f_{12}^4}$	$\frac{\eta(6z)^4\eta(24z)^4\eta(36z)^9}{\eta(12z)^5\eta(18z)^4\eta(72z)^4}$	2	432	IV
84	$\frac{f_2^7f_3^4}{f_1^4f_6^3}$	$\frac{\eta(12z)^7\eta(18z)^4}{\eta(6z)^4\eta(36z)^3}$	2	432	IV
85	$\frac{f_1^2f_4^5f_6^{16}}{f_2^6f_3^6f_{12}^7}$	$\frac{\eta(6z)^2\eta(24z)^5\eta(36z)^{16}}{\eta(12z)^6\eta(18z)^6\eta(72z)^7}$	2	432	V
86	$\frac{f_3^6f_4^3}{f_1^2f_6^2f_{12}}$	$\frac{\eta(18z)^6\eta(24z)^3}{\eta(6z)^2\eta(36z)^2\eta(72z)}$	2	432	V
87	$\frac{f_1f_4^7f_6^9}{f_2^5f_3^3f_{12}^5}$	$\frac{\eta(6z)\eta(24z)^7\eta(36z)^9}{\eta(12z)^5\eta(18z)^3\eta(72z)^5}$	2	432	VI
88	$\frac{f_3^3f_4^6}{f_1f_2^2f_{12}^2}$	$\frac{\eta(18z)^3\eta(24z)^6}{\eta(6z)\eta(12z)^2\eta(72z)^2}$	2	432	VI
89	$\frac{f_1^3f_4^6f_6^2}{f_2^4f_3f_{12}^2}$	$\frac{\eta(6z)^3\eta(24z)^6\eta(36z)^2}{\eta(12z)^4\eta(18z)\eta(72z)^2}$	2	432	VI
90	$\frac{f_2^5f_3f_4^3}{f_1^3f_6f_{12}}$	$\frac{\eta(12z)^5\eta(18z)\eta(24z)^3}{\eta(6z)^3\eta(36z)\eta(72z)}$	2	432	VI
91	$\frac{f_2f_3f_4^2f_{12}^3}{f_1f_6^2f_8f_{24}}$	$\frac{\eta(12z)\eta(18z)\eta(24z)^2\eta(72z)^3}{\eta(6z)\eta(36z)^2\eta(48z)\eta(144z)}$	1	1728	VII
92	$\frac{f_1f_4^3f_6f_{12}^2}{f_2^2f_3f_8f_{24}}$	$\frac{\eta(6z)\eta(24z)^3\eta(36z)\eta(72z)^2}{\eta(12z)^2\eta(18z)\eta(48z)\eta(144z)}$	1	1728	VII
93	$\frac{f_1f_6^3}{f_3f_{12}}$	$\frac{\eta(6z)\eta(36z)^3}{\eta(18z)\eta(72z)}$	1	1728	VII
94	$\frac{f_2^3f_3}{f_1f_4}$	$\frac{\eta(12z)^3\eta(18z)}{\eta(6z)\eta(24z)}$	1	1728	VII
95	$\frac{f_1^3f_4f_6^6}{f_2^3f_3^3f_{12}^2}$	$\frac{\eta(6z)^3\eta(24z)\eta(36z)^6}{\eta(12z)^3\eta(18z)^3\eta(72z)^2}$	1	1728	VII

96	$\frac{f_2^6f_3^3f_{12}}{f_1^3f_4^2f_6^3}$	$\frac{\eta(12z)^6\eta(18z)^3\eta(72z)}{\eta(6z)^3\eta(24z)^2\eta(36z)^3}$	1	1728	VII
97	$\frac{f_2^3f_3f_8f_{12}^8}{f_1f_4^3f_6^4f_{24}^3}$	$\frac{\eta(12z)^3\eta(18z)\eta(48z)\eta(72z)^8}{\eta(6z)\eta(24z)^3\eta(36z)^4\eta(144z)^3}$	1	1728	VIII
98	$\frac{f_1f_8f_{12}^7}{f_3f_4^2f_6f_{24}^3}$	$\frac{\eta(6z)\eta(48z)\eta(72z)^7}{\eta(18z)\eta(24z)^2\eta(36z)\eta(144z)^3}$	1	1728	VIII
99	$\frac{f_1f_4^8f_6^3f_{24}}{f_2^4f_3f_8^3f_{12}^3}$	$\frac{\eta(6z)\eta(24z)^8\eta(36z)^3\eta(144z)}{\eta(12z)^4\eta(18z)\eta(48z)^3\eta(72z)^3}$	1	1728	VIII
100	$\frac{f_3f_4^7f_{24}}{f_1f_2f_8^3f_{12}^2}$	$\frac{\eta(18z)\eta(24z)^7\eta(144z)}{\eta(6z)\eta(12z)\eta(48z)^3\eta(72z)^2}$	1	1728	VIII
101	$\frac{f_1f_2^2f_6}{f_3f_4}$	$\frac{\eta(6z)\eta(12z)^2\eta(36z)}{\eta(18z)\eta(24z)}$	1	1728	VIII
102	$\frac{f_2^5f_3f_{12}}{f_1f_4^2f_6^2}$	$\frac{\eta(12z)^5\eta(18z)\eta(72z)}{\eta(6z)\eta(24z)^2\eta(36z)^2}$	1	1728	VIII
103	$\frac{f_1f_4f_6^5}{f_2^2f_3f_{12}^2}$	$\frac{\eta(6z)\eta(24z)\eta(36z)^5}{\eta(12z)^2\eta(18z)\eta(72z)^2}$	1	1728	VIII
104	$\frac{f_2f_3f_6^2}{f_1f_{12}}$	$\frac{\eta(12z)\eta(18z)\eta(36z)^2}{\eta(6z)\eta(72z)}$	1	1728	VIII
105	$\frac{f_1f_4f_6^8}{f_2^2f_3^3f_{12}^3}$	$\frac{\eta(6z)\eta(24z)\eta(36z)^8}{\eta(12z)^2\eta(18z)^3\eta(72z)^3}$	1	432	IX
106	$\frac{f_2f_3^3}{f_1f_6}$	$\frac{\eta(12z)\eta(18z)^3}{\eta(6z)\eta(36z)}$	1	432	IX
107	$\frac{f_1^3f_6}{f_2f_3}$	$\frac{\eta(6z)^3\eta(36z)}{\eta(12z)\eta(18z)}$	1	432	IX
108	$\frac{f_2^8f_3f_{12}}{f_1^3f_4^3f_6^2}$	$\frac{\eta(12z)^8\eta(18z)\eta(72z)}{\eta(6z)^3\eta(24z)^3\eta(36z)^2}$	1	432	IX
109	$\frac{f_1^2f_8^{10}}{f_2f_4^3f_{16}^4}$	$\frac{\eta(6z)^2\eta(48z)^{10}}{\eta(12z)\eta(24z)^3\eta(96z)^4}$	2	576	X
110	$\frac{f_2^5f_8^{10}}{f_1^2f_4^5f_{16}^4}$	$\frac{\eta(12z)^5\eta(48z)^{10}}{\eta(6z)^2\eta(24z)^5\eta(96z)^4}$	2	576	X
111	$\frac{f_1f_4^5f_6^2}{f_2f_3f_{12}^2}$	$\frac{\eta(6z)\eta(24z)^5\eta(36z)^2}{\eta(12z)\eta(18z)\eta(72z)^2}$	2	1728	X
112	$\frac{f_2^2f_3f_4^4}{f_1f_6f_{12}}$	$\frac{\eta(12z)^2\eta(18z)\eta(24z)^4}{\eta(6z)\eta(36z)\eta(72z)}$	2	1728	X
113	$\frac{f_2f_8^{14}f_{12}^2}{f_4^6f_6f_{16}^5f_{24}}$	$\frac{\eta(12z)\eta(48z)^{14}\eta(72z)^2}{\eta(24z)^6\eta(36z)\eta(96z)^5\eta(144z)}$	2	6912	XI
114	$\frac{f_6f_8^{13}}{f_2f_4^3f_{12}f_{16}^5}$	$\frac{\eta(36z)\eta(48z)^{13}}{\eta(12z)\eta(24z)^3\eta(72z)\eta(96z)^5}$	2	6912	XI
115	$\frac{f_2^2f_8f_{12}^2}{f_4^2f_{24}}$	$\frac{\eta(12z)^2\eta(48z)\eta(72z)^2}{\eta(24z)^2\eta(144z)}$	1	576	XI
116	$\frac{f_8^{11}}{f_2^2f_{16}^5}$	$\frac{\eta(48z)^{11}}{\eta(12z)^2\eta(96z)^5}$	2	2304	XI
117	$\frac{f_4^4f_{12}^2}{f_2^2f_8f_{24}}$	$\frac{\eta(24z)^4\eta(72z)^2}{\eta(12z)^2\eta(48z)\eta(144z)}$	1	144	XI
118	$\frac{f_2^2f_8^{13}}{f_4^6f_{16}^5}$	$\frac{\eta(12z)^2\eta(48z)^{13}}{\eta(24z)^6\eta(96z)^5}$	2	2304	XI
119	$\frac{f_4^{15}f_6f_{24}}{f_2^5f_8^5f_{12}^3}$	$\frac{\eta(24z)^{15}\eta(36z)\eta(144z)}{\eta(12z)^5\eta(48z)^5\eta(72z)^3}$	2	1728	XI
120	$\frac{f_2^5}{f_6}$	$\frac{\eta(12z)^5}{\eta(36z)}$	2	432	XI

121	$\frac{f_2^2 f_4^4}{f_8^2}$	$\frac{\eta(12z)^2\eta(24z)^4}{\eta(48z)^2}$	2	576	XI
122	$\frac{f_2 f_4^4 f_{12}^2}{f_6 f_8 f_{24}}$	$\frac{\eta(12z)\eta(24z)^4\eta(72z)^2}{\eta(36z)\eta(48z)\eta(144z)}$	2	432	XI
123	$\frac{f_4^7 f_6}{f_2 f_8^2 f_{12}}$	$\frac{\eta(24z)^7\eta(36z)}{\eta(12z)\eta(48z)^2\eta(72z)}$	2	1728	XI
124	$\frac{f_4^{10}}{f_2^2 f_8^4}$	$\frac{\eta(24z)^{10}}{\eta(12z)^2\eta(48z)^4}$	2	144	XI
125	$\frac{f_2^3 f_8^3 f_{12}^{17}}{f_4^5 f_6^7 f_{24}^7}$	$\frac{\eta(12z)^3\eta(48z)^3\eta(72z)^{17}}{\eta(24z)^5\eta(36z)^7\eta(144z)^7}$	2	1728	XI
126	$\frac{f_4^4 f_6^7}{f_2^3 f_{12}^4}$	$\frac{\eta(24z)^4\eta(36z)^7}{\eta(12z)^3\eta(72z)^4}$	2	432	XI
127	$\frac{f_2^4 f_8^4 f_{12}^3}{f_4^3 f_6^2 f_{24}^2}$	$\frac{\eta(12z)^4\eta(48z)^4\eta(72z)^3}{\eta(24z)^3\eta(36z)^2\eta(144z)^2}$	2	1728	XII
128	$\frac{f_4^9 f_6^2}{f_2^4 f_{12}^3}$	$\frac{\eta(24z)^9\eta(36z)^2}{\eta(12z)^4\eta(72z)^3}$	2	432	XII
129	$\frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2}$	$\frac{\eta(12z)\eta(48z)^2\eta(72z)^4}{\eta(24z)^2\eta(36z)\eta(144z)^2}$	1	432	XIII
130	$\frac{f_4 f_6 f_8 f_{12}}{f_2 f_{24}}$	$\frac{\eta(24z)\eta(36z)\eta(48z)\eta(72z)}{\eta(12z)\eta(144z)}$	1	1728	XIII
131	$\frac{f_2^2 f_3 f_8^3 f_{12}}{f_1 f_4^2 f_6 f_{24}}$	$\frac{\eta(12z)^2\eta(18z)\eta(48z)^3\eta(72z)}{\eta(6z)\eta(24z)^2\eta(36z)\eta(144z)}$	1	1728	XIV
132	$\frac{f_1 f_6^2 f_8^3}{f_2 f_3 f_4 f_{24}}$	$\frac{\eta(6z)\eta(36z)^2\eta(48z)^3}{\eta(12z)\eta(18z)\eta(24z)\eta(144z)}$	1	1728	XIV
133	$\frac{f_2 f_8^4 f_{12}^2}{f_4^2 f_6 f_{16} f_{24}}$	$\frac{\eta(12z)\eta(48z)^4\eta(72z)^2}{\eta(24z)^2\eta(36z)\eta(96z)\eta(144z)}$	1	6912	XV
134	$\frac{f_4 f_6 f_8^3}{f_2 f_{12} f_{16}}$	$\frac{\eta(24z)\eta(36z)\eta(48z)^3}{\eta(12z)\eta(72z)\eta(96z)}$	1	6912	XV
135	$\frac{f_2 f_8 f_{12}^2}{f_6 f_{24}}$	$\frac{\eta(12z)\eta(48z)\eta(72z)^2}{\eta(36z)\eta(144z)}$	1	1728	XV
136	$\frac{f_4^3 f_6}{f_2 f_{12}}$	$\frac{\eta(24z)^3\eta(36z)}{\eta(12z)\eta(72z)}$	1	432	XV
137	$\frac{f_1^2 f_4}{f_2}$	$\frac{\eta(6z)^2\eta(24z)}{\eta(12z)}$	1	576	XVI
138	$\frac{f_2^5}{f_1^2 f_4}$	$\frac{\eta(12z)^5}{\eta(6z)^2\eta(24z)}$	1	576	XVI
139	$\frac{f_4^4 f_8}{f_2^2 f_{16}}$	$\frac{\eta(24z)^4\eta(48z)}{\eta(12z)^2\eta(96z)}$	1	2304	XVII
140	$\frac{f_2^2 f_8^3}{f_4^2 f_{16}}$	$\frac{\eta(12z)^2\eta(48z)^3}{\eta(24z)^2\eta(96z)}$	1	2304	XVII
141	f_2^2	$\eta(12z)^2$	1	144	XVII
142	$\frac{f_4^6}{f_2^2 f_8^2}$	$\frac{\eta(24z)^6}{\eta(12z)^2\eta(48z)^2}$	1	576	XVII
143	$\frac{f_3^2 f_4}{f_6}$	$\frac{\eta(18z)^2\eta(24z)}{\eta(36z)}$	1	576	XVIII
144	$\frac{f_4 f_6^5}{f_3^2 f_{12}^2}$	$\frac{\eta(24z)\eta(36z)^5}{\eta(18z)^2\eta(72z)^2}$	1	576	XVIII
145	$\frac{f_4 f_{16} f_{24}^5}{f_8^2 f_{12}^2 f_{48}^2}$	$\frac{\eta(24z)\eta(96z)\eta(144z)^5}{\eta(48z)^2\eta(72z)^2\eta(288z)^2}$	$\frac{1}{2}$	2304	XIX

146	$\frac{f_8 f_{12}^2}{f_4 f_{24}}$	$\frac{\eta(48z)\eta(72z)^2}{\eta(24z)\eta(144z)}$	$\frac{1}{2}$	144	XIX
147	$\frac{f_8^{13}}{f_4^5 f_{16}^5}$	$\frac{\eta(48z)^{13}}{\eta(24z)^5\eta(96z)^5}$	$\frac{3}{2}$	2304	XIX
148	$\frac{f_4^5}{f_8^2}$	$\frac{\eta(24z)^5}{\eta(48z)^2}$	$\frac{3}{2}$	144	XIX
149	$\frac{f_8^3}{f_4 f_{16}}$	$\frac{\eta(48z)^3}{\eta(24z)\eta(96z)}$	$\frac{1}{2}$	2304	XIX
150	f_4	$\eta(24z)$	$\frac{1}{2}$	576	XIX

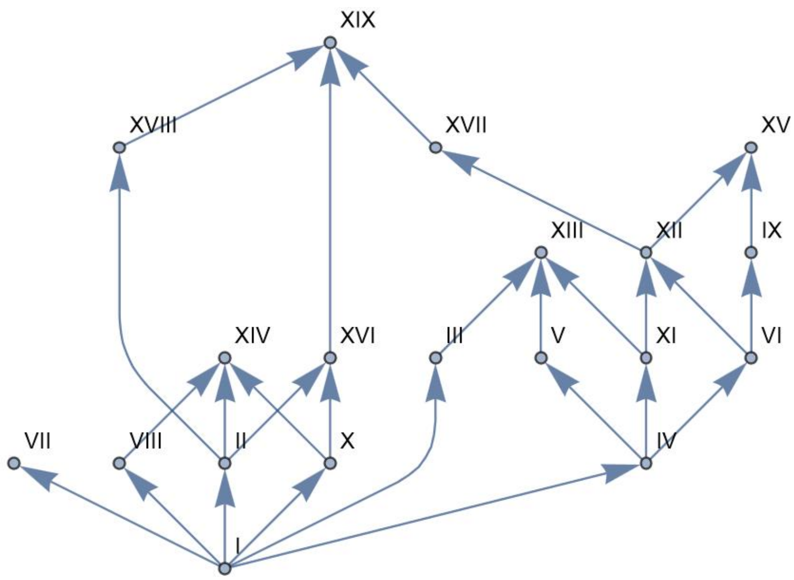

FIGURE 2. The grouping of eta quotients in Table 7, which have vanishing coefficient behaviour similar to f_1^4

Table 8: Eta quotients in Table 7 with expansions as double theta series

Number	Modular Form	Weight	Theta Series
1 I	$\eta(6z)^4$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{-4}{n}\right)q^{\frac{1}{4}(m^2+3n^2)}$
1 I	$\eta(6z)^4$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{2}\right)^2\left(\frac{n}{12}\right)q^{\frac{1}{4}(3m^2+n^2)}$
2 I	$\frac{\eta(12z)^{12}}{\eta(6z)^4\eta(24z)^4}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{24}{m}\right)\left(\frac{-8}{n}\right)q^{\frac{1}{4}(m^2+3n^2)}$
2 I	$\frac{\eta(12z)^{12}}{\eta(6z)^4\eta(24z)^4}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{8}{m}\right)\left(\frac{-6}{n}\right)q^{\frac{1}{4}(3m^2+n^2)}$
5 I	$\frac{\eta(12z)^{10}\eta(18z)^2}{\eta(6z)^4\eta(24z)^3\eta(36z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{6}\right)^2\left(\frac{-8}{n}\right)q^{\frac{1}{4}(m^2+3n^2)}$
6 I	$\frac{\eta(6z)^4\eta(24z)\eta(36z)^5}{\eta(12z)^2\eta(18z)^2\eta(72z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{18}{m}\right)\left(\frac{-4}{n}\right)q^{\frac{1}{4}(m^2+3n^2)}$
13 I	$\frac{\eta(12z)^{12}\eta(36z)}{\eta(6z)^3\eta(18z)\eta(24z)^5}$	2	$\sum_{m,n=1}^{\infty} m\left(\frac{-6}{m}\right)\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{\frac{1}{4}(m^2+3n^2)}$

19 I	$\frac{\eta(12z)^3\eta(18z)^2}{\eta(6z)^2\eta(36z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{m}{2}\right)^2\left(\frac{n}{6}\right)^2 q^{\frac{1}{4}(3m^2+n^2)}$
20 I	$\frac{\eta(6z)^2\eta(24z)^2\eta(36z)^5}{\eta(12z)^3\eta(18z)^2\eta(72z)^2}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{8}{m}\right)\left(\frac{18}{n}\right) q^{\frac{1}{4}(3m^2+n^2)}$
23 I	$\frac{\eta(12z)^7\eta(36z)^5}{\eta(6z)^2\eta(18z)^2\eta(24z)^2\eta(72z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{18}{m}\right)\left(\frac{-8}{n}\right) q^{\frac{1}{4}(m^2+3n^2)}$
24 I	$\frac{\eta(6z)^2\eta(12z)\eta(18z)^2}{\eta(36z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{6}\right)^2\left(\frac{-4}{n}\right) q^{\frac{1}{4}(m^2+3n^2)}$
25 I	$\frac{\eta(12z)^9}{\eta(6z)^2\eta(24z)^3}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{-8}{n}\right) q^{\frac{1}{4}(m^2+3n^2)}$
26 I	$\frac{\eta(6z)^2\eta(12z)^3}{\eta(24z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{24}{m}\right)\left(\frac{-4}{n}\right) q^{\frac{1}{4}(m^2+3n^2)}$
29 I	$\frac{\eta(12z)\eta(36z)^{12}}{\eta(6z)\eta(18z)^3\eta(72z)^5}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{6}\right)^2\left(\frac{-6}{n}\right) q^{\frac{1}{4}(m^2+3n^2)}$
30 I	$\frac{\eta(6z)\eta(18z)^3\eta(24z)\eta(36z)^3}{\eta(12z)^2\eta(72z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{18}{m}\right)\left(\frac{n}{12}\right) q^{\frac{1}{4}(m^2+3n^2)}$
31 I	$\frac{\eta(12z)\eta(18z)^7}{\eta(6z)\eta(36z)^3}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{6}\right)^2\left(\frac{n}{12}\right) q^{\frac{1}{4}(m^2+3n^2)}$
32 I	$\frac{\eta(6z)\eta(24z)\eta(36z)^{18}}{\eta(12z)^2\eta(18z)^7\eta(72z)^7}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{18}{m}\right)\left(\frac{-6}{n}\right) q^{\frac{1}{4}(m^2+3n^2)}$
33 I	$\frac{\eta(12z)^3\eta(18z)^5}{\eta(6z)\eta(24z)\eta(36z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{24}{m}\right)\left(\frac{n}{12}\right) q^{\frac{1}{4}(m^2+3n^2)}$
34 I	$\frac{\eta(6z)\eta(36z)^{13}}{\eta(18z)^5\eta(72z)^5}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{-6}{n}\right) q^{\frac{1}{4}(m^2+3n^2)}$
39 I	$\frac{\eta(6z)\eta(18z)^5}{\eta(36z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{n}{12}\right) q^{\frac{1}{4}(m^2+3n^2)}$
40 I	$\frac{\eta(12z)^3\eta(36z)^{13}}{\eta(6z)\eta(18z)^5\eta(24z)\eta(72z)^5}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{24}{m}\right)\left(\frac{-6}{n}\right) q^{\frac{1}{4}(m^2+3n^2)}$
41 I	$\frac{\eta(6z)^2\eta(12z)^7}{\eta(24z)^3}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-8}{m}\right)\left(\frac{n}{12}\right) q^{\frac{1}{4}(3m^2+n^2)}$
42 I	$\frac{\eta(12z)^{13}}{\eta(6z)^2\eta(24z)^5}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-4}{m}\right)\left(\frac{-6}{n}\right) q^{\frac{1}{4}(3m^2+n^2)}$
45 I	$\frac{\eta(6z)^5\eta(36z)^{13}}{\eta(12z)^2\eta(18z)^5\eta(72z)^5}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-6}{m}\right)\left(\frac{n}{12}\right) q^{\frac{1}{4}(3m^2+n^2)}$
46 I	$\frac{\eta(12z)^{13}\eta(18z)^5}{\eta(6z)^5\eta(24z)^5\eta(36z)^2}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-6}{m}\right)\left(\frac{n}{12}\right) q^{\frac{1}{4}(m^2+3n^2)}$
47 I	$\frac{\eta(6z)^5\eta(36z)^3}{\eta(12z)^2\eta(18z)\eta(72z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{24}{m}\right)\left(\frac{n}{12}\right) q^{\frac{1}{4}(3m^2+n^2)}$
48 I	$\frac{\eta(12z)^{13}\eta(18z)}{\eta(6z)^5\eta(24z)^5}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{-6}{n}\right) q^{\frac{1}{4}(3m^2+n^2)}$
49 I	$\frac{\eta(6z)^5\eta(18z)}{\eta(12z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{n}{12}\right) q^{\frac{1}{4}(3m^2+n^2)}$
50 I	$\frac{\eta(12z)^{13}\eta(36z)^3}{\eta(6z)^5\eta(18z)\eta(24z)^5\eta(72z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{24}{m}\right)\left(\frac{-6}{n}\right) q^{\frac{1}{4}(3m^2+n^2)}$
51 I	$\frac{\eta(6z)^5\eta(18z)^5}{\eta(12z)^2\eta(36z)^2}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{m}{12}\right)\left(\frac{n}{12}\right) q^{\frac{1}{4}(3m^2+n^2)}$
51 I	$\frac{\eta(6z)^5\eta(18z)^5}{\eta(12z)^2\eta(36z)^2}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{m}{12}\right)\left(\frac{n}{12}\right) q^{\frac{1}{4}(m^2+3n^2)}$
52 I	$\frac{\eta(12z)^{13}\eta(36z)^{13}}{\eta(6z)^5\eta(18z)^5\eta(24z)^5\eta(72z)^5}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-6}{m}\right)\left(\frac{-6}{n}\right) q^{\frac{1}{4}(3m^2+n^2)}$
52 I	$\frac{\eta(12z)^{13}\eta(36z)^{13}}{\eta(6z)^5\eta(18z)^5\eta(24z)^5\eta(72z)^5}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-6}{m}\right)\left(\frac{-6}{n}\right) q^{\frac{1}{4}(m^2+3n^2)}$
53 I	$\frac{\eta(6z)^6\eta(24z)}{\eta(12z)^3}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{8}{m}\right)\left(\frac{n}{12}\right) q^{\frac{1}{4}(3m^2+n^2)}$
54 I	$\frac{\eta(12z)^{15}}{\eta(6z)^6\eta(24z)^5}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{2}\right)^2\left(\frac{-6}{n}\right) q^{\frac{1}{4}(3m^2+n^2)}$
57 I	$\frac{\eta(6z)^7\eta(36z)}{\eta(12z)^3\eta(18z)}$	2	$\sum_{m,n=1}^{\infty} m\left(\frac{m}{12}\right)\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right) q^{\frac{1}{4}(m^2+3n^2)}$
59 I	$\frac{\eta(6z)^8}{\eta(12z)^2}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-4}{m}\right)\left(\frac{n}{12}\right) q^{\frac{1}{4}(3m^2+n^2)}$
60 I	$\frac{\eta(12z)^{22}}{\eta(6z)^8\eta(24z)^8}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-8}{m}\right)\left(\frac{-6}{n}\right) q^{\frac{1}{4}(3m^2+n^2)}$
77 III	$\eta(6z)\eta(18z)$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{12}{n}\right) q^{\frac{1}{4}(3m^2+n^2)}$

77 III	$\eta(6z)\eta(18z)$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{12}{n}\right)q^{\frac{1}{4}(m^2+3n^2)}$
77 III	$\eta(6z)\eta(18z)$	1	$\sum_{m,n=1}^{\infty}\left(\frac{m}{6}\right)^2\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{\frac{1}{4}(m^2+3n^2)}$
78 III	$\frac{\eta(12z)^3\eta(36z)^3}{\eta(6z)\eta(18z)\eta(24z)\eta(72z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{24}{m}\right)\left(\frac{24}{n}\right)q^{\frac{1}{4}(3m^2+n^2)}$
78 III	$\frac{\eta(12z)^3\eta(36z)^3}{\eta(6z)\eta(18z)\eta(24z)\eta(72z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{24}{m}\right)\left(\frac{24}{n}\right)q^{\frac{1}{4}(m^2+3n^2)}$
93 VII	$\frac{\eta(6z)\eta(36z)^3}{\eta(18z)\eta(72z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{24}{n}\right)q^{\frac{1}{4}(m^2+3n^2)}$
94 VII	$\frac{\eta(12z)^3\eta(18z)}{\eta(6z)\eta(24z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{24}{n}\right)q^{\frac{1}{4}(3m^2+n^2)}$
95 VII	$\frac{\eta(6z)^3\eta(24z)\eta(36z)^6}{\eta(12z)^3\eta(18z)^3\eta(72z)^2}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{18}{m}\right)\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{\frac{1}{4}(m^2+3n^2)}$
101 VIII	$\frac{\eta(6z)\eta(12z)^2\eta(36z)}{\eta(18z)\eta(24z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{24}{m}\right)\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{\frac{1}{4}(m^2+3n^2)}$
103 VIII	$\frac{\eta(6z)\eta(24z)\eta(36z)^5}{\eta(12z)^2\eta(18z)\eta(72z)^2}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{18}{n}\right)q^{\frac{1}{4}(3m^2+n^2)}$
104 VIII	$\frac{\eta(12z)\eta(18z)\eta(36z)^2}{\eta(6z)\eta(72z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{24}{m}\right)\left(\frac{n}{6}\right)^2q^{\frac{1}{4}(3m^2+n^2)}$
105 IX	$\frac{\eta(6z)\eta(24z)\eta(36z)^8}{\eta(12z)^2\eta(18z)^3\eta(72z)^3}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{24}{m}\right)\left(\frac{18}{n}\right)q^{\frac{1}{4}(3m^2+n^2)}$
106 IX	$\frac{\eta(12z)\eta(18z)^3}{\eta(6z)\eta(36z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{n}{6}\right)^2q^{\frac{1}{4}(3m^2+n^2)}$
107 IX	$\frac{\eta(6z)^3\eta(36z)}{\eta(12z)\eta(18z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{\frac{1}{4}(m^2+3n^2)}$
113 XI	$\frac{\eta(12z)\eta(48z)^{14}\eta(72z)^2}{\eta(24z)^6\eta(36z)\eta(96z)^5\eta(144z)}$	2	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}m\left(\frac{-6}{m}\right)\left(1-\frac{3\left(\frac{n}{3}\right)^2}{2}\right)q^{m^2+12n^2}$
114 XI	$\frac{\eta(36z)\eta(48z)^{13}}{\eta(12z)\eta(24z)^3\eta(72z)\eta(96z)^5}$	2	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}m\left(\frac{-6}{m}\right)\left(-2\left(\frac{n}{2}\right)^2-\frac{3\left(\frac{n}{3}\right)^2}{2}+3\left(\frac{n}{6}\right)^2+1\right)\times q^{m^2+12n^2}$
115 XI	$\frac{\eta(12z)^2\eta(48z)\eta(72z)^2}{\eta(24z)^2\eta(144z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}(-1)^n\left(\frac{m}{6}\right)^2q^{m^2+12n^2}$
116 XI	$\frac{\eta(48z)^{11}}{\eta(12z)^2\eta(96z)^5}$	2	$\sum_{\substack{m=-\infty\\n=1}}^{\infty}n\left(\frac{-6}{n}\right)q^{12m^2+n^2}$
117 XI	$\frac{\eta(24z)^4\eta(72z)^2}{\eta(12z)^2\eta(48z)\eta(144z)}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty}\left(\frac{n}{6}\right)^2q^{12m^2+n^2}$
118 XI	$\frac{\eta(12z)^2\eta(48z)^{13}}{\eta(24z)^6\eta(96z)^5}$	2	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}m(-1)^n\left(\frac{-6}{m}\right)q^{m^2+12n^2}$
121 XI	$\frac{\eta(12z)^2\eta(24z)^4}{\eta(48z)^2}$	2	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}m(-1)^n\left(\frac{m}{12}\right)q^{m^2+12n^2}$
122 XI	$\frac{\eta(12z)\eta(24z)^4\eta(72z)^2}{\eta(36z)\eta(48z)\eta(144z)}$	2	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}m\left(\frac{m}{12}\right)\left(1-\frac{3\left(\frac{n}{3}\right)^2}{2}\right)q^{m^2+12n^2}$
123 XI	$\frac{\eta(24z)^7\eta(36z)}{\eta(12z)\eta(48z)^2\eta(72z)}$	2	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}m\left(\frac{m}{12}\right)\left(-2\left(\frac{n}{2}\right)^2-\frac{3\left(\frac{n}{3}\right)^2}{2}+3\left(\frac{n}{6}\right)^2+1\right)\times q^{m^2+12n^2}$
124 XI	$\frac{\eta(24z)^{10}}{\eta(12z)^2\eta(48z)^4}$	2	$\sum_{\substack{m=-\infty\\n=1}}^{\infty}n\left(\frac{n}{12}\right)q^{12m^2+n^2}$
129 XIII	$\frac{\eta(12z)\eta(48z)^2\eta(72z)^4}{\eta(24z)^2\eta(36z)\eta(144z)^2}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}\left(\frac{m}{6}\right)^2\left(1-\frac{3\left(\frac{n}{3}\right)^2}{2}\right)q^{m^2+12n^2}$
130 XIII	$\frac{\eta(24z)\eta(36z)\eta(48z)\eta(72z)}{\eta(12z)\eta(144z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}\left(\frac{m}{6}\right)^2\left(-2\left(\frac{n}{2}\right)^2-\frac{3\left(\frac{n}{3}\right)^2}{2}+3\left(\frac{n}{6}\right)^2+1\right)\times q^{m^2+12n^2}$
133 XV	$\frac{\eta(12z)\eta(48z)^4\eta(72z)^2}{\eta(24z)^2\eta(36z)\eta(96z)\eta(144z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}\left(\frac{24}{m}\right)\left(1-\frac{3\left(\frac{n}{3}\right)^2}{2}\right)q^{m^2+12n^2}$

134 XV	$\frac{\eta(24z)\eta(36z)\eta(48z)^3}{\eta(12z)\eta(72z)\eta(96z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}\left(\frac{24}{m}\right)\left(-2\left(\frac{n}{2}\right)^2-\frac{3\left(\frac{n}{3}\right)^2}{2}+3\left(\frac{n}{6}\right)^2+1\right)\times q^{m^2+12n^2}$
135 XV	$\frac{\eta(12z)\eta(48z)\eta(72z)^2}{\eta(36z)\eta(144z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}\left(\frac{12}{m}\right)\left(1-\frac{3\left(\frac{n}{3}\right)^2}{2}\right)q^{m^2+12n^2}$
136 XV	$\frac{\eta(24z)^3\eta(36z)}{\eta(12z)\eta(72z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}\left(\frac{12}{m}\right)\left(-2\left(\frac{n}{2}\right)^2-\frac{3\left(\frac{n}{3}\right)^2}{2}+3\left(\frac{n}{6}\right)^2+1\right)\times q^{m^2+12n^2}$
137 XVI	$\frac{\eta(6z)^2\eta(24z)}{\eta(12z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}(-1)^n\left(\frac{12}{m}\right)q^{m^2+6n^2}$
137 XVI	$\frac{\eta(6z)^2\eta(24z)}{\eta(12z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{8}{m}\right)\left(\frac{12}{n}\right)q^{\frac{1}{4}(3m^2+n^2)}$
138 XVI	$\frac{\eta(12z)^5}{\eta(6z)^2\eta(24z)}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty}\left(\frac{12}{n}\right)q^{6m^2+n^2}$
138 XVI	$\frac{\eta(12z)^5}{\eta(6z)^2\eta(24z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{m}{2}\right)^2\left(\frac{24}{n}\right)q^{\frac{1}{4}(3m^2+n^2)}$
139 XVII	$\frac{\eta(24z)^4\eta(48z)}{\eta(12z)^2\eta(96z)}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty}\left(\frac{24}{n}\right)q^{12m^2+n^2}$
140 XVII	$\frac{\eta(12z)^2\eta(48z)^3}{\eta(24z)^2\eta(96z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}(-1)^n\left(\frac{24}{m}\right)q^{m^2+12n^2}$
141 XVII	$\eta(12z)^2$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}(-1)^n\left(\frac{12}{m}\right)q^{m^2+12n^2}$
141 XVII	$\eta(12z)^2$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{12}{n}\right)q^{\frac{1}{2}(m^2+n^2)}$
141 XVII	$\eta(12z)^2$	1	$\sum_{m,n=1}^{\infty}\left(\frac{m}{2}\right)^2\left(\frac{12}{n}\right)q^{\frac{1}{4}(3m^2+n^2)}$
141 XVII	$\eta(12z)^2$	1	$\sum_{m,n=1}^{\infty}\left(\frac{8}{m}\right)\left(\frac{24}{n}\right)q^{\frac{1}{4}(3m^2+n^2)}$
142 XVII	$\frac{\eta(24z)^6}{\eta(12z)^2\eta(48z)^2}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty}\left(\frac{12}{n}\right)q^{12m^2+n^2}$
142 XVII	$\frac{\eta(24z)^6}{\eta(12z)^2\eta(48z)^2}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{24}{m}\right)\left(\frac{24}{n}\right)q^{\frac{1}{2}(m^2+n^2)}$
143 XVIII	$\frac{\eta(18z)^2\eta(24z)}{\eta(36z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}(-1)^n\left(\frac{12}{m}\right)q^{m^2+18n^2}$
143 XVIII	$\frac{\eta(18z)^2\eta(24z)}{\eta(36z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{8}{m}\right)\left(\frac{n}{6}\right)^2q^{\frac{1}{4}(3m^2+n^2)}$
144 XVIII	$\frac{\eta(24z)\eta(36z)^5}{\eta(18z)^2\eta(72z)^2}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty}\left(\frac{12}{n}\right)q^{18m^2+n^2}$
144 XVIII	$\frac{\eta(24z)\eta(36z)^5}{\eta(18z)^2\eta(72z)^2}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{m}{2}\right)^2\left(\frac{18}{n}\right)q^{\frac{1}{4}(3m^2+n^2)}$

Table 9: Eta quotients with vanishing behaviour similar to f_1^6

Number	q-Product	Modular Form	Weight	Level	Group
1	f_1^6	$\eta(4z)^6$	3	16	I
2	$\frac{f_2^{18}}{f_1^6 f_4^6}$	$\frac{\eta(8z)^{18}}{\eta(4z)^6\eta(16z)^6}$	3	64	I
3	$\frac{f_2^8 f_3^2}{f_1^4 f_6^2}$	$\frac{\eta(8z)^8\eta(12z)^2}{\eta(4z)^4\eta(24z)^2}$	2	288	I
4	$\frac{f_1^4 f_4^4 f_6^4}{f_2^4 f_3^2 f_{12}^2}$	$\frac{\eta(4z)^4\eta(16z)^4\eta(24z)^4}{\eta(8z)^4\eta(12z)^2\eta(48z)^2}$	2	576	I
5	$\frac{f_2^{10} f_3^2 f_{12}^2}{f_1^4 f_4^4 f_6^4}$	$\frac{\eta(8z)^{10}\eta(12z)^2\eta(48z)^2}{\eta(4z)^4\eta(16z)^4\eta(24z)^4}$	1	576	I

6	$\frac{f_1^4 f_6^2}{f_2^2 f_3^2}$	$\frac{\eta(4z)^4\eta(24z)^2}{\eta(8z)^2\eta(12z)^2}$	1	72	I
7	$\frac{f_2^{11}}{f_1^4 f_4^3}$	$\frac{\eta(8z)^{11}}{\eta(4z)^4\eta(16z)^3}$	2	128	I
8	$\frac{f_1^4 f_4}{f_2}$	$\frac{\eta(4z)^4\eta(16z)}{\eta(8z)}$	2	128	I
9	$\frac{f_3^{12}}{f_1^3 f_9^3}$	$\frac{\eta(12z)^{12}}{\eta(4z)^3\eta(36z)^3}$	3	144	I
10	$\frac{f_1^3 f_4^3 f_6^{36} f_9^3 f_{36}^3}{f_2^9 f_3^{12} f_{12}^{12} f_{18}^9}$	$\frac{\eta(4z)^3\eta(16z)^3\eta(24z)^{36}\eta(36z)^3\eta(144z)^3}{\eta(8z)^9\eta(12z)^{12}\eta(48z)^{12}\eta(72z)^9}$	3	576	I
11	$\frac{f_2^3 f_3^7}{f_1^3 f_6^3}$	$\frac{\eta(8z)^3\eta(12z)^7}{\eta(4z)^3\eta(24z)^3}$	2	288	I
12	$\frac{f_1^3 f_4^3 f_6^{18}}{f_2^6 f_3^7 f_{12}^7}$	$\frac{\eta(4z)^3\eta(16z)^3\eta(24z)^{18}}{\eta(8z)^6\eta(12z)^7\eta(48z)^7}$	2	576	I
13	$\frac{f_2^2 f_4^7}{f_1^2 f_8^3}$	$\frac{\eta(8z)^2\eta(16z)^7}{\eta(4z)^2\eta(32z)^3}$	2	256	I
14	$\frac{f_1^2 f_4^9}{f_2^4 f_8^3}$	$\frac{\eta(4z)^2\eta(16z)^9}{\eta(8z)^4\eta(32z)^3}$	2	256	I
15	$\frac{f_2^2 f_3^2 f_4^4}{f_1^2 f_6 f_{12}}$	$\frac{\eta(8z)^2\eta(12z)^2\eta(16z)^4}{\eta(4z)^2\eta(24z)\eta(48z)}$	2	1152	I
16	$\frac{f_1^2 f_4^6 f_6^5}{f_2^4 f_3^2 f_{12}^3}$	$\frac{\eta(4z)^2\eta(16z)^6\eta(24z)^5}{\eta(8z)^4\eta(12z)^2\eta(48z)^3}$	2	1152	I
17	$\frac{f_2^4}{f_1^2}$	$\frac{\eta(8z)^4}{\eta(4z)^2}$	1	8	I
18	$\frac{f_1^2 f_4^2}{f_2^2}$	$\frac{\eta(4z)^2\eta(16z)^2}{\eta(8z)^2}$	1	64	I
19	$\frac{f_2^6 f_8}{f_1^2 f_4^3}$	$\frac{\eta(8z)^6\eta(32z)}{\eta(4z)^2\eta(16z)^3}$	1	256	I
20	$\frac{f_1^2 f_8}{f_4}$	$\frac{\eta(4z)^2\eta(32z)}{\eta(16z)}$	1	256	I
21	$\frac{f_2^8}{f_1^2 f_4^2}$	$\frac{\eta(8z)^8}{\eta(4z)^2\eta(16z)^2}$	2	64	I
22	$f_1^2 f_2^2$	$\eta(4z)^2\eta(8z)^2$	2	32	I
23	$\frac{f_3 f_4^9}{f_1 f_2 f_6 f_8^3}$	$\frac{\eta(12z)\eta(16z)^9}{\eta(4z)\eta(8z)\eta(24z)\eta(32z)^3}$	2	2304	I
24	$\frac{f_1 f_4^{10} f_6^2}{f_2^4 f_3 f_8^3 f_{12}}$	$\frac{\eta(4z)\eta(16z)^{10}\eta(24z)^2}{\eta(8z)^4\eta(12z)\eta(32z)^3\eta(48z)}$	2	2304	I
25	$\frac{f_2 f_5^3}{f_1 f_{10}}$	$\frac{\eta(8z)\eta(20z)^3}{\eta(4z)\eta(40z)}$	1	40	I
26	$\frac{f_1 f_4 f_{10}^8}{f_2^2 f_5^3 f_{20}^3}$	$\frac{\eta(4z)\eta(16z)\eta(40z)^8}{\eta(8z)^2\eta(20z)^3\eta(80z)^3}$	1	320	I
27	$\frac{f_2^4 f_3 f_{12}}{f_1 f_4 f_6^2}$	$\frac{\eta(8z)^4\eta(12z)\eta(48z)}{\eta(4z)\eta(16z)\eta(24z)^2}$	1	576	I
28	$\frac{f_1 f_2 f_6}{f_3}$	$\frac{\eta(4z)\eta(8z)\eta(24z)}{\eta(12z)}$	1	72	I
29	$\frac{f_2^5 f_3}{f_1 f_6}$	$\frac{\eta(8z)^5\eta(12z)}{\eta(4z)\eta(24z)}$	2	288	I
30	$\frac{f_1 f_2^2 f_4 f_6^2}{f_3 f_{12}}$	$\frac{\eta(4z)\eta(8z)^2\eta(16z)\eta(24z)^2}{\eta(12z)\eta(48z)}$	2	576	I

31	$\frac{f_2^8 f_6}{f_1 f_3 f_4^3}$	$\frac{\eta(8z)^8\eta(24z)}{\eta(4z)\eta(12z)\eta(16z)^3}$	2	1152	I
32	$\frac{f_1 f_2^5 f_3 f_{12}}{f_4^2 f_6^2}$	$\frac{\eta(4z)\eta(8z)^5\eta(12z)\eta(48z)}{\eta(16z)^2\eta(24z)^2}$	2	1152	I
33	$\frac{f_1^3 f_{10}}{f_2 f_5}$	$\frac{\eta(4z)^3\eta(40z)}{\eta(8z)\eta(20z)}$	1	40	I
34	$\frac{f_2^8 f_5 f_{20}}{f_1^3 f_4^3 f_{10}^2}$	$\frac{\eta(8z)^8\eta(20z)\eta(80z)}{\eta(4z)^3\eta(16z)^3\eta(40z)^2}$	1	320	I
35	$\frac{f_1^5 f_6}{f_2 f_3}$	$\frac{\eta(4z)^5\eta(24z)}{\eta(8z)\eta(12z)}$	2	288	I
36	$\frac{f_2^{14} f_3 f_{12}}{f_1^5 f_4^5 f_6^2}$	$\frac{\eta(8z)^{14}\eta(12z)\eta(48z)}{\eta(4z)^5\eta(16z)^5\eta(24z)^2}$	2	576	I
37	$\frac{f_1^8 f_6^2}{f_2^4 f_3^2}$	$\frac{\eta(4z)^8\eta(24z)^2}{\eta(8z)^4\eta(12z)^2}$	2	288	I
38	$\frac{f_2^{20} f_3^2 f_{12}^2}{f_1^8 f_4^8 f_6^4}$	$\frac{\eta(8z)^{20}\eta(12z)^2\eta(48z)^2}{\eta(4z)^8\eta(16z)^8\eta(24z)^4}$	2	576	I
39	$\frac{f_1^{14}}{f_2^4}$	$\frac{\eta(4z)^{14}}{\eta(8z)^4}$	5	8	I
40	$\frac{f_2^{38}}{f_1^{14} f_4^{14}}$	$\frac{\eta(8z)^{38}}{\eta(4z)^{14}\eta(16z)^{14}}$	5	64	I
41	$\frac{f_1 f_4^6 f_6^3 f_{24}}{f_2^3 f_3 f_8^2 f_{12}^3}$	$\frac{\eta(4z)\eta(16z)^6\eta(24z)^3\eta(96z)}{\eta(8z)^3\eta(12z)\eta(32z)^2\eta(48z)^3}$	1	2304	I
42	$\frac{f_3 f_4^5 f_{24}}{f_1 f_8^2 f_{12}^2}$	$\frac{\eta(12z)\eta(16z)^5\eta(96z)}{\eta(4z)\eta(32z)^2\eta(48z)^2}$	1	2304	I
43	$\frac{f_1^2 f_4^5 f_6 f_{24}}{f_2^3 f_8^2 f_{12}^2}$	$\frac{\eta(4z)^2\eta(16z)^5\eta(24z)\eta(96z)}{\eta(8z)^3\eta(32z)^2\eta(48z)^2}$	1	2304	II
44	$\frac{f_2^3 f_4^3 f_6 f_{24}}{f_1^2 f_8^2 f_{12}^2}$	$\frac{\eta(8z)^3\eta(16z)^3\eta(24z)\eta(96z)}{\eta(4z)^2\eta(32z)^2\eta(48z)^2}$	1	2304	II
45	$\frac{f_1^2 f_2 f_{12}}{f_4 f_6}$	$\frac{\eta(4z)^2\eta(8z)\eta(48z)}{\eta(16z)\eta(24z)}$	1	576	II
46	$\frac{f_2^7 f_{12}}{f_1^2 f_4^3 f_6}$	$\frac{\eta(8z)^7\eta(48z)}{\eta(4z)^2\eta(16z)^3\eta(24z)}$	1	144	II
47	$\frac{f_1 f_6^2 f_8}{f_3 f_{12}}$	$\frac{\eta(4z)\eta(24z)^2\eta(32z)}{\eta(12z)\eta(48z)}$	1	2304	III
48	$\frac{f_2^3 f_3 f_8}{f_1 f_4 f_6}$	$\frac{\eta(8z)^3\eta(12z)\eta(32z)}{\eta(4z)\eta(16z)\eta(24z)}$	1	2304	III
49	$\frac{f_1 f_4^3 f_6^2}{f_2^2 f_3 f_{12}}$	$\frac{\eta(4z)\eta(16z)^3\eta(24z)^2}{\eta(8z)^2\eta(12z)\eta(48z)}$	1	144	III
50	$\frac{f_2 f_3 f_4^2}{f_1 f_6}$	$\frac{\eta(8z)\eta(12z)\eta(16z)^2}{\eta(4z)\eta(24z)}$	1	576	III
51	$\frac{f_1 f_4 f_6^{13}}{f_2 f_3^5 f_{12}^5}$	$\frac{\eta(4z)\eta(16z)\eta(24z)^{13}}{\eta(8z)\eta(12z)^5\eta(48z)^5}$	2	576	IV
52	$\frac{f_2^2 f_3^5}{f_1 f_6^2}$	$\frac{\eta(8z)^2\eta(12z)^5}{\eta(4z)\eta(24z)^2}$	2	72	IV
53	$\frac{f_1 f_4 f_6^{10}}{f_2 f_3^3 f_{12}^4}$	$\frac{\eta(4z)\eta(16z)\eta(24z)^{10}}{\eta(8z)\eta(12z)^3\eta(48z)^4}$	2	1152	IV
54	$\frac{f_2^2 f_3^3 f_6}{f_1 f_{12}}$	$\frac{\eta(8z)^2\eta(12z)^3\eta(24z)}{\eta(4z)\eta(48z)}$	2	1152	IV
55	$\frac{f_1 f_3^5 f_4}{f_2 f_6^2}$	$\frac{\eta(4z)\eta(12z)^5\eta(16z)}{\eta(8z)\eta(24z)^2}$	2	1152	IV

56	$\frac{f_2^2 f_6^{13}}{f_1 f_3^5 f_{12}^5}$	$\frac{\eta(8z)^2\eta(24z)^{13}}{\eta(4z)\eta(12z)^5\eta(48z)^5}$	2	1152	IV
57	$\frac{f_1^2 f_6^{14}}{f_2 f_3^6 f_{12}^5}$	$\frac{\eta(4z)^2\eta(24z)^{14}}{\eta(8z)\eta(12z)^6\eta(48z)^5}$	2	1152	IV
58	$\frac{f_2^5 f_3^6 f_{12}}{f_1^2 f_4^2 f_6^4}$	$\frac{\eta(8z)^5\eta(12z)^6\eta(48z)}{\eta(4z)^2\eta(16z)^2\eta(24z)^4}$	2	1152	IV
59	$\frac{f_1^2 f_3^2 f_6^2}{f_2 f_{12}}$	$\frac{\eta(4z)^2\eta(12z)^2\eta(24z)^2}{\eta(8z)\eta(48z)}$	2	1152	IV
60	$\frac{f_2^5 f_6^8}{f_1^2 f_3^2 f_4^2 f_{12}^3}$	$\frac{\eta(8z)^5\eta(24z)^8}{\eta(4z)^2\eta(12z)^2\eta(16z)^2\eta(48z)^3}$	2	1152	IV
61	$\frac{f_1^2 f_3^4}{f_2 f_6}$	$\frac{\eta(4z)^2\eta(12z)^4}{\eta(8z)\eta(24z)}$	2	72	IV
62	$\frac{f_2^5 f_6^{11}}{f_1^2 f_3^4 f_4^2 f_{12}^4}$	$\frac{\eta(8z)^5\eta(24z)^{11}}{\eta(4z)^2\eta(12z)^4\eta(16z)^2\eta(48z)^4}$	2	576	IV
63	$\frac{f_1^3 f_6^3}{f_3 f_{12}}$	$\frac{\eta(4z)^3\eta(24z)^3}{\eta(12z)\eta(48z)}$	2	1152	IV
64	$\frac{f_2^9 f_3}{f_1^3 f_4^3}$	$\frac{\eta(8z)^9\eta(12z)}{\eta(4z)^3\eta(16z)^3}$	2	1152	IV
65	$f_1^3 f_3$	$\eta(4z)^3\eta(12z)$	2	144	IV
66	$\frac{f_2^9 f_6^3}{f_1^3 f_3 f_4^3 f_{12}}$	$\frac{\eta(8z)^9\eta(24z)^3}{\eta(4z)^3\eta(12z)\eta(16z)^3\eta(48z)}$	2	576	IV
67	$\frac{f_1^2 f_4 f_6^9}{f_2^3 f_3^4 f_{12}^3}$	$\frac{\eta(4z)^2\eta(16z)\eta(24z)^9}{\eta(8z)^3\eta(12z)^4\eta(48z)^3}$	1	144	V
68	$\frac{f_2^3 f_3^4 f_{12}}{f_1^2 f_4 f_6^3}$	$\frac{\eta(8z)^3\eta(12z)^4\eta(48z)}{\eta(4z)^2\eta(16z)\eta(24z)^3}$	1	576	V
69	$\frac{f_1^3 f_6^6}{f_3^3 f_{12}^2}$	$\frac{\eta(4z)^3\eta(24z)^6}{\eta(12z)^3\eta(48z)^2}$	2	576	VI
70	$\frac{f_2^9 f_3^3 f_{12}}{f_1^3 f_4^3 f_6^3}$	$\frac{\eta(8z)^9\eta(12z)^3\eta(48z)}{\eta(4z)^3\eta(16z)^3\eta(24z)^3}$	2	144	VI
71	$\frac{f_4^3 f_6^4}{f_3^2 f_{24}}$	$\frac{\eta(16z)^3\eta(24z)^4}{\eta(12z)^2\eta(96z)}$	2	2304	VII
72	$\frac{f_3^2 f_4^3 f_{12}^2}{f_6^2 f_{24}}$	$\frac{\eta(12z)^2\eta(16z)^3\eta(48z)^2}{\eta(24z)^2\eta(96z)}$	2	2304	VII
73	$\frac{f_3^2 f_4^3}{f_{12}}$	$\frac{\eta(12z)^2\eta(16z)^3}{\eta(48z)}$	2	144	VII
74	$\frac{f_4^3 f_6^6}{f_3^2 f_{12}^3}$	$\frac{\eta(16z)^3\eta(24z)^6}{\eta(12z)^2\eta(48z)^3}$	2	576	VII
75	$\frac{f_2^5 f_3^2 f_{12}^{11}}{f_1^2 f_4^2 f_6^6 f_{24}^4}$	$\frac{\eta(8z)^5\eta(12z)^2\eta(48z)^{11}}{\eta(4z)^2\eta(16z)^2\eta(24z)^6\eta(96z)^4}$	2	1152	VIII
76	$\frac{f_1^2 f_{12}^9}{f_2 f_3^2 f_{24}^4}$	$\frac{\eta(4z)^2\eta(48z)^9}{\eta(8z)\eta(12z)^2\eta(96z)^4}$	2	1152	VIII
77	$\frac{f_2^2 f_{12}^9}{f_1 f_3 f_6 f_{24}^4}$	$\frac{\eta(8z)^2\eta(48z)^9}{\eta(4z)\eta(12z)\eta(24z)\eta(96z)^4}$	2	1152	VIII
78	$\frac{f_1 f_3 f_4 f_{12}^{10}}{f_2 f_6^4 f_{24}^4}$	$\frac{\eta(4z)\eta(12z)\eta(16z)\eta(48z)^{10}}{\eta(8z)\eta(24z)^4\eta(96z)^4}$	2	1152	VIII
79	$\frac{f_6^9 f_8^2}{f_3^4 f_4 f_{24}^2}$	$\frac{\eta(24z)^9\eta(32z)^2}{\eta(12z)^4\eta(16z)\eta(96z)^2}$	2	1152	IX
80	$\frac{f_3^4 f_8^2 f_{12}^4}{f_4 f_6^3 f_{24}^2}$	$\frac{\eta(12z)^4\eta(32z)^2\eta(48z)^4}{\eta(16z)\eta(24z)^3\eta(96z)^2}$	2	1152	IX

81	$\frac{f_4^2 f_6^9}{f_3^4 f_8 f_{12} f_{24}}$	$\frac{\eta(16z)^2\eta(24z)^9}{\eta(12z)^4\eta(32z)\eta(48z)\eta(96z)}$	2	1152	IX
82	$\frac{f_3^4 f_4^2 f_{12}^3}{f_6^3 f_8 f_{24}}$	$\frac{\eta(12z)^4\eta(16z)^2\eta(48z)^3}{\eta(24z)^3\eta(32z)\eta(96z)}$	2	1152	IX
83	$\frac{f_2^2 f_3 f_{12}^{13}}{f_1 f_6^6 f_{24}^5}$	$\frac{\eta(8z)^2\eta(12z)\eta(48z)^{13}}{\eta(4z)\eta(24z)^6\eta(96z)^5}$	2	2304	X
84	$\frac{f_1 f_4 f_{12}^{12}}{f_2 f_3 f_6^3 f_{24}^5}$	$\frac{\eta(4z)\eta(16z)\eta(48z)^{12}}{\eta(8z)\eta(12z)\eta(24z)^3\eta(96z)^5}$	2	2304	X
85	$\frac{f_2^5 f_{12}^{13}}{f_1^2 f_4^2 f_6^5 f_{24}^5}$	$\frac{\eta(8z)^5\eta(48z)^{13}}{\eta(4z)^2\eta(16z)^2\eta(24z)^5\eta(96z)^5}$	2	2304	X
86	$\frac{f_1^2 f_{12}^{13}}{f_2 f_6^5 f_{24}^5}$	$\frac{\eta(4z)^2\eta(48z)^{13}}{\eta(8z)\eta(24z)^5\eta(96z)^5}$	2	2304	X
87	$\frac{f_2^9 f_6^{13}}{f_1^3 f_3^5 f_4^3 f_{12}^5}$	$\frac{\eta(8z)^9\eta(24z)^{13}}{\eta(4z)^3\eta(12z)^5\eta(16z)^3\eta(48z)^5}$	3	576	X
88	$\frac{f_1^3 f_3^5}{f_6^2}$	$\frac{\eta(4z)^3\eta(12z)^5}{\eta(24z)^2}$	3	288	X
89	$\frac{f_1 f_4 f_6^7}{f_2 f_3 f_{12}^3}$	$\frac{\eta(4z)\eta(16z)\eta(24z)^7}{\eta(8z)\eta(12z)\eta(48z)^3}$	2	144	X
90	$\frac{f_2^2 f_3 f_6^4}{f_1 f_{12}^2}$	$\frac{\eta(8z)^2\eta(12z)\eta(24z)^4}{\eta(4z)\eta(48z)^2}$	2	576	X
91	$\frac{f_1^2 f_6^5}{f_2 f_{12}^2}$	$\frac{\eta(4z)^2\eta(24z)^5}{\eta(8z)\eta(48z)^2}$	2	576	X
92	$\frac{f_2^5 f_6^5}{f_1^2 f_4^2 f_{12}^2}$	$\frac{\eta(8z)^5\eta(24z)^5}{\eta(4z)^2\eta(16z)^2\eta(48z)^2}$	2	144	X
93	$\frac{f_1^3 f_6^{13}}{f_3^5 f_{12}^5}$	$\frac{\eta(4z)^3\eta(24z)^{13}}{\eta(12z)^5\eta(48z)^5}$	3	1152	XI
94	$\frac{f_2^9 f_3^5}{f_1^3 f_4^3 f_6^2}$	$\frac{\eta(8z)^9\eta(12z)^5}{\eta(4z)^3\eta(16z)^3\eta(24z)^2}$	3	1152	XI
95	$\frac{f_1 f_3 f_4}{f_2}$	$\frac{\eta(4z)\eta(12z)\eta(16z)}{\eta(8z)}$	1	1152	XII
96	$\frac{f_2^2 f_6^3}{f_1 f_3 f_{12}}$	$\frac{\eta(8z)^2\eta(24z)^3}{\eta(4z)\eta(12z)\eta(48z)}$	1	1152	XII
97	$\frac{f_1^2 f_6^4}{f_2 f_3^2 f_{12}}$	$\frac{\eta(4z)^2\eta(24z)^4}{\eta(8z)\eta(12z)^2\eta(48z)}$	1	1152	XII
98	$\frac{f_2^5 f_3^2 f_{12}}{f_1^2 f_4^2 f_6^2}$	$\frac{\eta(8z)^5\eta(12z)^2\eta(48z)}{\eta(4z)^2\eta(16z)^2\eta(24z)^2}$	1	1152	XII
99	$\frac{f_2^2 f_3 f_{12}^3}{f_1 f_6^2 f_{24}}$	$\frac{\eta(8z)^2\eta(12z)\eta(48z)^3}{\eta(4z)\eta(24z)^2\eta(96z)}$	1	2304	XIII
100	$\frac{f_1 f_4 f_6 f_{12}^2}{f_2 f_3 f_{24}}$	$\frac{\eta(4z)\eta(16z)\eta(24z)\eta(48z)^2}{\eta(8z)\eta(12z)\eta(96z)}$	1	2304	XIII
101	$\frac{f_2^5 f_{12}^3}{f_1^2 f_4^2 f_6 f_{24}}$	$\frac{\eta(8z)^5\eta(48z)^3}{\eta(4z)^2\eta(16z)^2\eta(24z)\eta(96z)}$	1	2304	XIII
102	$\frac{f_1^2 f_{12}^3}{f_2 f_6 f_{24}}$	$\frac{\eta(4z)^2\eta(48z)^3}{\eta(8z)\eta(24z)\eta(96z)}$	1	2304	XIII
103	$\frac{f_1 f_4 f_6^3}{f_2 f_3 f_{12}}$	$\frac{\eta(4z)\eta(16z)\eta(24z)^3}{\eta(8z)\eta(12z)\eta(48z)}$	1	576	XIII
104	$\frac{f_2^2 f_3}{f_1}$	$\frac{\eta(8z)^2\eta(12z)}{\eta(4z)}$	1	288	XIII
105	$\frac{f_1^2 f_6}{f_2}$	$\frac{\eta(4z)^2\eta(24z)}{\eta(8z)}$	1	288	XIII

106	$\frac{f_2^5 f_6}{f_1^2 f_4^2}$	$\frac{\eta(8z)^5\eta(24z)}{\eta(4z)^2\eta(16z)^2}$	1	576	XIII
107	$\frac{f_1 f_4^2 f_6^8}{f_2^3 f_3^3 f_{12}^3}$	$\frac{\eta(4z)\eta(16z)^2\eta(24z)^8}{\eta(8z)^3\eta(12z)^3\eta(48z)^3}$	1	1152	XIV
108	$\frac{f_3^3 f_4}{f_1 f_6}$	$\frac{\eta(12z)^3\eta(16z)}{\eta(4z)\eta(24z)}$	1	1152	XIV
109	$\frac{f_1^3 f_4 f_6}{f_2^2 f_3}$	$\frac{\eta(4z)^3\eta(16z)\eta(24z)}{\eta(8z)^2\eta(12z)}$	1	1152	XIV
110	$\frac{f_2^7 f_3 f_{12}}{f_1^3 f_4^2 f_6^2}$	$\frac{\eta(8z)^7\eta(12z)\eta(48z)}{\eta(4z)^3\eta(16z)^2\eta(24z)^2}$	1	1152	XIV
111	$\frac{f_4^8 f_{12}}{f_2^3 f_8 f_{24}}$	$\frac{\eta(16z)^8\eta(48z)}{\eta(8z)^3\eta(32z)\eta(96z)}$	2	1152	XV
112	$\frac{f_2^6 f_8^2 f_{12}^5}{f_4^4 f_6^3 f_{24}^2}$	$\frac{\eta(8z)^6\eta(32z)^2\eta(48z)^5}{\eta(16z)^4\eta(24z)^3\eta(96z)^2}$	2	1152	XV
113	$\frac{f_4^{14} f_6^3 f_{24}}{f_2^6 f_8^4 f_{12}^4}$	$\frac{\eta(16z)^{14}\eta(24z)^3\eta(96z)}{\eta(8z)^6\eta(32z)^4\eta(48z)^4}$	2	2304	XV
114	$\frac{f_2^3 f_8^2 f_{12}}{f_4 f_{24}}$	$\frac{\eta(8z)^3\eta(32z)^2\eta(48z)}{\eta(16z)\eta(96z)}$	2	2304	XV
115	$\frac{f_2^9}{f_4^3}$	$\frac{\eta(8z)^9}{\eta(16z)^3}$	3	128	XV
116	$\frac{f_2^3 f_4^2}{f_8}$	$\frac{\eta(8z)^3\eta(16z)^2}{\eta(32z)}$	2	256	XV
117	$\frac{f_4^{11}}{f_2^3 f_8^4}$	$\frac{\eta(16z)^{11}}{\eta(8z)^3\eta(32z)^4}$	2	128	XV
118	$\frac{f_4^{24}}{f_2^9 f_8^9}$	$\frac{\eta(16z)^{24}}{\eta(8z)^9\eta(32z)^9}$	3	256	XV
119	$\frac{f_4^3 f_{12}^7}{f_6^3 f_{24}^3}$	$\frac{\eta(16z)^3\eta(48z)^7}{\eta(24z)^3\eta(96z)^3}$	2	2304	XVI
120	$\frac{f_4^3 f_6^3}{f_{12}^2}$	$\frac{\eta(16z)^3\eta(24z)^3}{\eta(48z)^2}$	2	1152	XVI
121	$\frac{f_3^2 f_{12}^2 f_{18}^{10}}{f_6^4 f_9^4 f_{36}^4}$	$\frac{\eta(12z)^2\eta(48z)^2\eta(72z)^{10}}{\eta(24z)^4\eta(36z)^4\eta(144z)^4}$	1	576	XVII
122	$\frac{f_6^2 f_9^4}{f_3^2 f_{18}^2}$	$\frac{\eta(24z)^2\eta(36z)^4}{\eta(12z)^2\eta(72z)^2}$	1	72	XVII
123	$\frac{f_3^4 f_6}{f_{12}}$	$\frac{\eta(12z)^4\eta(24z)}{\eta(48z)}$	2	1152	XVII
124	$\frac{f_6^{13}}{f_3^4 f_{12}^5}$	$\frac{\eta(24z)^{13}}{\eta(12z)^4\eta(48z)^5}$	2	1152	XVII
125	$\frac{f_3^6}{f_6^2}$	$\frac{\eta(12z)^6}{\eta(24z)^2}$	2	288	XVII
126	$\frac{f_6^{16}}{f_3^6 f_{12}^6}$	$\frac{\eta(24z)^{16}}{\eta(12z)^6\eta(48z)^6}$	2	576	XVII
127	$\frac{f_3^{10}}{f_6^4}$	$\frac{\eta(12z)^{10}}{\eta(24z)^4}$	3	72	XVII
128	$\frac{f_6^{26}}{f_3^{10} f_{12}^{10}}$	$\frac{\eta(24z)^{26}}{\eta(12z)^{10}\eta(48z)^{10}}$	3	576	XVII
129	$\frac{f_2^3 f_{10}^3}{f_1 f_4 f_5 f_{20}}$	$\frac{\eta(8z)^3\eta(40z)^3}{\eta(4z)\eta(16z)\eta(20z)\eta(80z)}$	1	320	XVIII
130	$f_1 f_5$	$\eta(4z)\eta(20z)$	1	80	XVIII

131	$\frac{f_4^7 f_6 f_{24}}{f_2^2 f_8^3 f_{12}^2}$	$\frac{\eta(16z)^7\eta(24z)\eta(96z)}{\eta(8z)^2\eta(32z)^3\eta(48z)^2}$	1	1152	XIX
132	$\frac{f_2^2 f_4 f_{12}}{f_6 f_8}$	$\frac{\eta(8z)^2\eta(16z)\eta(48z)}{\eta(24z)\eta(32z)}$	1	2304	XIX
133	$\frac{f_2 f_8^3 f_{12}}{f_4^2 f_{24}}$	$\frac{\eta(8z)\eta(32z)^3\eta(48z)}{\eta(16z)^2\eta(96z)}$	1	1152	XX
134	$\frac{f_4 f_8^2 f_{12}}{f_2 f_{24}}$	$\frac{\eta(16z)\eta(32z)^2\eta(48z)}{\eta(8z)\eta(96z)}$	1	2304	XX
135	$\frac{f_8^2 f_{12}^{14}}{f_4 f_6^5 f_{24}^6}$	$\frac{\eta(32z)^2\eta(48z)^{14}}{\eta(16z)\eta(24z)^5\eta(96z)^6}$	2	1152	XXI
136	$\frac{f_6^5 f_8^2}{f_4 f_{12} f_{24}}$	$\frac{\eta(24z)^5\eta(32z)^2}{\eta(16z)\eta(48z)\eta(96z)}$	2	2304	XXI
137	$\frac{f_4^2 f_{12}^{13}}{f_6^5 f_8 f_{24}^5}$	$\frac{\eta(16z)^2\eta(48z)^{13}}{\eta(24z)^5\eta(32z)\eta(96z)^5}$	2	1152	XXI
138	$\frac{f_4^2 f_6^5}{f_8 f_{12}^2}$	$\frac{\eta(16z)^2\eta(24z)^5}{\eta(32z)\eta(48z)^2}$	2	2304	XXI
139	$\frac{f_4^{10} f_6 f_{24}}{f_2^4 f_8^4 f_{12}^2}$	$\frac{\eta(16z)^{10}\eta(24z)\eta(96z)}{\eta(8z)^4\eta(32z)^4\eta(48z)^2}$	1	2304	XXII
140	$\frac{f_2^4 f_{12}}{f_4^2 f_6}$	$\frac{\eta(8z)^4\eta(48z)}{\eta(16z)^2\eta(24z)}$	1	1152	XXII
141	$f_2 f_4$	$\eta(8z)\eta(16z)$	1	128	XXII
142	$\frac{f_4^4}{f_2 f_8}$	$\frac{\eta(16z)^4}{\eta(8z)\eta(32z)}$	1	256	XXII
143	$\frac{f_2^2 f_8^2 f_{12}^2}{f_4^2 f_6 f_{24}}$	$\frac{\eta(8z)^2\eta(32z)^2\eta(48z)^2}{\eta(16z)^2\eta(24z)\eta(96z)}$	1	2304	XXII
144	$\frac{f_4^4 f_6}{f_2^2 f_{12}}$	$\frac{\eta(16z)^4\eta(24z)}{\eta(8z)^2\eta(48z)}$	1	1152	XXII
145	$\frac{f_2^2 f_8 f_{12}^8}{f_4^3 f_6^3 f_{24}^3}$	$\frac{\eta(8z)^2\eta(32z)\eta(48z)^8}{\eta(16z)^3\eta(24z)^3\eta(96z)^3}$	1	1152	XXIII
146	$\frac{f_4^3 f_6^3}{f_2^2 f_8 f_{12}}$	$\frac{\eta(16z)^3\eta(24z)^3}{\eta(8z)^2\eta(32z)\eta(48z)}$	1	2304	XXIII
147	$\frac{f_8^2 f_{12}^4}{f_4 f_6 f_{24}^2}$	$\frac{\eta(32z)^2\eta(48z)^4}{\eta(16z)\eta(24z)\eta(96z)^2}$	1	1152	XXIV
148	$\frac{f_6 f_8^2 f_{12}}{f_4 f_{24}}$	$\frac{\eta(24z)\eta(32z)^2\eta(48z)}{\eta(16z)\eta(96z)}$	1	2304	XXIV
149	$\frac{f_4^2 f_{12}^3}{f_6 f_8 f_{24}}$	$\frac{\eta(16z)^2\eta(48z)^3}{\eta(24z)\eta(32z)\eta(96z)}$	1	1152	XXIV
150	$\frac{f_4^2 f_6}{f_8}$	$\frac{\eta(16z)^2\eta(24z)}{\eta(32z)}$	1	2304	XXIV
151	$\frac{f_6^4 f_{12}}{f_3^2 f_{24}}$	$\frac{\eta(24z)^4\eta(48z)}{\eta(12z)^2\eta(96z)}$	1	2304	XXV
152	$\frac{f_3^2 f_{12}^3}{f_6^2 f_{24}}$	$\frac{\eta(12z)^2\eta(48z)^3}{\eta(24z)^2\eta(96z)}$	1	2304	XXV
153	f_3^2	$\eta(12z)^2$	1	144	XXV
154	$\frac{f_6^6}{f_3^2 f_{12}^2}$	$\frac{\eta(24z)^6}{\eta(12z)^2\eta(48z)^2}$	1	576	XXV
155	$\frac{f_6 f_{12}^6}{f_{24}^3}$	$\frac{\eta(24z)\eta(48z)^6}{\eta(96z)^3}$	2	2304	XXVI
156	$\frac{f_{12}^9}{f_6 f_{24}^4}$	$\frac{\eta(48z)^9}{\eta(24z)\eta(96z)^4}$	2	1152	XXVI

157	$\frac{f_{12}^{28}}{f_6^{11}f_{24}^{11}}$	$\frac{\eta(48z)^{28}}{\eta(24z)^{11}\eta(96z)^{11}}$	3	2304	XXVI
158	$\frac{f_6^{11}}{f_{12}^5}$	$\frac{\eta(24z)^{11}}{\eta(48z)^5}$	3	1152	XXVI
159	$\frac{f_{12}^8}{f_6^3f_{24}^3}$	$\frac{\eta(48z)^8}{\eta(24z)^3\eta(96z)^3}$	1	2304	XXVII
160	$\frac{f_6^3}{f_{12}}$	$\frac{\eta(24z)^3}{\eta(48z)}$	1	1152	XXVII
161	f_2^3	$\eta(8z)^3$	$\frac{3}{2}$	64	XXVIII
162	$\frac{f_4^5f_6f_{24}}{f_2^2f_8^2f_{12}^2}$	$\frac{\eta(16z)^5\eta(24z)\eta(96z)}{\eta(8z)^2\eta(32z)^2\eta(48z)^2}$	$\frac{1}{2}$	2304	XXVIII
163	$\frac{f_2^2f_{12}}{f_4f_6}$	$\frac{\eta(8z)^2\eta(48z)}{\eta(16z)\eta(24z)}$	$\frac{1}{2}$	144	XXVIII
164	$\frac{f_2f_8}{f_4}$	$\frac{\eta(8z)\eta(32z)}{\eta(16z)}$	$\frac{1}{2}$	256	XXVIII
165	$\frac{f_4^2}{f_2}$	$\frac{\eta(16z)^2}{\eta(8z)}$	$\frac{1}{2}$	16	XXVIII
166	$\frac{f_4^9}{f_2^3f_8^3}$	$\frac{\eta(16z)^9}{\eta(8z)^3\eta(32z)^3}$	$\frac{3}{2}$	256	XXVIII
167	$\frac{f_{12}^{13}}{f_6^5f_{24}^5}$	$\frac{\eta(48z)^{13}}{\eta(24z)^5\eta(96z)^5}$	$\frac{3}{2}$	2304	XXIX
168	$\frac{f_6^5}{f_{12}^2}$	$\frac{\eta(24z)^5}{\eta(48z)^2}$	$\frac{3}{2}$	144	XXIX
169	$\frac{f_{12}^3}{f_6f_{24}}$	$\frac{\eta(48z)^3}{\eta(24z)\eta(96z)}$	$\frac{1}{2}$	2304	XXIX
170	f_6	$\eta(24z)$	$\frac{1}{2}$	576	XXIX
171	$\frac{f_{12}f_{18}^2}{f_6f_{36}}$	$\frac{\eta(48z)\eta(72z)^2}{\eta(24z)\eta(144z)}$	$\frac{1}{2}$	144	XXIX
172	$\frac{f_6f_{24}f_{36}^5}{f_{12}^2f_{18}^2f_{72}^2}$	$\frac{\eta(24z)\eta(96z)\eta(144z)^5}{\eta(48z)^2\eta(72z)^2\eta(288z)^2}$	$\frac{1}{2}$	2304	XXIX

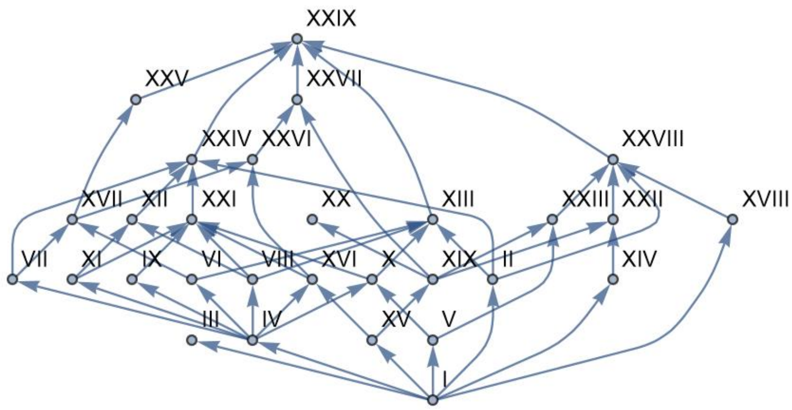

FIGURE 3. The grouping of eta quotients in Table 9, which have vanishing coefficient behaviour similar to f_1^6

Table 10: Eta quotients in Table 9 with expansions as double theta series

Number	Modular Form	Weight	Theta Series
1 I	$\eta(4z)^6$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-4}{m}\right)\left(\frac{-4}{n}\right)q^{\frac{1}{2}(m^2+n^2)}$
2 I	$\frac{\eta(8z)^{18}}{\eta(4z)^6\eta(16z)^6}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-8}{m}\right)\left(\frac{-8}{n}\right)q^{\frac{1}{2}(m^2+n^2)}$
6 I	$\frac{\eta(4z)^4\eta(24z)^2}{\eta(8z)^2\eta(12z)^2}$	1	$\sum_{m,n=1}^{\infty}\left(3\left(\frac{m}{6}\right)^2-2\left(\frac{m}{2}\right)^2\right)\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)$ $\times q^{\frac{1}{2}(m^2+n^2)}$
7 I	$\frac{\eta(8z)^{11}}{\eta(4z)^4\eta(16z)^3}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{2}\right)^2\left(\frac{-8}{n}\right)q^{\frac{1}{2}(m^2+n^2)}$
8 I	$\frac{\eta(4z)^4\eta(16z)}{\eta(8z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{8}{m}\right)\left(\frac{-4}{n}\right)q^{\frac{1}{2}(m^2+n^2)}$
13 I	$\frac{\eta(8z)^2\eta(16z)^7}{\eta(4z)^2\eta(32z)^3}$	2	$\sum_{\substack{m=-\infty \\ n=1}}^{\infty} n\left(\frac{-8}{n}\right)q^{4m^2+n^2}$
14 I	$\frac{\eta(4z)^2\eta(16z)^9}{\eta(8z)^4\eta(32z)^3}$	2	$\sum_{\substack{m=1 \\ n=-\infty}}^{\infty} m(-1)^n\left(\frac{-8}{m}\right)q^{m^2+4n^2}$
17 I	$\frac{\eta(8z)^4}{\eta(4z)^2}$	1	$\sum_{\substack{m=-\infty \\ n=1}}^{\infty}\left(\frac{n}{2}\right)^2 q^{4m^2+n^2}$
17 I	$\frac{\eta(8z)^4}{\eta(4z)^2}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{m}{2}\right)^2\left(\frac{n}{2}\right)^2 q^{\frac{1}{2}(m^2+n^2)}$
18 I	$\frac{\eta(4z)^2\eta(16z)^2}{\eta(8z)^2}$	1	$\sum_{\substack{m=1 \\ n=-\infty}}^{\infty}(-1)^n\left(\frac{m}{2}\right)^2 q^{m^2+4n^2}$
18 I	$\frac{\eta(4z)^2\eta(16z)^2}{\eta(8z)^2}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{8}{m}\right)\left(\frac{8}{n}\right)q^{\frac{1}{2}(m^2+n^2)}$
19 I	$\frac{\eta(8z)^6\eta(32z)}{\eta(4z)^2\eta(16z)^3}$	1	$\sum_{\substack{m=-\infty \\ n=1}}^{\infty}\left(\frac{8}{n}\right)q^{4m^2+n^2}$
20 I	$\frac{\eta(4z)^2\eta(32z)}{\eta(16z)}$	1	$\sum_{\substack{m=1 \\ n=-\infty}}^{\infty}(-1)^n\left(\frac{8}{m}\right)q^{m^2+4n^2}$
21 I	$\frac{\eta(8z)^8}{\eta(4z)^2\eta(16z)^2}$	2	$\sum_{\substack{m=-\infty \\ n=1}}^{\infty} n\left(\frac{-4}{n}\right)q^{4m^2+n^2}$
21 I	$\frac{\eta(8z)^8}{\eta(4z)^2\eta(16z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{8}{m}\right)\left(\frac{-8}{n}\right)q^{\frac{1}{2}(m^2+n^2)}$
22 I	$\eta(4z)^2\eta(8z)^2$	2	$\sum_{\substack{m=1 \\ n=-\infty}}^{\infty} m(-1)^n\left(\frac{-4}{m}\right)q^{m^2+4n^2}$
22 I	$\eta(4z)^2\eta(8z)^2$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{2}\right)^2\left(\frac{-4}{n}\right)q^{\frac{1}{2}(m^2+n^2)}$

23 I	$\frac{\eta(12z)\eta(16z)^9}{\eta(4z)\eta(8z)\eta(24z)\eta(32z)^3}$	2	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} m\left(\frac{-8}{m}\right)\left(-2\left(\frac{n}{2}\right)^2-\frac{3\left(\frac{n}{3}\right)^2}{2}+3\left(\frac{n}{6}\right)^2+1\right) \times q^{m^2+4n^2}$
24 I	$\frac{\eta(4z)\eta(16z)^{10}\eta(24z)^2}{\eta(8z)^4\eta(12z)\eta(32z)^3\eta(48z)}$	2	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} m\left(\frac{-8}{m}\right)\left(1-\frac{3\left(\frac{n}{3}\right)^2}{2}\right)q^{m^2+4n^2}$
27 I	$\frac{\eta(8z)^4\eta(12z)\eta(48z)}{\eta(4z)\eta(16z)\eta(24z)^2}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty}\left(-2\left(\frac{m}{2}\right)^2-\frac{3\left(\frac{m}{3}\right)^2}{2}+3\left(\frac{m}{6}\right)^2+1\right) \times\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{4m^2+n^2}$
28 I	$\frac{\eta(4z)\eta(8z)\eta(24z)}{\eta(12z)}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty}\left(1-\frac{3\left(\frac{m}{3}\right)^2}{2}\right)\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{4m^2+n^2}$
28 I	$\frac{\eta(4z)\eta(8z)\eta(24z)}{\eta(12z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{m}{2}\right)^2\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{\frac{1}{2}(m^2+n^2)}$
29 I	$\frac{\eta(8z)^5\eta(12z)}{\eta(4z)\eta(24z)}$	2	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} m\left(\frac{-4}{m}\right)\left(-2\left(\frac{n}{2}\right)^2-\frac{3\left(\frac{n}{3}\right)^2}{2}+3\left(\frac{n}{6}\right)^2+1\right) \times q^{m^2+4n^2}$
30 I	$\frac{\eta(4z)\eta(8z)^2\eta(16z)\eta(24z)^2}{\eta(12z)\eta(48z)}$	2	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} m\left(\frac{-4}{m}\right)\left(1-\frac{3\left(\frac{n}{3}\right)^2}{2}\right)q^{m^2+4n^2}$
31 I	$\frac{\eta(8z)^8\eta(24z)}{\eta(4z)\eta(12z)\eta(16z)^3}$	2	$\sum_{m,n=1}^{\infty} m\left(\frac{-8}{m}\right)\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{\frac{1}{2}(m^2+n^2)}$
35 I	$\frac{\eta(4z)^5\eta(24z)}{\eta(8z)\eta(12z)}$	2	$\sum_{m,n=1}^{\infty} m\left(\frac{-4}{m}\right)\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{\frac{1}{2}(m^2+n^2)}$
45 II	$\frac{\eta(4z)^2\eta(8z)\eta(48z)}{\eta(16z)\eta(24z)}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty}(-1)^m\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{4m^2+n^2}$
46 II	$\frac{\eta(8z)^7\eta(48z)}{\eta(4z)^2\eta(16z)^3\eta(24z)}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty}\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{4m^2+n^2}$
47 III	$\frac{\eta(4z)\eta(24z)^2\eta(32z)}{\eta(12z)\eta(48z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}\left(\frac{8}{m}\right)\left(1-\frac{3\left(\frac{n}{3}\right)^2}{2}\right)q^{m^2+4n^2}$
48 III	$\frac{\eta(8z)^3\eta(12z)\eta(32z)}{\eta(4z)\eta(16z)\eta(24z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}\left(\frac{8}{m}\right)\left(-2\left(\frac{n}{2}\right)^2-\frac{3\left(\frac{n}{3}\right)^2}{2}+3\left(\frac{n}{6}\right)^2+1\right) \times q^{m^2+4n^2}$
49 III	$\frac{\eta(4z)\eta(16z)^3\eta(24z)^2}{\eta(8z)^2\eta(12z)\eta(48z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}\left(\frac{m}{2}\right)^2\left(1-\frac{3\left(\frac{n}{3}\right)^2}{2}\right)q^{m^2+4n^2}$
50 III	$\frac{\eta(8z)\eta(12z)\eta(16z)^2}{\eta(4z)\eta(24z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}\left(\frac{m}{2}\right)^2\left(-2\left(\frac{n}{2}\right)^2-\frac{3\left(\frac{n}{3}\right)^2}{2}+3\left(\frac{n}{6}\right)^2+1\right) \times q^{m^2+4n^2}$
51 IV	$\frac{\eta(4z)\eta(16z)\eta(24z)^{13}}{\eta(8z)\eta(12z)^5\eta(48z)^5}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{8}{m}\right)\left(\frac{-6}{n}\right)q^{\frac{1}{2}(m^2+n^2)}$
52 IV	$\frac{\eta(8z)^2\eta(12z)^5}{\eta(4z)\eta(24z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{2}\right)^2\left(\frac{n}{12}\right)q^{\frac{1}{2}(m^2+n^2)}$
55 IV	$\frac{\eta(4z)\eta(12z)^5\eta(16z)}{\eta(8z)\eta(24z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{8}{m}\right)\left(\frac{n}{12}\right)q^{\frac{1}{2}(m^2+n^2)}$
56 IV	$\frac{\eta(8z)^2\eta(24z)^{13}}{\eta(4z)\eta(12z)^5\eta(48z)^5}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{2}\right)^2\left(\frac{-6}{n}\right)q^{\frac{1}{2}(m^2+n^2)}$
57 IV	$\frac{\eta(4z)^2\eta(24z)^{14}}{\eta(8z)\eta(12z)^6\eta(48z)^5}$	2	$\sum_{m,n=1}^{\infty} m\left(\frac{-6}{m}\right)\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{\frac{1}{2}(m^2+n^2)}$
61 IV	$\frac{\eta(4z)^2\eta(12z)^4}{\eta(8z)\eta(24z)}$	2	$\sum_{m,n=1}^{\infty} m\left(\frac{m}{12}\right)\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{\frac{1}{2}(m^2+n^2)}$
63 IV	$\frac{\eta(4z)^3\eta(24z)^3}{\eta(12z)\eta(48z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{24}{m}\right)\left(\frac{-4}{n}\right)q^{\frac{1}{2}(m^2+n^2)}$
64 IV	$\frac{\eta(8z)^9\eta(12z)}{\eta(4z)^3\eta(16z)^3}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{-8}{n}\right)q^{\frac{1}{2}(m^2+n^2)}$

65 IV	$\eta(4z)^3\eta(12z)$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{-4}{n}\right)q^{\frac{1}{2}(m^2+n^2)}$
66 IV	$\frac{\eta(8z)^9\eta(24z)^3}{\eta(4z)^3\eta(12z)\eta(16z)^3\eta(48z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{24}{m}\right)\left(\frac{-8}{n}\right)q^{\frac{1}{2}(m^2+n^2)}$
83 X	$\frac{\eta(8z)^2\eta(12z)\eta(48z)^{13}}{\eta(4z)\eta(24z)^6\eta(96z)^5}$	2	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} m\left(\frac{-6}{m}\right)\left(-2\left(\frac{n}{2}\right)^2-\frac{3\left(\frac{n}{3}\right)^2}{2}+3\left(\frac{n}{6}\right)^2+1\right) \times q^{m^2+4n^2}$
84 X	$\frac{\eta(4z)\eta(16z)\eta(48z)^{12}}{\eta(8z)\eta(12z)\eta(24z)^3\eta(96z)^5}$	2	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} m\left(\frac{-6}{m}\right)\left(1-\frac{3\left(\frac{n}{3}\right)^2}{2}\right)q^{m^2+4n^2}$
85 X	$\frac{\eta(8z)^5\eta(48z)^{13}}{\eta(4z)^2\eta(16z)^2\eta(24z)^5\eta(96z)^5}$	2	$\sum_{\substack{m=-\infty\\n=1}}^{\infty} n\left(\frac{-6}{n}\right)q^{4m^2+n^2}$
86 X	$\frac{\eta(4z)^2\eta(48z)^{13}}{\eta(8z)\eta(24z)^5\eta(96z)^5}$	2	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} m(-1)^n\left(\frac{-6}{m}\right)q^{m^2+4n^2}$
87 X	$\frac{\eta(8z)^9\eta(24z)^{13}}{\eta(4z)^3\eta(12z)^5\eta(16z)^3\eta(48z)^5}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-8}{m}\right)\left(\frac{-6}{n}\right)q^{\frac{1}{2}(m^2+n^2)}$
88 X	$\frac{\eta(4z)^3\eta(12z)^5}{\eta(24z)^2}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-4}{m}\right)\left(\frac{n}{12}\right)q^{\frac{1}{2}(m^2+n^2)}$
89 X	$\frac{\eta(4z)\eta(16z)\eta(24z)^7}{\eta(8z)\eta(12z)\eta(48z)^3}$	2	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} m\left(\frac{m}{12}\right)\left(1-\frac{3\left(\frac{n}{3}\right)^2}{2}\right)q^{m^2+4n^2}$
90 X	$\frac{\eta(8z)^2\eta(12z)\eta(24z)^4}{\eta(4z)\eta(48z)^2}$	2	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} m\left(\frac{m}{12}\right)\left(-2\left(\frac{n}{2}\right)^2-\frac{3\left(\frac{n}{3}\right)^2}{2}+3\left(\frac{n}{6}\right)^2+1\right) \times q^{m^2+4n^2}$
91 X	$\frac{\eta(4z)^2\eta(24z)^5}{\eta(8z)\eta(48z)^2}$	2	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} m(-1)^n\left(\frac{m}{12}\right)q^{m^2+4n^2}$
92 X	$\frac{\eta(8z)^5\eta(24z)^5}{\eta(4z)^2\eta(16z)^2\eta(48z)^2}$	2	$\sum_{\substack{m=-\infty\\n=1}}^{\infty} n\left(\frac{n}{12}\right)q^{4m^2+n^2}$
93 XI	$\frac{\eta(4z)^3\eta(24z)^{13}}{\eta(12z)^5\eta(48z)^5}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-4}{m}\right)\left(\frac{-6}{n}\right)q^{\frac{1}{2}(m^2+n^2)}$
94 XI	$\frac{\eta(8z)^9\eta(12z)^5}{\eta(4z)^3\eta(16z)^3\eta(24z)^2}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-8}{m}\right)\left(\frac{n}{12}\right)q^{\frac{1}{2}(m^2+n^2)}$
95 XII	$\frac{\eta(4z)\eta(12z)\eta(16z)}{\eta(8z)}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{8}{m}\right)\left(\frac{12}{n}\right)q^{\frac{1}{2}(m^2+n^2)}$
96 XII	$\frac{\eta(8z)^2\eta(24z)^3}{\eta(4z)\eta(12z)\eta(48z)}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{m}{2}\right)^2\left(\frac{24}{n}\right)q^{\frac{1}{2}(m^2+n^2)}$
97 XII	$\frac{\eta(4z)^2\eta(24z)^4}{\eta(8z)\eta(12z)^2\eta(48z)}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{24}{m}\right)\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{\frac{1}{2}(m^2+n^2)}$
99 XIII	$\frac{\eta(8z)^2\eta(12z)\eta(48z)^3}{\eta(4z)\eta(24z)^2\eta(96z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} \left(\frac{24}{m}\right)\left(-2\left(\frac{n}{2}\right)^2-\frac{3\left(\frac{n}{3}\right)^2}{2}+3\left(\frac{n}{6}\right)^2+1\right) \times q^{m^2+4n^2}$
100 XIII	$\frac{\eta(4z)\eta(16z)\eta(24z)\eta(48z)^2}{\eta(8z)\eta(12z)\eta(96z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} \left(\frac{24}{m}\right)\left(1-\frac{3\left(\frac{n}{3}\right)^2}{2}\right)q^{m^2+4n^2}$
101 XIII	$\frac{\eta(8z)^5\eta(48z)^3}{\eta(4z)^2\eta(16z)^2\eta(24z)\eta(96z)}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty} \left(\frac{24}{n}\right)q^{4m^2+n^2}$
102 XIII	$\frac{\eta(4z)^2\eta(48z)^3}{\eta(8z)\eta(24z)\eta(96z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} (-1)^n\left(\frac{24}{m}\right)q^{m^2+4n^2}$
103 XIII	$\frac{\eta(4z)\eta(16z)\eta(24z)^3}{\eta(8z)\eta(12z)\eta(48z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} \left(\frac{12}{m}\right)\left(1-\frac{3\left(\frac{n}{3}\right)^2}{2}\right)q^{m^2+4n^2}$
103 XIII	$\frac{\eta(4z)\eta(16z)\eta(24z)^3}{\eta(8z)\eta(12z)\eta(48z)}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{8}{m}\right)\left(\frac{24}{n}\right)q^{\frac{1}{2}(m^2+n^2)}$
104 XIII	$\frac{\eta(8z)^2\eta(12z)}{\eta(4z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} \left(\frac{12}{m}\right)\left(-2\left(\frac{n}{2}\right)^2-\frac{3\left(\frac{n}{3}\right)^2}{2}+3\left(\frac{n}{6}\right)^2+1\right) \times q^{m^2+4n^2}$
104 XIII	$\frac{\eta(8z)^2\eta(12z)}{\eta(4z)}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{m}{2}\right)^2\left(\frac{12}{n}\right)q^{\frac{1}{2}(m^2+n^2)}$
105 XIII	$\frac{\eta(4z)^2\eta(24z)}{\eta(8z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} (-1)^n\left(\frac{12}{m}\right)q^{m^2+4n^2}$

105 XIII	$\frac{\eta(4z)^2\eta(24z)}{\eta(8z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{\frac{1}{2}(m^2+n^2)}$
106 XIII	$\frac{\eta(8z)^5\eta(24z)}{\eta(4z)^2\eta(16z)^2}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty}\left(\frac{12}{n}\right)q^{4m^2+n^2}$
109 XIV	$\frac{\eta(4z)^3\eta(16z)\eta(24z)}{\eta(8z)^2\eta(12z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{8}{m}\right)\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{\frac{1}{2}(m^2+n^2)}$
111 XV	$\frac{\eta(16z)^8\eta(48z)}{\eta(8z)^3\eta(32z)\eta(96z)}$	2	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} m\left(\frac{-8}{m}\right)\left(-2\left(\frac{n}{2}\right)^2-\frac{3\left(\frac{n}{3}\right)^2}{2}+3\left(\frac{n}{6}\right)^2+1\right)\times q^{m^2+16n^2}$
114 XV	$\frac{\eta(8z)^3\eta(32z)^2\eta(48z)}{\eta(16z)\eta(96z)}$	2	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} m\left(\frac{-4}{m}\right)\left(-2\left(\frac{n}{2}\right)^2-\frac{3\left(\frac{n}{3}\right)^2}{2}+3\left(\frac{n}{6}\right)^2+1\right)\times q^{m^2+16n^2}$
115 XV	$\frac{\eta(8z)^9}{\eta(16z)^3}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-4}{m}\right)\left(\frac{-8}{n}\right)q^{\frac{1}{2}(m^2+n^2)}$
116 XV	$\frac{\eta(8z)^3\eta(16z)^2}{\eta(32z)}$	2	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} m(-1)^n\left(\frac{-4}{m}\right)q^{m^2+16n^2}$
117 XV	$\frac{\eta(16z)^{11}}{\eta(8z)^3\eta(32z)^4}$	2	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} m(-1)^n\left(\frac{-8}{m}\right)q^{m^2+16n^2}$
121 XVII	$\frac{\eta(12z)^2\eta(48z)^2\eta(72z)^{10}}{\eta(24z)^4\eta(36z)^4\eta(144z)^4}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{18}{m}\right)\left(\frac{18}{n}\right)q^{\frac{m^2}{2}+\frac{n^2}{2}}$
122 XVII	$\frac{\eta(24z)^2\eta(36z)^4}{\eta(12z)^2\eta(72z)^2}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{m}{6}\right)^2\left(\frac{n}{6}\right)^2 q^{\frac{m^2}{2}+\frac{n^2}{2}}$
123 XVII	$\frac{\eta(12z)^4\eta(24z)}{\eta(48z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{24}{m}\right)\left(\frac{n}{12}\right)q^{\frac{m^2}{2}+\frac{n^2}{2}}$
124 XVII	$\frac{\eta(24z)^{13}}{\eta(12z)^4\eta(48z)^5}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{-6}{n}\right)q^{\frac{m^2}{2}+\frac{n^2}{2}}$
125 XVII	$\frac{\eta(12z)^6}{\eta(24z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{n}{12}\right)q^{\frac{m^2}{2}+\frac{n^2}{2}}$
126 XVII	$\frac{\eta(24z)^{16}}{\eta(12z)^6\eta(48z)^6}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{24}{m}\right)\left(\frac{-6}{n}\right)q^{\frac{m^2}{2}+\frac{n^2}{2}}$
127 XVII	$\frac{\eta(12z)^{10}}{\eta(24z)^4}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{m}{12}\right)\left(\frac{n}{12}\right)q^{\frac{m^2}{2}+\frac{n^2}{2}}$
128 XVII	$\frac{\eta(24z)^{26}}{\eta(12z)^{10}\eta(48z)^{10}}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-6}{m}\right)\left(\frac{-6}{n}\right)q^{\frac{m^2}{2}+\frac{n^2}{2}}$
129 XVIII	$\frac{\eta(8z)^3\eta(40z)^3}{\eta(4z)\eta(16z)\eta(20z)\eta(80z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{24}{m}\right)\left(\frac{24}{n}\right)q^{\frac{1}{6}(m^2+5n^2)}$
130 XVIII	$\eta(4z)\eta(20z)$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{12}{n}\right)q^{\frac{1}{6}(5m^2+n^2)}$
132 XIX	$\frac{\eta(8z)^2\eta(16z)\eta(48z)}{\eta(24z)\eta(32z)}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty}(-1)^m\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{16m^2+n^2}$
133 XX	$\frac{\eta(8z)\eta(32z)^3\eta(48z)}{\eta(16z)^2\eta(96z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}\left(\frac{8}{m}\right)\left(-2\left(\frac{n}{2}\right)^2-\frac{3\left(\frac{n}{3}\right)^2}{2}+3\left(\frac{n}{6}\right)^2+1\right)\times q^{m^2+16n^2}$
134 XX	$\frac{\eta(16z)\eta(32z)^2\eta(48z)}{\eta(8z)\eta(96z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}\left(\frac{m}{2}\right)^2\left(-2\left(\frac{n}{2}\right)^2-\frac{3\left(\frac{n}{3}\right)^2}{2}+3\left(\frac{n}{6}\right)^2+1\right)\times q^{m^2+16n^2}$
135 XXI	$\frac{\eta(32z)^2\eta(48z)^{14}}{\eta(16z)\eta(24z)^5\eta(96z)^6}$	2	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} m\left(\frac{-6}{m}\right)\left(-2\left(\frac{n}{2}\right)^2-\frac{3\left(\frac{n}{3}\right)^2}{2}+3\left(\frac{n}{6}\right)^2+1\right)\times q^{m^2+16n^2}$
136 XXI	$\frac{\eta(24z)^5\eta(32z)^2}{\eta(16z)\eta(48z)\eta(96z)}$	2	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} m\left(\frac{m}{12}\right)\left(-2\left(\frac{n}{2}\right)^2-\frac{3\left(\frac{n}{3}\right)^2}{2}+3\left(\frac{n}{6}\right)^2+1\right)\times q^{m^2+16n^2}$
137 XXI	$\frac{\eta(16z)^2\eta(48z)^{13}}{\eta(24z)^5\eta(32z)\eta(96z)^5}$	2	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} m(-1)^n\left(\frac{-6}{m}\right)q^{m^2+16n^2}$

138 XXI	$\frac{\eta(16z)^2\eta(24z)^5}{\eta(32z)\eta(48z)^2}$	2	$\sum_{m=1,\, n=-\infty}^{\infty} m(-1)^n \left(\frac{m}{12}\right) q^{m^2+16n^2}$
140 XXII	$\frac{\eta(8z)^4\eta(48z)}{\eta(16z)^2\eta(24z)}$	1	$\sum_{m=-\infty,\, n=1}^{\infty}(-1)^m\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{8m^2+n^2}$
141 XXII	$\eta(8z)\eta(16z)$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{12}{n}\right)q^{\frac{1}{3}(2m^2+n^2)}$
141 XXII	$\eta(8z)\eta(16z)$	1	$\sum_{m=1,\, n=-\infty}^{\infty}(-1)^n\left(\frac{8}{m}\right)q^{m^2+16n^2}$
141 XXII	$\eta(8z)\eta(16z)$	1	$\sum_{m=-\infty,\, n=1}^{\infty}\left(-2\left(\frac{m}{2}\right)^2-\frac{3\left(\frac{m}{3}\right)^2}{2}+3\left(\frac{m}{6}\right)^2+1\right) \times\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{8m^2+n^2}$
141 XXII	$\eta(8z)\eta(16z)$	1	$\sum_{m=1,\, n=-\infty}^{\infty}(-1)^n\left(\frac{m}{2}\right)^2 q^{m^2+8n^2}$
141 XXII	$\eta(8z)\eta(16z)$	1	$\sum_{m,n=1}^{\infty}\left(\frac{8}{m}\right)\left(\frac{n}{2}\right)^2 q^{\frac{1}{2}(m^2+n^2)}$
142 XXII	$\frac{\eta(16z)^4}{\eta(8z)\eta(32z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{24}{n}\right)q^{\frac{1}{3}(2m^2+n^2)}$
142 XXII	$\frac{\eta(16z)^4}{\eta(8z)\eta(32z)}$	1	$\sum_{m=1,\, n=-\infty}^{\infty}(-1)^n\left(\frac{m}{2}\right)^2 q^{m^2+16n^2}$
142 XXII	$\frac{\eta(16z)^4}{\eta(8z)\eta(32z)}$	1	$\sum_{m=-\infty,\, n=1}^{\infty}\left(\frac{8}{n}\right)q^{8m^2+n^2}$
143 XXII	$\frac{\eta(8z)^2\eta(32z)^2\eta(48z)^2}{\eta(16z)^2\eta(24z)\eta(96z)}$	1	$\sum_{m=-\infty,\, n=1}^{\infty}\left(-2\left(\frac{m}{2}\right)^2-\frac{3\left(\frac{m}{3}\right)^2}{2}+3\left(\frac{m}{6}\right)^2+1\right) \times\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{16m^2+n^2}$
143 XXII	$\frac{\eta(8z)^2\eta(32z)^2\eta(48z)^2}{\eta(16z)^2\eta(24z)\eta(96z)}$	1	$\sum_{m=1,\, n=-\infty}^{\infty}\left(\frac{8}{m}\right)\left(1-\frac{3\left(\frac{n}{3}\right)^2}{2}\right)q^{m^2+8n^2}$
144 XXII	$\frac{\eta(16z)^4\eta(24z)}{\eta(8z)^2\eta(48z)}$	1	$\sum_{m=1,\, n=-\infty}^{\infty}\left(\frac{m}{2}\right)^2\left(-2\left(\frac{n}{2}\right)^2-\frac{3\left(\frac{n}{3}\right)^2}{2}+3\left(\frac{n}{6}\right)^2+1\right) \times q^{m^2+8n^2}$
147 XXIV	$\frac{\eta(32z)^2\eta(48z)^4}{\eta(16z)\eta(24z)\eta(96z)^2}$	1	$\sum_{m=1,\, n=-\infty}^{\infty}\left(\frac{24}{m}\right)\left(-2\left(\frac{n}{2}\right)^2-\frac{3\left(\frac{n}{3}\right)^2}{2}+3\left(\frac{n}{6}\right)^2+1\right) \times q^{m^2+16n^2}$
148 XXIV	$\frac{\eta(24z)\eta(32z)^2\eta(48z)}{\eta(16z)\eta(96z)}$	1	$\sum_{m=1,\, n=-\infty}^{\infty}\left(\frac{12}{m}\right)\left(-2\left(\frac{n}{2}\right)^2-\frac{3\left(\frac{n}{3}\right)^2}{2}+3\left(\frac{n}{6}\right)^2+1\right) \times q^{m^2+16n^2}$
149 XXIV	$\frac{\eta(16z)^2\eta(48z)^3}{\eta(24z)\eta(32z)\eta(96z)}$	1	$\sum_{m=1,\, n=-\infty}^{\infty}(-1)^n\left(\frac{24}{m}\right)q^{m^2+16n^2}$
150 XXIV	$\frac{\eta(16z)^2\eta(24z)}{\eta(32z)}$	1	$\sum_{m=1,\, n=-\infty}^{\infty}(-1)^n\left(\frac{12}{m}\right)q^{m^2+16n^2}$
151 XXV	$\frac{\eta(24z)^4\eta(48z)}{\eta(12z)^2\eta(96z)}$	1	$\sum_{m=-\infty,\, n=1}^{\infty}\left(\frac{24}{n}\right)q^{12m^2+n^2}$
152 XXV	$\frac{\eta(12z)^2\eta(48z)^3}{\eta(24z)^2\eta(96z)}$	1	$\sum_{m=1,\, n=-\infty}^{\infty}(-1)^n\left(\frac{24}{m}\right)q^{m^2+12n^2}$
153 XXV	$\eta(12z)^2$	1	$\sum_{m=1,\, n=-\infty}^{\infty}(-1)^n\left(\frac{12}{m}\right)q^{m^2+12n^2}$
153 XXV	$\eta(12z)^2$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{12}{n}\right)q^{\frac{m^2}{2}+\frac{n^2}{2}}$
154 XXV	$\frac{\eta(24z)^6}{\eta(12z)^2\eta(48z)^2}$	1	$\sum_{m=-\infty,\, n=1}^{\infty}\left(\frac{12}{n}\right)q^{12m^2+n^2}$
154 XXV	$\frac{\eta(24z)^6}{\eta(12z)^2\eta(48z)^2}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{24}{m}\right)\left(\frac{24}{n}\right)q^{\frac{m^2}{2}+\frac{n^2}{2}}$
158 XXVI	$\frac{\eta(24z)^{11}}{\eta(48z)^5}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-6}{m}\right)\left(\frac{n}{12}\right)q^{\frac{m^2}{2}+\frac{n^2}{2}}$
159 XXVII	$\frac{\eta(48z)^8}{\eta(24z)^3\eta(96z)^3}$	1	$\sum_{m=-\infty,\, n=1}^{\infty}\left(\frac{24}{n}\right)q^{24m^2+n^2}$

160 XXVII	$\frac{\eta(24z)^3}{\eta(48z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}(-1)^n\left(\frac{12}{m}\right)q^{m^2+24n^2}$
160 XXVII	$\frac{\eta(24z)^3}{\eta(48z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{24}{n}\right)q^{\frac{m^2}{2}+\frac{n^2}{2}}$

Table 11: Eta quotients vanishing behaviour similar to f_1^8

Number	q-Product	Modular Form	Weight	Level	Group
1	f_1^8	$\eta(3z)^8$	4	9	I
2	$\frac{f_2^{24}}{f_1^8f_4^8}$	$\frac{\eta(6z)^{24}}{\eta(3z)^8\eta(12z)^8}$	4	36	I
3	$\frac{f_2^9f_3^4f_{12}}{f_1^4f_4^3f_6^3}$	$\frac{\eta(6z)^9\eta(9z)^4\eta(36z)}{\eta(3z)^4\eta(12z)^3\eta(18z)^3}$	2	108	I
4	$\frac{f_1^4f_4f_6^9}{f_2^3f_3^4f_{12}^3}$	$\frac{\eta(3z)^4\eta(12z)\eta(18z)^9}{\eta(6z)^3\eta(9z)^4\eta(36z)^3}$	2	108	I
5	$\frac{f_2^{11}f_6^{11}}{f_1^4f_3^4f_4^4f_{12}^4}$	$\frac{\eta(6z)^{11}\eta(18z)^{11}}{\eta(3z)^4\eta(9z)^4\eta(12z)^4\eta(36z)^4}$	3	108	I
6	$\frac{f_1^4f_3^4}{f_2f_6}$	$\frac{\eta(3z)^4\eta(9z)^4}{\eta(6z)\eta(18z)}$	3	108	I
7	$\frac{f_2^7f_3}{f_1^3f_6}$	$\frac{\eta(6z)^7\eta(9z)}{\eta(3z)^3\eta(18z)}$	2	108	I
8	$\frac{f_1^3f_4^3f_6^2}{f_2^2f_3f_{12}}$	$\frac{\eta(3z)^3\eta(12z)^3\eta(18z)^2}{\eta(6z)^2\eta(9z)\eta(36z)}$	2	108	I
9	$\frac{f_2^7f_3f_{12}}{f_1^3f_4^3f_6}$	$\frac{\eta(6z)^7\eta(9z)\eta(36z)}{\eta(3z)^3\eta(12z)^3\eta(18z)}$	1	36	I
10	$\frac{f_1^3f_6^2}{f_2^2f_3}$	$\frac{\eta(3z)^3\eta(18z)^2}{\eta(6z)^2\eta(9z)}$	1	36	I
11	$\frac{f_2^2f_3^6}{f_1^2f_6^2}$	$\frac{\eta(6z)^2\eta(9z)^6}{\eta(3z)^2\eta(18z)^2}$	2	108	I
12	$\frac{f_1^2f_4^2f_6^{16}}{f_2^4f_3^6f_{12}^6}$	$\frac{\eta(3z)^2\eta(12z)^2\eta(18z)^{16}}{\eta(6z)^4\eta(9z)^6\eta(36z)^6}$	2	108	I
13	$\frac{f_2^6f_6^6}{f_1^2f_3^2f_4^2f_{12}^2}$	$\frac{\eta(6z)^6\eta(18z)^6}{\eta(3z)^2\eta(9z)^2\eta(12z)^2\eta(36z)^2}$	2	108	I
14	$f_1^2f_3^2$	$\eta(3z)^2\eta(9z)^2$	2	27	I
15	$\frac{f_3^3}{f_1}$	$\frac{\eta(9z)^3}{\eta(3z)}$	1	9	I
16	$\frac{f_1f_4f_6^9}{f_2^3f_3^3f_{12}^3}$	$\frac{\eta(3z)\eta(12z)\eta(18z)^9}{\eta(6z)^3\eta(9z)^3\eta(36z)^3}$	1	36	I
17	$\frac{f_3^3f_4^3}{f_1f_{12}}$	$\frac{\eta(9z)^3\eta(12z)^3}{\eta(3z)\eta(36z)}$	2	108	I
18	$\frac{f_1f_4^4f_6^9}{f_2^3f_3^3f_{12}^4}$	$\frac{\eta(3z)\eta(12z)^4\eta(18z)^9}{\eta(6z)^3\eta(9z)^3\eta(36z)^4}$	2	108	I
19	$\frac{f_2^4f_6^2}{f_1f_3f_4^2}$	$\frac{\eta(6z)^4\eta(18z)^2}{\eta(3z)\eta(9z)\eta(12z)^2}$	1	144	I
20	$\frac{f_1f_2f_3f_{12}}{f_4f_6}$	$\frac{\eta(3z)\eta(6z)\eta(9z)\eta(36z)}{\eta(12z)\eta(18z)}$	1	144	I
21	$\frac{f_2^4f_4f_6^2}{f_1f_3f_{12}}$	$\frac{\eta(6z)^4\eta(12z)\eta(18z)^2}{\eta(3z)\eta(9z)\eta(36z)}$	2	432	I

22	$\frac{f_1f_2f_3f_4^2}{f_6}$	$\frac{\eta(3z)\eta(6z)\eta(9z)\eta(12z)^2}{\eta(18z)}$	2	432	I
23	$\frac{f_1^6f_6^2}{f_2^2f_3^2}$	$\frac{\eta(3z)^6\eta(18z)^2}{\eta(6z)^2\eta(9z)^2}$	2	108	I
24	$\frac{f_2^{16}f_3^2f_{12}^2}{f_1^6f_4^6f_6^4}$	$\frac{\eta(6z)^{16}\eta(9z)^2\eta(36z)^2}{\eta(3z)^6\eta(12z)^6\eta(18z)^4}$	2	108	I
25	$\frac{f_1^4f_4^2}{f_2^2}$	$\frac{\eta(3z)^4\eta(12z)^2}{\eta(6z)^2}$	2	144	II
26	$\frac{f_2^{10}}{f_1^4f_4^2}$	$\frac{\eta(6z)^{10}}{\eta(3z)^4\eta(12z)^2}$	2	144	II
27	$\frac{f_1^2f_4f_6^4}{f_2^2f_3^2f_{12}}$	$\frac{\eta(3z)^2\eta(12z)\eta(18z)^4}{\eta(6z)^2\eta(9z)^2\eta(36z)}$	1	432	III
28	$\frac{f_2^4f_3^2f_{12}}{f_1^2f_4f_6^2}$	$\frac{\eta(6z)^4\eta(9z)^2\eta(36z)}{\eta(3z)^2\eta(12z)\eta(18z)^2}$	1	432	III
29	$\frac{f_2^5f_8^2f_{12}^{16}}{f_4^6f_6^7f_{24}^6}$	$\frac{\eta(6z)^5\eta(24z)^2\eta(36z)^{16}}{\eta(12z)^6\eta(18z)^7\eta(72z)^6}$	2	432	IV
30	$\frac{f_2f_8^2f_{12}^{14}}{f_4^4f_6^3f_{24}^6}$	$\frac{\eta(6z)\eta(24z)^2\eta(36z)^{14}}{\eta(12z)^4\eta(18z)^3\eta(72z)^6}$	2	432	IV
31	$\frac{f_4^6f_6f_{12}^4}{f_2^3f_8^2f_{24}^2}$	$\frac{\eta(12z)^6\eta(18z)\eta(36z)^4}{\eta(6z)^3\eta(24z)^2\eta(72z)^2}$	2	432	IV
32	$\frac{f_2f_6^3f_8f_{12}^3}{f_4^2f_{24}^2}$	$\frac{\eta(6z)\eta(18z)^3\eta(24z)\eta(36z)^3}{\eta(12z)^2\eta(72z)^2}$	2	1728	IV
33	$\frac{f_4^9f_6^7f_{24}}{f_2^5f_8^3f_{12}^5}$	$\frac{\eta(12z)^9\eta(18z)^7\eta(72z)}{\eta(6z)^5\eta(24z)^3\eta(36z)^5}$	2	432	IV
34	$\frac{f_6^3f_8f_{12}^5}{f_2f_4f_{24}^3}$	$\frac{\eta(18z)^3\eta(24z)\eta(36z)^5}{\eta(6z)\eta(12z)\eta(72z)^3}$	2	432	IV
35	$\frac{f_4^{16}f_6^5f_{24}^2}{f_2^7f_8^6f_{12}^6}$	$\frac{\eta(12z)^{16}\eta(18z)^5\eta(72z)^2}{\eta(6z)^7\eta(24z)^6\eta(36z)^6}$	2	432	IV
36	$\frac{f_4^7f_6^3f_{24}}{f_2f_8^3f_{12}^3}$	$\frac{\eta(12z)^7\eta(18z)^3\eta(72z)}{\eta(6z)\eta(24z)^3\eta(36z)^3}$	2	432	IV
37	$\frac{f_2^3f_8f_{12}^7}{f_4^3f_6f_{24}^3}$	$\frac{\eta(6z)^3\eta(24z)\eta(36z)^7}{\eta(12z)^3\eta(18z)\eta(72z)^3}$	2	432	IV
38	$\frac{f_2^3f_6f_{12}^2}{f_4f_{24}}$	$\frac{\eta(6z)^3\eta(18z)\eta(36z)^2}{\eta(12z)\eta(72z)}$	2	1728	IV
39	$\frac{f_2f_4^4f_{12}^6}{f_6^3f_8^2f_{24}^2}$	$\frac{\eta(6z)\eta(12z)^4\eta(36z)^6}{\eta(18z)^3\eta(24z)^2\eta(72z)^2}$	2	432	IV
40	$\frac{f_4^8f_{12}^5}{f_2^3f_6f_8^3f_{24}^2}$	$\frac{\eta(12z)^8\eta(36z)^5}{\eta(6z)^3\eta(18z)\eta(24z)^3\eta(72z)^2}$	2	1728	IV
41	$\frac{f_4f_{12}^{12}}{f_2f_6^3f_{24}^5}$	$\frac{\eta(12z)\eta(36z)^{12}}{\eta(6z)\eta(18z)^3\eta(72z)^5}$	2	1728	IV
42	$\frac{f_4^{14}f_6f_{24}^2}{f_2^3f_8^6f_{12}^4}$	$\frac{\eta(12z)^{14}\eta(18z)\eta(72z)^2}{\eta(6z)^3\eta(24z)^6\eta(36z)^4}$	2	432	IV
43	$\frac{f_2^3f_4^5f_{24}}{f_6f_8^3f_{12}}$	$\frac{\eta(6z)^3\eta(12z)^5\eta(72z)}{\eta(18z)\eta(24z)^3\eta(36z)}$	2	432	IV
44	$\frac{f_2^7f_8f_{12}^9}{f_4^5f_6^5f_{24}^3}$	$\frac{\eta(6z)^7\eta(24z)\eta(36z)^9}{\eta(12z)^5\eta(18z)^5\eta(72z)^3}$	2	432	IV
45	$\frac{f_2^5f_{12}^{13}}{f_4^2f_6^5f_{24}^5}$	$\frac{\eta(6z)^5\eta(36z)^{13}}{\eta(12z)^2\eta(18z)^5\eta(72z)^5}$	3	1728	IV

46	$\frac{f_4^{13}f_6^5}{f_2^5f_8^5f_{12}^2}$	$\frac{\eta(12z)^{13}\eta(18z)^5}{\eta(6z)^5\eta(24z)^5\eta(36z)^2}$	3	1728	IV
47	$\frac{f_4^{30}f_6^2f_{24}^2}{f_2^{10}f_8^{10}f_{12}^6}$	$\frac{\eta(12z)^{30}\eta(18z)^2\eta(72z)^2}{\eta(6z)^{10}\eta(24z)^{10}\eta(36z)^6}$	4	432	IV
48	$\frac{f_2^{10}}{f_6^2}$	$\frac{\eta(6z)^{10}}{\eta(18z)^2}$	4	108	IV
49	$\frac{f_4^{23}f_6^3f_{24}^3}{f_2^9f_8^9f_{12}^7}$	$\frac{\eta(12z)^{23}\eta(18z)^3\eta(72z)^3}{\eta(6z)^9\eta(24z)^9\eta(36z)^7}$	2	432	IV
50	$\frac{f_2^9f_{12}^2}{f_4^4f_6^3}$	$\frac{\eta(6z)^9\eta(36z)^2}{\eta(12z)^4\eta(18z)^3}$	2	432	IV
51	$\frac{f_2^8}{f_4^2}$	$\frac{\eta(6z)^8}{\eta(12z)^2}$	3	144	IV
52	$\frac{f_4^{18}f_6f_{24}}{f_2^7f_8^7f_{12}^2}$	$\frac{\eta(12z)^{18}\eta(18z)\eta(72z)}{\eta(6z)^7\eta(24z)^7\eta(36z)^2}$	2	432	IV
53	$\frac{f_2^7f_{12}}{f_4^3f_6}$	$\frac{\eta(6z)^7\eta(36z)}{\eta(12z)^3\eta(18z)}$	2	108	IV
54	$\frac{f_2^6f_8}{f_4^3}$	$\frac{\eta(6z)^6\eta(24z)}{\eta(12z)^3}$	2	576	IV
55	$\frac{f_4^{13}f_{12}^{13}}{f_2^5f_6^5f_8^5f_{24}^5}$	$\frac{\eta(12z)^{13}\eta(36z)^{13}}{\eta(6z)^5\eta(18z)^5\eta(24z)^5\eta(72z)^5}$	3	432	IV
56	$\frac{f_2^5f_6^5}{f_4^2f_{12}^2}$	$\frac{\eta(6z)^5\eta(18z)^5}{\eta(12z)^2\eta(36z)^2}$	3	108	IV
57	$\frac{f_4^{13}f_{12}^3}{f_2^5f_6f_8^5f_{24}}$	$\frac{\eta(12z)^{13}\eta(36z)^3}{\eta(6z)^5\eta(18z)\eta(24z)^5\eta(72z)}$	2	432	IV
58	$\frac{f_2^5f_6}{f_4^2}$	$\frac{\eta(6z)^5\eta(18z)}{\eta(12z)^2}$	2	432	IV
59	f_2^4	$\eta(6z)^4$	2	36	IV
60	$\frac{f_4^7f_{12}^5}{f_2^2f_6^2f_8^2f_{24}^2}$	$\frac{\eta(12z)^7\eta(36z)^5}{\eta(6z)^2\eta(18z)^2\eta(24z)^2\eta(72z)^2}$	2	144	IV
61	$\frac{f_2^2f_4f_6^2}{f_{12}}$	$\frac{\eta(6z)^2\eta(12z)\eta(18z)^2}{\eta(36z)}$	2	144	IV
62	$\frac{f_2^2f_4^3}{f_8}$	$\frac{\eta(6z)^2\eta(12z)^3}{\eta(24z)}$	2	576	IV
63	$\frac{f_2^2f_4^7}{f_8^3}$	$\frac{\eta(6z)^2\eta(12z)^7}{\eta(24z)^3}$	3	576	IV
64	$\frac{f_4^5f_{12}^8}{f_2f_6^3f_8^2f_{24}^3}$	$\frac{\eta(12z)^5\eta(36z)^8}{\eta(6z)\eta(18z)^3\eta(24z)^2\eta(72z)^3}$	2	1728	IV
65	$\frac{f_2f_4^2f_6^3}{f_8f_{12}}$	$\frac{\eta(6z)\eta(12z)^2\eta(18z)^3}{\eta(24z)\eta(36z)}$	2	1728	IV
66	$\frac{f_4^3f_{12}^{13}}{f_2f_6^5f_8f_{24}^5}$	$\frac{\eta(12z)^3\eta(36z)^{13}}{\eta(6z)\eta(18z)^5\eta(24z)\eta(72z)^5}$	2	432	IV
67	$\frac{f_2f_6^5}{f_{12}^2}$	$\frac{\eta(6z)\eta(18z)^5}{\eta(36z)^2}$	2	432	IV
68	$\frac{f_2f_{12}^{13}}{f_6^5f_{24}^5}$	$\frac{\eta(6z)\eta(36z)^{13}}{\eta(18z)^5\eta(72z)^5}$	2	1728	IV
69	$\frac{f_4^3f_6^5}{f_2f_8f_{12}^2}$	$\frac{\eta(12z)^3\eta(18z)^5}{\eta(6z)\eta(24z)\eta(36z)^2}$	2	1728	IV
70	$\frac{f_2f_8f_{12}^{18}}{f_4^2f_6^7f_{24}^7}$	$\frac{\eta(6z)\eta(24z)\eta(36z)^{18}}{\eta(12z)^2\eta(18z)^7\eta(72z)^7}$	2	432	IV

71	$\frac{f_4 f_6^7}{f_2 f_{12}^3}$	$\frac{\eta(12z)\eta(18z)^7}{\eta(6z)\eta(36z)^3}$	2	108	IV
72	$\frac{f_2^2 f_8^2 f_{12}^5}{f_4^3 f_6^2 f_{24}^2}$	$\frac{\eta(6z)^2\eta(24z)^2\eta(36z)^5}{\eta(12z)^3\eta(18z)^2\eta(72z)^2}$	1	144	IV
73	$\frac{f_4^3 f_6^2}{f_2^2 f_{12}}$	$\frac{\eta(12z)^3\eta(18z)^2}{\eta(6z)^2\eta(36z)}$	1	36	IV
74	$\frac{f_2^2 f_8^2 f_{12}^4}{f_6^2 f_{24}^2}$	$\frac{\eta(6z)^2\eta(24z)^2\eta(36z)^4}{\eta(18z)^2\eta(72z)^2}$	2	432	IV
75	$\frac{f_4^6 f_6^2}{f_2^2 f_{12}^2}$	$\frac{\eta(12z)^6\eta(18z)^2}{\eta(6z)^2\eta(36z)^2}$	2	108	IV
76	$\frac{f_4^9}{f_2^2 f_8^3}$	$\frac{\eta(12z)^9}{\eta(6z)^2\eta(24z)^3}$	2	576	IV
77	$\frac{f_4^{13}}{f_2^2 f_8^5}$	$\frac{\eta(12z)^{13}}{\eta(6z)^2\eta(24z)^5}$	3	576	IV
78	$\frac{f_2^3 f_4^3 f_6 f_{24}}{f_8^2 f_{12}^2}$	$\frac{\eta(6z)^3\eta(12z)^3\eta(18z)\eta(72z)}{\eta(24z)^2\eta(36z)^2}$	2	1728	IV
79	$\frac{f_4^{12} f_{12}}{f_2^3 f_6 f_8^5}$	$\frac{\eta(12z)^{12}\eta(36z)}{\eta(6z)^3\eta(18z)\eta(24z)^5}$	2	1728	IV
80	$\frac{f_2^3 f_8^3 f_{12}^{23}}{f_4^7 f_6^9 f_{24}^9}$	$\frac{\eta(6z)^3\eta(24z)^3\eta(36z)^{23}}{\eta(12z)^7\eta(18z)^9\eta(72z)^9}$	2	432	IV
81	$\frac{f_4^2 f_6^9}{f_2^3 f_{12}^4}$	$\frac{\eta(12z)^2\eta(18z)^9}{\eta(6z)^3\eta(36z)^4}$	2	432	IV
82	$\frac{f_2^4 f_8 f_{12}^5}{f_4^2 f_6^2 f_{24}^2}$	$\frac{\eta(6z)^4\eta(24z)\eta(36z)^5}{\eta(12z)^2\eta(18z)^2\eta(72z)^2}$	2	576	IV
83	$\frac{f_4^{10} f_6^2}{f_2^4 f_8^3 f_{12}}$	$\frac{\eta(12z)^{10}\eta(18z)^2}{\eta(6z)^4\eta(24z)^3\eta(36z)}$	2	576	IV
84	$\frac{f_4^{12}}{f_2^4 f_8^4}$	$\frac{\eta(12z)^{12}}{\eta(6z)^4\eta(24z)^4}$	2	144	IV
85	$\frac{f_2^5 f_{12}^3}{f_4^2 f_6 f_{24}}$	$\frac{\eta(6z)^5\eta(36z)^3}{\eta(12z)^2\eta(18z)\eta(72z)}$	2	1728	IV
86	$\frac{f_4^{13} f_6}{f_2^5 f_8^5}$	$\frac{\eta(12z)^{13}\eta(18z)}{\eta(6z)^5\eta(24z)^5}$	2	1728	IV
87	$\frac{f_4^{15}}{f_2^6 f_8^5}$	$\frac{\eta(12z)^{15}}{\eta(6z)^6\eta(24z)^5}$	2	576	IV
88	$\frac{f_4^{22}}{f_2^8 f_8^8}$	$\frac{\eta(12z)^{22}}{\eta(6z)^8\eta(24z)^8}$	3	144	IV
89	$\frac{f_4^3 f_{12}^3}{f_2 f_6 f_8 f_{24}}$	$\frac{\eta(12z)^3\eta(36z)^3}{\eta(6z)\eta(18z)\eta(24z)\eta(72z)}$	1	432	V
90	$f_2 f_6$	$\eta(6z)\eta(18z)$	1	108	V
91	$\frac{f_2^6 f_8^3 f_{12}^2}{f_4^4 f_6^2 f_{24}}$	$\frac{\eta(6z)^6\eta(24z)^3\eta(36z)^2}{\eta(12z)^4\eta(18z)^2\eta(72z)}$	2	432	VI
92	$\frac{f_4^{14} f_6^2 f_{24}}{f_2^6 f_8^3 f_{12}^4}$	$\frac{\eta(12z)^{14}\eta(18z)^2\eta(72z)}{\eta(6z)^6\eta(24z)^3\eta(36z)^4}$	2	432	VI
93	$\frac{f_4^4 f_8 f_{12}^6}{f_2^2 f_6^2 f_{24}^3}$	$\frac{\eta(12z)^4\eta(24z)\eta(36z)^6}{\eta(6z)^2\eta(18z)^2\eta(72z)^3}$	2	432	VI
94	$\frac{f_2^2 f_6^2 f_8^3}{f_4^2 f_{24}}$	$\frac{\eta(6z)^2\eta(18z)^2\eta(24z)^3}{\eta(12z)^2\eta(72z)}$	2	432	VI
95	$\frac{f_2^4 f_8^4 f_{12}^9}{f_4^5 f_6^4 f_{24}^4}$	$\frac{\eta(6z)^4\eta(24z)^4\eta(36z)^9}{\eta(12z)^5\eta(18z)^4\eta(72z)^4}$	2	432	VI

96	$\frac{f_4^7 f_6^4}{f_2^4 f_{12}^3}$	$\frac{\eta(12z)^7\eta(18z)^4}{\eta(6z)^4\eta(36z)^3}$	2	432	VI
97	$\frac{f_2^2 f_8^5 f_{12}^{16}}{f_4^6 f_6^6 f_{24}^7}$	$\frac{\eta(6z)^2\eta(24z)^5\eta(36z)^{16}}{\eta(12z)^6\eta(18z)^6\eta(72z)^7}$	2	432	VII
98	$\frac{f_6^6 f_8^3}{f_2^2 f_{12}^2 f_{24}}$	$\frac{\eta(18z)^6\eta(24z)^3}{\eta(6z)^2\eta(36z)^2\eta(72z)}$	2	432	VII
99	$\frac{f_2^2 f_3 f_8^5 f_{12}}{f_1 f_4^3 f_6 f_{16}^2}$	$\frac{\eta(6z)^2\eta(9z)\eta(24z)^5\eta(36z)}{\eta(3z)\eta(12z)^3\eta(18z)\eta(48z)^2}$	1	144	VIII
100	$\frac{f_1 f_6^2 f_8^5}{f_2 f_3 f_4^2 f_{16}^2}$	$\frac{\eta(3z)\eta(18z)^2\eta(24z)^5}{\eta(6z)\eta(9z)\eta(12z)^2\eta(48z)^2}$	1	144	VIII
101	$\frac{f_2^2 f_3 f_4 f_{12}}{f_1 f_6 f_8}$	$\frac{\eta(6z)^2\eta(9z)\eta(12z)\eta(36z)}{\eta(3z)\eta(18z)\eta(24z)}$	1	576	VIII
102	$\frac{f_1 f_4^2 f_6^2}{f_2 f_3 f_8}$	$\frac{\eta(3z)\eta(12z)^2\eta(18z)^2}{\eta(6z)\eta(9z)\eta(24z)}$	1	576	VIII
103	$\frac{f_1^2 f_4^4}{f_2 f_8}$	$\frac{\eta(3z)^2\eta(12z)^4}{\eta(6z)\eta(24z)}$	2	576	IX
104	$\frac{f_2^5 f_4^2}{f_1^2 f_8}$	$\frac{\eta(6z)^5\eta(12z)^2}{\eta(3z)^2\eta(24z)}$	2	576	IX
105	$\frac{f_2^2 f_3 f_8^2 f_{12}^2}{f_1 f_4^2 f_6 f_{24}}$	$\frac{\eta(6z)^2\eta(9z)\eta(24z)^2\eta(36z)^2}{\eta(3z)\eta(12z)^2\eta(18z)\eta(72z)}$	1	1728	X
106	$\frac{f_1 f_6^2 f_8^2 f_{12}}{f_2 f_3 f_4 f_{24}}$	$\frac{\eta(3z)\eta(18z)^2\eta(24z)^2\eta(36z)}{\eta(6z)\eta(9z)\eta(12z)\eta(72z)}$	1	1728	X
107	$\frac{f_4^6 f_6^3 f_{24}}{f_2^3 f_8^2 f_{12}^3}$	$\frac{\eta(12z)^6\eta(18z)^3\eta(72z)}{\eta(6z)^3\eta(24z)^2\eta(36z)^3}$	1	1728	XI
108	$\frac{f_2^3 f_8 f_{12}^6}{f_4^3 f_6^3 f_{24}^2}$	$\frac{\eta(6z)^3\eta(24z)\eta(36z)^6}{\eta(12z)^3\eta(18z)^3\eta(72z)^2}$	1	1728	XI
109	$\frac{f_2 f_{12}^3}{f_6 f_{24}}$	$\frac{\eta(6z)\eta(36z)^3}{\eta(18z)\eta(72z)}$	1	1728	XI
110	$\frac{f_4^3 f_6}{f_2 f_8}$	$\frac{\eta(12z)^3\eta(18z)}{\eta(6z)\eta(24z)}$	1	1728	XI
111	$\frac{f_4 f_6 f_{12}^2}{f_2 f_{24}}$	$\frac{\eta(12z)\eta(18z)\eta(36z)^2}{\eta(6z)\eta(72z)}$	1	1728	XII
112	$\frac{f_2 f_8 f_{12}^5}{f_4^2 f_6 f_{24}^2}$	$\frac{\eta(6z)\eta(24z)\eta(36z)^5}{\eta(12z)^2\eta(18z)\eta(72z)^2}$	1	1728	XII
113	$\frac{f_4^5 f_6 f_{24}}{f_2 f_8^2 f_{12}^2}$	$\frac{\eta(12z)^5\eta(18z)\eta(72z)}{\eta(6z)\eta(24z)^2\eta(36z)^2}$	1	1728	XII
114	$\frac{f_2 f_4^2 f_{12}}{f_6 f_8}$	$\frac{\eta(6z)\eta(12z)^2\eta(36z)}{\eta(18z)\eta(24z)}$	1	1728	XII
115	$\frac{f_4^8 f_6 f_{24}}{f_2^3 f_8^3 f_{12}^2}$	$\frac{\eta(12z)^8\eta(18z)\eta(72z)}{\eta(6z)^3\eta(24z)^3\eta(36z)^2}$	1	432	XIII
116	$\frac{f_2^3 f_{12}}{f_4 f_6}$	$\frac{\eta(6z)^3\eta(36z)}{\eta(12z)\eta(18z)}$	1	432	XIII
117	$\frac{f_2 f_8 f_{12}^8}{f_4^2 f_6^3 f_{24}^3}$	$\frac{\eta(6z)\eta(24z)\eta(36z)^8}{\eta(12z)^2\eta(18z)^3\eta(72z)^3}$	1	432	XIII
118	$\frac{f_4 f_6^3}{f_2 f_{12}}$	$\frac{\eta(12z)\eta(18z)^3}{\eta(6z)\eta(36z)}$	1	432	XIII
119	$\frac{f_1 f_4 f_6^2 f_8^5}{f_2 f_3 f_{12} f_{16}^2}$	$\frac{\eta(3z)\eta(12z)\eta(18z)^2\eta(24z)^5}{\eta(6z)\eta(9z)\eta(36z)\eta(48z)^2}$	2	432	XIV
120	$\frac{f_2^2 f_3 f_8^5}{f_1 f_6 f_{16}^2}$	$\frac{\eta(6z)^2\eta(9z)\eta(24z)^5}{\eta(3z)\eta(18z)\eta(48z)^2}$	2	432	XIV

121	$\frac{f_2^5f_8^5}{f_1^2f_4^2f_{16}^2}$	$\frac{\eta(6z)^5\eta(24z)^5}{\eta(3z)^2\eta(12z)^2\eta(48z)^2}$	2	144	XIV
122	$\frac{f_1^2f_8^5}{f_2f_{16}^2}$	$\frac{\eta(3z)^2\eta(24z)^5}{\eta(6z)\eta(48z)^2}$	2	144	XIV
123	$\frac{f_1f_4f_6^2f_{16}^3}{f_2f_3f_8f_{12}f_{32}}$	$\frac{\eta(3z)\eta(12z)\eta(18z)^2\eta(48z)^3}{\eta(6z)\eta(9z)\eta(24z)\eta(36z)\eta(96z)}$	1	6912	XV
124	$\frac{f_2^2f_3f_{16}^3}{f_1f_6f_8f_{32}}$	$\frac{\eta(6z)^2\eta(9z)\eta(48z)^3}{\eta(3z)\eta(18z)\eta(24z)\eta(96z)}$	1	6912	XV
125	$\frac{f_1f_4f_6^2f_8}{f_2f_3f_{12}}$	$\frac{\eta(3z)\eta(12z)\eta(18z)^2\eta(24z)}{\eta(6z)\eta(9z)\eta(36z)}$	1	1728	XVI
126	$\frac{f_2^2f_3f_8}{f_1f_6}$	$\frac{\eta(6z)^2\eta(9z)\eta(24z)}{\eta(3z)\eta(18z)}$	1	1728	XVI
127	$\frac{f_2^3f_{14}^3}{f_1f_4f_7f_{28}}$	$\frac{\eta(6z)^3\eta(42z)^3}{\eta(3z)\eta(12z)\eta(21z)\eta(84z)}$	1	252	XVII
128	f_1f_7	$\eta(3z)\eta(21z)$	1	63	XVII
129	$\frac{f_2^5f_{16}^3}{f_1^2f_4^2f_8f_{32}}$	$\frac{\eta(6z)^5\eta(48z)^3}{\eta(3z)^2\eta(12z)^2\eta(24z)\eta(96z)}$	1	2304	XVIII
130	$\frac{f_1^2f_{16}^3}{f_2f_8f_{32}}$	$\frac{\eta(3z)^2\eta(48z)^3}{\eta(6z)\eta(24z)\eta(96z)}$	1	2304	XVIII
131	$\frac{f_8^{10}}{f_4^2f_{16}^4}$	$\frac{\eta(24z)^{10}}{\eta(12z)^2\eta(48z)^4}$	2	144	XIX
132	$\frac{f_4^2f_8^4}{f_{16}^2}$	$\frac{\eta(12z)^2\eta(24z)^4}{\eta(48z)^2}$	2	576	XIX
133	$\frac{f_4^3f_{16}^3f_{24}^{17}}{f_8^5f_{12}^7f_{48}^7}$	$\frac{\eta(12z)^3\eta(48z)^3\eta(72z)^{17}}{\eta(24z)^5\eta(36z)^7\eta(144z)^7}$	2	1728	XIX
134	$\frac{f_8^4f_{12}^7}{f_4^3f_{24}^4}$	$\frac{\eta(24z)^4\eta(36z)^7}{\eta(12z)^3\eta(72z)^4}$	2	432	XIX
135	$\frac{f_8^{15}f_{12}f_{48}}{f_4^5f_{16}^5f_{24}^3}$	$\frac{\eta(24z)^{15}\eta(36z)\eta(144z)}{\eta(12z)^5\eta(48z)^5\eta(72z)^3}$	2	1728	XIX
136	$\frac{f_4^5}{f_{12}}$	$\frac{\eta(12z)^5}{\eta(36z)}$	2	432	XIX
137	$\frac{f_1^2f_8}{f_2}$	$\frac{\eta(3z)^2\eta(24z)}{\eta(6z)}$	1	576	XX
138	$\frac{f_2^5f_8}{f_1^2f_4^2}$	$\frac{\eta(6z)^5\eta(24z)}{\eta(3z)^2\eta(12z)^2}$	1	576	XX
139	$\frac{f_2^2f_8}{f_4}$	$\frac{\eta(6z)^2\eta(24z)}{\eta(12z)}$	1	576	XXI
140	$\frac{f_4^5}{f_2^2f_8}$	$\frac{\eta(12z)^5}{\eta(6z)^2\eta(24z)}$	1	576	XXI
141	$\frac{f_8^4f_{16}}{f_4^2f_{32}}$	$\frac{\eta(24z)^4\eta(48z)}{\eta(12z)^2\eta(96z)}$	1	2304	XXII
142	$\frac{f_4^2f_{16}^3}{f_8^2f_{32}}$	$\frac{\eta(12z)^2\eta(48z)^3}{\eta(24z)^2\eta(96z)}$	1	2304	XXII
143	$\frac{f_8^6}{f_4^2f_{16}^2}$	$\frac{\eta(24z)^6}{\eta(12z)^2\eta(48z)^2}$	1	576	XXII
144	f_4^2	$\eta(12z)^2$	1	144	XXII
145	$\frac{f_8f_{12}^5}{f_6^2f_{24}^2}$	$\frac{\eta(24z)\eta(36z)^5}{\eta(18z)^2\eta(72z)^2}$	1	576	XXIII
146	$\frac{f_6^2f_8}{f_{12}}$	$\frac{\eta(18z)^2\eta(24z)}{\eta(36z)}$	1	576	XXIII

147	$\frac{f_1 f_6^2}{f_2 f_3}$	$\frac{\eta(3z)\eta(18z)^2}{\eta(6z)\eta(9z)}$	$\frac{1}{2}$	144	XXIV
148	$\frac{f_2^2 f_3 f_{12}}{f_1 f_4 f_6}$	$\frac{\eta(6z)^2\eta(9z)\eta(36z)}{\eta(3z)\eta(12z)\eta(18z)}$	$\frac{1}{2}$	36	XXIV
149	$\frac{f_1^2 f_4^2}{f_2}$	$\frac{\eta(3z)^2\eta(12z)^2}{\eta(6z)}$	$\frac{3}{2}$	36	XXIV
150	$\frac{f_2^5}{f_1^2}$	$\frac{\eta(6z)^5}{\eta(3z)^2}$	$\frac{3}{2}$	144	XXIV
151	$\frac{f_{16}^{13}}{f_8^5 f_{32}^5}$	$\frac{\eta(48z)^{13}}{\eta(24z)^5\eta(96z)^5}$	$\frac{3}{2}$	2304	XXV
152	$\frac{f_8^5}{f_{16}^2}$	$\frac{\eta(24z)^5}{\eta(48z)^2}$	$\frac{3}{2}$	144	XXV
153	$\frac{f_{16} f_{24}^2}{f_8 f_{48}}$	$\frac{\eta(48z)\eta(72z)^2}{\eta(24z)\eta(144z)}$	$\frac{1}{2}$	144	XXV
154	$\frac{f_8 f_{32} f_{48}^5}{f_{16}^2 f_{24}^2 f_{96}^2}$	$\frac{\eta(24z)\eta(96z)\eta(144z)^5}{\eta(48z)^2\eta(72z)^2\eta(288z)^2}$	$\frac{1}{2}$	2304	XXV
155	$\frac{f_{16}^3}{f_8 f_{32}}$	$\frac{\eta(48z)^3}{\eta(24z)\eta(96z)}$	$\frac{1}{2}$	2304	XXV
156	f_8	$\eta(24z)$	$\frac{1}{2}$	576	XXV

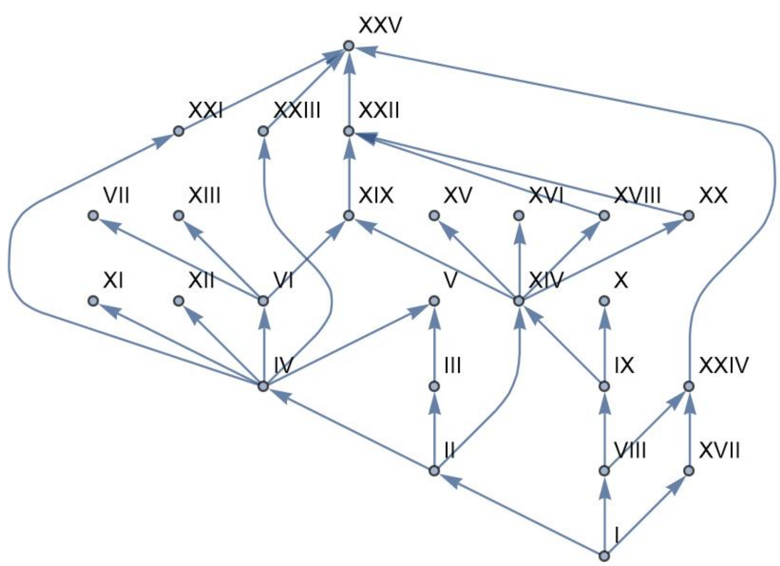

FIGURE 4. The grouping of eta quotients in Table 11, which have vanishing coefficient behaviour similar to f_1^8

Table 12: Eta quotients in Table 11 with expansions as double theta series

Number	Modular Form	Weight	Theta Series
7 I	$\frac{\eta(6z)^7\eta(9z)}{\eta(3z)^3\eta(18z)}$	2	$\sum_{\substack{m=-\infty \\ n=1}}^{\infty} n\left(-2\left(\frac{m}{2}\right)^2-\frac{3\left(\frac{m}{3}\right)^2}{2}+3\left(\frac{m}{6}\right)^2+1\right) \times\left(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right)\right)q^{3m^2+n^2}$
8 I	$\frac{\eta(3z)^3\eta(12z)^3\eta(18z)^2}{\eta(6z)^2\eta(9z)\eta(36z)}$	2	$\sum_{\substack{m=1 \\ n=-\infty}}^{\infty} m\left(\frac{m}{3}\right)\left(1-\frac{3\left(\frac{n}{3}\right)^2}{2}\right)q^{m^2+3n^2}$
9 I	$\frac{\eta(6z)^7\eta(9z)\eta(36z)}{\eta(3z)^3\eta(12z)^3\eta(18z)}$	1	$\sum_{\substack{m=-\infty \\ n=1}}^{\infty}\left(\frac{n}{3}\right)^2 q^{3m^2+n^2}$
10 I	$\frac{\eta(3z)^3\eta(18z)^2}{\eta(6z)^2\eta(9z)}$	1	$\sum_{\substack{m=-\infty \\ n=1}}^{\infty}(-1)^m\left(2\left(\frac{n}{6}\right)^2-\left(\frac{n}{3}\right)^2\right)q^{3m^2+n^2}$
19 I	$\frac{\eta(6z)^4\eta(18z)^2}{\eta(3z)\eta(9z)\eta(12z)^2}$	1	$\sum_{\substack{m=-\infty \\ n=1}}^{\infty}\left(2\left(\frac{n}{6}\right)^2-\left(\frac{n}{3}\right)^2\right)q^{3m^2+n^2}$
20 I	$\frac{\eta(3z)\eta(6z)\eta(9z)\eta(36z)}{\eta(12z)\eta(18z)}$	1	$\sum_{\substack{m=1 \\ n=-\infty}}^{\infty}(-1)^n\left(\frac{m}{3}\right)^2 q^{m^2+3n^2}$
21 I	$\frac{\eta(6z)^4\eta(12z)\eta(18z)^2}{\eta(3z)\eta(9z)\eta(36z)}$	2	$\sum_{\substack{m=-\infty \\ n=1}}^{\infty} n\left(1-\frac{3\left(\frac{m}{3}\right)^2}{2}\right)(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right))q^{3m^2+n^2}$
22 I	$\frac{\eta(3z)\eta(6z)\eta(9z)\eta(12z)^2}{\eta(18z)}$	2	$\sum_{\substack{m=1 \\ n=-\infty}}^{\infty} m\left(\frac{m}{3}\right) \times\left(-2\left(\frac{n}{2}\right)^2-\frac{3\left(\frac{n}{3}\right)^2}{2}+3\left(\frac{n}{6}\right)^2+1\right)q^{m^2+3n^2}$
25 II	$\frac{\eta(3z)^4\eta(12z)^2}{\eta(6z)^2}$	2	$\sum_{\substack{m=1 \\ n=-\infty}}^{\infty} m(-1)^n\left(\frac{m}{3}\right)q^{m^2+3n^2}$
26 II	$\frac{\eta(6z)^{10}}{\eta(3z)^4\eta(12z)^2}$	2	$\sum_{\substack{m=-\infty \\ n=1}}^{\infty} n(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right))q^{3m^2+n^2}$
27 III	$\frac{\eta(3z)^2\eta(12z)\eta(18z)^4}{\eta(6z)^2\eta(9z)^2\eta(36z)}$	1	$\sum_{\substack{m=-\infty \\ n=1}}^{\infty}\left(1-\frac{3\left(\frac{m}{3}\right)^2}{2}\right)\left(2\left(\frac{n}{6}\right)^2-\left(\frac{n}{3}\right)^2\right)q^{3m^2+n^2}$
28 III	$\frac{\eta(6z)^4\eta(9z)^2\eta(36z)}{\eta(3z)^2\eta(12z)\eta(18z)^2}$	1	$\sum_{\substack{m=1 \\ n=-\infty}}^{\infty}\left(\frac{m}{3}\right)^2\left(-2\left(\frac{n}{2}\right)^2-\frac{3\left(\frac{n}{3}\right)^2}{2}+3\left(\frac{n}{6}\right)^2+1\right) \times q^{m^2+3n^2}$
32 IV	$\frac{\eta(6z)\eta(18z)^3\eta(24z)\eta(36z)^3}{\eta(12z)^2\eta(72z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{18}{m}\right)\left(\frac{n}{12}\right)q^{\frac{1}{4}(m^2+3n^2)}$
41 IV	$\frac{\eta(12z)\eta(36z)^{12}}{\eta(6z)\eta(18z)^3\eta(72z)^5}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{6}\right)^2\left(\frac{-6}{n}\right)q^{\frac{1}{4}(m^2+3n^2)}$
45 IV	$\frac{\eta(6z)^5\eta(36z)^{13}}{\eta(12z)^2\eta(18z)^5\eta(72z)^5}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-6}{m}\right)\left(\frac{n}{12}\right)q^{\frac{1}{4}(3m^2+n^2)}$
46 IV	$\frac{\eta(12z)^{13}\eta(18z)^5}{\eta(6z)^5\eta(24z)^5\eta(36z)^2}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-6}{m}\right)\left(\frac{n}{12}\right)q^{\frac{1}{4}(m^2+3n^2)}$
51 IV	$\frac{\eta(6z)^8}{\eta(12z)^2}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-4}{m}\right)\left(\frac{n}{12}\right)q^{\frac{1}{4}(3m^2+n^2)}$
53 IV	$\frac{\eta(6z)^7\eta(36z)}{\eta(12z)^3\eta(18z)}$	2	$\sum_{m,n=1}^{\infty} m\left(\frac{m}{12}\right)\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{\frac{1}{4}(m^2+3n^2)}$
54 IV	$\frac{\eta(6z)^6\eta(24z)}{\eta(12z)^3}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{8}{m}\right)\left(\frac{n}{12}\right)q^{\frac{1}{4}(3m^2+n^2)}$
55 IV	$\frac{\eta(12z)^{13}\eta(36z)^{13}}{\eta(6z)^5\eta(18z)^5\eta(24z)^5\eta(72z)^5}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-6}{m}\right)\left(\frac{-6}{n}\right)q^{\frac{1}{4}(3m^2+n^2)}$
55 IV	$\frac{\eta(12z)^{13}\eta(36z)^{13}}{\eta(6z)^5\eta(18z)^5\eta(24z)^5\eta(72z)^5}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-6}{m}\right)\left(\frac{-6}{n}\right)q^{\frac{1}{4}(m^2+3n^2)}$
56 IV	$\frac{\eta(6z)^5\eta(18z)^5}{\eta(12z)^2\eta(36z)^2}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{m}{12}\right)\left(\frac{n}{12}\right)q^{\frac{1}{4}(3m^2+n^2)}$
56 IV	$\frac{\eta(6z)^5\eta(18z)^5}{\eta(12z)^2\eta(36z)^2}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{m}{12}\right)\left(\frac{n}{12}\right)q^{\frac{1}{4}(m^2+3n^2)}$
57 IV	$\frac{\eta(12z)^{13}\eta(36z)^3}{\eta(6z)^5\eta(18z)\eta(24z)^5\eta(72z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{24}{m}\right)\left(\frac{-6}{n}\right)q^{\frac{1}{4}(3m^2+n^2)}$
58 IV	$\frac{\eta(6z)^5\eta(18z)}{\eta(12z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{n}{12}\right)q^{\frac{1}{4}(3m^2+n^2)}$

59 IV	$\eta(6z)^4$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{-4}{n}\right)q^{\frac{1}{4}(m^2+3n^2)}$
59 IV	$\eta(6z)^4$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{2}\right)^2\left(\frac{n}{12}\right)q^{\frac{1}{4}(3m^2+n^2)}$
59 IV	$\eta(6z)^4$	2	$\sum_{\substack{m=-\infty \\ n=1}}^{\infty}(-1)^m n(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right))q^{3m^2+n^2}$
59 IV	$\eta(6z)^4$	2	$\sum_{\substack{m=-\infty \\ n=1}}^{\infty} n\left(\frac{n}{3}\right)q^{3m^2+n^2}$
60 IV	$\frac{\eta(12z)^7\eta(36z)^5}{\eta(6z)^2\eta(18z)^2\eta(24z)^2\eta(72z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{18}{m}\right)\left(\frac{-8}{n}\right)q^{\frac{1}{4}(m^2+3n^2)}$
61 IV	$\frac{\eta(6z)^2\eta(12z)\eta(18z)^2}{\eta(36z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{6}\right)^2\left(\frac{-4}{n}\right)q^{\frac{1}{4}(m^2+3n^2)}$
62 IV	$\frac{\eta(6z)^2\eta(12z)^3}{\eta(24z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{24}{m}\right)\left(\frac{-4}{n}\right)q^{\frac{1}{4}(m^2+3n^2)}$
63 IV	$\frac{\eta(6z)^2\eta(12z)^7}{\eta(24z)^3}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-8}{m}\right)\left(\frac{n}{12}\right)q^{\frac{1}{4}(3m^2+n^2)}$
66 IV	$\frac{\eta(12z)^3\eta(36z)^{13}}{\eta(6z)\eta(18z)^5\eta(24z)\eta(72z)^5}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{24}{m}\right)\left(\frac{-6}{n}\right)q^{\frac{1}{4}(m^2+3n^2)}$
67 IV	$\frac{\eta(6z)\eta(18z)^5}{\eta(36z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{n}{12}\right)q^{\frac{1}{4}(m^2+3n^2)}$
68 IV	$\frac{\eta(6z)\eta(36z)^{13}}{\eta(18z)^5\eta(72z)^5}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{-6}{n}\right)q^{\frac{1}{4}(m^2+3n^2)}$
69 IV	$\frac{\eta(12z)^3\eta(18z)^5}{\eta(6z)\eta(24z)\eta(36z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{24}{m}\right)\left(\frac{n}{12}\right)q^{\frac{1}{4}(m^2+3n^2)}$
70 IV	$\frac{\eta(6z)\eta(24z)\eta(36z)^{18}}{\eta(12z)^2\eta(18z)^7\eta(72z)^7}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{18}{m}\right)\left(\frac{-6}{n}\right)q^{\frac{1}{4}(m^2+3n^2)}$
71 IV	$\frac{\eta(12z)\eta(18z)^7}{\eta(6z)\eta(36z)^3}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{6}\right)^2\left(\frac{n}{12}\right)q^{\frac{1}{4}(m^2+3n^2)}$
72 IV	$\frac{\eta(6z)^2\eta(24z)^2\eta(36z)^5}{\eta(12z)^3\eta(18z)^2\eta(72z)^2}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{8}{m}\right)\left(\frac{18}{n}\right)q^{\frac{1}{4}(3m^2+n^2)}$
73 IV	$\frac{\eta(12z)^3\eta(18z)^2}{\eta(6z)^2\eta(36z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{m}{2}\right)^2\left(\frac{n}{6}\right)^2 q^{\frac{1}{4}(3m^2+n^2)}$
76 IV	$\frac{\eta(12z)^9}{\eta(6z)^2\eta(24z)^3}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{-8}{n}\right)q^{\frac{1}{4}(m^2+3n^2)}$
77 IV	$\frac{\eta(12z)^{13}}{\eta(6z)^2\eta(24z)^5}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-4}{m}\right)\left(\frac{-6}{n}\right)q^{\frac{1}{4}(3m^2+n^2)}$
79 IV	$\frac{\eta(12z)^{12}\eta(36z)}{\eta(6z)^3\eta(18z)\eta(24z)^5}$	2	$\sum_{m,n=1}^{\infty} m\left(\frac{-6}{m}\right)\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{\frac{1}{4}(m^2+3n^2)}$
82 IV	$\frac{\eta(6z)^4\eta(24z)\eta(36z)^5}{\eta(12z)^2\eta(18z)^2\eta(72z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{18}{m}\right)\left(\frac{-4}{n}\right)q^{\frac{1}{4}(m^2+3n^2)}$
83 IV	$\frac{\eta(12z)^{10}\eta(18z)^2}{\eta(6z)^4\eta(24z)^3\eta(36z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{6}\right)^2\left(\frac{-8}{n}\right)q^{\frac{1}{4}(m^2+3n^2)}$
84 IV	$\frac{\eta(12z)^{12}}{\eta(6z)^4\eta(24z)^4}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{24}{m}\right)\left(\frac{-8}{n}\right)q^{\frac{1}{4}(m^2+3n^2)}$
84 IV	$\frac{\eta(12z)^{12}}{\eta(6z)^4\eta(24z)^4}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{8}{m}\right)\left(\frac{-6}{n}\right)q^{\frac{1}{4}(3m^2+n^2)}$
85 IV	$\frac{\eta(6z)^5\eta(36z)^3}{\eta(12z)^2\eta(18z)\eta(72z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{24}{m}\right)\left(\frac{n}{12}\right)q^{\frac{1}{4}(3m^2+n^2)}$
86 IV	$\frac{\eta(12z)^{13}\eta(18z)}{\eta(6z)^5\eta(24z)^5}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{-6}{n}\right)q^{\frac{1}{4}(3m^2+n^2)}$
87 IV	$\frac{\eta(12z)^{15}}{\eta(6z)^6\eta(24z)^5}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{2}\right)^2\left(\frac{-6}{n}\right)q^{\frac{1}{4}(3m^2+n^2)}$
88 IV	$\frac{\eta(12z)^{22}}{\eta(6z)^8\eta(24z)^8}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-8}{m}\right)\left(\frac{-6}{n}\right)q^{\frac{1}{4}(3m^2+n^2)}$
89 V	$\frac{\eta(12z)^3\eta(36z)^3}{\eta(6z)\eta(18z)\eta(24z)\eta(72z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{24}{m}\right)\left(\frac{24}{n}\right)q^{\frac{1}{4}(3m^2+n^2)}$
89 V	$\frac{\eta(12z)^3\eta(36z)^3}{\eta(6z)\eta(18z)\eta(24z)\eta(72z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{24}{m}\right)\left(\frac{24}{n}\right)q^{\frac{1}{4}(m^2+3n^2)}$
90 V	$\eta(6z)\eta(18z)$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{12}{n}\right)q^{\frac{1}{4}(3m^2+n^2)}$
90 V	$\eta(6z)\eta(18z)$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{12}{n}\right)q^{\frac{1}{4}(m^2+3n^2)}$
90 V	$\eta(6z)\eta(18z)$	1	$\sum_{m,n=1}^{\infty}\left(\frac{m}{6}\right)^2\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{\frac{1}{4}(m^2+3n^2)}$

90 V	$\eta(6z)\eta(18z)$	1	$\sum_{\substack{m=-\infty\\ n=1}}^{\infty}\left(-2\left(\frac{m}{2}\right)^2-\frac{3\left(\frac{m}{3}\right)^2}{2}+3\left(\frac{m}{6}\right)^2+1\right)\times\left(2\left(\frac{n}{6}\right)^2-\left(\frac{n}{3}\right)^2\right)q^{3m^2+n^2}$
90 V	$\eta(6z)\eta(18z)$	1	$\sum_{\substack{m=1\\ n=-\infty}}^{\infty}\left(\frac{m}{3}\right)^2\left(1-\frac{3\left(\frac{n}{3}\right)^2}{2}\right)q^{m^2+3n^2}$
99 VIII	$\frac{\eta(6z)^2\eta(9z)\eta(24z)^5\eta(36z)}{\eta(3z)\eta(12z)^3\eta(18z)\eta(48z)^2}$	1	$\sum_{\substack{m=-\infty\\ n=1}}^{\infty}\left(\frac{n}{3}\right)^2q^{12m^2+n^2}$
100 VIII	$\frac{\eta(3z)\eta(18z)^2\eta(24z)^5}{\eta(6z)\eta(9z)\eta(12z)^2\eta(48z)^2}$	1	$\sum_{\substack{m=-\infty\\ n=1}}^{\infty}\left(2\left(\frac{n}{6}\right)^2-\left(\frac{n}{3}\right)^2\right)q^{12m^2+n^2}$
101 VIII	$\frac{\eta(6z)^2\eta(9z)\eta(12z)\eta(36z)}{\eta(3z)\eta(18z)\eta(24z)}$	1	$\sum_{\substack{m=1\\ n=-\infty}}^{\infty}(-1)^n\left(\frac{m}{3}\right)^2q^{m^2+12n^2}$
102 VIII	$\frac{\eta(3z)\eta(12z)^2\eta(18z)^2}{\eta(6z)\eta(9z)\eta(24z)}$	1	$\sum_{\substack{m=-\infty\\ n=1}}^{\infty}(-1)^m\left(2\left(\frac{n}{6}\right)^2-\left(\frac{n}{3}\right)^2\right)q^{12m^2+n^2}$
103 IX	$\frac{\eta(3z)^2\eta(12z)^4}{\eta(6z)\eta(24z)}$	2	$\sum_{\substack{m=1\\ n=-\infty}}^{\infty}m(-1)^n\left(\frac{m}{3}\right)q^{m^2+12n^2}$
104 IX	$\frac{\eta(6z)^5\eta(12z)^2}{\eta(3z)^2\eta(24z)}$	2	$\sum_{\substack{m=-\infty\\ n=1}}^{\infty}(-1)^m n(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right))q^{12m^2+n^2}$
105 X	$\frac{\eta(6z)^2\eta(9z)\eta(24z)^2\eta(36z)^2}{\eta(3z)\eta(12z)^2\eta(18z)\eta(72z)}$	1	$\sum_{\substack{m=1\\ n=-\infty}}^{\infty}\left(\frac{m}{3}\right)^2\left(-2\left(\frac{n}{2}\right)^2-\frac{3\left(\frac{n}{3}\right)^2}{2}+3\left(\frac{n}{6}\right)^2+1\right)\times q^{m^2+12n^2}$
106 X	$\frac{\eta(3z)\eta(18z)^2\eta(24z)^2\eta(36z)}{\eta(6z)\eta(9z)\eta(12z)\eta(72z)}$	1	$\sum_{\substack{m=-\infty\\ n=1}}^{\infty}\left(-2\left(\frac{m}{2}\right)^2-\frac{3\left(\frac{m}{3}\right)^2}{2}+3\left(\frac{m}{6}\right)^2+1\right)\times\left(2\left(\frac{n}{6}\right)^2-\left(\frac{n}{3}\right)^2\right)q^{12m^2+n^2}$
108 XI	$\frac{\eta(6z)^3\eta(24z)\eta(36z)^6}{\eta(12z)^3\eta(18z)^3\eta(72z)^2}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{18}{m}\right)\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{\frac{1}{4}(m^2+3n^2)}$
109 XI	$\frac{\eta(6z)\eta(36z)^3}{\eta(18z)\eta(72z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{24}{n}\right)q^{\frac{1}{4}(m^2+3n^2)}$
110 XI	$\frac{\eta(12z)^3\eta(18z)}{\eta(6z)\eta(24z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{24}{n}\right)q^{\frac{1}{4}(3m^2+n^2)}$
111 XII	$\frac{\eta(12z)\eta(18z)\eta(36z)^2}{\eta(6z)\eta(72z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{24}{m}\right)\left(\frac{n}{6}\right)^2q^{\frac{1}{4}(3m^2+n^2)}$
112 XII	$\frac{\eta(6z)\eta(24z)\eta(36z)^5}{\eta(12z)^2\eta(18z)\eta(72z)^2}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{18}{n}\right)q^{\frac{1}{4}(3m^2+n^2)}$
114 XII	$\frac{\eta(6z)\eta(12z)^2\eta(36z)}{\eta(18z)\eta(24z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{24}{m}\right)\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{\frac{1}{4}(m^2+3n^2)}$
116 XIII	$\frac{\eta(6z)^3\eta(36z)}{\eta(12z)\eta(18z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{\frac{1}{4}(m^2+3n^2)}$
117 XIII	$\frac{\eta(6z)\eta(24z)\eta(36z)^8}{\eta(12z)^2\eta(18z)^3\eta(72z)^3}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{24}{m}\right)\left(\frac{18}{n}\right)q^{\frac{1}{4}(3m^2+n^2)}$
118 XIII	$\frac{\eta(12z)\eta(18z)^3}{\eta(6z)\eta(36z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{n}{6}\right)^2q^{\frac{1}{4}(3m^2+n^2)}$
119 XIV	$\frac{\eta(3z)\eta(12z)\eta(18z)^2\eta(24z)^5}{\eta(6z)\eta(9z)\eta(36z)\eta(48z)^2}$	2	$\sum_{\substack{m=1\\ n=-\infty}}^{\infty}m\left(\frac{m}{12}\right)\left(1-\frac{3\left(\frac{n}{3}\right)^2}{2}\right)q^{m^2+3n^2}$
120 XIV	$\frac{\eta(6z)^2\eta(9z)\eta(24z)^5}{\eta(3z)\eta(18z)\eta(48z)^2}$	2	$\sum_{\substack{m=1\\ n=-\infty}}^{\infty}m\left(\frac{m}{12}\right)\left(-2\left(\frac{n}{2}\right)^2-\frac{3\left(\frac{n}{3}\right)^2}{2}+3\left(\frac{n}{6}\right)^2+1\right)\times q^{m^2+3n^2}$
121 XIV	$\frac{\eta(6z)^5\eta(24z)^5}{\eta(3z)^2\eta(12z)^2\eta(48z)^2}$	2	$\sum_{\substack{m=-\infty\\ n=1}}^{\infty}n(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right))q^{12m^2+n^2}$
121 XIV	$\frac{\eta(6z)^5\eta(24z)^5}{\eta(3z)^2\eta(12z)^2\eta(48z)^2}$	2	$\sum_{\substack{m=-\infty\\ n=1}}^{\infty}n\left(\frac{n}{12}\right)q^{3m^2+n^2}$
122 XIV	$\frac{\eta(3z)^2\eta(24z)^5}{\eta(6z)\eta(48z)^2}$	2	$\sum_{\substack{m=1\\ n=-\infty}}^{\infty}m(-1)^n\left(\frac{m}{12}\right)q^{m^2+3n^2}$
122 XIV	$\frac{\eta(3z)^2\eta(24z)^5}{\eta(6z)\eta(48z)^2}$	2	$\sum_{\substack{m=-\infty\\ n=1}}^{\infty}n\left(\frac{n}{3}\right)q^{12m^2+n^2}$

123 XV	$\frac{\eta(3z)\eta(12z)\eta(18z)^2\eta(48z)^3}{\eta(6z)\eta(9z)\eta(24z)\eta(36z)\eta(96z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} \left(\frac{24}{m}\right)\left(1-\frac{3\left(\frac{n}{3}\right)^2}{2}\right) q^{m^2+3n^2}$
124 XV	$\frac{\eta(6z)^2\eta(9z)\eta(48z)^3}{\eta(3z)\eta(18z)\eta(24z)\eta(96z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} \left(\frac{24}{m}\right)\left(-2\left(\frac{n}{2}\right)^2-\frac{3\left(\frac{n}{3}\right)^2}{2}+3\left(\frac{n}{6}\right)^2+1\right) \times q^{m^2+3n^2}$
125 XVI	$\frac{\eta(3z)\eta(12z)\eta(18z)^2\eta(24z)}{\eta(6z)\eta(9z)\eta(36z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} \left(\frac{12}{m}\right)\left(1-\frac{3\left(\frac{n}{3}\right)^2}{2}\right) q^{m^2+3n^2}$
126 XVI	$\frac{\eta(6z)^2\eta(9z)\eta(24z)}{\eta(3z)\eta(18z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} \left(\frac{12}{m}\right)\left(-2\left(\frac{n}{2}\right)^2-\frac{3\left(\frac{n}{3}\right)^2}{2}+3\left(\frac{n}{6}\right)^2+1\right) \times q^{m^2+3n^2}$
127 XVII	$\frac{\eta(6z)^3\eta(42z)^3}{\eta(3z)\eta(12z)\eta(21z)\eta(84z)}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{24}{m}\right)\left(\frac{24}{n}\right) q^{\frac{1}{8}(7m^2+n^2)}$
128 XVII	$\eta(3z)\eta(21z)$	1	$\sum_{m,n=1}^{\infty} \left(\frac{12}{m}\right)\left(\frac{12}{n}\right) q^{\frac{1}{8}(m^2+7n^2)}$
129 XVIII	$\frac{\eta(6z)^5\eta(48z)^3}{\eta(3z)^2\eta(12z)^2\eta(24z)\eta(96z)}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty} \left(\frac{24}{n}\right) q^{3m^2+n^2}$
130 XVIII	$\frac{\eta(3z)^2\eta(48z)^3}{\eta(6z)\eta(24z)\eta(96z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} (-1)^n \left(\frac{24}{m}\right) q^{m^2+3n^2}$
131 XIX	$\frac{\eta(24z)^{10}}{\eta(12z)^2\eta(48z)^4}$	2	$\sum_{\substack{m=-\infty\\n=1}}^{\infty} n \left(\frac{n}{12}\right) q^{12m^2+n^2}$
132 XIX	$\frac{\eta(12z)^2\eta(24z)^4}{\eta(48z)^2}$	2	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} m(-1)^n \left(\frac{m}{12}\right) q^{m^2+12n^2}$
137 XX	$\frac{\eta(3z)^2\eta(24z)}{\eta(6z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} (-1)^n \left(\frac{12}{m}\right) q^{m^2+3n^2}$
138 XX	$\frac{\eta(6z)^5\eta(24z)}{\eta(3z)^2\eta(12z)^2}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty} \left(\frac{12}{n}\right) q^{3m^2+n^2}$
139 XXI	$\frac{\eta(6z)^2\eta(24z)}{\eta(12z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} (-1)^n \left(\frac{12}{m}\right) q^{m^2+6n^2}$
139 XXI	$\frac{\eta(6z)^2\eta(24z)}{\eta(12z)}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{8}{m}\right)\left(\frac{12}{n}\right) q^{\frac{1}{4}(3m^2+n^2)}$
140 XXI	$\frac{\eta(12z)^5}{\eta(6z)^2\eta(24z)}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty} \left(\frac{12}{n}\right) q^{6m^2+n^2}$
140 XXI	$\frac{\eta(12z)^5}{\eta(6z)^2\eta(24z)}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{m}{2}\right)^2\left(\frac{24}{n}\right) q^{\frac{1}{4}(3m^2+n^2)}$
141 XXII	$\frac{\eta(24z)^4\eta(48z)}{\eta(12z)^2\eta(96z)}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty} \left(\frac{24}{n}\right) q^{12m^2+n^2}$
142 XXII	$\frac{\eta(12z)^2\eta(48z)^3}{\eta(24z)^2\eta(96z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} (-1)^n \left(\frac{24}{m}\right) q^{m^2+12n^2}$
143 XXII	$\frac{\eta(24z)^6}{\eta(12z)^2\eta(48z)^2}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty} \left(\frac{12}{n}\right) q^{12m^2+n^2}$
143 XXII	$\frac{\eta(24z)^6}{\eta(12z)^2\eta(48z)^2}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{24}{m}\right)\left(\frac{24}{n}\right) q^{\frac{1}{2}(m^2+n^2)}$
144 XXII	$\eta(12z)^2$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} (-1)^n \left(\frac{12}{m}\right) q^{m^2+12n^2}$
144 XXII	$\eta(12z)^2$	1	$\sum_{m,n=1}^{\infty} \left(\frac{12}{m}\right)\left(\frac{12}{n}\right) q^{\frac{1}{2}(m^2+n^2)}$
144 XXII	$\eta(12z)^2$	1	$\sum_{m,n=1}^{\infty} \left(\frac{m}{2}\right)^2\left(\frac{12}{n}\right) q^{\frac{1}{4}(3m^2+n^2)}$
144 XXII	$\eta(12z)^2$	1	$\sum_{m,n=1}^{\infty} \left(\frac{8}{m}\right)\left(\frac{24}{n}\right) q^{\frac{1}{4}(3m^2+n^2)}$
145 XXIII	$\frac{\eta(24z)\eta(36z)^5}{\eta(18z)^2\eta(72z)^2}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty} \left(\frac{12}{n}\right) q^{18m^2+n^2}$
145 XXIII	$\frac{\eta(24z)\eta(36z)^5}{\eta(18z)^2\eta(72z)^2}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{m}{2}\right)^2\left(\frac{18}{n}\right) q^{\frac{1}{4}(3m^2+n^2)}$
146 XXIII	$\frac{\eta(18z)^2\eta(24z)}{\eta(36z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} (-1)^n \left(\frac{12}{m}\right) q^{m^2+18n^2}$
146 XXIII	$\frac{\eta(18z)^2\eta(24z)}{\eta(36z)}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{8}{m}\right)\left(\frac{n}{6}\right)^2 q^{\frac{1}{4}(3m^2+n^2)}$

Table 13: Eta quotients with vanishing behaviour similar to f_1^{10}

Number	q-Product	Modular Form	Weight	Level	Group
1	f_1^{10}	$\eta(12z)^{10}$	5	144	I
2	$\frac{f_2^{30}}{f_1^{10}f_4^{10}}$	$\frac{\eta(24z)^{30}}{\eta(12z)^{10}\eta(48z)^{10}}$	5	576	I
3	$\frac{f_2^{9}}{f_1^{4}f_4}$	$\frac{\eta(24z)^{9}}{\eta(12z)^{4}\eta(48z)}$	2	1152	I
4	$\frac{f_1^{4}f_4^{3}}{f_2^{3}}$	$\frac{\eta(12z)^{4}\eta(48z)^{3}}{\eta(24z)^{3}}$	2	1152	I
5	$\frac{f_2^{10}f_3f_{12}}{f_1^{3}f_4^{4}f_6}$	$\frac{\eta(24z)^{10}\eta(36z)\eta(144z)}{\eta(12z)^{3}\eta(48z)^{4}\eta(72z)}$	2	1152	I
6	$\frac{f_1^{3}f_2f_6^{2}}{f_3f_4}$	$\frac{\eta(12z)^{3}\eta(24z)\eta(72z)^{2}}{\eta(36z)\eta(48z)}$	2	1152	I
7	$\frac{f_4^{13}}{f_1^{2}f_8^{5}}$	$\frac{\eta(48z)^{13}}{\eta(12z)^{2}\eta(96z)^{5}}$	3	2304	I
8	$\frac{f_1^{2}f_4^{15}}{f_2^{6}f_8^{5}}$	$\frac{\eta(12z)^{2}\eta(48z)^{15}}{\eta(24z)^{6}\eta(96z)^{5}}$	3	2304	I
9	$\frac{f_2^{4}f_4^{3}}{f_1^{2}f_8}$	$\frac{\eta(24z)^{4}\eta(48z)^{3}}{\eta(12z)^{2}\eta(96z)}$	2	2304	I
10	$\frac{f_1^{2}f_4^{5}}{f_2^{2}f_8}$	$\frac{\eta(12z)^{2}\eta(48z)^{5}}{\eta(24z)^{2}\eta(96z)}$	2	2304	I
11	$\frac{f_2^{6}}{f_1^{2}}$	$\frac{\eta(24z)^{6}}{\eta(12z)^{2}}$	2	288	I
12	$f_1^{2}f_4^{2}$	$\eta(12z)^{2}\eta(48z)^{2}$	2	576	I
13	$\frac{f_2^{10}}{f_1^{2}f_4^{2}}$	$\frac{\eta(24z)^{10}}{\eta(12z)^{2}\eta(48z)^{2}}$	3	576	I
14	$f_1^{2}f_2^{4}$	$\eta(12z)^{2}\eta(24z)^{4}$	3	72	I
15	$\frac{f_2f_3f_6}{f_1}$	$\frac{\eta(24z)\eta(36z)\eta(72z)}{\eta(12z)}$	1	72	I
16	$\frac{f_1f_4f_6^{4}}{f_2^{2}f_3f_{12}}$	$\frac{\eta(12z)\eta(48z)\eta(72z)^{4}}{\eta(24z)^{2}\eta(36z)\eta(144z)}$	1	576	I
17	$\frac{f_2f_3^{5}}{f_1f_6}$	$\frac{\eta(24z)\eta(36z)^{5}}{\eta(12z)\eta(72z)}$	2	288	I
18	$\frac{f_1f_4f_6^{14}}{f_2^{2}f_3^{5}f_{12}^{5}}$	$\frac{\eta(12z)\eta(48z)\eta(72z)^{14}}{\eta(24z)^{2}\eta(36z)^{5}\eta(144z)^{5}}$	2	576	I
19	$\frac{f_2^{3}f_6^{9}}{f_1f_3^{3}f_4f_{12}^{3}}$	$\frac{\eta(24z)^{3}\eta(72z)^{9}}{\eta(12z)\eta(36z)^{3}\eta(48z)\eta(144z)^{3}}$	2	576	I
20	$f_1f_3^{3}$	$\eta(12z)\eta(36z)^{3}$	2	144	I
21	$\frac{f_2^{3}f_3^{3}}{f_1f_4}$	$\frac{\eta(24z)^{3}\eta(36z)^{3}}{\eta(12z)\eta(48z)}$	2	1152	I
22	$\frac{f_1f_6^{9}}{f_3^{3}f_{12}^{3}}$	$\frac{\eta(12z)\eta(72z)^{9}}{\eta(36z)^{3}\eta(144z)^{3}}$	2	1152	I
23	$\frac{f_1^{4}f_9}{f_3}$	$\frac{\eta(12z)^{4}\eta(108z)}{\eta(36z)}$	2	1296	I
24	$\frac{f_2^{12}f_3f_{12}f_{18}^{3}}{f_1^{4}f_4^{4}f_6^{3}f_9f_{36}}$	$\frac{\eta(24z)^{12}\eta(36z)\eta(144z)\eta(216z)^{3}}{\eta(12z)^{4}\eta(48z)^{4}\eta(72z)^{3}\eta(108z)\eta(432z)}$	2	5184	I
25	$\frac{f_1^{5}f_6^{9}}{f_2^{2}f_3^{3}f_{12}^{3}}$	$\frac{\eta(12z)^{5}\eta(72z)^{9}}{\eta(24z)^{2}\eta(36z)^{3}\eta(144z)^{3}}$	3	1152	I
26	$\frac{f_2^{13}f_3^{3}}{f_1^{5}f_4^{5}}$	$\frac{\eta(24z)^{13}\eta(36z)^{3}}{\eta(12z)^{5}\eta(48z)^{5}}$	3	1152	I
27	$\frac{f_1^{5}f_6^{2}}{f_2^{2}f_3}$	$\frac{\eta(12z)^{5}\eta(72z)^{2}}{\eta(24z)^{2}\eta(36z)}$	2	72	I
28	$\frac{f_2^{13}f_3f_{12}}{f_1^{5}f_4^{5}f_6}$	$\frac{\eta(24z)^{13}\eta(36z)\eta(144z)}{\eta(12z)^{5}\eta(48z)^{5}\eta(72z)}$	2	576	I
29	$\frac{f_1^{5}f_3f_{12}}{f_2^{2}f_6}$	$\frac{\eta(12z)^{5}\eta(36z)\eta(144z)}{\eta(24z)^{2}\eta(72z)}$	2	1152	I
30	$\frac{f_2^{13}f_6^{2}}{f_1^{5}f_3f_4^{5}}$	$\frac{\eta(24z)^{13}\eta(72z)^{2}}{\eta(12z)^{5}\eta(36z)\eta(48z)^{5}}$	2	1152	I

31	$\frac{f_1^5 f_3^3}{f_2^2}$	$\frac{\eta(12z)^5\eta(36z)^3}{\eta(24z)^2}$	3	288	I
32	$\frac{f_2^{13} f_6^9}{f_1^5 f_3^3 f_4^5 f_{12}^3}$	$\frac{\eta(24z)^{13}\eta(72z)^9}{\eta(12z)^5\eta(36z)^3\eta(48z)^5\eta(144z)^3}$	3	576	I
33	$f_1^5 f_5$	$\eta(12z)^5\eta(60z)$	3	720	I
34	$\frac{f_2^{15} f_{10}^3}{f_1^5 f_4^5 f_5 f_{20}}$	$\frac{\eta(24z)^{15}\eta(120z)^3}{\eta(12z)^5\eta(48z)^5\eta(60z)\eta(240z)}$	3	2880	I
35	$\frac{f_1^7 f_6^3}{f_2^3 f_3^3}$	$\frac{\eta(12z)^7\eta(72z)^3}{\eta(24z)^3\eta(36z)^3}$	2	288	I
36	$\frac{f_2^{18} f_3^3 f_{12}^3}{f_1^7 f_4^7 f_6^6}$	$\frac{\eta(24z)^{18}\eta(36z)^3\eta(144z)^3}{\eta(12z)^7\eta(48z)^7\eta(72z)^6}$	2	576	I
37	$\frac{f_2^3 f_3 f_8 f_{12}^6}{f_1 f_4^3 f_6^3 f_{24}^2}$	$\frac{\eta(24z)^3\eta(36z)\eta(96z)\eta(144z)^6}{\eta(12z)\eta(48z)^3\eta(72z)^3\eta(288z)^2}$	1	2304	I
38	$\frac{f_1 f_8 f_{12}^5}{f_3 f_4^2 f_{24}^2}$	$\frac{\eta(12z)\eta(96z)\eta(144z)^5}{\eta(36z)\eta(48z)^2\eta(288z)^2}$	1	2304	I
39	$\frac{f_2^6 f_8 f_{12}^5}{f_1^2 f_4^2 f_6^2 f_{24}^2}$	$\frac{\eta(24z)^6\eta(96z)\eta(144z)^5}{\eta(12z)^2\eta(48z)^2\eta(72z)^2\eta(288z)^2}$	2	2304	II
40	$\frac{f_1^2 f_8 f_{12}^5}{f_6^2 f_{24}^2}$	$\frac{\eta(12z)^2\eta(96z)\eta(144z)^5}{\eta(72z)^2\eta(288z)^2}$	2	2304	II
41	$\frac{f_1^2 f_4^3 f_6^2}{f_2^2 f_{12}}$	$\frac{\eta(12z)^2\eta(48z)^3\eta(72z)^2}{\eta(24z)^2\eta(144z)}$	2	144	II
42	$\frac{f_2^4 f_4 f_6^2}{f_1^2 f_{12}}$	$\frac{\eta(24z)^4\eta(48z)\eta(72z)^2}{\eta(12z)^2\eta(144z)}$	2	576	II
43	$\frac{f_1 f_3 f_4 f_6^5}{f_2^2 f_{12}^2}$	$\frac{\eta(12z)\eta(36z)\eta(48z)\eta(72z)^5}{\eta(24z)^2\eta(144z)^2}$	2	1152	III
44	$\frac{f_2 f_6^8}{f_1 f_3 f_{12}^3}$	$\frac{\eta(24z)\eta(72z)^8}{\eta(12z)\eta(36z)\eta(144z)^3}$	2	1152	III
45	$\frac{f_3 f_4^{12} f_{12}}{f_1 f_2^3 f_6 f_8^5}$	$\frac{\eta(36z)\eta(48z)^{12}\eta(144z)}{\eta(12z)\eta(24z)^3\eta(72z)\eta(96z)^5}$	2	2304	IV
46	$\frac{f_1 f_4^{13} f_6^2}{f_2^6 f_3 f_8^5}$	$\frac{\eta(12z)\eta(48z)^{13}\eta(72z)^2}{\eta(24z)^6\eta(36z)\eta(96z)^5}$	2	2304	IV
47	$\frac{f_2^7 f_3 f_{12}}{f_1 f_4^3 f_6}$	$\frac{\eta(24z)^7\eta(36z)\eta(144z)}{\eta(12z)\eta(48z)^3\eta(72z)}$	2	144	IV
48	$\frac{f_1 f_2^4 f_6^2}{f_3 f_4^2}$	$\frac{\eta(12z)\eta(24z)^4\eta(72z)^2}{\eta(36z)\eta(48z)^2}$	2	576	IV
49	$\frac{f_2^{11} f_8^2 f_{12}^5}{f_1^4 f_4^6 f_6^2 f_{24}^2}$	$\frac{\eta(24z)^{11}\eta(96z)^2\eta(144z)^5}{\eta(12z)^4\eta(48z)^6\eta(72z)^2\eta(288z)^2}$	2	1152	V
50	$\frac{f_1^4 f_8^2 f_{12}^5}{f_2 f_4^2 f_6^2 f_{24}^2}$	$\frac{\eta(12z)^4\eta(96z)^2\eta(144z)^5}{\eta(24z)\eta(48z)^2\eta(72z)^2\eta(288z)^2}$	2	1152	V
51	$\frac{f_1 f_3 f_4^{10} f_{12}}{f_2^4 f_6 f_8^4}$	$\frac{\eta(12z)\eta(36z)\eta(48z)^{10}\eta(144z)}{\eta(24z)^4\eta(72z)\eta(96z)^4}$	2	1152	V
52	$\frac{f_4^9 f_6^2}{f_1 f_2 f_3 f_8^4}$	$\frac{\eta(48z)^9\eta(72z)^2}{\eta(12z)\eta(24z)\eta(36z)\eta(96z)^4}$	2	1152	V
53	$\frac{f_1^3 f_4 f_6^9}{f_2^3 f_3^3 f_{12}^3}$	$\frac{\eta(12z)^3\eta(48z)\eta(72z)^9}{\eta(24z)^3\eta(36z)^3\eta(144z)^3}$	2	144	VI
54	$\frac{f_2^6 f_3^3}{f_1^3 f_4^2}$	$\frac{\eta(24z)^6\eta(36z)^3}{\eta(12z)^3\eta(48z)^2}$	2	576	VI
55	$\frac{f_1^5 f_{18}^3}{f_2^2 f_9 f_{36}}$	$\frac{\eta(12z)^5\eta(216z)^3}{\eta(24z)^2\eta(108z)\eta(432z)}$	2	10368	VII
56	$\frac{f_2^{13} f_9}{f_1^5 f_4^5}$	$\frac{\eta(24z)^{13}\eta(108z)}{\eta(12z)^5\eta(48z)^5}$	2	10368	VII
57	$\frac{f_2^{13} f_{18}^3}{f_1^5 f_4^5 f_9 f_{36}}$	$\frac{\eta(24z)^{13}\eta(216z)^3}{\eta(12z)^5\eta(48z)^5\eta(108z)\eta(432z)}$	2	5184	VII
58	$\frac{f_1^5 f_9}{f_2^2}$	$\frac{\eta(12z)^5\eta(108z)}{\eta(24z)^2}$	2	2592	VII
59	$\frac{f_1^2 f_4^2 f_6^{30} f_9^2 f_{36}^2}{f_2^6 f_3^{10} f_{12}^{10} f_{18}^6}$	$\frac{\eta(12z)^2\eta(48z)^2\eta(72z)^{30}\eta(108z)^2\eta(432z)^2}{\eta(24z)^6\eta(36z)^{10}\eta(144z)^{10}\eta(216z)^6}$	3	5184	VII
60	$\frac{f_3^{10}}{f_1^2 f_9^2}$	$\frac{\eta(36z)^{10}}{\eta(12z)^2\eta(108z)^2}$	3	1296	VII

61	$\frac{f_1f_3f_{12}}{f_6}$	$\frac{\eta(12z)\eta(36z)\eta(144z)}{\eta(72z)}$	1	1152	VIII
62	$\frac{f_2^3f_6^2}{f_1f_3f_4}$	$\frac{\eta(24z)^3\eta(72z)^2}{\eta(12z)\eta(36z)\eta(48z)}$	1	1152	VIII
63	$\frac{f_2f_3f_4^2f_{12}}{f_1f_6f_8}$	$\frac{\eta(24z)\eta(36z)\eta(48z)^2\eta(144z)}{\eta(12z)\eta(72z)\eta(96z)}$	1	2304	IX
64	$\frac{f_1f_4^3f_6^2}{f_2^2f_3f_8}$	$\frac{\eta(12z)\eta(48z)^3\eta(72z)^2}{\eta(24z)^2\eta(36z)\eta(96z)}$	1	2304	IX
65	$\frac{f_1f_6^2}{f_3}$	$\frac{\eta(12z)\eta(72z)^2}{\eta(36z)}$	1	288	IX
66	$\frac{f_2^3f_3f_{12}}{f_1f_4f_6}$	$\frac{\eta(24z)^3\eta(36z)\eta(144z)}{\eta(12z)\eta(48z)\eta(72z)}$	1	576	IX
67	$\frac{f_1f_4f_6^2f_{10}^5}{f_2^2f_3f_5^2f_{20}^2}$	$\frac{\eta(12z)\eta(48z)\eta(72z)^2\eta(120z)^5}{\eta(24z)^2\eta(36z)\eta(60z)^2\eta(240z)^2}$	1	5760	X
68	$\frac{f_2f_3f_5^2f_{12}}{f_1f_6f_{10}}$	$\frac{\eta(24z)\eta(36z)\eta(60z)^2\eta(144z)}{\eta(12z)\eta(72z)\eta(120z)}$	1	5760	X
69	$\frac{f_1f_4f_6^7}{f_2^2f_3^3f_{12}^2}$	$\frac{\eta(12z)\eta(48z)\eta(72z)^7}{\eta(24z)^2\eta(36z)^3\eta(144z)^2}$	1	1152	XI
70	$\frac{f_2f_3^3f_{12}}{f_1f_6^2}$	$\frac{\eta(24z)\eta(36z)^3\eta(144z)}{\eta(12z)\eta(72z)^2}$	1	1152	XI
71	$\frac{f_1^3f_{12}}{f_2f_3}$	$\frac{\eta(12z)^3\eta(144z)}{\eta(24z)\eta(36z)}$	1	1152	XI
72	$\frac{f_2^8f_3f_{12}^2}{f_1^3f_4^3f_6^3}$	$\frac{\eta(24z)^8\eta(36z)\eta(144z)^2}{\eta(12z)^3\eta(48z)^3\eta(72z)^3}$	1	1152	XI
73	$\frac{f_1f_{18}^3}{f_9f_{36}}$	$\frac{\eta(12z)\eta(216z)^3}{\eta(108z)\eta(432z)}$	1	10368	XII
74	$\frac{f_2^3f_9}{f_1f_4}$	$\frac{\eta(24z)^3\eta(108z)}{\eta(12z)\eta(48z)}$	1	10368	XII
75	$\frac{f_2^3f_{18}^3}{f_1f_4f_9f_{36}}$	$\frac{\eta(24z)^3\eta(216z)^3}{\eta(12z)\eta(48z)\eta(108z)\eta(432z)}$	1	5184	XIII
76	f_1f_9	$\eta(12z)\eta(108z)$	1	1296	XIII
77	$\frac{f_1f_4f_6^5f_{18}^3}{f_2^2f_3^2f_9f_{12}^2f_{36}}$	$\frac{\eta(12z)\eta(48z)\eta(72z)^5\eta(216z)^3}{\eta(24z)^2\eta(36z)^2\eta(108z)\eta(144z)^2\eta(432z)}$	1	5184	XIV
78	$\frac{f_2f_3^2f_9}{f_1f_6}$	$\frac{\eta(24z)\eta(36z)^2\eta(108z)}{\eta(12z)\eta(72z)}$	1	2592	XIV
79	$\frac{f_2f_3^2f_{18}^3}{f_1f_6f_9f_{36}}$	$\frac{\eta(24z)\eta(36z)^2\eta(216z)^3}{\eta(12z)\eta(72z)\eta(108z)\eta(432z)}$	1	10368	XV
80	$\frac{f_1f_4f_6^5f_9}{f_2^2f_3^2f_{12}^2}$	$\frac{\eta(12z)\eta(48z)\eta(72z)^5\eta(108z)}{\eta(24z)^2\eta(36z)^2\eta(144z)^2}$	1	10368	XV
81	$\frac{f_3f_4f_{18}^3}{f_6f_9f_{36}}$	$\frac{\eta(36z)\eta(48z)\eta(216z)^3}{\eta(72z)\eta(108z)\eta(432z)}$	1	1296	XVI
82	$\frac{f_4f_6^2f_9}{f_3f_{12}}$	$\frac{\eta(48z)\eta(72z)^2\eta(108z)}{\eta(36z)\eta(144z)}$	1	5184	XVI
83	$\frac{f_2f_8^3f_{12}^{14}}{f_4^4f_6^4f_{24}^6}$	$\frac{\eta(24z)\eta(96z)^3\eta(144z)^{14}}{\eta(48z)^4\eta(72z)^4\eta(288z)^6}$	2	2304	XVII
84	$\frac{f_6^4f_8^2f_{12}^2}{f_2f_4f_{24}^2}$	$\frac{\eta(72z)^4\eta(96z)^2\eta(144z)^2}{\eta(24z)\eta(48z)\eta(288z)^2}$	2	1152	XVII
85	$\frac{f_2^7}{f_4}$	$\frac{\eta(24z)^7}{\eta(48z)}$	3	1152	XVII
86	$\frac{f_2^5f_8}{f_4^2}$	$\frac{\eta(24z)^5\eta(96z)}{\eta(48z)^2}$	2	2304	XVII
87	$\frac{f_4^5f_{12}^5}{f_2f_6^2f_8f_{24}^2}$	$\frac{\eta(48z)^5\eta(144z)^5}{\eta(24z)\eta(72z)^2\eta(96z)\eta(288z)^2}$	2	2304	XVII
88	$\frac{f_2f_4^2f_6^2}{f_{12}}$	$\frac{\eta(24z)\eta(48z)^2\eta(72z)^2}{\eta(144z)}$	2	1152	XVII
89	$\frac{f_4^{13}}{f_2^5f_8^4}$	$\frac{\eta(48z)^{13}}{\eta(24z)^5\eta(96z)^4}$	2	1152	XVII
90	$\frac{f_4^{20}}{f_2^7f_8^7}$	$\frac{\eta(48z)^{20}}{\eta(24z)^7\eta(96z)^7}$	3	2304	XVII
91	$\frac{f_4^8f_{12}^5}{f_2^3f_6^2f_8^2f_{24}^2}$	$\frac{\eta(48z)^8\eta(144z)^5}{\eta(24z)^3\eta(72z)^2\eta(96z)^2\eta(288z)^2}$	2	1152	XVIII

92	$\frac{f_2^3 f_6^2 f_8}{f_4 f_{12}}$	$\frac{\eta(24z)^3\eta(72z)^2\eta(96z)}{\eta(48z)\eta(144z)}$	2	2304	XVIII
93	$\frac{f_1^2 f_{20}^3}{f_2 f_{10} f_{40}}$	$\frac{\eta(12z)^2\eta(240z)^3}{\eta(24z)\eta(120z)\eta(480z)}$	1	11520	XIX
94	$\frac{f_2^5 f_{20}^3}{f_1^2 f_4^2 f_{10} f_{40}}$	$\frac{\eta(24z)^5\eta(240z)^3}{\eta(12z)^2\eta(48z)^2\eta(120z)\eta(480z)}$	1	11520	XIX
95	$\frac{f_1^2 f_{10}}{f_2}$	$\frac{\eta(12z)^2\eta(120z)}{\eta(24z)}$	1	360	XIX
96	$\frac{f_2^5 f_{10}}{f_1^2 f_4^2}$	$\frac{\eta(24z)^5\eta(120z)}{\eta(12z)^2\eta(48z)^2}$	1	2880	XIX
97	$\frac{f_2 f_8^2 f_{12}^5}{f_4^2 f_6^2 f_{24}^2}$	$\frac{\eta(24z)\eta(96z)^2\eta(144z)^5}{\eta(48z)^2\eta(72z)^2\eta(288z)^2}$	1	1152	XX
98	$\frac{f_4 f_6^2 f_8}{f_2 f_{12}}$	$\frac{\eta(48z)\eta(72z)^2\eta(96z)}{\eta(24z)\eta(144z)}$	1	2304	XX
99	$f_2 f_8$	$\eta(24z)\eta(96z)$	1	2304	XXI
100	$\frac{f_4^3}{f_2}$	$\frac{\eta(48z)^3}{\eta(24z)}$	1	1152	XXI
101	$\frac{f_4^5 f_{20}^3}{f_2^2 f_8^2 f_{10} f_{40}}$	$\frac{\eta(48z)^5\eta(240z)^3}{\eta(24z)^2\eta(96z)^2\eta(120z)\eta(480z)}$	1	11520	XXII
102	$\frac{f_2^2 f_{10}}{f_4}$	$\frac{\eta(24z)^2\eta(120z)}{\eta(48z)}$	1	5760	XXII
103	$\frac{f_{10}^{26}}{f_5^{10} f_{20}^{10}}$	$\frac{\eta(120z)^{26}}{\eta(60z)^{10}\eta(240z)^{10}}$	3	2880	XXIII
104	$\frac{f_5^{10}}{f_{10}^4}$	$\frac{\eta(60z)^{10}}{\eta(120z)^4}$	3	360	XXIII
105	$\frac{f_{10}^{16}}{f_5^6 f_{20}^6}$	$\frac{\eta(120z)^{16}}{\eta(60z)^6\eta(240z)^6}$	2	2880	XXIII
106	$\frac{f_5^6}{f_{10}^2}$	$\frac{\eta(60z)^6}{\eta(120z)^2}$	2	1440	XXIII
107	$\frac{f_{10}^4 f_{20}}{f_5^2 f_{40}}$	$\frac{\eta(120z)^4\eta(240z)}{\eta(60z)^2\eta(480z)}$	1	11520	XXIV
108	$\frac{f_5^2 f_{20}^3}{f_{10}^2 f_{40}}$	$\frac{\eta(60z)^2\eta(240z)^3}{\eta(120z)^2\eta(480z)}$	1	11520	XXIV
109	$\frac{f_{10}^6}{f_5^2 f_{20}^2}$	$\frac{\eta(120z)^6}{\eta(60z)^2\eta(240z)^2}$	1	2880	XXIV
110	f_5^2	$\eta(60z)^2$	1	720	XXIV
111	$\frac{f_{10} f_{40} f_{60}^5}{f_{20}^2 f_{30}^2 f_{120}^2}$	$\frac{\eta(120z)\eta(480z)\eta(720z)^5}{\eta(240z)^2\eta(360z)^2\eta(1440z)^2}$	$\frac{1}{2}$	11520	XXV
112	$\frac{f_{20} f_{30}^2}{f_{10} f_{60}}$	$\frac{\eta(240z)\eta(360z)^2}{\eta(120z)\eta(720z)}$	$\frac{1}{2}$	720	XXV
113	$\frac{f_{20}^{13}}{f_{10}^5 f_{40}^5}$	$\frac{\eta(240z)^{13}}{\eta(120z)^5\eta(480z)^5}$	$\frac{3}{2}$	11520	XXV
114	$\frac{f_{10}^5}{f_{20}^2}$	$\frac{\eta(120z)^5}{\eta(240z)^2}$	$\frac{3}{2}$	720	XXV
115	$\frac{f_{20}^3}{f_{10} f_{40}}$	$\frac{\eta(240z)^3}{\eta(120z)\eta(480z)}$	$\frac{1}{2}$	11520	XXV
116	f_{10}	$\eta(120z)$	$\frac{1}{2}$	2880	XXV

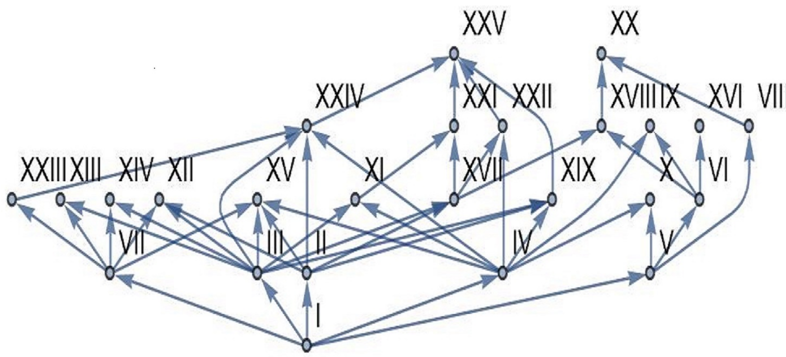

FIGURE 5. The grouping of eta quotients in Table 13, which have vanishing coefficient behaviour similar to f_1^{10}

Table 14: Eta quotients in Table 13 with expansions as double theta series

Number	Modular Form	Weight	Theta Series
7 I	$\frac{\eta(48z)^{13}}{\eta(12z)^2\eta(96z)^5}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-6}{m}\right)(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right))q^{m^2+4n^2}$
8 I	$\frac{\eta(12z)^2\eta(48z)^{15}}{\eta(24z)^6\eta(96z)^5}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{m}{3}\right)\left(\frac{-6}{n}\right)q^{4m^2+n^2}$
9 I	$\frac{\eta(24z)^4\eta(48z)^3}{\eta(12z)^2\eta(96z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{24}{m}\right)(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right))q^{m^2+4n^2}$
10 I	$\frac{\eta(12z)^2\eta(48z)^5}{\eta(24z)^2\eta(96z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{24}{m}\right)\left(\frac{n}{3}\right)q^{m^2+4n^2}$
11 I	$\frac{\eta(24z)^6}{\eta(12z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right))q^{m^2+4n^2}$
12 I	$\eta(12z)^2\eta(48z)^2$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{n}{3}\right)q^{m^2+4n^2}$
13 I	$\frac{\eta(24z)^{10}}{\eta(12z)^2\eta(48z)^2}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{m}{12}\right)(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right))q^{m^2+4n^2}$
14 I	$\eta(12z)^2\eta(24z)^4$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{m}{3}\right)\left(\frac{n}{12}\right)q^{4m^2+n^2}$
15 I	$\frac{\eta(24z)\eta(36z)\eta(72z)}{\eta(12z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{m}{2}\right)^2\left(\frac{n}{6}\right)^2 q^{\frac{1}{2}(9m^2+n^2)}$
15 I	$\frac{\eta(24z)\eta(36z)\eta(72z)}{\eta(12z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{m}{3}\right)^2\left(\frac{n}{6}\right)^2 q^{4m^2+n^2}$
16 I	$\frac{\eta(12z)\eta(48z)\eta(72z)^4}{\eta(24z)^2\eta(36z)\eta(144z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{8}{m}\right)\left(\frac{18}{n}\right)q^{\frac{1}{2}(9m^2+n^2)}$
16 I	$\frac{\eta(12z)\eta(48z)\eta(72z)^4}{\eta(24z)^2\eta(36z)\eta(144z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{m}{6}\right)^2\left(2\left(\frac{n}{6}\right)^2-\left(\frac{n}{3}\right)^2\right)q^{m^2+4n^2}$
17 I	$\frac{\eta(24z)\eta(36z)^5}{\eta(12z)\eta(72z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{6}\right)^2\left(\frac{-4}{n}\right)q^{\frac{1}{2}(m^2+9n^2)}$
18 I	$\frac{\eta(12z)\eta(48z)\eta(72z)^{14}}{\eta(24z)^2\eta(36z)^5\eta(144z)^5}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{18}{m}\right)\left(\frac{-8}{n}\right)q^{\frac{1}{2}(m^2+9n^2)}$
19 I	$\frac{\eta(24z)^3\eta(72z)^9}{\eta(12z)\eta(36z)^3\eta(48z)\eta(144z)^3}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{24}{m}\right)\left(\frac{-8}{n}\right)q^{\frac{1}{2}(m^2+9n^2)}$
20 I	$\eta(12z)\eta(36z)^3$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{-4}{n}\right)q^{\frac{1}{2}(m^2+9n^2)}$
21 I	$\frac{\eta(24z)^3\eta(36z)^3}{\eta(12z)\eta(48z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{24}{m}\right)\left(\frac{-4}{n}\right)q^{\frac{1}{2}(m^2+9n^2)}$
22 I	$\frac{\eta(12z)\eta(72z)^9}{\eta(36z)^3\eta(144z)^3}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{-8}{n}\right)q^{\frac{1}{2}(m^2+9n^2)}$
25 I	$\frac{\eta(12z)^5\eta(72z)^9}{\eta(24z)^2\eta(36z)^3\eta(144z)^3}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-8}{m}\right)\left(\frac{n}{12}\right)q^{\frac{1}{2}(9m^2+n^2)}$
26 I	$\frac{\eta(24z)^{13}\eta(36z)^3}{\eta(12z)^5\eta(48z)^5}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-4}{m}\right)\left(\frac{-6}{n}\right)q^{\frac{1}{2}(9m^2+n^2)}$

27 I	$\frac{\eta(12z)^5\eta(72z)^2}{\eta(24z)^2\eta(36z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{2}\right)^2\left(\frac{n}{12}\right)q^{\frac{1}{2}(9m^2+n^2)}$
28 I	$\frac{\eta(24z)^{13}\eta(36z)\eta(144z)}{\eta(12z)^5\eta(48z)^5\eta(72z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{8}{m}\right)\left(\frac{-6}{n}\right)q^{\frac{1}{2}(9m^2+n^2)}$
29 I	$\frac{\eta(12z)^5\eta(36z)\eta(144z)}{\eta(24z)^2\eta(72z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{8}{m}\right)\left(\frac{n}{12}\right)q^{\frac{1}{2}(9m^2+n^2)}$
30 I	$\frac{\eta(24z)^{13}\eta(72z)^2}{\eta(12z)^5\eta(36z)\eta(48z)^5}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{2}\right)^2\left(\frac{-6}{n}\right)q^{\frac{1}{2}(9m^2+n^2)}$
31 I	$\frac{\eta(12z)^5\eta(36z)^3}{\eta(24z)^2}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-4}{m}\right)\left(\frac{n}{12}\right)q^{\frac{1}{2}(9m^2+n^2)}$
32 I	$\frac{\eta(24z)^{13}\eta(72z)^9}{\eta(12z)^5\eta(36z)^3\eta(48z)^5\eta(144z)^3}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-8}{m}\right)\left(\frac{-6}{n}\right)q^{\frac{1}{2}(9m^2+n^2)}$
37 I	$\frac{\eta(24z)^3\eta(36z)\eta(96z)\eta(144z)^6}{\eta(12z)\eta(48z)^3\eta(72z)^3\eta(288z)^2}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{m}{3}\right)^2\left(\frac{18}{n}\right)q^{4m^2+n^2}$
38 I	$\frac{\eta(12z)\eta(96z)\eta(144z)^5}{\eta(36z)\eta(48z)^2\eta(288z)^2}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{18}{m}\right)\left(2\left(\frac{n}{6}\right)^2-\left(\frac{n}{3}\right)^2\right)q^{m^2+4n^2}$
39 II	$\frac{\eta(24z)^6\eta(96z)\eta(144z)^5}{\eta(12z)^2\eta(48z)^2\eta(72z)^2\eta(288z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{18}{m}\right)(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right))q^{m^2+4n^2}$
40 II	$\frac{\eta(12z)^2\eta(96z)\eta(144z)^5}{\eta(72z)^2\eta(288z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{18}{m}\right)\left(\frac{n}{3}\right)q^{m^2+4n^2}$
41 II	$\frac{\eta(12z)^2\eta(48z)^3\eta(72z)^2}{\eta(24z)^2\eta(144z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{6}\right)^2\left(\frac{n}{3}\right)q^{m^2+4n^2}$
42 II	$\frac{\eta(24z)^4\eta(48z)\eta(72z)^2}{\eta(12z)^2\eta(144z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{6}\right)^2(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right))q^{m^2+4n^2}$
43 III	$\frac{\eta(12z)\eta(36z)\eta(48z)\eta(72z)^5}{\eta(24z)^2\eta(144z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{18}{m}\right)\left(\frac{-4}{n}\right)q^{\frac{1}{2}(m^2+9n^2)}$
44 III	$\frac{\eta(24z)\eta(72z)^8}{\eta(12z)\eta(36z)\eta(144z)^3}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{6}\right)^2\left(\frac{-8}{n}\right)q^{\frac{1}{2}(m^2+9n^2)}$
45 IV	$\frac{\eta(36z)\eta(48z)^{12}\eta(144z)}{\eta(12z)\eta(24z)^3\eta(72z)\eta(96z)^5}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{3}\right)^2\left(\frac{-6}{n}\right)q^{4m^2+n^2}$
46 IV	$\frac{\eta(12z)\eta(48z)^{13}\eta(72z)^2}{\eta(24z)^6\eta(36z)\eta(96z)^5}$	2	$\sum_{m,n=1}^{\infty} m\left(\frac{-6}{m}\right)\left(2\left(\frac{n}{6}\right)^2-\left(\frac{n}{3}\right)^2\right)q^{m^2+4n^2}$
47 IV	$\frac{\eta(24z)^7\eta(36z)\eta(144z)}{\eta(12z)\eta(48z)^3\eta(72z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{3}\right)^2\left(\frac{n}{12}\right)q^{4m^2+n^2}$
48 IV	$\frac{\eta(12z)\eta(24z)^4\eta(72z)^2}{\eta(36z)\eta(48z)^2}$	2	$\sum_{m,n=1}^{\infty} m\left(\frac{m}{12}\right)\left(2\left(\frac{n}{6}\right)^2-\left(\frac{n}{3}\right)^2\right)q^{m^2+4n^2}$
55 VII	$\frac{\eta(12z)^5\eta(216z)^3}{\eta(24z)^2\eta(108z)\eta(432z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{24}{m}\right)\left(\frac{n}{12}\right)q^{\frac{1}{2}(9m^2+n^2)}$
56 VII	$\frac{\eta(24z)^{13}\eta(108z)}{\eta(12z)^5\eta(48z)^5}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{-6}{n}\right)q^{\frac{1}{2}(9m^2+n^2)}$
57 VII	$\frac{\eta(24z)^{13}\eta(216z)^3}{\eta(12z)^5\eta(48z)^5\eta(108z)\eta(432z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{24}{m}\right)\left(\frac{-6}{n}\right)q^{\frac{1}{2}(9m^2+n^2)}$
58 VII	$\frac{\eta(12z)^5\eta(108z)}{\eta(24z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{n}{12}\right)q^{\frac{1}{2}(9m^2+n^2)}$
61 VIII	$\frac{\eta(12z)\eta(36z)\eta(144z)}{\eta(72z)}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{8}{m}\right)\left(\frac{12}{n}\right)q^{\frac{1}{2}(9m^2+n^2)}$
62 VIII	$\frac{\eta(24z)^3\eta(72z)^2}{\eta(12z)\eta(36z)\eta(48z)}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{m}{2}\right)^2\left(\frac{24}{n}\right)q^{\frac{1}{2}(9m^2+n^2)}$
63 IX	$\frac{\eta(24z)\eta(36z)\eta(48z)^2\eta(144z)}{\eta(12z)\eta(72z)\eta(96z)}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{m}{3}\right)^2\left(\frac{24}{n}\right)q^{4m^2+n^2}$
64 IX	$\frac{\eta(12z)\eta(48z)^3\eta(72z)^2}{\eta(24z)^2\eta(36z)\eta(96z)}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{24}{m}\right)\left(2\left(\frac{n}{6}\right)^2-\left(\frac{n}{3}\right)^2\right)q^{m^2+4n^2}$
65 IX	$\frac{\eta(12z)\eta(72z)^2}{\eta(36z)}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{m}{2}\right)^2\left(\frac{12}{n}\right)q^{\frac{1}{2}(9m^2+n^2)}$
65 IX	$\frac{\eta(12z)\eta(72z)^2}{\eta(36z)}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{12}{m}\right)\left(2\left(\frac{n}{6}\right)^2-\left(\frac{n}{3}\right)^2\right)q^{m^2+4n^2}$
66 IX	$\frac{\eta(24z)^3\eta(36z)\eta(144z)}{\eta(12z)\eta(48z)\eta(72z)}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{8}{m}\right)\left(\frac{24}{n}\right)q^{\frac{1}{2}(9m^2+n^2)}$
66 IX	$\frac{\eta(24z)^3\eta(36z)\eta(144z)}{\eta(12z)\eta(48z)\eta(72z)}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{m}{3}\right)^2\left(\frac{12}{n}\right)q^{4m^2+n^2}$
69 XI	$\frac{\eta(12z)\eta(48z)\eta(72z)^7}{\eta(24z)^2\eta(36z)^3\eta(144z)^2}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{m}{2}\right)^2\left(\frac{18}{n}\right)q^{\frac{1}{2}(9m^2+n^2)}$

70 XI	$\frac{\eta(24z)\eta(36z)^3\eta(144z)}{\eta(12z)\eta(72z)^2}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{8}{m}\right)\left(\frac{n}{6}\right)^2 q^{\frac{1}{2}(9m^2+n^2)}$
73 XII	$\frac{\eta(12z)\eta(216z)^3}{\eta(108z)\eta(432z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{24}{n}\right) q^{\frac{1}{2}(m^2+9n^2)}$
74 XII	$\frac{\eta(24z)^3\eta(108z)}{\eta(12z)\eta(48z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{24}{n}\right) q^{\frac{1}{2}(9m^2+n^2)}$
75 XIII	$\frac{\eta(24z)^3\eta(216z)^3}{\eta(12z)\eta(48z)\eta(108z)\eta(432z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{24}{m}\right)\left(\frac{24}{n}\right) q^{\frac{1}{2}(m^2+9n^2)}$
75 XIII	$\frac{\eta(24z)^3\eta(216z)^3}{\eta(12z)\eta(48z)\eta(108z)\eta(432z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{24}{m}\right)\left(\frac{24}{n}\right) q^{\frac{1}{2}(9m^2+n^2)}$
76 XIII	$\eta(12z)\eta(108z)$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{12}{n}\right) q^{\frac{1}{2}(m^2+9n^2)}$
76 XIII	$\eta(12z)\eta(108z)$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{12}{n}\right) q^{\frac{1}{2}(9m^2+n^2)}$
77 XIV	$\frac{\eta(12z)\eta(48z)\eta(72z)^5\eta(216z)^3}{\eta(24z)^2\eta(36z)^2\eta(108z)\eta(144z)^2\eta(432z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{24}{m}\right)\left(\frac{18}{n}\right) q^{\frac{1}{2}(9m^2+n^2)}$
78 XIV	$\frac{\eta(24z)\eta(36z)^2\eta(108z)}{\eta(12z)\eta(72z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{n}{6}\right)^2 q^{\frac{1}{2}(9m^2+n^2)}$
79 XV	$\frac{\eta(24z)\eta(36z)^2\eta(216z)^3}{\eta(12z)\eta(72z)\eta(108z)\eta(432z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{24}{m}\right)\left(\frac{n}{6}\right)^2 q^{\frac{1}{2}(9m^2+n^2)}$
80 XV	$\frac{\eta(12z)\eta(48z)\eta(72z)^5\eta(108z)}{\eta(24z)^2\eta(36z)^2\eta(144z)^2}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{18}{n}\right) q^{\frac{1}{2}(9m^2+n^2)}$
86 XVII	$\frac{\eta(24z)^5\eta(96z)}{\eta(48z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{n}{12}\right) q^{4m^2+n^2}$
89 XVII	$\frac{\eta(48z)^{13}}{\eta(24z)^5\eta(96z)^4}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{-6}{n}\right) q^{4m^2+n^2}$
93XIX	$\frac{\eta(12z)^2\eta(240z)^3}{\eta(24z)\eta(120z)\eta(480z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}(-1)^n\left(\frac{24}{m}\right) q^{5m^2+12n^2}$
94XIX	$\frac{\eta(24z)^5\eta(240z)^3}{\eta(12z)^2\eta(48z)^2\eta(120z)\eta(480z)}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty}\left(\frac{24}{n}\right) q^{12m^2+5n^2}$
95XIX	$\frac{\eta(12z)^2\eta(120z)}{\eta(24z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}(-1)^n\left(\frac{12}{m}\right) q^{5m^2+12n^2}$
96XIX	$\frac{\eta(24z)^5\eta(120z)}{\eta(12z)^2\eta(48z)^2}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty}\left(\frac{12}{n}\right) q^{12m^2+5n^2}$
97 XX	$\frac{\eta(24z)\eta(96z)^2\eta(144z)^5}{\eta(48z)^2\eta(72z)^2\eta(288z)^2}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{18}{n}\right) q^{4m^2+n^2}$
98 XX	$\frac{\eta(48z)\eta(72z)^2\eta(96z)}{\eta(24z)\eta(144z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{n}{6}\right)^2 q^{4m^2+n^2}$
99 XXI	$\eta(24z)\eta(96z)$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{12}{n}\right) q^{4m^2+n^2}$
99 XXI	$\eta(24z)\eta(96z)$	1	$\sum_{m,n=1}^{\infty}\left(\frac{8}{m}\right)\left(\frac{12}{n}\right) q^{3m^2+2n^2}$
100 XXI	$\frac{\eta(48z)^3}{\eta(24z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{24}{n}\right) q^{4m^2+n^2}$
100 XXI	$\frac{\eta(48z)^3}{\eta(24z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{m}{2}\right)^2\left(\frac{12}{n}\right) q^{3m^2+2n^2}$
101 XXII	$\frac{\eta(48z)^5\eta(240z)^3}{\eta(24z)^2\eta(96z)^2\eta(120z)\eta(480z)}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty}\left(\frac{24}{n}\right) q^{24m^2+5n^2}$
102 XXII	$\frac{\eta(24z)^2\eta(120z)}{\eta(48z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}(-1)^n\left(\frac{12}{m}\right) q^{5m^2+24n^2}$
103 XXIII	$\frac{\eta(120z)^{26}}{\eta(60z)^{10}\eta(240z)^{10}}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-6}{m}\right)\left(\frac{-6}{n}\right) q^{\frac{5}{2}(m^2+n^2)}$
104 XXIII	$\frac{\eta(60z)^{10}}{\eta(120z)^4}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{m}{12}\right)\left(\frac{n}{12}\right) q^{\frac{5}{2}(m^2+n^2)}$
105 XXIII	$\frac{\eta(120z)^{16}}{\eta(60z)^6\eta(240z)^6}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{24}{m}\right)\left(\frac{-6}{n}\right) q^{\frac{5}{2}(m^2+n^2)}$
106 XXIII	$\frac{\eta(60z)^6}{\eta(120z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{n}{12}\right) q^{\frac{5}{2}(m^2+n^2)}$
107 XXIV	$\frac{\eta(120z)^4\eta(240z)}{\eta(60z)^2\eta(480z)}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty}\left(\frac{24}{n}\right) q^{60m^2+5n^2}$
108 XXIV	$\frac{\eta(60z)^2\eta(240z)^3}{\eta(120z)^2\eta(480z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}(-1)^n\left(\frac{24}{m}\right) q^{5(m^2+12n^2)}$
109 XXIV	$\frac{\eta(120z)^6}{\eta(60z)^2\eta(240z)^2}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{24}{m}\right)\left(\frac{24}{n}\right) q^{\frac{5}{2}(m^2+n^2)}$
109 XXIV	$\frac{\eta(120z)^6}{\eta(60z)^2\eta(240z)^2}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty}\left(\frac{12}{n}\right) q^{60m^2+5n^2}$

110 XXIV	$\eta(60z)^2$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{12}{n}\right)q^{\frac{5}{2}(m^2+n^2)}$
110 XXIV	$\eta(60z)^2$	1	$\sum_{\substack{m=1\\ n=-\infty}}^{\infty}(-1)^n\left(\frac{12}{m}\right)q^{5(m^2+12n^2)}$

Table 15: Eta quotients with vanishing behaviour similar to f_1^{14}

Number	q-Product	Modular Form	Weight	Level	Group
1	f_1^{14}	$\eta(12z)^{14}$	7	144	I
2	$\frac{f_2^{42}}{f_1^{14}f_4^{14}}$	$\frac{\eta(24z)^{42}}{\eta(12z)^{14}\eta(48z)^{14}}$	7	576	I
3	$\frac{f_2^{11}f_8^2}{f_1^4f_4^5}$	$\frac{\eta(24z)^{11}\eta(96z)^2}{\eta(12z)^4\eta(48z)^5}$	2	576	I
4	$\frac{f_1^4f_8^2}{f_2f_4}$	$\frac{\eta(12z)^4\eta(96z)^2}{\eta(24z)\eta(48z)}$	2	576	I
5	$\frac{f_2^2f_4^9}{f_1^2f_8^3}$	$\frac{\eta(24z)^2\eta(48z)^9}{\eta(12z)^2\eta(96z)^3}$	3	2304	I
6	$\frac{f_1^2f_4^{11}}{f_2^4f_8^3}$	$\frac{\eta(12z)^2\eta(48z)^{11}}{\eta(24z)^4\eta(96z)^3}$	3	2304	I
7	$\frac{f_2^4f_4^2}{f_1^2}$	$\frac{\eta(24z)^4\eta(48z)^2}{\eta(12z)^2}$	2	576	I
8	$\frac{f_1^2f_4^4}{f_2^2}$	$\frac{\eta(12z)^2\eta(48z)^4}{\eta(24z)^2}$	2	144	I
9	$\frac{f_2^6f_8}{f_1^2f_4}$	$\frac{\eta(24z)^6\eta(96z)}{\eta(12z)^2\eta(48z)}$	2	2304	I
10	$f_1^2f_4f_8$	$\eta(12z)^2\eta(48z)\eta(96z)$	2	2304	I
11	$\frac{f_2^7f_{12}}{f_1^2f_4f_6}$	$\frac{\eta(24z)^7\eta(144z)}{\eta(12z)^2\eta(48z)\eta(72z)}$	2	1728	I
12	$\frac{f_1^2f_2f_4f_{12}}{f_6}$	$\frac{\eta(12z)^2\eta(24z)\eta(48z)\eta(144z)}{\eta(72z)}$	2	432	I
13	$\frac{f_2^8}{f_1^2}$	$\frac{\eta(24z)^8}{\eta(12z)^2}$	3	144	I
14	$f_1^2f_2^2f_4^2$	$\eta(12z)^2\eta(24z)^2\eta(48z)^2$	3	576	I
15	$\frac{f_3^5}{f_1}$	$\frac{\eta(36z)^5}{\eta(12z)}$	2	432	I
16	$\frac{f_1f_4f_6^{15}}{f_2^3f_3^5f_{12}^5}$	$\frac{\eta(12z)\eta(48z)\eta(72z)^{15}}{\eta(24z)^3\eta(36z)^5\eta(144z)^5}$	2	1728	I
17	$\frac{f_2f_3f_4f_{12}}{f_1f_6}$	$\frac{\eta(24z)\eta(36z)\eta(48z)\eta(144z)}{\eta(12z)\eta(72z)}$	1	144	I
18	$\frac{f_1f_4^2f_6^2}{f_2^2f_3}$	$\frac{\eta(12z)\eta(48z)^2\eta(72z)^2}{\eta(24z)^2\eta(36z)}$	1	576	I
19	$\frac{f_2^5f_3f_{12}}{f_1f_4f_6}$	$\frac{\eta(24z)^5\eta(36z)\eta(144z)}{\eta(12z)\eta(48z)\eta(72z)}$	2	576	I
20	$\frac{f_1f_2^2f_6^2}{f_3}$	$\frac{\eta(12z)\eta(24z)^2\eta(72z)^2}{\eta(36z)}$	2	144	I
21	$\frac{f_1f_4^9f_6^2}{f_2^4f_3f_8^3}$	$\frac{\eta(12z)\eta(48z)^9\eta(72z)^2}{\eta(24z)^4\eta(36z)\eta(96z)^3}$	2	2304	I
22	$\frac{f_3f_4^8f_{12}}{f_1f_2f_6f_8^3}$	$\frac{\eta(36z)\eta(48z)^8\eta(144z)}{\eta(12z)\eta(24z)\eta(72z)\eta(96z)^3}$	2	2304	I

23	$\frac{f_1f_6^2f_8}{f_3f_4}$	$\frac{\eta(12z)\eta(72z)^2\eta(96z)}{\eta(36z)\eta(48z)}$	1	2304	I
24	$\frac{f_2^3f_3f_8f_{12}}{f_1f_4^2f_6}$	$\frac{\eta(24z)^3\eta(36z)\eta(96z)\eta(144z)}{\eta(12z)\eta(48z)^2\eta(72z)}$	1	2304	I
25	$f_1^5f_3^3$	$\eta(12z)^5\eta(36z)^3$	4	144	I
26	$\frac{f_2^{15}f_6^9}{f_1^5f_3^3f_4^5f_{12}^3}$	$\frac{\eta(24z)^{15}\eta(72z)^9}{\eta(12z)^5\eta(36z)^3\eta(48z)^5\eta(144z)^3}$	4	576	I
27	$\frac{f_1^7f_6^4}{f_2^4f_3^3}$	$\frac{\eta(12z)^7\eta(72z)^4}{\eta(24z)^4\eta(36z)^3}$	2	432	I
28	$\frac{f_2^{17}f_3^3f_{12}^3}{f_1^7f_4^7f_6^5}$	$\frac{\eta(24z)^{17}\eta(36z)^3\eta(144z)^3}{\eta(12z)^7\eta(48z)^7\eta(72z)^5}$	2	1728	I
29	$\frac{f_1^8f_8^2}{f_2^3f_4}$	$\frac{\eta(12z)^8\eta(96z)^2}{\eta(24z)^3\eta(48z)}$	3	576	I
30	$\frac{f_2^{21}f_8^2}{f_1^8f_4^9}$	$\frac{\eta(24z)^{21}\eta(96z)^2}{\eta(12z)^8\eta(48z)^9}$	3	576	I
31	$\frac{f_2^3f_4^5f_6f_{24}}{f_1^2f_8^2f_{12}^2}$	$\frac{\eta(24z)^3\eta(48z)^5\eta(72z)\eta(288z)}{\eta(12z)^2\eta(96z)^2\eta(144z)^2}$	2	6912	I
32	$\frac{f_1^2f_4^7f_6f_{24}}{f_2^3f_8^2f_{12}^2}$	$\frac{\eta(12z)^2\eta(48z)^7\eta(72z)\eta(288z)}{\eta(24z)^3\eta(96z)^2\eta(144z)^2}$	2	6912	I
33	$\frac{f_3f_4^4f_{24}}{f_1f_8^2f_{12}}$	$\frac{\eta(36z)\eta(48z)^4\eta(288z)}{\eta(12z)\eta(96z)^2\eta(144z)}$	1	6912	II
34	$\frac{f_1f_4^5f_6^3f_{24}}{f_2^3f_3f_8^2f_{12}^2}$	$\frac{\eta(12z)\eta(48z)^5\eta(72z)^3\eta(288z)}{\eta(24z)^3\eta(36z)\eta(96z)^2\eta(144z)^2}$	1	6912	II
35	$\frac{f_2^4f_3f_{12}^2}{f_1f_4^2f_6^2}$	$\frac{\eta(24z)^4\eta(36z)\eta(144z)^2}{\eta(12z)\eta(48z)^2\eta(72z)^2}$	1	432	II
36	$\frac{f_1f_2f_6f_{12}}{f_3f_4}$	$\frac{\eta(12z)\eta(24z)\eta(72z)\eta(144z)}{\eta(36z)\eta(48z)}$	1	1728	II
37	$\frac{f_2^7f_3f_{12}^{10}}{f_1^3f_4f_6^6f_8f_{24}^3}$	$\frac{\eta(24z)^7\eta(36z)\eta(144z)^{10}}{\eta(12z)^3\eta(48z)\eta(72z)^6\eta(96z)\eta(288z)^3}$	2	1728	III
38	$\frac{f_1^3f_4^2f_{12}^9}{f_2^2f_3f_6^3f_8f_{24}^3}$	$\frac{\eta(12z)^3\eta(48z)^2\eta(144z)^9}{\eta(24z)^2\eta(36z)\eta(72z)^3\eta(96z)\eta(288z)^3}$	2	1728	III
39	$\frac{f_3^3f_4^2f_{12}^9}{f_1f_6^5f_8f_{24}^3}$	$\frac{\eta(36z)^3\eta(48z)^2\eta(144z)^9}{\eta(12z)\eta(72z)^5\eta(96z)\eta(288z)^3}$	2	1728	III
40	$\frac{f_1f_4^3f_6^4f_{12}^6}{f_2^3f_3^3f_8f_{24}^3}$	$\frac{\eta(12z)\eta(48z)^3\eta(72z)^4\eta(144z)^6}{\eta(24z)^3\eta(36z)^3\eta(96z)\eta(288z)^3}$	2	1728	III
41	$\frac{f_3^3f_4^2f_{24}}{f_1f_6f_8f_{12}}$	$\frac{\eta(36z)^3\eta(48z)^2\eta(288z)}{\eta(12z)\eta(72z)\eta(96z)\eta(144z)}$	1	1728	IV
42	$\frac{f_1f_4^3f_6^8f_{24}}{f_2^3f_3^3f_8f_{12}^4}$	$\frac{\eta(12z)\eta(48z)^3\eta(72z)^8\eta(288z)}{\eta(24z)^3\eta(36z)^3\eta(96z)\eta(144z)^4}$	1	1728	IV
43	$\frac{f_1^3f_4^2f_6f_{24}}{f_2^2f_3f_8f_{12}}$	$\frac{\eta(12z)^3\eta(48z)^2\eta(72z)\eta(288z)}{\eta(24z)^2\eta(36z)\eta(96z)\eta(144z)}$	1	1728	IV
44	$\frac{f_2^7f_3f_{24}}{f_1^3f_4f_6^2f_8}$	$\frac{\eta(24z)^7\eta(36z)\eta(288z)}{\eta(12z)^3\eta(48z)\eta(72z)^2\eta(96z)}$	1	1728	IV
45	$\frac{f_1f_3f_4^2f_{24}}{f_2f_8f_{12}}$	$\frac{\eta(12z)\eta(36z)\eta(48z)^2\eta(288z)}{\eta(24z)\eta(96z)\eta(144z)}$	1	1728	V
46	$\frac{f_2^2f_4f_6^3f_{24}}{f_1f_3f_8f_{12}^2}$	$\frac{\eta(24z)^2\eta(48z)\eta(72z)^3\eta(288z)}{\eta(12z)\eta(36z)\eta(96z)\eta(144z)^2}$	1	1728	V
47	$\frac{f_1^2f_4^2f_{12}^3}{f_2f_6f_{24}}$	$\frac{\eta(12z)^2\eta(48z)^2\eta(144z)^3}{\eta(24z)\eta(72z)\eta(288z)}$	2	6912	VI

48	$\frac{f_2^5 f_{12}^3}{f_1^2 f_6 f_{24}}$	$\frac{\eta(24z)^5\eta(144z)^3}{\eta(12z)^2\eta(72z)\eta(288z)}$	2	6912	VI
49	$\frac{f_2^2 f_3 f_{12}^{14}}{f_1 f_4 f_6^6 f_{24}^5}$	$\frac{\eta(24z)^2\eta(36z)\eta(144z)^{14}}{\eta(12z)\eta(48z)\eta(72z)^6\eta(288z)^5}$	2	6912	VI
50	$\frac{f_1 f_{12}^{13}}{f_2 f_3 f_6^3 f_{24}^5}$	$\frac{\eta(12z)\eta(144z)^{13}}{\eta(24z)\eta(36z)\eta(72z)^3\eta(288z)^5}$	2	6912	VI
51	$\frac{f_1^2 f_4^2 f_{12}^{13}}{f_2 f_6^5 f_{24}^5}$	$\frac{\eta(12z)^2\eta(48z)^2\eta(144z)^{13}}{\eta(24z)\eta(72z)^5\eta(288z)^5}$	3	6912	VI
52	$\frac{f_2^5 f_{12}^{13}}{f_1^2 f_6^5 f_{24}^5}$	$\frac{\eta(24z)^5\eta(144z)^{13}}{\eta(12z)^2\eta(72z)^5\eta(288z)^5}$	3	6912	VI
53	$\frac{f_2^2 f_3 f_6^4}{f_1 f_4 f_{12}}$	$\frac{\eta(24z)^2\eta(36z)\eta(72z)^4}{\eta(12z)\eta(48z)\eta(144z)}$	2	432	VI
54	$\frac{f_1 f_6^7}{f_2 f_3 f_{12}^2}$	$\frac{\eta(12z)\eta(72z)^7}{\eta(24z)\eta(36z)\eta(144z)^2}$	2	1728	VI
55	$\frac{f_1^2 f_4^2 f_6^5}{f_2 f_{12}^2}$	$\frac{\eta(12z)^2\eta(48z)^2\eta(72z)^5}{\eta(24z)\eta(144z)^2}$	3	432	VI
56	$\frac{f_2^5 f_6^5}{f_1^2 f_{12}^2}$	$\frac{\eta(24z)^5\eta(72z)^5}{\eta(12z)^2\eta(144z)^2}$	3	1728	VI
57	$\frac{f_1^2 f_4^2 f_6}{f_2}$	$\frac{\eta(12z)^2\eta(48z)^2\eta(72z)}{\eta(24z)}$	2	1728	VI
58	$\frac{f_2^5 f_6}{f_1^2}$	$\frac{\eta(24z)^5\eta(72z)}{\eta(12z)^2}$	2	432	VI
59	$\frac{f_2^2 f_3 f_{12}^4}{f_1 f_4 f_6^2 f_{24}}$	$\frac{\eta(24z)^2\eta(36z)\eta(144z)^4}{\eta(12z)\eta(48z)\eta(72z)^2\eta(288z)}$	1	6912	VII
60	$\frac{f_1 f_6 f_{12}^3}{f_2 f_3 f_{24}}$	$\frac{\eta(12z)\eta(72z)\eta(144z)^3}{\eta(24z)\eta(36z)\eta(288z)}$	1	6912	VII
61	$\frac{f_2^2 f_3 f_{12}}{f_1 f_4}$	$\frac{\eta(24z)^2\eta(36z)\eta(144z)}{\eta(12z)\eta(48z)}$	1	1728	VII
62	$\frac{f_1 f_6^3}{f_2 f_3}$	$\frac{\eta(12z)\eta(72z)^3}{\eta(24z)\eta(36z)}$	1	432	VII
63	$\frac{f_4^{14} f_6 f_{24}^2}{f_2^4 f_8^5 f_{12}^4}$	$\frac{\eta(48z)^{14}\eta(72z)\eta(288z)^2}{\eta(24z)^4\eta(96z)^5\eta(144z)^4}$	2	6912	VIII
64	$\frac{f_2^4 f_4^2 f_{24}}{f_6 f_8 f_{12}}$	$\frac{\eta(24z)^4\eta(48z)^2\eta(288z)}{\eta(72z)\eta(96z)\eta(144z)}$	2	1728	VIII
65	$\frac{f_2^2 f_6 f_8 f_{12}^2}{f_4 f_{24}}$	$\frac{\eta(24z)^2\eta(72z)\eta(96z)\eta(144z)^2}{\eta(48z)\eta(288z)}$	2	1728	VIII
66	$\frac{f_4^5 f_{12}^5}{f_2^2 f_6 f_8 f_{24}^2}$	$\frac{\eta(48z)^5\eta(144z)^5}{\eta(24z)^2\eta(72z)\eta(96z)\eta(288z)^2}$	2	6912	VIII
67	$f_2^3 f_8$	$\eta(24z)^3\eta(96z)$	2	2304	VIII
68	$\frac{f_4^9}{f_2^3 f_8^2}$	$\frac{\eta(48z)^9}{\eta(24z)^3\eta(96z)^2}$	2	576	VIII
69	$\frac{f_2^5 f_4^5}{f_8^2}$	$\frac{\eta(24z)^5\eta(48z)^5}{\eta(96z)^2}$	4	576	VIII
70	$\frac{f_4^{20}}{f_2^5 f_8^7}$	$\frac{\eta(48z)^{20}}{\eta(24z)^5\eta(96z)^7}$	4	2304	VIII
71	$\frac{f_4^5 f_{12}^8}{f_2^2 f_6^3 f_8 f_{24}^3}$	$\frac{\eta(48z)^5\eta(144z)^8}{\eta(24z)^2\eta(72z)^3\eta(96z)\eta(288z)^3}$	2	1728	IX
72	$\frac{f_2^2 f_6^3 f_8}{f_4 f_{12}}$	$\frac{\eta(24z)^2\eta(72z)^3\eta(96z)}{\eta(48z)\eta(144z)}$	2	6912	IX

73	$\frac{f_4^5 f_6 f_{24}}{f_2^2 f_8 f_{12}^2}$	$\frac{\eta(48z)^5\eta(72z)\eta(288z)}{\eta(24z)^2\eta(96z)\eta(144z)^2}$	1	1728	X
74	$\frac{f_2^2 f_8 f_{12}}{f_4 f_6}$	$\frac{\eta(24z)^2\eta(96z)\eta(144z)}{\eta(48z)\eta(72z)}$	1	6912	X
75	$\frac{f_8 f_{12}^{13}}{f_6^5 f_{24}^5}$	$\frac{\eta(96z)\eta(144z)^{13}}{\eta(72z)^5\eta(288z)^5}$	2	1728	XI
76	$\frac{f_6^5 f_8}{f_{12}^2}$	$\frac{\eta(72z)^5\eta(96z)}{\eta(144z)^2}$	2	6912	XI
77	$\frac{f_2 f_8^2}{f_4}$	$\frac{\eta(24z)\eta(96z)^2}{\eta(48z)}$	1	576	XII
78	$\frac{f_4^2 f_8}{f_2}$	$\frac{\eta(48z)^2\eta(96z)}{\eta(24z)}$	1	2304	XII
79	$\frac{f_8 f_{12}^3}{f_6 f_{24}}$	$\frac{\eta(96z)\eta(144z)^3}{\eta(72z)\eta(288z)}$	1	1728	XIII
80	$f_6 f_8$	$\eta(72z)\eta(96z)$	1	6912	XIII
81	$\frac{f_7^2 f_{28}^3}{f_{14}^2 f_{56}}$	$\frac{\eta(84z)^2\eta(336z)^3}{\eta(168z)^2\eta(672z)}$	1	16128	XIV
82	$\frac{f_{14}^4 f_{28}}{f_7^2 f_{56}}$	$\frac{\eta(168z)^4\eta(336z)}{\eta(84z)^2\eta(672z)}$	1	16128	XIV
83	$\frac{f_{14}^6}{f_7^2 f_{28}^2}$	$\frac{\eta(168z)^6}{\eta(84z)^2\eta(336z)^2}$	1	4032	XIV
84	f_7^2	$\eta(84z)^2$	1	1008	XIV
85	$\frac{f_{14} f_{56} f_{84}^5}{f_{28}^2 f_{42}^2 f_{168}^2}$	$\frac{\eta(168z)\eta(672z)\eta(1008z)^5}{\eta(336z)^2\eta(504z)^2\eta(2016z)^2}$	$\frac{1}{2}$	16128	XV
86	$\frac{f_{28} f_{42}^2}{f_{14} f_{84}}$	$\frac{\eta(336z)\eta(504z)^2}{\eta(168z)\eta(1008z)}$	$\frac{1}{2}$	1008	XV
87	$\frac{f_{28}^3}{f_{14} f_{56}}$	$\frac{\eta(336z)^3}{\eta(168z)\eta(672z)}$	$\frac{1}{2}$	16128	XV
88	$\frac{f_{28}^{13}}{f_{14}^5 f_{56}^5}$	$\frac{\eta(336z)^{13}}{\eta(168z)^5\eta(672z)^5}$	$\frac{3}{2}$	16128	XV
89	$\frac{f_{14}^5}{f_{28}^2}$	$\frac{\eta(168z)^5}{\eta(336z)^2}$	$\frac{3}{2}$	1008	XV
90	f_{14}	$\eta(168z)$	$\frac{1}{2}$	4032	XV

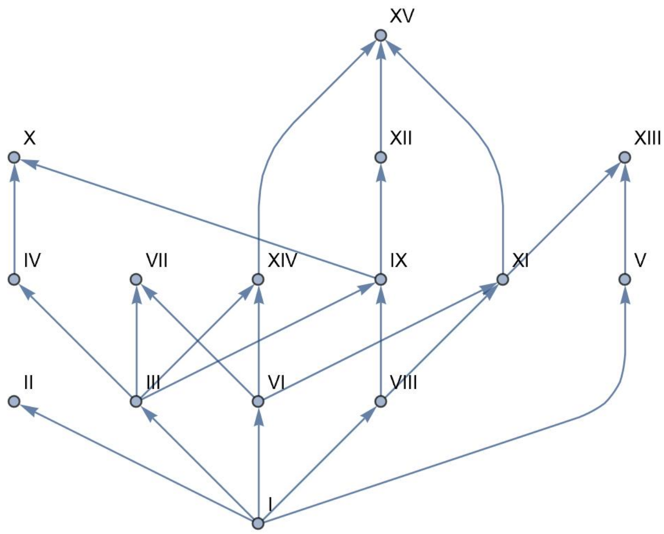

FIGURE 6. The grouping of eta quotients in Table 15, which have vanishing coefficient behaviour similar to f_1^{14}

Table 16: Eta quotients in Table 15 with expansions as double theta series

Number	Modular Form	Weight	Theta Series
5 I	$\frac{\eta(24z)^2\eta(48z)^9}{\eta(12z)^2\eta(96z)^3}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-8}{m}\right)(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right))q^{3m^2+4n^2}$
6 I	$\frac{\eta(12z)^2\eta(48z)^{11}}{\eta(24z)^4\eta(96z)^3}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-8}{m}\right)\left(\frac{n}{3}\right)q^{3m^2+4n^2}$
7 I	$\frac{\eta(24z)^4\eta(48z)^2}{\eta(12z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{2}\right)^2(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right))q^{3m^2+4n^2}$
8 I	$\frac{\eta(12z)^2\eta(48z)^4}{\eta(24z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{2}\right)^2\left(\frac{n}{3}\right)q^{3m^2+4n^2}$
9 I	$\frac{\eta(24z)^6\eta(96z)}{\eta(12z)^2\eta(48z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{8}{m}\right)(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right))q^{3m^2+4n^2}$
10 I	$\eta(12z)^2\eta(48z)\eta(96z)$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{8}{m}\right)\left(\frac{n}{3}\right)q^{3m^2+4n^2}$
11 I	$\frac{\eta(24z)^7\eta(144z)}{\eta(12z)^2\eta(48z)\eta(72z)}$	2	$\sum_{m,n=1}^{\infty} n\left(3\left(\frac{m}{6}\right)^2-2\left(\frac{m}{2}\right)^2\right)(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right))$ $\times q^{3m^2+4n^2}$
12 I	$\frac{\eta(12z)^2\eta(24z)\eta(48z)\eta(144z)}{\eta(72z)}$	2	$\sum_{m,n=1}^{\infty} m\left(\frac{m}{3}\right)\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{4m^2+3n^2}$
13 I	$\frac{\eta(24z)^8}{\eta(12z)^2}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-4}{m}\right)(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right))q^{3m^2+4n^2}$
14 I	$\eta(12z)^2\eta(24z)^2\eta(48z)^2$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-4}{m}\right)\left(\frac{n}{3}\right)q^{3m^2+4n^2}$
17 I	$\frac{\eta(24z)\eta(36z)\eta(48z)\eta(144z)}{\eta(12z)\eta(72z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{m}{2}\right)^2\left(\frac{n}{3}\right)^2q^{3m^2+4n^2}$
18 I	$\frac{\eta(12z)\eta(48z)^2\eta(72z)^2}{\eta(24z)^2\eta(36z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{m}{2}\right)^2\left(2\left(\frac{n}{6}\right)^2-\left(\frac{n}{3}\right)^2\right)q^{3m^2+4n^2}$
19 I	$\frac{\eta(24z)^5\eta(36z)\eta(144z)}{\eta(12z)\eta(48z)\eta(72z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{3}\right)^2\left(\frac{-4}{n}\right)q^{4m^2+3n^2}$
20 I	$\frac{\eta(12z)\eta(24z)^2\eta(72z)^2}{\eta(36z)}$	2	$\sum_{m,n=1}^{\infty} m\left(\frac{-4}{m}\right)\left(2\left(\frac{n}{6}\right)^2-\left(\frac{n}{3}\right)^2\right)q^{3m^2+4n^2}$
21 I	$\frac{\eta(12z)\eta(48z)^9\eta(72z)^2}{\eta(24z)^4\eta(36z)\eta(96z)^3}$	2	$\sum_{m,n=1}^{\infty} m\left(\frac{-8}{m}\right)\left(2\left(\frac{n}{6}\right)^2-\left(\frac{n}{3}\right)^2\right)q^{3m^2+4n^2}$

22 I	$\frac{\eta(36z)\eta(48z)^8\eta(144z)}{\eta(12z)\eta(24z)\eta(72z)\eta(96z)^3}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{3}\right)^2\left(\frac{-8}{n}\right)q^{4m^2+3n^2}$
23 I	$\frac{\eta(12z)\eta(72z)^2\eta(96z)}{\eta(36z)\eta(48z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{8}{m}\right)\left(2\left(\frac{n}{6}\right)^2-\left(\frac{n}{3}\right)^2\right)q^{3m^2+4n^2}$
24 I	$\frac{\eta(24z)^3\eta(36z)\eta(96z)\eta(144z)}{\eta(12z)\eta(48z)^2\eta(72z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{8}{m}\right)\left(\frac{n}{3}\right)^2 q^{3m^2+4n^2}$
35 II	$\frac{\eta(24z)^4\eta(36z)\eta(144z)^2}{\eta(12z)\eta(48z)^2\eta(72z)^2}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{m}{3}\right)^2\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{4m^2+3n^2}$
36 II	$\frac{\eta(12z)\eta(24z)\eta(72z)\eta(144z)}{\eta(36z)\eta(48z)}$	1	$\sum_{m,n=1}^{\infty}\left(2\left(\frac{m}{6}\right)^2-\left(\frac{m}{3}\right)^2\right)\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)$ $\times q^{4m^2+3n^2}$
47 VI	$\frac{\eta(12z)^2\eta(48z)^2\eta(144z)^3}{\eta(24z)\eta(72z)\eta(288z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{24}{m}\right)\left(\frac{n}{3}\right)q^{3m^2+4n^2}$
48 VI	$\frac{\eta(24z)^5\eta(144z)^3}{\eta(12z)^2\eta(72z)\eta(288z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{24}{m}\right)(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right))q^{3m^2+4n^2}$
49 VI	$\frac{\eta(24z)^2\eta(36z)\eta(144z)^{14}}{\eta(12z)\eta(48z)\eta(72z)^6\eta(288z)^5}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{3}\right)^2\left(\frac{-6}{n}\right)q^{4m^2+3n^2}$
50 VI	$\frac{\eta(12z)\eta(144z)^{13}}{\eta(24z)\eta(36z)\eta(72z)^3\eta(288z)^5}$	2	$\sum_{m,n=1}^{\infty} m\left(\frac{-6}{m}\right)\left(2\left(\frac{n}{6}\right)^2-\left(\frac{n}{3}\right)^2\right)q^{3m^2+4n^2}$
51 VI	$\frac{\eta(12z)^2\eta(48z)^2\eta(144z)^{13}}{\eta(24z)\eta(72z)^5\eta(288z)^5}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{m}{3}\right)\left(\frac{-6}{n}\right)q^{4m^2+3n^2}$
52 VI	$\frac{\eta(24z)^5\eta(144z)^{13}}{\eta(12z)^2\eta(72z)^5\eta(288z)^5}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-6}{m}\right)(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right))q^{3m^2+4n^2}$
53 VI	$\frac{\eta(24z)^2\eta(36z)\eta(72z)^4}{\eta(12z)\eta(48z)\eta(144z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{3}\right)^2\left(\frac{n}{12}\right)q^{4m^2+3n^2}$
54 VI	$\frac{\eta(12z)\eta(72z)^7}{\eta(24z)\eta(36z)\eta(144z)^2}$	2	$\sum_{m,n=1}^{\infty} m\left(\frac{m}{12}\right)\left(2\left(\frac{n}{6}\right)^2-\left(\frac{n}{3}\right)^2\right)q^{3m^2+4n^2}$
55 VI	$\frac{\eta(12z)^2\eta(48z)^2\eta(72z)^5}{\eta(24z)\eta(144z)^2}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{m}{3}\right)\left(\frac{n}{12}\right)q^{4m^2+3n^2}$
56 VI	$\frac{\eta(24z)^5\eta(72z)^5}{\eta(12z)^2\eta(144z)^2}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{m}{12}\right)(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right))q^{3m^2+4n^2}$
57 VI	$\frac{\eta(12z)^2\eta(48z)^2\eta(72z)}{\eta(24z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{n}{3}\right)q^{3m^2+4n^2}$
58 VI	$\frac{\eta(24z)^5\eta(72z)}{\eta(12z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right))q^{3m^2+4n^2}$
59 VII	$\frac{\eta(24z)^2\eta(36z)\eta(144z)^4}{\eta(12z)\eta(48z)\eta(72z)^2\eta(288z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{m}{3}\right)^2\left(\frac{24}{n}\right)q^{4m^2+3n^2}$
60 VII	$\frac{\eta(12z)\eta(72z)\eta(144z)^3}{\eta(24z)\eta(36z)\eta(288z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{24}{m}\right)\left(2\left(\frac{n}{6}\right)^2-\left(\frac{n}{3}\right)^2\right)q^{3m^2+4n^2}$
61 VII	$\frac{\eta(24z)^2\eta(36z)\eta(144z)}{\eta(12z)\eta(48z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{m}{3}\right)^2\left(\frac{12}{n}\right)q^{4m^2+3n^2}$
62 VII	$\frac{\eta(12z)\eta(72z)^3}{\eta(24z)\eta(36z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(2\left(\frac{n}{6}\right)^2-\left(\frac{n}{3}\right)^2\right)q^{3m^2+4n^2}$
67 VIII	$\eta(24z)^3\eta(96z)$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{-4}{n}\right)q^{4m^2+3n^2}$
68 VIII	$\frac{\eta(48z)^9}{\eta(24z)^3\eta(96z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{-8}{n}\right)q^{4m^2+3n^2}$
74 X	$\frac{\eta(24z)^2\eta(96z)\eta(144z)}{\eta(48z)\eta(72z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{4m^2+3n^2}$
75 XI	$\frac{\eta(96z)\eta(144z)^{13}}{\eta(72z)^5\eta(288z)^5}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{-6}{n}\right)q^{4m^2+3n^2}$
76 XI	$\frac{\eta(72z)^5\eta(96z)}{\eta(144z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{n}{12}\right)q^{4m^2+3n^2}$
77 XII	$\frac{\eta(24z)\eta(96z)^2}{\eta(48z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{m}{2}\right)^2\left(\frac{12}{n}\right)q^{6m^2+n^2}$
77 XII	$\frac{\eta(24z)\eta(96z)^2}{\eta(48z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{8}{m}\right)\left(\frac{12}{n}\right)q^{3m^2+4n^2}$
78 XII	$\frac{\eta(48z)^2\eta(96z)}{\eta(24z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{m}{2}\right)^2\left(\frac{24}{n}\right)q^{6m^2+n^2}$
78 XII	$\frac{\eta(48z)^2\eta(96z)}{\eta(24z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{m}{2}\right)^2\left(\frac{12}{n}\right)q^{3m^2+4n^2}$
79 XIII	$\frac{\eta(96z)\eta(144z)^3}{\eta(72z)\eta(288z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{24}{n}\right)q^{4m^2+3n^2}$

80 XIII	$\eta(72z)\eta(96z)$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{12}{n}\right)q^{4m^2+3n^2}$
81 XIV	$\frac{\eta(84z)^2\eta(336z)^3}{\eta(168z)^2\eta(672z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}(-1)^n\left(\frac{24}{m}\right)q^{7(m^2+12n^2)}$
82 XIV	$\frac{\eta(168z)^4\eta(336z)}{\eta(84z)^2\eta(672z)}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty}\left(\frac{24}{n}\right)q^{84m^2+7n^2}$
83 XIV	$\frac{\eta(168z)^6}{\eta(84z)^2\eta(336z)^2}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty}\left(\frac{12}{n}\right)q^{84m^2+7n^2}$
83 XIV	$\frac{\eta(168z)^6}{\eta(84z)^2\eta(336z)^2}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{24}{m}\right)\left(\frac{24}{n}\right)q^{\frac{7}{2}(m^2+n^2)}$
84 XIV	$\eta(84z)^2$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}(-1)^n\left(\frac{12}{m}\right)q^{7(m^2+12n^2)}$
84 XIV	$\eta(84z)^2$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{12}{n}\right)q^{\frac{7}{2}(m^2+n^2)}$

Table 17: Eta quotients with vanishing behaviour similar to f_1^{26}

Number	q-Product	Modular Form	Weight	Level	Group
1	f_1^{26}	$\eta(12z)^{26}$	13	144	I
2	$\frac{f_2^{78}}{f_1^{26}f_4^{26}}$	$\frac{\eta(24z)^{78}}{\eta(12z)^{26}\eta(48z)^{26}}$	13	576	I
3	$\frac{f_2^2f_4^6}{f_1^2}$	$\frac{\eta(24z)^2\eta(48z)^6}{\eta(12z)^2}$	3	576	I
4	$\frac{f_1^2f_4^8}{f_2^4}$	$\frac{\eta(12z)^2\eta(48z)^8}{\eta(24z)^4}$	3	144	I
5	$\frac{f_2^8f_8^3}{f_1^2f_4^3}$	$\frac{\eta(24z)^8\eta(96z)^3}{\eta(12z)^2\eta(48z)^3}$	3	2304	I
6	$\frac{f_1^2f_2^2f_8^3}{f_4}$	$\frac{\eta(12z)^2\eta(24z)^2\eta(96z)^3}{\eta(48z)}$	3	2304	I
7	$\frac{f_3^9}{f_1}$	$\frac{\eta(36z)^9}{\eta(12z)}$	4	144	I
8	$\frac{f_1f_4f_6^{27}}{f_2^3f_3^9f_{12}^9}$	$\frac{\eta(12z)\eta(48z)\eta(72z)^{27}}{\eta(24z)^3\eta(36z)^9\eta(144z)^9}$	4	576	I
9	$\frac{f_1^6f_4^6}{f_2^2}$	$\frac{\eta(12z)^6\eta(48z)^6}{\eta(24z)^2}$	5	576	I
10	$\frac{f_2^{16}}{f_1^6}$	$\frac{\eta(24z)^{16}}{\eta(12z)^6}$	5	144	I
11	$\frac{f_1^{11}f_{12}^3}{f_2^4f_3^3f_4}$	$\frac{\eta(12z)^{11}\eta(144z)^3}{\eta(24z)^4\eta(36z)^3\eta(48z)}$	3	144	I
12	$\frac{f_2^{29}f_3^3f_{12}^6}{f_1^{11}f_4^{12}f_6^9}$	$\frac{\eta(24z)^{29}\eta(36z)^3\eta(144z)^6}{\eta(12z)^{11}\eta(48z)^{12}\eta(72z)^9}$	3	576	I
13	$\frac{f_2^{14}f_3f_{12}^4}{f_1^5f_4^6f_6^4}$	$\frac{\eta(24z)^{14}\eta(36z)\eta(144z)^4}{\eta(12z)^5\eta(48z)^6\eta(72z)^4}$	2	576	II
14	$\frac{f_1^5f_{12}^3}{f_2f_3f_4f_6}$	$\frac{\eta(12z)^5\eta(144z)^3}{\eta(24z)\eta(36z)\eta(48z)\eta(72z)}$	2	144	II
15	$\frac{f_1f_2^5f_{12}^3}{f_3f_4^3f_6}$	$\frac{\eta(12z)\eta(24z)^5\eta(144z)^3}{\eta(36z)\eta(48z)^3\eta(72z)}$	2	576	II
16	$\frac{f_2^8f_3f_{12}^4}{f_1f_4^4f_6^4}$	$\frac{\eta(24z)^8\eta(36z)\eta(144z)^4}{\eta(12z)\eta(48z)^4\eta(72z)^4}$	2	144	II
17	$\frac{f_1^2f_8^4}{f_4^2}$	$\frac{\eta(12z)^2\eta(96z)^4}{\eta(48z)^2}$	2	288	III
18	$\frac{f_2^6f_8^4}{f_1^2f_4^4}$	$\frac{\eta(24z)^6\eta(96z)^4}{\eta(12z)^2\eta(48z)^4}$	2	576	III
19	$\frac{f_1^2f_4f_8^3}{f_2^2}$	$\frac{\eta(12z)^2\eta(48z)\eta(96z)^3}{\eta(24z)^2}$	2	2304	III
20	$\frac{f_2^4f_8^3}{f_1^2f_4}$	$\frac{\eta(24z)^4\eta(96z)^3}{\eta(12z)^2\eta(48z)}$	2	2304	III
21	$\frac{f_2^6f_3f_{12}f_{18}^9}{f_1^2f_4^2f_6^3f_9^3f_{36}^3}$	$\frac{\eta(24z)^6\eta(36z)\eta(144z)\eta(216z)^9}{\eta(12z)^2\eta(48z)^2\eta(72z)^3\eta(108z)^3\eta(432z)^3}$	2	1728	IV

22	$\frac{f_1^2 f_9^3}{f_3}$	$\frac{\eta(12z)^2\eta(108z)^3}{\eta(36z)}$	2	432	IV
23	$\frac{f_1 f_2 f_3^3 f_{12}^3}{f_4 f_6^3}$	$\frac{\eta(12z)\eta(24z)\eta(36z)^3\eta(144z)^3}{\eta(48z)\eta(72z)^3}$	2	144	V
24	$\frac{f_2^4 f_6^6}{f_1 f_3^3 f_4^2}$	$\frac{\eta(24z)^4\eta(72z)^6}{\eta(12z)\eta(36z)^3\eta(48z)^2}$	2	576	V
25	$\frac{f_1^3 f_6^6}{f_2^2 f_3^3}$	$\frac{\eta(12z)^3\eta(72z)^6}{\eta(24z)^2\eta(36z)^3}$	2	144	VI
26	$\frac{f_2^7 f_3^3 f_{12}^3}{f_1^3 f_4^3 f_6^3}$	$\frac{\eta(24z)^7\eta(36z)^3\eta(144z)^3}{\eta(12z)^3\eta(48z)^3\eta(72z)^3}$	2	576	VI
27	$\frac{f_1^2 f_6^5 f_8^4}{f_2^2 f_3^2 f_4 f_{12}^2}$	$\frac{\eta(12z)^2\eta(72z)^5\eta(96z)^4}{\eta(24z)^2\eta(36z)^2\eta(48z)\eta(144z)^2}$	2	1152	VII
28	$\frac{f_2^4 f_3^2 f_8^4}{f_1^2 f_4^3 f_6}$	$\frac{\eta(24z)^4\eta(36z)^2\eta(96z)^4}{\eta(12z)^2\eta(48z)^3\eta(72z)}$	2	1152	VII
29	$\frac{f_2^6 f_6^6 f_{18}^6}{f_1^2 f_3^2 f_4^2 f_9^2 f_{12}^2 f_{36}^2}$	$\frac{\eta(24z)^6\eta(72z)^6\eta(216z)^6}{\eta(12z)^2\eta(36z)^2\eta(48z)^2\eta(108z)^2\eta(144z)^2\eta(432z)^2}$	3	5184	VIII
30	$f_1^2 f_3^2 f_9^2$	$\eta(12z)^2\eta(36z)^2\eta(108z)^2$	3	1296	VIII
31	$\frac{f_1 f_4 f_6^9 f_{18}^6}{f_2^3 f_3^3 f_9^2 f_{12}^3 f_{36}^2}$	$\frac{\eta(12z)\eta(48z)\eta(72z)^9\eta(216z)^6}{\eta(24z)^3\eta(36z)^3\eta(108z)^2\eta(144z)^3\eta(432z)^2}$	2	5184	VIII
32	$\frac{f_3^3 f_9^2}{f_1}$	$\frac{\eta(36z)^3\eta(108z)^2}{\eta(12z)}$	2	1296	VIII
33	$\frac{f_2^5 f_3^2 f_{12}^2}{f_4^2 f_6}$	$\frac{\eta(24z)^5\eta(36z)^2\eta(144z)^2}{\eta(48z)^2\eta(72z)}$	3	432	IX
34	$\frac{f_2^5 f_6^5}{f_3^2 f_4^2}$	$\frac{\eta(24z)^5\eta(72z)^5}{\eta(36z)^2\eta(48z)^2}$	3	1728	IX
35	$\frac{f_2 f_3^2 f_{12}^2}{f_6}$	$\frac{\eta(24z)\eta(36z)^2\eta(144z)^2}{\eta(72z)}$	2	1728	IX
36	$\frac{f_2 f_6^5}{f_3^2}$	$\frac{\eta(24z)\eta(72z)^5}{\eta(36z)^2}$	2	432	IX
37	$\frac{f_3^2 f_4^3 f_{12}^2}{f_2 f_6 f_8}$	$\frac{\eta(36z)^2\eta(48z)^3\eta(144z)^2}{\eta(24z)\eta(72z)\eta(96z)}$	2	6912	IX
38	$\frac{f_4^3 f_6^5}{f_2 f_3^2 f_8}$	$\frac{\eta(48z)^3\eta(72z)^5}{\eta(24z)\eta(36z)^2\eta(96z)}$	2	6912	IX
39	$\frac{f_3^2 f_4^{13} f_{12}^2}{f_2^5 f_6 f_8^5}$	$\frac{\eta(36z)^2\eta(48z)^{13}\eta(144z)^2}{\eta(24z)^5\eta(72z)\eta(96z)^5}$	3	6912	IX
40	$\frac{f_4^{13} f_6^5}{f_2^5 f_3^2 f_8^5}$	$\frac{\eta(48z)^{13}\eta(72z)^5}{\eta(24z)^5\eta(36z)^2\eta(96z)^5}$	3	6912	IX
41	$\frac{f_3^2 f_4 f_6 f_{12}}{f_2}$	$\frac{\eta(36z)^2\eta(48z)\eta(72z)\eta(144z)}{\eta(24z)}$	2	432	X
42	$\frac{f_4 f_6^7}{f_2 f_3^2 f_{12}}$	$\frac{\eta(48z)\eta(72z)^7}{\eta(24z)\eta(36z)^2\eta(144z)}$	2	1728	X
43	$\frac{f_1 f_6^5}{f_2 f_3}$	$\frac{\eta(12z)\eta(72z)^5}{\eta(24z)\eta(36z)}$	2	288	XI
44	$\frac{f_2^2 f_3 f_6^2 f_{12}}{f_1 f_4}$	$\frac{\eta(24z)^2\eta(36z)\eta(72z)^2\eta(144z)}{\eta(12z)\eta(48z)}$	2	576	XI
45	$\frac{f_1 f_6 f_{12}^2}{f_2 f_3}$	$\frac{\eta(12z)\eta(72z)\eta(144z)^2}{\eta(24z)\eta(36z)}$	1	576	XI
46	$\frac{f_2^2 f_3 f_{12}^3}{f_1 f_4 f_6^2}$	$\frac{\eta(24z)^2\eta(36z)\eta(144z)^3}{\eta(12z)\eta(48z)\eta(72z)^2}$	1	144	XI
47	$\frac{f_1^2 f_4^2 f_{12}^2}{f_2 f_6}$	$\frac{\eta(12z)^2\eta(48z)^2\eta(144z)^2}{\eta(24z)\eta(72z)}$	2	144	XI
48	$\frac{f_2^5 f_{12}^2}{f_1^2 f_6}$	$\frac{\eta(24z)^5\eta(144z)^2}{\eta(12z)^2\eta(72z)}$	2	576	XI
49	$\frac{f_1^2 f_4^2 f_6^3}{f_2}$	$\frac{\eta(12z)^2\eta(48z)^2\eta(72z)^3}{\eta(24z)}$	3	576	XI
50	$\frac{f_2^5 f_6^3}{f_1^2}$	$\frac{\eta(24z)^5\eta(72z)^3}{\eta(12z)^2}$	3	288	XI
51	$\frac{f_2^3 f_{10}^{15}}{f_1 f_4 f_5^5 f_{20}^5}$	$\frac{\eta(24z)^3\eta(120z)^{15}}{\eta(12z)\eta(48z)\eta(60z)^5\eta(240z)^5}$	3	2880	XII
52	$f_1 f_5^5$	$\eta(12z)\eta(60z)^5$	3	720	XII

53	$f_2^7 f_4^3$	$\eta(24z)^7\eta(48z)^3$	5	1152	XIII
54	$\frac{f_4^{10} f_6 f_{24}}{f_2^4 f_8^3 f_{12}}$	$\frac{\eta(48z)^{10}\eta(72z)\eta(288z)}{\eta(24z)^4\eta(96z)^3\eta(144z)}$	2	1152	XIII
55	$\frac{f_2^4 f_8 f_{12}^2}{f_4^2 f_6}$	$\frac{\eta(24z)^4\eta(96z)\eta(144z)^2}{\eta(48z)^2\eta(72z)}$	2	2304	XIII
56	$\frac{f_2^3 f_8^4}{f_4^3}$	$\frac{\eta(24z)^3\eta(96z)^4}{\eta(48z)^3}$	2	1152	XIII
57	$\frac{f_4^7 f_6 f_{24}}{f_2^2 f_8^2 f_{12}}$	$\frac{\eta(48z)^7\eta(72z)\eta(288z)}{\eta(24z)^2\eta(96z)^2\eta(144z)}$	2	2304	XIII
58	$\frac{f_2^2 f_4 f_{12}^2}{f_6}$	$\frac{\eta(24z)^2\eta(48z)\eta(144z)^2}{\eta(72z)}$	2	1152	XIII
59	$f_2 f_4^4 f_8$	$\eta(24z)\eta(48z)^4\eta(96z)$	3	2304	XIII
60	$\frac{f_4^7}{f_2}$	$\frac{\eta(48z)^7}{\eta(24z)}$	3	1152	XIII
61	$\frac{f_2^2 f_8^2 f_{12}^9}{f_4^3 f_6^3 f_{24}^3}$	$\frac{\eta(24z)^2\eta(96z)^2\eta(144z)^9}{\eta(48z)^3\eta(72z)^3\eta(288z)^3}$	2	2304	XIII
62	$\frac{f_4^3 f_6^3}{f_2^2}$	$\frac{\eta(48z)^3\eta(72z)^3}{\eta(24z)^2}$	2	1152	XIII
63	$\frac{f_4^6 f_8}{f_2^3}$	$\frac{\eta(48z)^6\eta(96z)}{\eta(24z)^3}$	2	2304	XIII
64	$\frac{f_4^{24}}{f_2^7 f_8^7}$	$\frac{\eta(48z)^{24}}{\eta(24z)^7\eta(96z)^7}$	5	2304	XIII
65	$\frac{f_4^{38}}{f_2^{11} f_8^{13}}$	$\frac{\eta(48z)^{38}}{\eta(24z)^{11}\eta(96z)^{13}}$	7	2304	XIV
66	$\frac{f_2^{11} f_4^5}{f_8^2}$	$\frac{\eta(24z)^{11}\eta(48z)^5}{\eta(96z)^2}$	7	576	XIV
67	$\frac{f_2^7 f_8^2}{f_4}$	$\frac{\eta(24z)^7\eta(96z)^2}{\eta(48z)}$	4	576	XIV
68	$\frac{f_2^5 f_8^3}{f_4^2}$	$\frac{\eta(24z)^5\eta(96z)^3}{\eta(48z)^2}$	3	2304	XIV
69	$f_2 f_8^3$	$\eta(24z)\eta(96z)^3$	2	2304	XIV
70	$\frac{f_2 f_8^4 f_{12}^5}{f_4^2 f_6^2 f_{24}^2}$	$\frac{\eta(24z)\eta(96z)^4\eta(144z)^5}{\eta(48z)^2\eta(72z)^2\eta(288z)^2}$	2	576	XIV
71	$\frac{f_4 f_6^2 f_8^3}{f_2 f_{12}}$	$\frac{\eta(48z)\eta(72z)^2\eta(96z)^3}{\eta(24z)\eta(144z)}$	2	2304	XIV
72	$\frac{f_4^3 f_8^2}{f_2}$	$\frac{\eta(48z)^3\eta(96z)^2}{\eta(24z)}$	2	576	XIV
73	$\frac{f_4^{13}}{f_2^5 f_8^2}$	$\frac{\eta(48z)^{13}}{\eta(24z)^5\eta(96z)^2}$	3	576	XIV
74	$\frac{f_4^{20}}{f_2^7 f_8^5}$	$\frac{\eta(48z)^{20}}{\eta(24z)^7\eta(96z)^5}$	4	2304	XIV
75	$\frac{f_8 f_{12}^9}{f_6^3 f_{24}^3}$	$\frac{\eta(96z)\eta(144z)^9}{\eta(72z)^3\eta(288z)^3}$	2	1152	XV
76	$f_6^3 f_8$	$\eta(72z)^3\eta(96z)$	2	2304	XV
77	$\frac{f_6 f_8 f_{24}}{f_{12}}$	$\frac{\eta(72z)\eta(96z)\eta(288z)}{\eta(144z)}$	1	1152	XVI
78	$\frac{f_8 f_{12}^2}{f_6}$	$\frac{\eta(96z)\eta(144z)^2}{\eta(72z)}$	1	2304	XVI
79	$\frac{f_{13}^2 f_{52}^3}{f_{26}^2 f_{104}}$	$\frac{\eta(156z)^2\eta(624z)^3}{\eta(312z)^2\eta(1248z)}$	1	29952	XVII
80	$\frac{f_{26}^4 f_{52}}{f_{13}^2 f_{104}}$	$\frac{\eta(312z)^4\eta(624z)}{\eta(156z)^2\eta(1248z)}$	1	29952	XVII
81	$\frac{f_{26}^6}{f_{13}^2 f_{52}^2}$	$\frac{\eta(312z)^6}{\eta(156z)^2\eta(624z)^2}$	1	7488	XVII
82	f_{13}^2	$\eta(156z)^2$	1	1872	XVII
83	$\frac{f_{52}^{13}}{f_{26}^5 f_{104}^5}$	$\frac{\eta(624z)^{13}}{\eta(312z)^5\eta(1248z)^5}$	$\frac{3}{2}$	29952	XVIII
84	$\frac{f_{26}^5}{f_{52}^2}$	$\frac{\eta(312z)^5}{\eta(624z)^2}$	$\frac{3}{2}$	1872	XVIII

85	$\frac{f_{26}f_{104}f_{156}^5}{f_{52}^2f_{78}^2f_{312}^2}$	$\frac{\eta(312z)\eta(1248z)\eta(1872z)^5}{\eta(624z)^2\eta(936z)^2\eta(3744z)^2}$	$\frac{1}{2}$	89856	XVIII
86	$\frac{f_{52}^3}{f_{26}f_{104}}$	$\frac{\eta(624z)^3}{\eta(312z)\eta(1248z)}$	$\frac{1}{2}$	29952	XVIII
87	$\frac{f_{52}f_{78}^2}{f_{26}f_{156}}$	$\frac{\eta(624z)\eta(936z)^2}{\eta(312z)\eta(1872z)}$	$\frac{1}{2}$	1872	XVIII
88	f_{26}	$\eta(312z)$	$\frac{1}{2}$	7488	XVIII

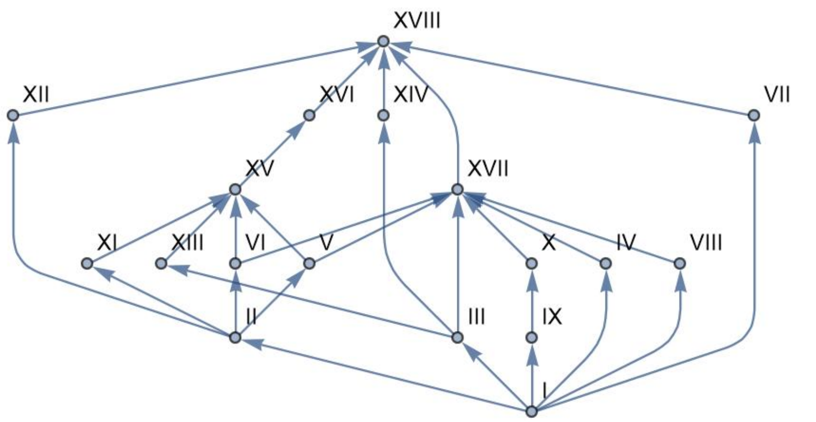

FIGURE 7. The grouping of eta quotients in Table 17, which have vanishing coefficient behaviour similar to f_1^{26}

Table 18: Eta quotients in Table 17 with expansions as double theta series

Number	Modular Form	Weight	Theta Series
33 IX	$\frac{\eta(24z)^5\eta(36z)^2\eta(144z)^2}{\eta(48z)^2\eta(72z)}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{m}{3}\right)\left(\frac{n}{12}\right)q^{12m^2+n^2}$
34 IX	$\frac{\eta(24z)^5\eta(72z)^5}{\eta(36z)^2\eta(48z)^2}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{m}{12}\right)(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right))q^{m^2+12n^2}$
35 IX	$\frac{\eta(24z)\eta(36z)^2\eta(144z)^2}{\eta(72z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{n}{3}\right)q^{m^2+12n^2}$
36 IX	$\frac{\eta(24z)\eta(72z)^5}{\eta(36z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right))q^{m^2+12n^2}$
37 IX	$\frac{\eta(36z)^2\eta(48z)^3\eta(144z)^2}{\eta(24z)\eta(72z)\eta(96z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{24}{m}\right)\left(\frac{n}{3}\right)q^{m^2+12n^2}$
38 IX	$\frac{\eta(48z)^3\eta(72z)^5}{\eta(24z)\eta(36z)^2\eta(96z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{24}{m}\right)(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right))q^{m^2+12n^2}$
39 IX	$\frac{\eta(36z)^2\eta(48z)^{13}\eta(144z)^2}{\eta(24z)^5\eta(72z)\eta(96z)^5}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{m}{3}\right)\left(\frac{-6}{n}\right)q^{12m^2+n^2}$
40 IX	$\frac{\eta(48z)^{13}\eta(72z)^5}{\eta(24z)^5\eta(36z)^2\eta(96z)^5}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-6}{m}\right)(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right))q^{m^2+12n^2}$
41 X	$\frac{\eta(36z)^2\eta(48z)\eta(72z)\eta(144z)}{\eta(24z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{6}\right)^2\left(\frac{n}{3}\right)q^{m^2+12n^2}$
42 X	$\frac{\eta(48z)\eta(72z)^7}{\eta(24z)\eta(36z)^2\eta(144z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{6}\right)^2(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right))q^{m^2+12n^2}$
43 XI	$\frac{\eta(12z)\eta(72z)^5}{\eta(24z)\eta(36z)}$	2	$\sum_{m,n=1}^{\infty} m\left(\frac{-4}{m}\right)\left(2\left(\frac{n}{6}\right)^2-\left(\frac{n}{3}\right)^2\right)q^{9m^2+4n^2}$
44 XI	$\frac{\eta(24z)^2\eta(36z)\eta(72z)^2\eta(144z)}{\eta(12z)\eta(48z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{3}\right)^2\left(\frac{-4}{n}\right)q^{4m^2+9n^2}$

45 XI	$\frac{\eta(12z)\eta(72z)\eta(144z)^2}{\eta(24z)\eta(36z)}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{m}{2}\right)^2 \left(2\left(\frac{n}{6}\right)^2 - \left(\frac{n}{3}\right)^2\right) q^{9m^2+4n^2}$
46 XI	$\frac{\eta(24z)^2\eta(36z)\eta(144z)^3}{\eta(12z)\eta(48z)\eta(72z)^2}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{m}{2}\right)^2 \left(\frac{n}{3}\right)^2 q^{9m^2+4n^2}$
47 XI	$\frac{\eta(12z)^2\eta(48z)^2\eta(144z)^2}{\eta(24z)\eta(72z)}$	2	$\sum_{m,n=1}^{\infty} n \left(\frac{m}{2}\right)^2 \left(\frac{n}{3}\right) q^{9m^2+4n^2}$
48 XI	$\frac{\eta(24z)^5\eta(144z)^2}{\eta(12z)^2\eta(72z)}$	2	$\sum_{m,n=1}^{\infty} n \left(\frac{m}{2}\right)^2 (2\left(\frac{n}{12}\right) - \left(\frac{n}{3}\right))q^{9m^2+4n^2}$
49 XI	$\frac{\eta(12z)^2\eta(48z)^2\eta(72z)^3}{\eta(24z)}$	3	$\sum_{m,n=1}^{\infty} mn \left(\frac{-4}{m}\right) \left(\frac{n}{3}\right) q^{9m^2+4n^2}$
50 XI	$\frac{\eta(24z)^5\eta(72z)^3}{\eta(12z)^2}$	3	$\sum_{m,n=1}^{\infty} mn \left(\frac{-4}{m}\right) (2\left(\frac{n}{12}\right) - \left(\frac{n}{3}\right))q^{9m^2+4n^2}$
68 XIV	$\frac{\eta(24z)^5\eta(96z)^3}{\eta(48z)^2}$	3	$\sum_{m,n=1}^{\infty} mn \left(\frac{-4}{m}\right) \left(\frac{n}{12}\right) q^{12m^2+n^2}$
69 XIV	$\eta(24z)\eta(96z)^3$	2	$\sum_{m,n=1}^{\infty} n \left(\frac{12}{m}\right) \left(\frac{-4}{n}\right) q^{m^2+12n^2}$
70 XIV	$\frac{\eta(24z)\eta(96z)^4\eta(144z)^5}{\eta(48z)^2\eta(72z)^2\eta(288z)^2}$	2	$\sum_{m,n=1}^{\infty} n \left(\frac{18}{m}\right) \left(\frac{-4}{n}\right) q^{m^2+12n^2}$
71 XIV	$\frac{\eta(48z)\eta(72z)^2\eta(96z)^3}{\eta(24z)\eta(144z)}$	2	$\sum_{m,n=1}^{\infty} n \left(\frac{m}{6}\right)^2 \left(\frac{-4}{n}\right) q^{m^2+12n^2}$
72 XIV	$\frac{\eta(48z)^3\eta(96z)^2}{\eta(24z)}$	2	$\sum_{m,n=1}^{\infty} n \left(\frac{24}{m}\right) \left(\frac{-4}{n}\right) q^{m^2+12n^2}$
73 XIV	$\frac{\eta(48z)^{13}}{\eta(24z)^5\eta(96z)^2}$	3	$\sum_{m,n=1}^{\infty} mn \left(\frac{-4}{m}\right) \left(\frac{-6}{n}\right) q^{12m^2+n^2}$
75 XV	$\frac{\eta(96z)\eta(144z)^9}{\eta(72z)^3\eta(288z)^3}$	2	$\sum_{m,n=1}^{\infty} n \left(\frac{12}{m}\right) \left(\frac{-8}{n}\right) q^{4m^2+9n^2}$
76 XV	$\eta(72z)^3\eta(96z)$	2	$\sum_{m,n=1}^{\infty} n \left(\frac{12}{m}\right) \left(\frac{-4}{n}\right) q^{4m^2+9n^2}$
77 XVI	$\frac{\eta(72z)\eta(96z)\eta(288z)}{\eta(144z)}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{8}{m}\right) \left(\frac{12}{n}\right) q^{9m^2+4n^2}$
78 XVI	$\frac{\eta(96z)\eta(144z)^2}{\eta(72z)}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{m}{2}\right)^2 \left(\frac{12}{n}\right) q^{9m^2+4n^2}$
81 XVII	$\frac{\eta(312z)^6}{\eta(156z)^2\eta(624z)^2}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{24}{m}\right) \left(\frac{24}{n}\right) q^{\frac{13}{2}(m^2+n^2)}$
82 XVII	$\eta(156z)^2$	1	$\sum_{m,n=1}^{\infty} \left(\frac{12}{m}\right) \left(\frac{12}{n}\right) q^{\frac{13}{2}(m^2+n^2)}$

Table 19: Eta quotients with vanishing behaviour similar to $f_1^3 f_2^3$

Number	q-Product	Modular Form	Weight	Level	Group
1	$f_1^3 f_2^3$	$\eta(8z)^3\eta(16z)^3$	3	128	I
2	$\frac{f_2^{12}}{f_1^3 f_4^3}$	$\frac{\eta(16z)^{12}}{\eta(8z)^3\eta(32z)^3}$	3	256	I
3	$\frac{f_2^{12} f_6^2}{f_1^4 f_3 f_4^5}$	$\frac{\eta(16z)^{12}\eta(48z)^2}{\eta(8z)^4\eta(24z)\eta(32z)^5}$	2	768	I
4	$\frac{f_1^4 f_3 f_{12}}{f_4 f_6}$	$\frac{\eta(8z)^4\eta(24z)\eta(96z)}{\eta(32z)\eta(48z)}$	2	96	I
5	$\frac{f_2^6 f_4^6}{f_1^3 f_8^3}$	$\frac{\eta(16z)^6\eta(32z)^6}{\eta(8z)^3\eta(64z)^3}$	3	512	I
6	$\frac{f_1^3 f_4^9}{f_2^3 f_8^3}$	$\frac{\eta(8z)^3\eta(32z)^9}{\eta(16z)^3\eta(64z)^3}$	3	512	I
7	$\frac{f_2^6 f_3^2}{f_1^3 f_6}$	$\frac{\eta(16z)^6\eta(24z)^2}{\eta(8z)^3\eta(48z)}$	2	384	I
8	$\frac{f_1^3 f_4^3 f_6^5}{f_2^3 f_3^2 f_{12}^2}$	$\frac{\eta(8z)^3\eta(32z)^3\eta(48z)^5}{\eta(16z)^3\eta(24z)^2\eta(96z)^2}$	2	768	I
9	$\frac{f_2^8}{f_1^3 f_4}$	$\frac{\eta(16z)^8}{\eta(8z)^3\eta(32z)}$	2	256	I
10	$\frac{f_1^3 f_4^2}{f_2}$	$\frac{\eta(8z)^3\eta(32z)^2}{\eta(16z)}$	2	64	I
11	$\frac{f_2^{10} f_8}{f_1^3 f_4^4}$	$\frac{\eta(16z)^{10}\eta(64z)}{\eta(8z)^3\eta(32z)^4}$	2	512	I

12	$\frac{f_1^3f_2f_8}{f_4}$	$\frac{\eta(8z)^3\eta(16z)\eta(64z)}{\eta(32z)}$	2	512	I
13	$\frac{f_2^{11}f_{12}}{f_1^3f_4^4f_6}$	$\frac{\eta(16z)^{11}\eta(96z)}{\eta(8z)^3\eta(32z)^4\eta(48z)}$	2	2304	I
14	$\frac{f_1^3f_2^2f_{12}}{f_4f_6}$	$\frac{\eta(8z)^3\eta(16z)^2\eta(96z)}{\eta(32z)\eta(48z)}$	2	576	I
15	$\frac{f_2^3f_3^3f_{12}}{f_1^2f_4f_6^2}$	$\frac{\eta(16z)^3\eta(24z)^3\eta(96z)}{\eta(8z)^2\eta(32z)\eta(48z)^2}$	1	96	I
16	$\frac{f_1^2f_4f_6^7}{f_2^3f_3^3f_{12}^2}$	$\frac{\eta(8z)^2\eta(32z)\eta(48z)^7}{\eta(16z)^3\eta(24z)^3\eta(96z)^2}$	1	768	I
17	$\frac{f_2^7f_3f_{12}^2}{f_1^2f_4^3f_6^3}$	$\frac{\eta(16z)^7\eta(24z)\eta(96z)^2}{\eta(8z)^2\eta(32z)^3\eta(48z)^3}$	1	768	I
18	$\frac{f_1^2f_2f_{12}}{f_3f_4}$	$\frac{\eta(8z)^2\eta(16z)\eta(96z)}{\eta(24z)\eta(32z)}$	1	96	I
19	$\frac{f_2^8f_3f_{12}}{f_1^2f_4^2f_6^2}$	$\frac{\eta(16z)^8\eta(24z)\eta(96z)}{\eta(8z)^2\eta(32z)^2\eta(48z)^2}$	2	2304	I
20	$\frac{f_1^2f_2^2f_6}{f_3}$	$\frac{\eta(8z)^2\eta(16z)^2\eta(48z)}{\eta(24z)}$	2	1152	I
21	$\frac{f_2^9f_3^3f_{12}^2}{f_1^2f_4^4f_6^4}$	$\frac{\eta(16z)^9\eta(24z)^3\eta(96z)^2}{\eta(8z)^2\eta(32z)^4\eta(48z)^4}$	2	96	I
22	$\frac{f_1^2f_2^3f_6^5}{f_3^3f_4^2f_{12}}$	$\frac{\eta(8z)^2\eta(16z)^3\eta(48z)^5}{\eta(24z)^3\eta(32z)^2\eta(96z)}$	2	768	I
23	$\frac{f_4^9}{f_1f_2f_8^3}$	$\frac{\eta(32z)^9}{\eta(8z)\eta(16z)\eta(64z)^3}$	2	512	I
24	$\frac{f_1f_4^{10}}{f_2^4f_8^3}$	$\frac{\eta(8z)\eta(32z)^{10}}{\eta(16z)^4\eta(64z)^3}$	2	512	I
25	$\frac{f_2f_4^2}{f_1}$	$\frac{\eta(16z)\eta(32z)^2}{\eta(8z)}$	1	32	I
26	$\frac{f_1f_4^3}{f_2^2}$	$\frac{\eta(8z)\eta(32z)^3}{\eta(16z)^2}$	1	256	I
27	$\frac{f_2^3f_4f_6^5}{f_1f_3^2f_{12}^2}$	$\frac{\eta(16z)^3\eta(32z)\eta(48z)^5}{\eta(8z)\eta(24z)^2\eta(96z)^2}$	2	768	I
28	$\frac{f_1f_3^2f_4^2}{f_6}$	$\frac{\eta(8z)\eta(24z)^2\eta(32z)^2}{\eta(48z)}$	2	96	I
29	$\frac{f_2^5}{f_1}$	$\frac{\eta(16z)^5}{\eta(8z)}$	2	128	I
30	$f_1f_2^2f_4$	$\eta(8z)\eta(16z)^2\eta(32z)$	2	256	I
31	$\frac{f_1^2f_4^9f_6}{f_2^4f_3f_8^3}$	$\frac{\eta(8z)^2\eta(32z)^9\eta(48z)}{\eta(16z)^4\eta(24z)\eta(64z)^3}$	2	4608	I
32	$\frac{f_2^2f_3f_4^7f_{12}}{f_1^2f_6^2f_8^3}$	$\frac{\eta(16z)^2\eta(24z)\eta(32z)^7\eta(96z)}{\eta(8z)^2\eta(48z)^2\eta(64z)^3}$	2	4608	I
33	$\frac{f_1^6f_6^2}{f_2^3f_3}$	$\frac{\eta(8z)^6\eta(48z)^2}{\eta(16z)^3\eta(24z)}$	2	384	I
34	$\frac{f_2^{15}f_3f_{12}}{f_1^6f_4^6f_6}$	$\frac{\eta(16z)^{15}\eta(24z)\eta(96z)}{\eta(8z)^6\eta(32z)^6\eta(48z)}$	2	768	I
35	$\frac{f_1^7f_4^2}{f_2^3}$	$\frac{\eta(8z)^7\eta(32z)^2}{\eta(16z)^3}$	3	32	I
36	$\frac{f_2^{18}}{f_1^7f_4^5}$	$\frac{\eta(16z)^{18}}{\eta(8z)^7\eta(32z)^5}$	3	256	I
37	$\frac{f_1^8f_6^3}{f_2^4f_3^3}$	$\frac{\eta(8z)^8\eta(48z)^3}{\eta(16z)^4\eta(24z)^3}$	2	384	I
38	$\frac{f_2^{20}f_3^3f_{12}^3}{f_1^8f_4^8f_6^6}$	$\frac{\eta(16z)^{20}\eta(24z)^3\eta(96z)^3}{\eta(8z)^8\eta(32z)^8\eta(48z)^6}$	2	768	I
39	$\frac{f_1f_4^2f_6^9f_9^2f_{36}}{f_2^3f_3^4f_{12}^3f_{18}^3}$	$\frac{\eta(8z)\eta(32z)^2\eta(48z)^9\eta(72z)^2\eta(288z)}{\eta(16z)^3\eta(24z)^4\eta(96z)^3\eta(144z)^3}$	1	288	I
40	$\frac{f_3^4f_4f_{12}f_{18}^3}{f_1f_6^3f_9^2f_{36}}$	$\frac{\eta(24z)^4\eta(32z)\eta(96z)\eta(144z)^3}{\eta(8z)\eta(48z)^3\eta(72z)^2\eta(288z)}$	1	2304	I
41	$\frac{f_1f_2f_{12}}{f_6}$	$\frac{\eta(8z)\eta(16z)\eta(96z)}{\eta(48z)}$	1	2304	II
42	$\frac{f_2^4f_{12}}{f_1f_4f_6}$	$\frac{\eta(16z)^4\eta(96z)}{\eta(8z)\eta(32z)\eta(48z)}$	1	288	II

43	$\frac{f_1^2f_4^2f_6}{f_2^2f_3}$	$\frac{\eta(8z)^2\eta(32z)^2\eta(48z)}{\eta(16z)^2\eta(24z)}$	1	288	II
44	$\frac{f_2^4f_3f_{12}}{f_1^2f_6^2}$	$\frac{\eta(16z)^4\eta(24z)\eta(96z)}{\eta(8z)^2\eta(48z)^2}$	1	2304	II
45	$\frac{f_1^4f_{12}}{f_2^2f_3}$	$\frac{\eta(8z)^4\eta(96z)}{\eta(16z)^2\eta(24z)}$	1	768	II
46	$\frac{f_2^{10}f_3f_{12}^2}{f_1^4f_4^4f_6^3}$	$\frac{\eta(16z)^{10}\eta(24z)\eta(96z)^2}{\eta(8z)^4\eta(32z)^4\eta(48z)^3}$	1	96	II
47	$\frac{f_2^{13}f_8}{f_1^5f_4^5}$	$\frac{\eta(16z)^{13}\eta(64z)}{\eta(8z)^5\eta(32z)^5}$	2	512	III
48	$\frac{f_1^5f_8}{f_2^2}$	$\frac{\eta(8z)^5\eta(64z)}{\eta(16z)^2}$	2	512	III
49	$\frac{f_1^2f_4^2f_6^{19}}{f_2^5f_3^7f_{12}^7}$	$\frac{\eta(8z)^2\eta(32z)^2\eta(48z)^{19}}{\eta(16z)^5\eta(24z)^7\eta(96z)^7}$	2	768	IV
50	$\frac{f_2f_3^7}{f_1^2f_6^2}$	$\frac{\eta(16z)\eta(24z)^7}{\eta(8z)^2\eta(48z)^2}$	2	384	IV
51	$\frac{f_2^6f_3f_8f_{12}}{f_1^2f_4^3f_6^2}$	$\frac{\eta(16z)^6\eta(24z)\eta(64z)\eta(96z)}{\eta(8z)^2\eta(32z)^3\eta(48z)^2}$	1	4608	V
52	$\frac{f_1^2f_6f_8}{f_3f_4}$	$\frac{\eta(8z)^2\eta(48z)\eta(64z)}{\eta(24z)\eta(32z)}$	1	4608	V
53	f_1f_8	$\eta(8z)\eta(64z)$	1	512	VI
54	$\frac{f_2^3f_8}{f_1f_4}$	$\frac{\eta(16z)^3\eta(64z)}{\eta(8z)\eta(32z)}$	1	512	VI
55	$\frac{f_1f_4f_6^5f_8}{f_2^2f_3^2f_{12}^2}$	$\frac{\eta(8z)\eta(32z)\eta(48z)^5\eta(64z)}{\eta(16z)^2\eta(24z)^2\eta(96z)^2}$	1	1536	VII
56	$\frac{f_2f_3^2f_8}{f_1f_6}$	$\frac{\eta(16z)\eta(24z)^2\eta(64z)}{\eta(8z)\eta(48z)}$	1	1536	VII
57	$\frac{f_1f_4f_6^5}{f_2f_{12}^2}$	$\frac{\eta(8z)\eta(32z)\eta(48z)^5}{\eta(16z)\eta(96z)^2}$	2	2304	VIII
58	$\frac{f_2^2f_6^5}{f_1f_{12}^2}$	$\frac{\eta(16z)^2\eta(48z)^5}{\eta(8z)\eta(96z)^2}$	2	288	VIII
59	$\frac{f_1^2f_6^6}{f_2f_3f_{12}^2}$	$\frac{\eta(8z)^2\eta(48z)^6}{\eta(16z)\eta(24z)\eta(96z)^2}$	2	96	VIII
60	$\frac{f_2^5f_3f_6^3}{f_1^2f_4^2f_{12}}$	$\frac{\eta(16z)^5\eta(24z)\eta(48z)^3}{\eta(8z)^2\eta(32z)^2\eta(96z)}$	2	768	VIII
61	$\frac{f_1^3f_6^5}{f_{12}^2}$	$\frac{\eta(8z)^3\eta(48z)^5}{\eta(96z)^2}$	3	576	VIII
62	$\frac{f_2^9f_6^5}{f_1^3f_4^3f_{12}^2}$	$\frac{\eta(16z)^9\eta(48z)^5}{\eta(8z)^3\eta(32z)^3\eta(96z)^2}$	3	2304	VIII
63	$f_1^3f_6$	$\eta(8z)^3\eta(48z)$	2	1152	VIII
64	$\frac{f_2^9f_6}{f_1^3f_4^3}$	$\frac{\eta(16z)^9\eta(48z)}{\eta(8z)^3\eta(32z)^3}$	2	2304	VIII
65	$f_2^3f_3$	$\eta(16z)^3\eta(24z)$	2	1152	IX
66	$\frac{f_2^3f_6^3}{f_3f_{12}}$	$\frac{\eta(16z)^3\eta(48z)^3}{\eta(24z)\eta(96z)}$	2	2304	IX
67	$\frac{f_2^3f_3^5}{f_6^2}$	$\frac{\eta(16z)^3\eta(24z)^5}{\eta(48z)^2}$	3	1152	IX
68	$\frac{f_2^3f_6^{13}}{f_3^5f_{12}^5}$	$\frac{\eta(16z)^3\eta(48z)^{13}}{\eta(24z)^5\eta(96z)^5}$	3	2304	IX
69	$\frac{f_2^2f_3^5f_{12}}{f_4f_6^3}$	$\frac{\eta(16z)^2\eta(24z)^5\eta(96z)}{\eta(32z)\eta(48z)^3}$	2	96	IX
70	$\frac{f_2^2f_6^{12}}{f_3^5f_4f_{12}^4}$	$\frac{\eta(16z)^2\eta(48z)^{12}}{\eta(24z)^5\eta(32z)\eta(96z)^4}$	2	768	IX
71	$\frac{f_2f_6^{13}f_8}{f_3^5f_4f_{12}^5}$	$\frac{\eta(16z)\eta(48z)^{13}\eta(64z)}{\eta(24z)^5\eta(32z)\eta(96z)^5}$	2	4608	IX
72	$\frac{f_2f_3^5f_8}{f_4f_6^2}$	$\frac{\eta(16z)\eta(24z)^5\eta(64z)}{\eta(32z)\eta(48z)^2}$	2	4608	IX
73	$\frac{f_4^2f_6^{13}}{f_2f_3^5f_{12}^5}$	$\frac{\eta(32z)^2\eta(48z)^{13}}{\eta(16z)\eta(24z)^5\eta(96z)^5}$	2	2304	IX

74	$\frac{f_3^5 f_4^2}{f_2 f_6^2}$	$\frac{\eta(24z)^5\eta(32z)^2}{\eta(16z)\eta(48z)^2}$	2	288	IX
75	$\frac{f_4^9 f_6^{13}}{f_2^3 f_3^5 f_8^3 f_{12}^5}$	$\frac{\eta(32z)^9\eta(48z)^{13}}{\eta(16z)^3\eta(24z)^5\eta(64z)^3\eta(96z)^5}$	3	4608	IX
76	$\frac{f_3^5 f_4^9}{f_2^3 f_6^2 f_8^3}$	$\frac{\eta(24z)^5\eta(32z)^9}{\eta(16z)^3\eta(48z)^2\eta(64z)^3}$	3	4608	IX
77	$\frac{f_4^9 f_6^3}{f_2^3 f_3 f_8^3 f_{12}}$	$\frac{\eta(32z)^9\eta(48z)^3}{\eta(16z)^3\eta(24z)\eta(64z)^3\eta(96z)}$	2	4608	IX
78	$\frac{f_3 f_4^9}{f_2^3 f_8^3}$	$\frac{\eta(24z)\eta(32z)^9}{\eta(16z)^3\eta(64z)^3}$	2	4608	IX
79	$\frac{f_2 f_6^3 f_8}{f_3 f_4 f_{12}}$	$\frac{\eta(16z)\eta(48z)^3\eta(64z)}{\eta(24z)\eta(32z)\eta(96z)}$	1	4608	X
80	$\frac{f_2 f_3 f_8}{f_4}$	$\frac{\eta(16z)\eta(24z)\eta(64z)}{\eta(32z)}$	1	4608	X
81	$\frac{f_1 f_4 f_6}{f_2}$	$\frac{\eta(8z)\eta(32z)\eta(48z)}{\eta(16z)}$	1	2304	XI
82	$\frac{f_2^2 f_6}{f_1}$	$\frac{\eta(16z)^2\eta(48z)}{\eta(8z)}$	1	1152	XI
83	$\frac{f_1^2 f_6^2}{f_2 f_3}$	$\frac{\eta(8z)^2\eta(48z)^2}{\eta(16z)\eta(24z)}$	1	384	XI
84	$\frac{f_2^5 f_3 f_{12}}{f_1^2 f_4^2 f_6}$	$\frac{\eta(16z)^5\eta(24z)\eta(96z)}{\eta(8z)^2\eta(32z)^2\eta(48z)}$	1	768	XI
85	$\frac{f_2^2 f_3 f_{12}}{f_4 f_6}$	$\frac{\eta(16z)^2\eta(24z)\eta(96z)}{\eta(32z)\eta(48z)}$	1	192	XII
86	$\frac{f_2^2 f_6^2}{f_3 f_4}$	$\frac{\eta(16z)^2\eta(48z)^2}{\eta(24z)\eta(32z)}$	1	768	XII
87	$\frac{f_3 f_4^2}{f_2}$	$\frac{\eta(24z)\eta(32z)^2}{\eta(16z)}$	1	576	XII
88	$\frac{f_4^2 f_6^3}{f_2 f_3 f_{12}}$	$\frac{\eta(32z)^2\eta(48z)^3}{\eta(16z)\eta(24z)\eta(96z)}$	1	2304	XII
89	$\frac{f_3^3 f_{12} f_{18}^3}{f_6^2 f_9^2 f_{36}}$	$\frac{\eta(24z)^3\eta(96z)\eta(144z)^3}{\eta(48z)^2\eta(72z)^2\eta(288z)}$	1	288	XIII
90	$\frac{f_6^7 f_9^2 f_{36}}{f_3^3 f_{12}^2 f_{18}^3}$	$\frac{\eta(48z)^7\eta(72z)^2\eta(288z)}{\eta(24z)^3\eta(96z)^2\eta(144z)^3}$	1	2304	XIII
91	$\frac{f_3 f_6^5}{f_{12}^2}$	$\frac{\eta(24z)\eta(48z)^5}{\eta(96z)^2}$	2	192	XIII
92	$\frac{f_6^8}{f_3 f_{12}^3}$	$\frac{\eta(48z)^8}{\eta(24z)\eta(96z)^3}$	2	768	XIII
93	$\frac{f_3^3 f_{12}}{f_6^2}$	$\frac{\eta(24z)^3\eta(96z)}{\eta(48z)^2}$	1	768	XIII
94	$\frac{f_6^7}{f_3^3 f_{12}^2}$	$\frac{\eta(48z)^7}{\eta(24z)^3\eta(96z)^2}$	1	96	XIII
95	$\frac{f_3^5}{f_6}$	$\frac{\eta(24z)^5}{\eta(48z)}$	2	384	XIII
96	$\frac{f_6^{14}}{f_3^5 f_{12}^5}$	$\frac{\eta(48z)^{14}}{\eta(24z)^5\eta(96z)^5}$	2	768	XIII
97	$\frac{f_3^5 f_6^3}{f_{12}^2}$	$\frac{\eta(24z)^5\eta(48z)^3}{\eta(96z)^2}$	3	96	XIII
98	$\frac{f_6^{18}}{f_3^5 f_{12}^7}$	$\frac{\eta(48z)^{18}}{\eta(24z)^5\eta(96z)^7}$	3	768	XIII
99	$f_3 f_6$	$\eta(24z)\eta(48z)$	1	384	XIV
100	$\frac{f_6^4}{f_3 f_{12}}$	$\frac{\eta(48z)^4}{\eta(24z)\eta(96z)}$	1	768	XIV
101	$\frac{f_6^5 f_{18}^3}{f_3^2 f_9 f_{12}^2 f_{36}}$	$\frac{\eta(48z)^5\eta(144z)^3}{\eta(24z)^2\eta(72z)\eta(96z)^2\eta(288z)}$	1	6912	XV
102	$\frac{f_3^2 f_9}{f_6}$	$\frac{\eta(24z)^2\eta(72z)}{\eta(48z)}$	1	3456	XV
103	$\frac{f_3^2 f_{18}^3}{f_6 f_9 f_{36}}$	$\frac{\eta(24z)^2\eta(144z)^3}{\eta(48z)\eta(72z)\eta(288z)}$	1	6912	XVI
104	$\frac{f_6^5 f_9}{f_3^2 f_{12}^2}$	$\frac{\eta(48z)^5\eta(72z)}{\eta(24z)^2\eta(96z)^2}$	1	1728	XVI

105	$\frac{f_3 f_{12}}{f_6}$	$\frac{\eta(24z)\eta(96z)}{\eta(48z)}$	$\frac{1}{2}$	768	XVII
106	$\frac{f_6^2}{f_3}$	$\frac{\eta(48z)^2}{\eta(24z)}$	$\frac{1}{2}$	48	XVII
107	$\frac{f_6^5 f_9 f_{36}}{f_3^2 f_{12}^2 f_{18}^2}$	$\frac{\eta(48z)^5\eta(72z)\eta(288z)}{\eta(24z)^2\eta(96z)^2\eta(144z)^2}$	$\frac{1}{2}$	6912	XVII
108	$\frac{f_3^2 f_{18}}{f_6 f_9}$	$\frac{\eta(24z)^2\eta(144z)}{\eta(48z)\eta(72z)}$	$\frac{1}{2}$	432	XVII
109	f_3^3	$\eta(24z)^3$	$\frac{3}{2}$	192	XVII
110	$\frac{f_6^9}{f_3^3 f_{12}^3}$	$\frac{\eta(48z)^9}{\eta(24z)^3\eta(96z)^3}$	$\frac{3}{2}$	768	XVII
111	$\frac{f_{18}^{13}}{f_9^5 f_{36}^5}$	$\frac{\eta(144z)^{13}}{\eta(72z)^5\eta(288z)^5}$	$\frac{3}{2}$	6912	XVIII
112	$\frac{f_9^5}{f_{18}^2}$	$\frac{\eta(72z)^5}{\eta(144z)^2}$	$\frac{3}{2}$	432	XVIII
113	$\frac{f_{18} f_{27}^2}{f_9 f_{54}}$	$\frac{\eta(144z)\eta(216z)^2}{\eta(72z)\eta(432z)}$	$\frac{1}{2}$	432	XVIII
114	$\frac{f_9 f_{36} f_{54}^5}{f_{18}^2 f_{27}^2 f_{108}^2}$	$\frac{\eta(72z)\eta(288z)\eta(432z)^5}{\eta(144z)^2\eta(216z)^2\eta(864z)^2}$	$\frac{1}{2}$	6912	XVIII
115	$\frac{f_{18}^3}{f_9 f_{36}}$	$\frac{\eta(144z)^3}{\eta(72z)\eta(288z)}$	$\frac{1}{2}$	6912	XVIII
116	f_9	$\eta(72z)$	$\frac{1}{2}$	1728	XVIII

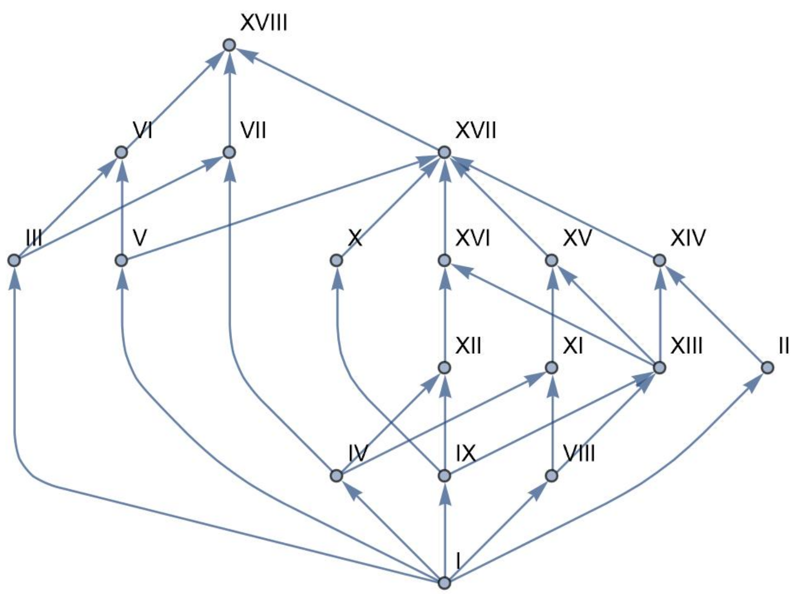

FIGURE 8. The grouping of eta quotients in Table 19, which have vanishing coefficient behaviour similar to $f_1^3 f_2^3$

Table 20: Eta quotients in Table 19 with expansions as double theta series

Number	Modular Form	Weight	Theta Series
1 I	$\eta(8z)^3\eta(16z)^3$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-4}{m}\right)\left(\frac{-4}{n}\right)q^{2m^2+n^2}$
1 I	$\eta(8z)^3\eta(16z)^3$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{m}{12}\right)(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right))q^{\frac{1}{3}(m^2+8n^2)}$
2 I	$\frac{\eta(16z)^{12}}{\eta(8z)^3\eta(32z)^3}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-4}{m}\right)\left(\frac{-8}{n}\right)q^{2m^2+n^2}$
2 I	$\frac{\eta(16z)^{12}}{\eta(8z)^3\eta(32z)^3}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{m}{3}\right)\left(\frac{-6}{n}\right)q^{\frac{1}{3}(8m^2+n^2)}$
3 I	$\frac{\eta(16z)^{12}\eta(48z)^2}{\eta(8z)^4\eta(24z)\eta(32z)^5}$	2	$\sum_{m,n=1}^{\infty} m\left(\frac{-6}{m}\right)\left(2\left(\frac{n}{6}\right)^2-\left(\frac{n}{3}\right)^2\right)q^{\frac{1}{3}(m^2+8n^2)}$
4 I	$\frac{\eta(8z)^4\eta(24z)\eta(96z)}{\eta(32z)\eta(48z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{3}\right)^2\left(\frac{n}{12}\right)q^{\frac{1}{3}(8m^2+n^2)}$
5 I	$\frac{\eta(16z)^6\eta(32z)^6}{\eta(8z)^3\eta(64z)^3}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-8}{m}\right)\left(\frac{-8}{n}\right)q^{2m^2+n^2}$
6 I	$\frac{\eta(8z)^3\eta(32z)^9}{\eta(16z)^3\eta(64z)^3}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-4}{m}\right)\left(\frac{-8}{n}\right)q^{m^2+2n^2}$
7 I	$\frac{\eta(16z)^6\eta(24z)^2}{\eta(8z)^3\eta(48z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{6}\right)^2(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right))q^{\frac{1}{3}(m^2+8n^2)}$
8 I	$\frac{\eta(8z)^3\eta(32z)^3\eta(48z)^5}{\eta(16z)^3\eta(24z)^2\eta(96z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{18}{m}\right)\left(\frac{n}{3}\right)q^{\frac{1}{3}(m^2+8n^2)}$
9 I	$\frac{\eta(16z)^8}{\eta(8z)^3\eta(32z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{2}\right)^2\left(\frac{-8}{n}\right)q^{2m^2+n^2}$
9 I	$\frac{\eta(16z)^8}{\eta(8z)^3\eta(32z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{24}{m}\right)(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right))q^{\frac{1}{3}(m^2+8n^2)}$
10 I	$\frac{\eta(8z)^3\eta(32z)^2}{\eta(16z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{2}\right)^2\left(\frac{-4}{n}\right)q^{2m^2+n^2}$
10 I	$\frac{\eta(8z)^3\eta(32z)^2}{\eta(16z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{n}{3}\right)q^{\frac{1}{3}(m^2+8n^2)}$
11 I	$\frac{\eta(16z)^{10}\eta(64z)}{\eta(8z)^3\eta(32z)^4}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{8}{m}\right)\left(\frac{-8}{n}\right)q^{2m^2+n^2}$
12 I	$\frac{\eta(8z)^3\eta(16z)\eta(64z)}{\eta(32z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{8}{m}\right)\left(\frac{-4}{n}\right)q^{2m^2+n^2}$
13 I	$\frac{\eta(16z)^{11}\eta(96z)}{\eta(8z)^3\eta(32z)^4\eta(48z)}$	2	$\sum_{m,n=1}^{\infty} m\left(\frac{-8}{m}\right)\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{m^2+2n^2}$
14 I	$\frac{\eta(8z)^3\eta(16z)^2\eta(96z)}{\eta(32z)\eta(48z)}$	2	$\sum_{m,n=1}^{\infty} m\left(\frac{-4}{m}\right)\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{m^2+2n^2}$
15 I	$\frac{\eta(16z)^3\eta(24z)^3\eta(96z)}{\eta(8z)^2\eta(32z)\eta(48z)^2}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{m}{3}\right)^2\left(\frac{n}{6}\right)^2q^{\frac{1}{3}(8m^2+n^2)}$
16 I	$\frac{\eta(8z)^2\eta(32z)\eta(48z)^7}{\eta(16z)^3\eta(24z)^3\eta(96z)^2}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{18}{m}\right)\left(2\left(\frac{n}{6}\right)^2-\left(\frac{n}{3}\right)^2\right)q^{\frac{1}{3}(m^2+8n^2)}$
18 I	$\frac{\eta(8z)^2\eta(16z)\eta(96z)}{\eta(24z)\eta(32z)}$	1	$\sum_{m,n=1}^{\infty}\left(3\left(\frac{m}{6}\right)^2-2\left(\frac{m}{2}\right)^2\right)$ $\times\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{2m^2+n^2}$
20 I	$\frac{\eta(8z)^2\eta(16z)^2\eta(48z)}{\eta(24z)}$	2	$\sum_{m,n=1}^{\infty} m\left(\frac{-4}{m}\right)\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{2m^2+n^2}$
23 I	$\frac{\eta(32z)^9}{\eta(8z)\eta(16z)\eta(64z)^3}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{2}\right)^2\left(\frac{-8}{n}\right)q^{m^2+2n^2}$
24 I	$\frac{\eta(8z)\eta(32z)^{10}}{\eta(16z)^4\eta(64z)^3}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{8}{m}\right)\left(\frac{-8}{n}\right)q^{m^2+2n^2}$
25 I	$\frac{\eta(16z)\eta(32z)^2}{\eta(8z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{m}{2}\right)^2\left(\frac{n}{2}\right)^2q^{2m^2+n^2}$
26 I	$\frac{\eta(8z)\eta(32z)^3}{\eta(16z)^2}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{8}{m}\right)\left(\frac{n}{2}\right)^2q^{m^2+2n^2}$
27 I	$\frac{\eta(16z)^3\eta(32z)\eta(48z)^5}{\eta(8z)\eta(24z)^2\eta(96z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{18}{m}\right)(2\left(\frac{n}{12}\right)-\left(\frac{n}{3}\right))q^{\frac{1}{3}(m^2+8n^2)}$
28 I	$\frac{\eta(8z)\eta(24z)^2\eta(32z)^2}{\eta(48z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{6}\right)^2\left(\frac{n}{3}\right)q^{\frac{1}{3}(m^2+8n^2)}$

29 I	$\frac{\eta(16z)^5}{\eta(8z)}$	2	$\sum_{m,n=1}^{\infty} n \left(\frac{m}{2}\right)^2 \left(\frac{-4}{n}\right) q^{m^2+2n^2}$
29 I	$\frac{\eta(16z)^5}{\eta(8z)}$	2	$\sum_{m,n=1}^{\infty} n \left(\frac{12}{m}\right) (2\left(\frac{n}{12}\right) - \left(\frac{n}{3}\right)) q^{\frac{1}{3}(m^2+8n^2)}$
30 I	$\eta(8z)\eta(16z)^2\eta(32z)$	2	$\sum_{m,n=1}^{\infty} n \left(\frac{8}{m}\right) \left(\frac{-4}{n}\right) q^{m^2+2n^2}$
30 I	$\eta(8z)\eta(16z)^2\eta(32z)$	2	$\sum_{m,n=1}^{\infty} n \left(\frac{24}{m}\right) \left(\frac{n}{3}\right) q^{\frac{1}{3}(m^2+8n^2)}$
31 I	$\frac{\eta(8z)^2\eta(32z)^9\eta(48z)}{\eta(16z)^4\eta(24z)\eta(64z)^3}$	2	$\sum_{m,n=1}^{\infty} m \left(\frac{-8}{m}\right) \left(3\left(\frac{n}{6}\right)^2 - 2\left(\frac{n}{2}\right)^2\right) q^{2m^2+n^2}$
33 I	$\frac{\eta(8z)^6\eta(48z)^2}{\eta(16z)^3\eta(24z)}$	2	$\sum_{m,n=1}^{\infty} m \left(\frac{m}{12}\right) \left(2\left(\frac{n}{6}\right)^2 - \left(\frac{n}{3}\right)^2\right) q^{\frac{1}{3}(m^2+8n^2)}$
34 I	$\frac{\eta(16z)^{15}\eta(24z)\eta(96z)}{\eta(8z)^6\eta(32z)^6\eta(48z)}$	2	$\sum_{m,n=1}^{\infty} n \left(\frac{m}{3}\right)^2 \left(\frac{-6}{n}\right) q^{\frac{1}{3}(8m^2+n^2)}$
35 I	$\frac{\eta(8z)^7\eta(32z)^2}{\eta(16z)^3}$	3	$\sum_{m,n=1}^{\infty} mn \left(\frac{m}{3}\right) \left(\frac{n}{12}\right) q^{\frac{1}{3}(8m^2+n^2)}$
36 I	$\frac{\eta(16z)^{18}}{\eta(8z)^7\eta(32z)^5}$	3	$\sum_{m,n=1}^{\infty} mn \left(\frac{-6}{m}\right) (2\left(\frac{n}{12}\right) - \left(\frac{n}{3}\right)) q^{\frac{1}{3}(m^2+8n^2)}$
41 II	$\frac{\eta(8z)\eta(16z)\eta(96z)}{\eta(48z)}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{8}{m}\right) \left(3\left(\frac{n}{6}\right)^2 - 2\left(\frac{n}{2}\right)^2\right) q^{m^2+2n^2}$
42 II	$\frac{\eta(16z)^4\eta(96z)}{\eta(8z)\eta(32z)\eta(48z)}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{m}{2}\right)^2 \left(3\left(\frac{n}{6}\right)^2 - 2\left(\frac{n}{2}\right)^2\right) q^{m^2+2n^2}$
43 II	$\frac{\eta(8z)^2\eta(32z)^2\eta(48z)}{\eta(16z)^2\eta(24z)}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{m}{2}\right)^2 \left(3\left(\frac{n}{6}\right)^2 - 2\left(\frac{n}{2}\right)^2\right) q^{2m^2+n^2}$
47 III	$\frac{\eta(16z)^{13}\eta(64z)}{\eta(8z)^5\eta(32z)^5}$	2	$\sum_{m,n=1}^{\infty} n \left(\frac{12}{m}\right) \left(\frac{-6}{n}\right) q^{\frac{1}{3}(8m^2+n^2)}$
48 III	$\frac{\eta(8z)^5\eta(64z)}{\eta(16z)^2}$	2	$\sum_{m,n=1}^{\infty} n \left(\frac{12}{m}\right) \left(\frac{n}{12}\right) q^{\frac{1}{3}(8m^2+n^2)}$
52 V	$\frac{\eta(8z)^2\eta(48z)\eta(64z)}{\eta(24z)\eta(32z)}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{8}{m}\right) \left(3\left(\frac{n}{6}\right)^2 - 2\left(\frac{n}{2}\right)^2\right) q^{2m^2+n^2}$
53 VI	$\eta(8z)\eta(64z)$	1	$\sum_{m,n=1}^{\infty} \left(\frac{12}{m}\right) \left(\frac{12}{n}\right) q^{\frac{1}{3}(8m^2+n^2)}$
53 VI	$\eta(8z)\eta(64z)$	1	$\sum_{m,n=1}^{\infty} \left(\frac{8}{m}\right) \left(\frac{8}{n}\right) q^{2m^2+n^2}$
54 VI	$\frac{\eta(16z)^3\eta(64z)}{\eta(8z)\eta(32z)}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{12}{m}\right) \left(\frac{24}{n}\right) q^{\frac{1}{3}(8m^2+n^2)}$
54 VI	$\frac{\eta(16z)^3\eta(64z)}{\eta(8z)\eta(32z)}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{8}{m}\right) \left(\frac{n}{2}\right)^2 q^{2m^2+n^2}$
55 VII	$\frac{\eta(8z)\eta(32z)\eta(48z)^5\eta(64z)}{\eta(16z)^2\eta(24z)^2\eta(96z)^2}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{12}{m}\right) \left(\frac{18}{n}\right) q^{\frac{1}{3}(8m^2+n^2)}$
56 VII	$\frac{\eta(16z)\eta(24z)^2\eta(64z)}{\eta(8z)\eta(48z)}$	1	$\sum_{m,n=1}^{\infty} \left(\frac{12}{m}\right) \left(\frac{n}{6}\right)^2 q^{\frac{1}{3}(8m^2+n^2)}$
57 VIII	$\frac{\eta(8z)\eta(32z)\eta(48z)^5}{\eta(16z)\eta(96z)^2}$	2	$\sum_{m,n=1}^{\infty} n \left(\frac{8}{m}\right) \left(\frac{n}{12}\right) q^{m^2+2n^2}$
58 VIII	$\frac{\eta(16z)^2\eta(48z)^5}{\eta(8z)\eta(96z)^2}$	2	$\sum_{m,n=1}^{\infty} n \left(\frac{m}{2}\right)^2 \left(\frac{n}{12}\right) q^{m^2+2n^2}$
59 VIII	$\frac{\eta(8z)^2\eta(48z)^6}{\eta(16z)\eta(24z)\eta(96z)^2}$	2	$\sum_{m,n=1}^{\infty} m \left(\frac{m}{12}\right) \left(3\left(\frac{n}{6}\right)^2 - 2\left(\frac{n}{2}\right)^2\right) q^{2m^2+n^2}$
61 VIII	$\frac{\eta(8z)^3\eta(48z)^5}{\eta(96z)^2}$	3	$\sum_{m,n=1}^{\infty} mn \left(\frac{-4}{m}\right) \left(\frac{n}{12}\right) q^{m^2+2n^2}$
62 VIII	$\frac{\eta(16z)^9\eta(48z)^5}{\eta(8z)^3\eta(32z)^3\eta(96z)^2}$	3	$\sum_{m,n=1}^{\infty} mn \left(\frac{-8}{m}\right) \left(\frac{n}{12}\right) q^{m^2+2n^2}$
63 VIII	$\eta(8z)^3\eta(48z)$	2	$\sum_{m,n=1}^{\infty} n \left(\frac{12}{m}\right) \left(\frac{-4}{n}\right) q^{2m^2+n^2}$
64 VIII	$\frac{\eta(16z)^9\eta(48z)}{\eta(8z)^3\eta(32z)^3}$	2	$\sum_{m,n=1}^{\infty} n \left(\frac{12}{m}\right) \left(\frac{-8}{n}\right) q^{2m^2+n^2}$
65 IX	$\eta(16z)^3\eta(24z)$	2	$\sum_{m,n=1}^{\infty} n \left(\frac{12}{m}\right) \left(\frac{-4}{n}\right) q^{m^2+2n^2}$
66 IX	$\frac{\eta(16z)^3\eta(48z)^3}{\eta(24z)\eta(96z)}$	2	$\sum_{m,n=1}^{\infty} n \left(\frac{24}{m}\right) \left(\frac{-4}{n}\right) q^{m^2+2n^2}$
67 IX	$\frac{\eta(16z)^3\eta(24z)^5}{\eta(48z)^2}$	3	$\sum_{m,n=1}^{\infty} mn \left(\frac{-4}{m}\right) \left(\frac{n}{12}\right) q^{2m^2+n^2}$
68 IX	$\frac{\eta(16z)^3\eta(48z)^{13}}{\eta(24z)^5\eta(96z)^5}$	3	$\sum_{m,n=1}^{\infty} mn \left(\frac{-4}{m}\right) \left(\frac{-6}{n}\right) q^{2m^2+n^2}$

69 IX	$\frac{\eta(16z)^2\eta(24z)^5\eta(96z)}{\eta(32z)\eta(48z)^3}$	2	$\sum_{m,n=1}^{\infty} m\left(\frac{m}{12}\right)\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{m^2+2n^2}$
70 IX	$\frac{\eta(16z)^2\eta(48z)^{12}}{\eta(24z)^5\eta(32z)\eta(96z)^4}$	2	$\sum_{m,n=1}^{\infty} m\left(\frac{-6}{m}\right)\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{m^2+2n^2}$
71 IX	$\frac{\eta(16z)\eta(48z)^{13}\eta(64z)}{\eta(24z)^5\eta(32z)\eta(96z)^5}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{8}{m}\right)\left(\frac{-6}{n}\right)q^{2m^2+n^2}$
72 IX	$\frac{\eta(16z)\eta(24z)^5\eta(64z)}{\eta(32z)\eta(48z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{8}{m}\right)\left(\frac{n}{12}\right)q^{2m^2+n^2}$
73 IX	$\frac{\eta(32z)^2\eta(48z)^{13}}{\eta(16z)\eta(24z)^5\eta(96z)^5}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{2}\right)^2\left(\frac{-6}{n}\right)q^{2m^2+n^2}$
74 IX	$\frac{\eta(24z)^5\eta(32z)^2}{\eta(16z)\eta(48z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{m}{2}\right)^2\left(\frac{n}{12}\right)q^{2m^2+n^2}$
75 IX	$\frac{\eta(32z)^9\eta(48z)^{13}}{\eta(16z)^3\eta(24z)^5\eta(64z)^3\eta(96z)^5}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-8}{m}\right)\left(\frac{-6}{n}\right)q^{2m^2+n^2}$
76 IX	$\frac{\eta(24z)^5\eta(32z)^9}{\eta(16z)^3\eta(48z)^2\eta(64z)^3}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-8}{m}\right)\left(\frac{n}{12}\right)q^{2m^2+n^2}$
77 IX	$\frac{\eta(32z)^9\eta(48z)^3}{\eta(16z)^3\eta(24z)\eta(64z)^3\eta(96z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{24}{m}\right)\left(\frac{-8}{n}\right)q^{m^2+2n^2}$
78 IX	$\frac{\eta(24z)\eta(32z)^9}{\eta(16z)^3\eta(64z)^3}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{-8}{n}\right)q^{m^2+2n^2}$
79 X	$\frac{\eta(16z)\eta(48z)^3\eta(64z)}{\eta(24z)\eta(32z)\eta(96z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{8}{m}\right)\left(\frac{24}{n}\right)q^{2m^2+n^2}$
80 X	$\frac{\eta(16z)\eta(24z)\eta(64z)}{\eta(32z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{8}{m}\right)\left(\frac{12}{n}\right)q^{2m^2+n^2}$
81 XI	$\frac{\eta(8z)\eta(32z)\eta(48z)}{\eta(16z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}\left(\frac{8}{m}\right)\left(1-\frac{3\left(\frac{n}{3}\right)^2}{2}\right)q^{3m^2+8n^2}$
81 XI	$\frac{\eta(8z)\eta(32z)\eta(48z)}{\eta(16z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{8}{m}\right)\left(\frac{12}{n}\right)q^{m^2+2n^2}$
82 XI	$\frac{\eta(16z)^2\eta(48z)}{\eta(8z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}\left(\frac{m}{2}\right)^2\left(-2\left(\frac{n}{2}\right)^2-\frac{3\left(\frac{n}{3}\right)^2}{2}+3\left(\frac{n}{6}\right)^2+1\right)\times q^{3m^2+8n^2}$
82 XI	$\frac{\eta(16z)^2\eta(48z)}{\eta(8z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{m}{2}\right)^2\left(\frac{12}{n}\right)q^{m^2+2n^2}$
83 XI	$\frac{\eta(8z)^2\eta(48z)^2}{\eta(16z)\eta(24z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{2m^2+n^2}$
83 XI	$\frac{\eta(8z)^2\eta(48z)^2}{\eta(16z)\eta(24z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}(-1)^n\left(\frac{m}{2}\right)^2 q^{3m^2+8n^2}$
83 XI	$\frac{\eta(8z)^2\eta(48z)^2}{\eta(16z)\eta(24z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(2\left(\frac{n}{6}\right)^2-\left(\frac{n}{3}\right)^2\right)q^{\frac{1}{3}(m^2+8n^2)}$
84 XI	$\frac{\eta(16z)^5\eta(24z)\eta(96z)}{\eta(8z)^2\eta(32z)^2\eta(48z)}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty}\left(\frac{8}{n}\right)q^{8m^2+3n^2}$
84 XI	$\frac{\eta(16z)^5\eta(24z)\eta(96z)}{\eta(8z)^2\eta(32z)^2\eta(48z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{m}{3}\right)^2\left(\frac{24}{n}\right)q^{\frac{1}{3}(8m^2+n^2)}$
85 XII	$\frac{\eta(16z)^2\eta(24z)\eta(96z)}{\eta(32z)\eta(48z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{m^2+2n^2}$
85 XII	$\frac{\eta(16z)^2\eta(24z)\eta(96z)}{\eta(32z)\eta(48z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}(-1)^n\left(\frac{8}{m}\right)q^{3m^2+16n^2}$
85 XII	$\frac{\eta(16z)^2\eta(24z)\eta(96z)}{\eta(32z)\eta(48z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{m}{3}\right)^2\left(\frac{12}{n}\right)q^{\frac{1}{3}(8m^2+n^2)}$
86 XII	$\frac{\eta(16z)^2\eta(48z)^2}{\eta(24z)\eta(32z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{24}{m}\right)\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{m^2+2n^2}$
86 XII	$\frac{\eta(16z)^2\eta(48z)^2}{\eta(24z)\eta(32z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}(-1)^n\left(\frac{m}{2}\right)^2 q^{3m^2+16n^2}$
86 XII	$\frac{\eta(16z)^2\eta(48z)^2}{\eta(24z)\eta(32z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{24}{m}\right)\left(2\left(\frac{n}{6}\right)^2-\left(\frac{n}{3}\right)^2\right)q^{\frac{1}{3}(m^2+8n^2)}$
87 XII	$\frac{\eta(24z)\eta(32z)^2}{\eta(16z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}\left(\frac{8}{m}\right)\left(-2\left(\frac{n}{2}\right)^2-\frac{3\left(\frac{n}{3}\right)^2}{2}+3\left(\frac{n}{6}\right)^2+1\right)\times q^{3m^2+16n^2}$
87 XII	$\frac{\eta(24z)\eta(32z)^2}{\eta(16z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{m}{2}\right)^2\left(\frac{12}{n}\right)q^{2m^2+n^2}$

88 XII	$\frac{\eta(32z)^2\eta(48z)^3}{\eta(16z)\eta(24z)\eta(96z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}\left(\frac{m}{2}\right)^2\left(-2\left(\frac{n}{2}\right)^2-\frac{3\left(\frac{n}{3}\right)^2}{2}+3\left(\frac{n}{6}\right)^2+1\right) \times q^{3m^2+16n^2}$
88 XII	$\frac{\eta(32z)^2\eta(48z)^3}{\eta(16z)\eta(24z)\eta(96z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{m}{2}\right)^2\left(\frac{24}{n}\right)q^{2m^2+n^2}$
89 XIII	$\frac{\eta(24z)^3\eta(96z)\eta(144z)^3}{\eta(48z)^2\eta(72z)^2\eta(288z)}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty}\left(1-\frac{3\left(\frac{m}{3}\right)^2}{2}\right)\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right) \times q^{3\left(8m^2+n^2\right)}$
91 XIII	$\frac{\eta(24z)\eta(48z)^5}{\eta(96z)^2}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{n}{12}\right)q^{m^2+2n^2}$
91 XIII	$\frac{\eta(24z)\eta(48z)^5}{\eta(96z)^2}$	2	$\sum_{\substack{m=-\infty\\n=1}}^{\infty} n\left(\frac{-4}{n}\right)q^{3\left(8m^2+n^2\right)}$
92 XIII	$\frac{\eta(48z)^8}{\eta(24z)\eta(96z)^3}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{24}{m}\right)\left(\frac{n}{12}\right)q^{m^2+2n^2}$
92 XIII	$\frac{\eta(48z)^8}{\eta(24z)\eta(96z)^3}$	2	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} m(-1)^n\left(\frac{-8}{m}\right)q^{3\left(m^2+8n^2\right)}$
93 XIII	$\frac{\eta(24z)^3\eta(96z)}{\eta(48z)^2}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}(-1)^n\left(\frac{8}{m}\right)q^{3\left(m^2+8n^2\right)}$
94 XIII	$\frac{\eta(48z)^7}{\eta(24z)^3\eta(96z)^2}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty}\left(\frac{n}{2}\right)^2 q^{3\left(8m^2+n^2\right)}$
95 XIII	$\frac{\eta(24z)^5}{\eta(48z)}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{n}{12}\right)q^{2m^2+n^2}$
95 XIII	$\frac{\eta(24z)^5}{\eta(48z)}$	2	$\sum_{\substack{m=1\\n=-\infty}}^{\infty} m(-1)^n\left(\frac{-4}{m}\right)q^{3\left(m^2+8n^2\right)}$
96 XIII	$\frac{\eta(48z)^{14}}{\eta(24z)^5\eta(96z)^5}$	2	$\sum_{m,n=1}^{\infty} n\left(\frac{12}{m}\right)\left(\frac{-6}{n}\right)q^{2m^2+n^2}$
96 XIII	$\frac{\eta(48z)^{14}}{\eta(24z)^5\eta(96z)^5}$	2	$\sum_{\substack{m=-\infty\\n=1}}^{\infty} n\left(\frac{-8}{n}\right)q^{3\left(8m^2+n^2\right)}$
97 XIII	$\frac{\eta(24z)^5\eta(48z)^3}{\eta(96z)^2}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{m}{12}\right)\left(\frac{n}{12}\right)q^{2m^2+n^2}$
97 XIII	$\frac{\eta(24z)^5\eta(48z)^3}{\eta(96z)^2}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{m}{12}\right)\left(\frac{n}{12}\right)q^{m^2+2n^2}$
98 XIII	$\frac{\eta(48z)^{18}}{\eta(24z)^5\eta(96z)^7}$	3	$\sum_{m,n=1}^{\infty} mn\left(\frac{-6}{m}\right)\left(\frac{n}{12}\right)q^{m^2+2n^2}$
99 XIV	$\eta(24z)\eta(48z)$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{12}{n}\right)q^{2m^2+n^2}$
99 XIV	$\eta(24z)\eta(48z)$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}(-1)^n\left(\frac{8}{m}\right)q^{3m^2+48n^2}$
99 XIV	$\eta(24z)\eta(48z)$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty}\left(-2\left(\frac{m}{2}\right)^2-\frac{3\left(\frac{m}{3}\right)^2}{2}+3\left(\frac{m}{6}\right)^2+1\right) \times\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{3\left(8m^2+n^2\right)}$
99 XIV	$\eta(24z)\eta(48z)$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}(-1)^n\left(\frac{m}{2}\right)^2 q^{3\left(m^2+8n^2\right)}$
99 XIV	$\eta(24z)\eta(48z)$	1	$\sum_{m,n=1}^{\infty}\left(\frac{m}{6}\right)^2\left(2\left(\frac{n}{6}\right)^2-\left(\frac{n}{3}\right)^2\right)q^{\frac{1}{3}\left(m^2+8n^2\right)}$
100 XIV	$\frac{\eta(48z)^4}{\eta(24z)\eta(96z)}$	1	$\sum_{m,n=1}^{\infty}\left(\frac{12}{m}\right)\left(\frac{24}{n}\right)q^{2m^2+n^2}$
100 XIV	$\frac{\eta(48z)^4}{\eta(24z)\eta(96z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}(-1)^n\left(\frac{m}{2}\right)^2 q^{3m^2+48n^2}$
100 XIV	$\frac{\eta(48z)^4}{\eta(24z)\eta(96z)}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty}\left(\frac{8}{n}\right)q^{3\left(8m^2+n^2\right)}$
100 XIV	$\frac{\eta(48z)^4}{\eta(24z)\eta(96z)}$	1	$\left(\frac{m}{3}\right)^2\left(\frac{18}{n}\right)q^{\frac{1}{3}\left(8m^2+n^2\right)}$
101 XV	$\frac{\eta(48z)^5\eta(144z)^3}{\eta(24z)^2\eta(72z)\eta(96z)^2\eta(288z)}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty}\left(\frac{24}{n}\right)q^{3\left(8m^2+n^2\right)}$
102 XV	$\frac{\eta(24z)^2\eta(72z)}{\eta(48z)}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty}(-1)^m\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{3\left(24m^2+n^2\right)}$
102 XV	$\frac{\eta(24z)^2\eta(72z)}{\eta(48z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}(-1)^n\left(\frac{12}{m}\right)q^{3\left(m^2+8n^2\right)}$
103 XVI	$\frac{\eta(24z)^2\eta(144z)^3}{\eta(48z)\eta(72z)\eta(288z)}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty}(-1)^m\left(3\left(\frac{n}{6}\right)^2-2\left(\frac{n}{2}\right)^2\right)q^{3\left(48m^2+n^2\right)}$

103 XVI	$\frac{\eta(24z)^2\eta(144z)^3}{\eta(48z)\eta(72z)\eta(288z)}$	1	$\sum_{\substack{m=1\\n=-\infty}}^{\infty}(-1)^n\left(\frac{24}{m}\right)q^{3\left(m^2+8n^2\right)}$
104 XVI	$\frac{\eta(48z)^5\eta(72z)}{\eta(24z)^2\eta(96z)^2}$	1	$\sum_{\substack{m=-\infty\\n=1}}^{\infty}\left(\frac{12}{n}\right)q^{3\left(8m^2+n^2\right)}$

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