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Integral representation of angular operators on the Bergman space over the upper half-plane

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ABSTRACT. Let Π denote the upper half-plane. In this article, we prove that every angular operator on the Bergman space $\mathcal{A}^2(\Pi)$ over the upper half-plane can be uniquely represented as an integral operator of the form

$$(A_\varphi f)(z) = \frac{1}{2\pi z^2}\int_\Pi f(w)\,\varphi\Big(\frac{z}{w}\Big)d\mu(w),\ \forall f \in \mathcal{A}^2(\Pi),\ z \in \Pi,$$

where φ is a function on $\mathbb{C}_- := \mathbb{C} - \{x \in \mathbb{R} : x \geq 0\}$ given by

$$\varphi(z) = \int_{\mathbb{R}} \sigma(t)\Big(\frac{2t}{1-e^{-2t\pi}}\Big)z^{1+it}dt,\ z \in \mathbb{C}_-$$

for some $\sigma \in L^\infty(\mathbb{R})$. Here $d\mu(w)$ is the Lebesgue measure on Π. Later on, with the help of above integral representation, we obtain various operator theoretic properties of the angular operators.

Also, we give integral representation of the form A_φ for all the operators in the C^*-algebra generated by Toeplitz operators $T_{\mathbf{a}}$ with angular symbols $\mathbf{a} \in L^\infty(\Pi)$.

CONTENTS

1. Introduction

Let $\Pi = \{z = x + iy \in \mathbb{C} : y > 0\}$ be the upper half-plane, and let $d\mu(z) = dxdy$ be the standard Lebesgue plane measure on Π. Let $\mathcal{A}^2(\Pi)$ be the Bergman

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space of all analytic functions on Π. This space is a reproducing kernel Hilbert space with the reproducing kernel given by

$$K_{\Pi,w}(z) = -\frac{1}{\pi(z-\overline{w})^2},\ \forall z, w \in \Pi.$$

In [21], K. Zhu defined a class of integral operators on the Fock space $F^2(\mathbb{C})$ and posed the question of characterizing all the integral kernels so that the operators are bounded. Cao et al. in [7] obtained a solution to this problem for the Fock space $F^2(\mathbb{C}^n)$ in all the dimensions by observing that the operators commute with a group of unitary operators on the Fock space. Recently, in [2, 3, 4], analogous results are obtained for various classes of integral operators on the Fock space $F^2(\mathbb{C}^n)$ and the Bergman space $\mathcal{A}^2(\Pi)$.

Let $\mathcal{B}(\mathcal{A}^2(\Pi))$ denote the collection of all bounded linear operators on $\mathcal{A}^2(\Pi)$. Since $\mathcal{A}^2(\Pi)$ is a reproducing kernel Hilbert space, every operator $T \in \mathcal{B}(\mathcal{A}^2(\Pi))$ can be uniquely written as an integral operator of the form

$$(Tf)(z) = \int_{\Pi} f(w)A_T(z,\overline{w})d\mu(w),\ z \in \Pi, \tag{1.1}$$

where $A_T(z,\overline{w}) := \overline{(T^*K_{\Pi,z})(w)} = \overline{\langle T^*K_{\Pi,z}, K_{\Pi,w}\rangle}_{\mathcal{A}^2} = \overline{\langle K_{\Pi,z}, TK_{\Pi,w}\rangle}_{\mathcal{A}^2} =: \overline{A_{T^*}(w,\overline{z})}$. It can be easily seen that $A_T(\cdot,\overline{(\cdot)})$ is defined on $\Pi\times\Pi$ and $A_T(\cdot,\overline{w})$, $\overline{A_T(z,\overline{(\cdot)})} \in \mathcal{A}^2(\Pi)$. Let $\mathbb{C}_- := \mathbb{C} - \{x \in \mathbb{R} : x \geq 0\}$. For a function φ on $\mathbb{C}_-$, we define

$$K_\varphi(z,\overline{w}) := \frac{1}{2\pi z^2}\,\varphi\Big(\frac{z}{\overline{w}}\Big),\ z, w \in \Pi.$$

Let $\mathcal{G}$ be the collection of all analytic functions φ on $\mathbb{C}_-$ such that $K_\varphi(\cdot,\overline{w})$, $\overline{K_\varphi(z,\overline{(\cdot)})} \in \mathcal{A}^2(\Pi)$ for each $z, w \in \Pi$. In this article, motivated by the works in [2, 3, 4, 7, 21], we consider the following class of integral operators on $\mathcal{A}^2(\Pi)$:

For $\varphi \in \mathcal{G}$, we formally define an integral operator $A_\varphi : \mathcal{A}^2(\Pi) \to \mathcal{A}^2(\Pi)$ by

$$(A_\varphi f)(z) = \frac{1}{2\pi z^2}\int_{\Pi} f(w)\varphi\Big(\frac{z}{\overline{w}}\Big)d\mu(w),\ z \in \Pi,\ f \in \mathcal{A}^2(\Pi). \tag{1.2}$$

We characterize all the symbols $\varphi \in \mathcal{G}$ for which the operator A_φ is bounded. Indeed, we prove the following theorem:

Theorem 1.1 (Main Theorem)**.** *Let $\varphi \in \mathcal{G}$. Then the integral operator A_φ defined by (1.2) is bounded on $\mathcal{A}^2(\Pi)$ if and only if there exists $\sigma \in L^\infty(\mathbb{R})$ such that*

$$\varphi(z) = \int_{\mathbb{R}} \sigma(t)\Big(\frac{2t}{1-e^{-2t\pi}}\Big)z^{1+it}dt,\ z \in \mathbb{C}_-. \tag{1.3}$$

Moreover, we have that

$$\|A_\varphi\|_{\mathcal{A}^2\to\mathcal{A}^2} = \|\sigma\|_{L^\infty(\mathbb{R})}.$$

We prove Theorem 1.1 by observing that $A_\varphi \in \mathcal{B}(\mathcal{A}^2(\Pi))$ commutes with a group of unitary operators on $\mathcal{A}^2(\Pi)$. Such operators are called angular operators and they are introduced in [10]. In fact, we obtain that the collection

$$\left\{ A_\varphi : \exists \sigma \in L^\infty(\mathbb{R}) \text{ and } \varphi(z) = \int_{\mathbb{R}} \sigma(t)\Big(\frac{2t}{1-e^{-2t\pi}}\Big) z^{1+it} dt,\ z \in \mathbb{C}_- \right\}$$

gives all angular operators in $\mathcal{B}(\mathcal{A}^2(\Pi))$. In other words, we provide integral representations of the form (1.1) for all the angular operators. Also, we prove various operator theoretic properties for the angular operators such as compactness, normality, C^*-algebra properties, etc..

In mathematics, Toeplitz operators are one of the widely studied operators on holomorphic function spaces (Hardy space, Bergman space, Fock space, etc.). For a better understanding, these operators are studied by restricting the defining symbols to a particular class (For example, see [10, 11, 12, 14, 15, 16, 17, 18, 20, 23]). In [10], C^*-algebra generated by Toeplitz operators on $\mathcal{A}^2(\Pi)$ with angular symbols from $L^\infty(\Pi)$ is described. As every Toeplitz operator $T_{\mathbf{a}}$ with angular symbol $\mathbf{a} \in L^\infty(\Pi)$ is an angular operator on $\mathcal{A}^2(\Pi)$, in Section 4, we represent $T_{\mathbf{a}}$ uniquely in the form (1.2) and give explicit representation for operators in the C^*-algebra generated by Toeplitz operators with angular symbols.

2. Preliminaries

Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{B}(\mathcal{H})$ be the collection of all bounded operators on $\mathcal{H}$. If $T \in \mathcal{B}(\mathcal{H})$, then the spectrum of T is defined by $\sigma(T) = \{\lambda \in \mathbb{C} : (T-\lambda I)^{-1} \notin \mathcal{B}(\mathcal{H})\}$ and the point spectrum of T is given by $\sigma_p(T) = \{\lambda \in \sigma(T) : (T-\lambda I) \text{ is not injective}\}$. A number $\lambda \in \sigma(T)$ is an approximate eigenvalue of T if there exists a sequence (x_n) of unit vectors such that $(T-\lambda I)x_n \to 0$ as $n \to \infty$. The approximate point spectrum of T, denoted by $\sigma_a(T)$, consists of all approximate eigenvalues of T. Clearly, $\sigma_p(T) \subseteq \sigma_a(T)$. Let $\operatorname{ran}(T) = \{Tx : x \in \mathcal{H}\}$ and $\ker(T) = \{x \in X : Tx = 0\}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be Fredholm if

(1) $\operatorname{ran}(T)$ is closed;

(2) $\ker(T)$ and $\ker(T^*)$ are finite dimensional.

The essential spectrum of T is defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\}.$$

For more details, we refer to [6, 9].

Let (X, M, ν) be a $\sigma-$finite measure space and $L^2(X,\nu) := L^2(X)$ be the Hilbert space of all $\nu-$measurable complex valued functions on X such that

$$\|f\|^2_{L^2(X)} = \int_X |f|^2 d\nu < \infty.$$

The inner product on $L^2(X)$ is given by

$$\langle f, g\rangle_{L^2(X)} = \int_X f\overline{g}d\nu$$

for all $f, g \in L^2(X)$. Let f be a $\nu-$measurable complex valued function on X. Then the essential range of f, denoted by $\text{ess}(f)$, is given by

$$\{a \in \mathbb{C} : \forall \epsilon > 0,\ \nu\{x \in X : |f(x) - a| < \epsilon\} > 0\}.$$

Let $L^\infty(X, \nu) := L^\infty(X)$ be the collection of all essentially bounded ν-measurable functions on X. It is a Banach space with the norm given by

$$\|f\|_{L^\infty(X)} = \sup\{|a| : a \in \text{ess}(f)\}.$$

It is known that the space $L^\infty(X)$ is a commutative C^*-algebra.

Let m be a ν-measurable function on X and $\mathcal{D}_m \subseteq L^2(X)$ be the set of all $f \in L^2(X)$ such that $m\cdot f \in L^2(X)$. The operator $M_m : \mathcal{D}_m \to L^2(X)$ defined by $M_m f = m\cdot f$ for all $f \in \mathcal{D}_m$ is called a multiplication operator. It is well known that M_m is bounded on $L^2(X)$ if and only if $m \in L^\infty(X)$. If $\mathcal{M}(L^2(X)) = \{M_m : m \in L^\infty(X)\}$, then the map $\Lambda : L^\infty(X) \to \mathcal{M}(L^2(X))$ defined by $\Lambda(m) = M_m$ is a $\star-$isometric isomorphism.

Theorem 2.1. [6, 8, 4] *For all $m, m_1, m_2 \in L^\infty(X, M, \nu)$, we have*

1. $M_m^* = M_{\overline{m}}$, *where* $\overline{m}(x) = \overline{m(x)}$ *for all* $x \in X$;
2. $M_{m_1}M_{m_2} = M_{m_1m_2} = M_{m_2m_1} = M_{m_2}M_{m_1}$;
3. *The collection $\mathcal{M}(L^2(X))$ is a maximal commutative C^*-subalgebra of $\mathcal{B}(L^2(X))$, where $\mathcal{B}(L^2(X))$ denote the set of all bounded linear operators on $L^2(X)$;*
4. $\lambda \in \sigma_p(M_m)$ *if and only if* $\nu(\{x : m(x) = \lambda\})$ *is positive;*
5. $\sigma(M_m) = \sigma_a(M_m) = \sigma_e(M_m) = \text{ess}(m)$;
6. *If ν is non-atomic measure on X, then M_m is compact if and only if $m = 0$* $\nu-$a.e. *on X.*

For $h \in \mathbb{R}_+$, let $D_h : \mathcal{A}^2(\Pi) \to \mathcal{A}^2(\Pi)$ be the dilation operator defined by

$$(D_h f)(z) = hf(hz), \qquad (f \in \mathcal{A}^2(\Pi),\ z \in \Pi).$$

It is easy to see that $(D_h)_{h\in\mathbb{R}_+}$ is a unitary representation of the group $\mathbb{R}_+$ on $\mathcal{A}^2(\Pi))$. An operator $T \in \mathcal{B}(\mathcal{A}^2(\Pi))$ is said to be angular if it commutes with all the dilations. That is,

$$TD_h = D_hT, \ \ \forall h \in \mathbb{R}_+.$$

In [11], an integral operator $R : \mathcal{A}^2(\Pi) \to L^2(\mathbb{R})$ defined by

$$(Rf)(t) = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2t}{1-e^{-2t\pi}}}\int_\Pi (\overline{z})^{-it-1} f(z)d\mu(z),\ f \in \mathcal{A}^2(\Pi),\ t \in \mathbb{R}$$

is considered and with the help of this transform it was proved that the C^*-algebra generated by Toeplitz operators on $\mathcal{A}^2(\Pi)$ with angular symbols is isomorphic to a C^*-subalgebra of $L^\infty(\mathbb{R})$. The operator R is shown to be an isometric isomorphism from $\mathcal{A}^2(\Pi)$ onto the space $L^2(\mathbb{R})$ and its inverse is given by

$$(R^*g)(z) = (R^{-1}g)(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \sqrt{\frac{2t}{1-e^{-2t\pi}}}\,(z)^{it-1} g(t)dt,\ g \in L^2(\mathbb{R}),\ z \in \Pi.$$

The operator R^* is a Bargmann type transform. One can refer to [1, 2, 3, 4, 5, 13, 20, 22] and references therin for various applications of the Bargmann type transforms.

If f is a suitable measurable function on $\mathbb{R}$, then its Fourier transform is defined by

$$(\mathcal{F}f)(x) = \frac{1}{(\pi)^{1/2}} \int_{\mathbb{R}} f(y)e^{-2ixy}dy.$$

The transform $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a unitary operator with the inverse defined by

$$(\mathcal{F}^{-1}f)(x) = \frac{1}{(\pi)^{1/2}} \int_{\mathbb{R}} f(y)e^{2ixy}dy.$$

Let $a, b \in \mathbb{R}$ and f be a measurable function on $\mathbb{R}^n$. Then the translation and modulation of f are given respectively by

$$(\tau_a f)(x) = f(x-a),\ (M_{e^{2\pi ib(\cdot)}} f)(x) = e^{2\pi ibx} f(x) \tag{2.1}$$

for all $x \in \mathbb{R}$. The operators τ_a and $M_{e^{2\pi ib(\cdot)}}$ defined above are unitary operators on $L^2(\mathbb{R})$.

The following theorem is well known.

Theorem 2.2 ([13]). *For any real numbers $a, b \in \mathbb{R}$, we have*

$$\mathcal{F}\tau_a\mathcal{F}^{-1} = M_{e^{2\pi ic(\cdot)}},\ \mathcal{F}M_{e^{2\pi ib(\cdot)}}\mathcal{F}^{-1} = \tau_{-\pi b},$$

where $c = -\frac{a}{\pi}$.

3. Integral representation of angular operators

In this section, we prove Theorem 1.1. As a consequence, we obtain various operator theoretic properties of the angular operators. We start with some auxiliary results which will be useful in proving Theorem 1.1.

Lemma 3.1. *Let $\sigma \in L^\infty(\mathbb{R})$. Then the function φ defined by (1.3) is analytic on $\mathbb{C}_-$.*

Proof. We are given that $\sigma \in L^\infty(\mathbb{R})$ such that

$$\varphi(z) = \int_{\mathbb{R}} \sigma(t)\Big(\frac{2t}{1-e^{-2t\pi}}\Big) z^{1+it}\, dt,\ z \in \mathbb{C}_-.$$

Let $z = |z|e^{i\arg z}$, where $\arg z \in (0, 2\pi)$ is the principal argument of z. Then we have

$$\begin{aligned}\varphi(z) &= \int_{\mathbb{R}} \sigma(t)\Big(\frac{2t}{1-e^{-2t\pi}}\Big)\, e^{\big(it+1\big)\big(\ln|z|+i\arg z\big)}\, dt \\ &= \int_{\mathbb{R}} \sigma(t)\Big(\frac{2t}{1-e^{-2t\pi}}\Big) e^{it\ln|z|} e^{-t\arg z} e^{\ln|z|} e^{i\arg z} dt.\end{aligned}$$

Therefore, we get

$$\begin{aligned}&|\varphi(z)| \\ &\le e^{\ln|z|}\|\sigma\|_{L^\infty} \int_{\mathbb{R}} \Big(\frac{2t}{1-e^{-2t\pi}}\Big) e^{-t\arg z} dt \\ &= e^{\ln|z|}\|\sigma\|_{L^\infty}\Big(\int_0^\infty \Big(\frac{2t}{1-e^{-2t\pi}}\Big) e^{-t\arg z} dt + \int_{-\infty}^0 \Big(\frac{2t}{1-e^{-2t\pi}}\Big) e^{-t\arg z} dt\Big) \\ &= e^{\ln|z|}\|\sigma\|_{L^\infty}\Big(\int_0^\infty \Big(\frac{2t}{1-e^{-2t\pi}}\Big) e^{-t\arg z} dt + \int_0^\infty \Big(\frac{2t}{e^{2t\pi}-1}\Big) e^{t\arg z} dt\Big) \\ &< +\infty.\end{aligned}$$

Thus, the integral in the definition of φ converges for all $z \in \mathbb{C}_-$. Now we show that φ is continuous.

Let $z = |z|e^{i\arg z} \in \mathbb{C}_-$ and let $\big\{z_n = |z_n|e^{i\arg z_n}\big\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}_-$ converging to z. Then for any $\sigma \in L^\infty(\mathbb{R})$,

$$\begin{aligned}&\sigma(t)2t(1-e^{-2t\pi})^{-1} e^{\ln|z_n|} e^{i\arg z_n} e^{it\ln|z_n|} e^{-t\arg z_n} \\ &\qquad \longrightarrow \sigma(t)2t(1-e^{-2t\pi})^{-1} e^{\ln|z|} e^{i\arg z} e^{it\ln|z|} e^{-t\arg z}\end{aligned}$$

pointwise a.e. on $\mathbb{R}$. Also,

$$\begin{aligned}&\Big|\sigma(t)\frac{2t}{1-e^{-2t\pi}} e^{\ln|z_n|} e^{i\arg z_n} e^{it\ln|z_n|} e^{-t\arg z_n}\Big| \\ &\qquad\qquad \le \|\sigma\|_{L^\infty} \frac{2t}{1-e^{-2t\pi}}\, e^{\ln|z_n|} e^{-t\arg z_n}\end{aligned}$$

Since $\{|z_n|\}$ converges to $|z| \neq 0$, the sequence $\{e^{ln|z_n|}\}$ is bounded. Let $c_1(z) > 0$ such that

$$e^{\ln|z_n|} \le c_1(z),\ \forall n \in \mathbb{N}.$$

If $t \in (0, \infty)$, then

$$\frac{2t}{1-e^{-2t\pi}} e^{\ln|z_n|} e^{-t\arg z_n} \le c_1(z)\, \frac{2t}{1-e^{-2t\pi}} e^{-t\arg z} \in L^1(\mathbb{R}_+).$$

If $t \in (-\infty, 0)$ and $u = -t$, then

$$\begin{aligned}\frac{2t}{1-e^{-2t\pi}} e^{\ln|z_n|} e^{-t\arg z_n} &= \frac{2u}{e^{2u\pi}-1} e^{\ln|z_n|} e^{u\arg z_n}\\ &\le c_2(z)\,\frac{2u}{e^{2u\pi}-1} e^{u\arg z} \in L^1(\mathbb{R}_+).\end{aligned}$$

Therefore, by the dominated convergence theorem, it follows that φ is continuous at each $z \in \mathbb{C}_-$. Finally, we now prove that φ is analytic on $\mathbb{C}_-$.

Let γ be a simple closed contour in $\mathbb{C}_-$. Then

$$\begin{aligned}&\int_\gamma \int_{\mathbb{R}} \left|\sigma(t)\frac{2t}{1-e^{-2t\pi}} e^{\ln|z|} e^{i\arg z} e^{it\ln|z|} e^{-t\arg z}\right| dt\,|d\gamma(z)|\\ &\le \|\sigma\|_{L^\infty} \int_\gamma \int_{\mathbb{R}} \frac{2t}{1-e^{-2t\pi}} e^{\ln|z|} e^{-t\arg z} dt\,|d\gamma(z)|\\ &= \|\sigma\|_{L^\infty}\Big(\int_\gamma \int_0^\infty \frac{2t}{1-e^{-2t\pi}} e^{\ln|z|} e^{-t\arg z} dt\,|d\gamma(z)|\\ &\qquad + \int_\gamma \int_{-\infty}^0 \frac{2t}{1-e^{-2t\pi}} e^{\ln|z|} e^{-t\arg z} dt\,|d\gamma(z)|\Big).\end{aligned}$$

Since γ is compact and the functions

$$\int_{-\infty}^0 \frac{2t}{1-e^{-2t\pi}} e^{\ln|z|} e^{-t\arg z} dt \quad \text{and} \quad \int_0^\infty \frac{2t}{1-e^{-2t\pi}} e^{\ln|z|} e^{-t\arg z} dt$$

are continuous functions of z, it follows that

$$\int_\gamma \int_{\mathbb{R}} \left|\sigma(t)\frac{2t}{1-e^{-2t\pi}} e^{\ln|z|} e^{i\arg z} e^{it\ln|z|} e^{-t\arg z}\right| dt\,|d\gamma(z)| < +\infty.$$

Therefore, by Fubini's theorem, we get

$$\begin{aligned}&\int_\gamma \int_{\mathbb{R}} \sigma(t)\frac{2t}{1-e^{-2t\pi}} e^{\ln|z|} e^{i\arg z} e^{it\ln|z|} e^{-t\arg z} dt\,d\gamma(z)\\ &\qquad = \int_{\mathbb{R}} \frac{2t}{1-e^{-2t\pi}} \int_\gamma z^{it+1} d\gamma(z) dt = \int_{\mathbb{R}} \frac{2t}{1-e^{-2t\pi}} (0) dt = 0.\end{aligned}$$

Since γ is arbitrary simple closed contour in $\mathbb{C}_-$, by Morera's theorem, it follows that the function φ is analytic on $\mathbb{C}_-$. This proves the lemma. □

Lemma 3.2. *Let $\sigma \in L^\infty(\mathbb{R})$ and*

$$F_\sigma(z, \overline{w}) = \frac{1}{2\pi z^2} \int_{\mathbb{R}} \sigma(t)\Big(\frac{2t}{1-e^{-2t\pi}}\Big)\Big(\frac{z}{\overline{w}}\Big)^{1+it} dt,\ z, w \in \Pi.$$

Then $F_\sigma(\cdot, \overline{w})$, $\overline{F_\sigma(z, \overline{(\cdot)})} \in \mathcal{A}^2(\Pi)$ for each $z, w \in \Pi$.

Proof. Let $z, w \in \Pi$. Then $\left(\frac{z}{\overline{w}}\right) \in \mathbb{C}_-$ and

$$F_\sigma(z, \overline{w}) = \frac{1}{2\pi z^2}\,\varphi\Big(\frac{z}{\overline{w}}\Big),$$

where the function φ is given by (1.3). By Lemma 3.1, we get

$$|F_\sigma(\cdot, \overline{w})| < +\infty,\ z, w \in \Pi.$$

Again by Lemma 3.1, it follows that the functions $F_\sigma(\cdot, \overline{w}),\ \overline{F_\sigma(z, \overline{(\cdot)})}$ are products of analytic functions on Π and hence they are analytic. Now, we show that $F_\sigma(\cdot, \overline{w}) \in \mathcal{A}^2(\Pi)$ for each $w \in \Pi$. Fix $w \in \Pi$ and consider

$$\int_\Pi |F_\sigma(z, \overline{w})|^2 d\mu(z) = \int_\Pi \Big|\frac{1}{2\pi z^2}\int_{\mathbb{R}} \sigma(t)\Big(\frac{2t}{1-e^{-2t\pi}}\Big)\Big(\frac{z}{\overline{w}}\Big)^{it+1} dt\Big|^2 d\mu(z).$$

Let $w = \rho e^{i\eta},\ z = re^{i\theta}$, where $r, \rho \in (0, \infty)$ and $\eta, \theta \in (0, \pi)$. Then we have

$$\begin{aligned}
&\int_\Pi |F_\sigma(z, \overline{w})|^2 d\mu(z) \\
&= \int_0^\pi \int_0^\infty |F_\sigma(re^{i\theta}, \rho e^{-i\eta})|^2\, r dr d\theta \\
&= \int_0^\pi \int_0^\infty \frac{1}{4\pi^2 r^4}\Big|\int_{\mathbb{R}} \sigma(t)\Big(\frac{2t}{1-e^{-2t\pi}}\Big)\Big(\frac{re^{i\theta}}{\rho e^{-i\eta}}\Big)^{it+1} dt\Big|^2 r dr d\theta.
\end{aligned}$$

Using the change of variable $r = e^u$, we get

$$\begin{aligned}
&\int_\Pi |F_\sigma(z, \overline{w})|^2 d\mu(z) \\
&= \int_0^\pi \int_{\mathbb{R}} \frac{1}{4\pi^2 e^{4u}}\Big|\int_{\mathbb{R}} \sigma(t)\Big(\frac{2t}{1-e^{-2t\pi}}\Big)\Big(\frac{e^u}{\rho}\Big)^{it+1} e^{i(\theta+\eta)(it+1)} dt\Big|^2 e^{2u} du d\theta \\
&= \frac{1}{4\pi^2\rho^2}\int_0^\pi \int_{\mathbb{R}} \Big|\int_{\mathbb{R}} \frac{\sigma(t)}{\rho^{it}}\Big(\frac{2t}{1-e^{-2t\pi}}\Big)e^{-t(\theta+\eta)} e^{itu} dt\Big|^2 du d\theta.
\end{aligned}$$

Since the Fourier transform is unitary on $L^2(\mathbb{R})$, we get

$$\begin{aligned}
\int_\Pi |F_\sigma(z, \overline{w})|^2 d\mu(z) &= \frac{1}{4\pi^2\rho^2}\int_0^\pi \int_{\mathbb{R}} \Big|\frac{\sigma(t)}{\rho^{it}}\Big(\frac{2t}{1-e^{-2t\pi}}\Big)e^{-t(\theta+\eta)}\Big|^2 dt d\theta \\
&\le \frac{\|\sigma\|_{L^\infty}}{4\pi^2\rho^2}\int_0^\pi \Big(\int_0^\infty \Big(\frac{2t}{1-e^{-2t\pi}}\Big)^2 e^{-2t(\theta+\eta)}\, dt \\
&\quad + \int_{-\infty}^0 \Big(\frac{2t}{1-e^{-2t\pi}}\Big)^2 e^{-2t(\theta+\eta)}\, dt\Big) d\theta.
\end{aligned}$$

Using the change of variable $t \to -t$ in the second integral, it follows that

$$\begin{aligned}
&\frac{\|\sigma\|_{L^\infty}}{4\pi^2\rho^2}\int_0^\pi \Big(\int_0^\infty \Big(\frac{2t}{1-e^{-2t\pi}}\Big)^2 e^{-2t(\theta+\eta)}\,dt \\
&\qquad + \int_{-\infty}^0 \Big(\frac{2t}{1-e^{-2t\pi}}\Big)^2 e^{-2t(\theta+\eta)}\,dt\Big)d\theta \\
&= \frac{\|\sigma\|_{L^\infty}}{4\pi^2\rho^2}\int_0^\pi \Big(\int_0^\infty \Big(\frac{2t}{1-e^{-2t\pi}}\Big)^2 e^{-2t(\theta+\eta)}\,dt \\
&\qquad + \int_0^\infty \Big(\frac{2t}{e^{2t\pi}-1}\Big)^2 e^{2t(\theta+\eta)}\,dt\Big)d\theta \\
&< +\infty.
\end{aligned}$$

Thus, the function $\overline{F_\sigma(\cdot,\overline{w})} \in \mathcal{A}^2(\Pi)$ for each $w \in \Pi$. In a similar way, we can show that $F_\sigma(z,\overline{(\cdot)}) \in \mathcal{A}^2(\Pi)$ for each $z \in \Pi$. Hence the lemma is proved. ☐

Lemma 3.3. *For $\sigma \in L^\infty(\mathbb{R})$, the function φ defined by (1.3) belongs to $\mathcal{G}$.*

Proof. Let φ be a function on $\mathbb{C}_-$ and $\sigma \in L^\infty(\mathbb{R})$ such that they satisfy (1.3). By Lemma 3.1, the function φ is analytic on $\mathbb{C}_-$ and Lemma 3.2 implies that the function

$$K_\varphi(z,\overline{w}) = \frac{1}{2\pi z^2}\varphi\Big(\frac{z}{\overline{w}}\Big),\ z,w \in \Pi$$

satisfies $K_\varphi(z,\overline{(\cdot)}), \overline{K_\varphi(\cdot,\overline{w})} \in \mathcal{A}^2(\Pi)$ for each $z,w \in \Pi$. Hence $\varphi \in \mathcal{G}$. ☐

Lemma 3.4. *Let $\sigma \in L^\infty(\mathbb{R})$. Then $R^*M_\sigma R = A_\psi$, where*

$$\psi(z) = \int_{\mathbb{R}} \sigma(t)\Big(\frac{2t}{1-e^{-2t\pi}}\Big)z^{1+it}dt,\ z \in \mathbb{C}_-.$$

Proof. Let $\sigma \in L^\infty(\mathbb{R})$ and $\mathcal{D} := \mathrm{Span}\{K_{\Pi,z} : z \in \Pi\}$. It is well-known that the set $\mathcal{D}$ is dense in $\mathcal{A}^2(\Pi)$. Then for $f \in \mathcal{D}$, we have

$$\begin{aligned}
(R^*M_\sigma Rf)(z) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\sqrt{\frac{2t}{1-e^{-2t\pi}}}\,(M_\sigma Rf)(t)z^{it-1}dt \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\sqrt{\frac{2t}{1-e^{-2t\pi}}}\,\sigma(t)(Rf)(t)z^{it-1}dt \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\Big(\frac{2t}{1-e^{-2t\pi}}\Big)\,\sigma(t)\int_\Pi (\overline{w})^{-it-1}f(w)d\mu(w)z^{it-1}dt.
\end{aligned}$$

We observe that for any $z \in \Pi$ the function $(\cdot)^{-1}K_{\Pi,z}(\cdot) \in L^1(\Pi)$. So for any $f \in \mathcal{D}$, the integral

$$\int_\Pi |w^{-1}f(w)|d\mu(w) < +\infty.$$

By Fubini's theorem, we get

$$\begin{aligned}(R^*M_\sigma Rf)(z) &= \int_\Pi f(w)\Big(\frac{1}{2\pi}\int_\mathbb{R} \sigma(t)\Big(\frac{2t}{1-e^{-2t\pi}}\Big)\frac{1}{z\overline{w}}\Big(\frac{z}{\overline{w}}\Big)^{it}dt\Big)d\mu(w)\\ &= \frac{1}{2\pi z^2}\int_\Pi f(w)\Big(\int_\mathbb{R} \sigma(t)\Big(\frac{2t}{1-e^{-2t\pi}}\Big)\Big(\frac{z}{\overline{w}}\Big)^{it+1}dt\Big)d\mu(w)\\ &= \frac{1}{2\pi z^2}\int_\Pi f(w)\psi\Big(\frac{z}{\overline{w}}\Big)d\mu(w)\\ &= (A_\psi f)(z),\ z\in\Pi,\end{aligned}$$

where

$$\psi(z) = \int_\mathbb{R} \sigma(t)\Big(\frac{2t}{1-e^{-2t\pi}}\Big)z^{1+it}dt,\ z\in\mathbb{C}_-.$$

From above, we get $R^*M_\sigma R = A_\psi$ on $\mathcal{D}$.

Now we show that $R^*M_\sigma R = A_\psi$ on $\mathcal{A}^2(\Pi)$. Let $g\in\mathcal{A}^2(\Pi)$ and $\{g_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}$ such that $g_n\to g$ in $\mathcal{A}^2(\Pi)$. For each $z\in\Pi$, let

$$q_z(w) := \frac{1}{2\pi z^2}\,\overline{\psi\Big(\frac{z}{\overline{w}}\Big)} = \overline{K_\psi(z,\overline{w})},\ w\in\Pi.$$

Then for each $z\in\Pi$, $q_z\in\mathcal{A}^2(\Pi)$ and $(A_\psi g_n)(z) = \langle g_n, q_z\rangle_{\mathcal{A}^2}\to\langle g, q_z\rangle_{\mathcal{A}^2} = (A_\psi g)(z)$. But $A_\psi g_n = R^*M_\sigma Rg_n$ for all $n\in\mathbb{N}$. This implies that

$$(R^*M_\sigma Rg_n)(z)\to(A_\psi g)(z)$$

for all $z\in\Pi$. As $R^*M_\sigma R$ is bounded on $\mathcal{A}^2(\Pi)$, we get $R^*M_\sigma Rg_n\to R^*M_\sigma Rg$ in $\mathcal{A}^2(\Pi)$. Since $\mathcal{A}^2(\Pi)$ is the reproducing kernel Hilbert space, $(R^*M_\sigma Rg_n)(z)\to(R^*M_\sigma Rg)(z)$ for all $z\in\Pi$. Hence $(R^*M_\sigma Rg)(z) = (A_\psi g)(z)$ for all $z\in\Pi$ and $g\in\mathcal{A}^2(\Pi)$. That is, $R^*M_\sigma Rg = A_\psi g$ for all $g\in\mathcal{A}^2(\Pi)$. Thus, we get $R^*M_\sigma R = A_\psi$ on $\mathcal{A}^2(\Pi)$. □

Remark 3.1. In Lemma 3.4, the choice of the dense set $\mathcal{D}$ is useful to apply Fubini's theorem for interchanging the order of integration.

Remark 3.2. For $h\in\mathbb{R}_+$, we consider $E_h(x) = h^{ix}$ for all $x\in L^2(\mathbb{R})$. Then by Lemma 3.4, we get $R^*M_{E_h}R = D_h$.

Lemma 3.5. *Let $M\in\mathcal{B}(L^2(\mathbb{R}))$ such that $MM_{E_h} = M_{E_h}M$ for all $h\in\mathbb{R}_+$. Then there exists $\sigma\in L^\infty(\mathbb{R})$ such that $M = M_\sigma$.*

Proof. Let $M\in\mathcal{B}(L^2(\mathbb{R}))$ such that $MM_{E_h} = M_{E_h}M$ for all $h\in\mathbb{R}_+$. That is,

$$MM_{e^{ix\ln(h)}} = M_{e^{ix\ln(h)}}M,\ \forall h\in\mathbb{R}_+.$$

As the map $h\mapsto\ln(h)$ is continuous from $\mathbb{R}_+$ onto $\mathbb{R}$, we have

$$MM_{e^{2\pi ib(\cdot)}} = M_{e^{2\pi ib(\cdot)}}M,\ \forall b\in\mathbb{R}.$$

By Theorem 2.2, we get $\mathcal{F}^{-1}M\mathcal{F}\tau_a = \tau_a\mathcal{F}^{-1}M\mathcal{F}$ for all $a\in\mathbb{R}$. By [19, Chapter 2, Proposition 2], there exixts $\sigma\in L^\infty(\mathbb{R})$ such that $M(\mathcal{F}f) = M_\sigma(\mathcal{F}f)$ for all $f\in L^2(\mathbb{R})$. Since the Fourier transform is unitary on $L^2(\mathbb{R})$, we get $Mf = \sigma f$ for all $f\in L^2(\mathbb{R})$. Hence $M = M_\sigma$. □

Theorem 3.6. *Let $T \in \mathcal{B}(\mathcal{A}^2(\Pi))$. Then T is an angular operator if and only if there exists $\sigma \in L^\infty(\mathbb{R})$ such that $T = R^* M_\sigma R$.*

Proof. Let $T \in \mathcal{B}(\mathcal{A}^2(\Pi))$ be angular operator. Then $TD_h = D_h T$, for all $h \in \mathbb{R}_+$. By Remark 3.2, we get $(RTR^*)M_{E_h} = M_{E_h}(RTR^*)$, for all $h \in \mathbb{R}_+$. By Lemma 3.5, it follows that $RTR^* = M_\sigma$ for some $\sigma \in L^\infty(\mathbb{R})$. That is,

$$T = R^* M_\sigma R.$$

Conversely, if $T = R^* M_\sigma R$ for some $\sigma \in L^\infty(\mathbb{R})$, then $RTR^* = M_\sigma$ commutes with all M_{E_h} for $h \in \mathbb{R}_+$. Hence, by Remark 3.2, T commutes with $R^* M_{E_h} R = D_h$ for all $h \in \mathbb{R}_+$. By definition of angular operators, we get that T is angular. This proves the theorem. □

Remark 3.3. The proof of Theorem 3.6 can also be found in [10, Theorem 2.2].

Lemma 3.7. *Let $\varphi \in \mathcal{G}$ and let A_φ be given by (1.2). If $A_\varphi \in \mathcal{B}(\mathcal{A}^2(\Pi))$, then there exists $\sigma \in L^\infty(\mathbb{R})$ such that $A_\varphi = R^* M_\sigma R$.*

Proof. Let $h \in \mathbb{R}_+$. Then

$$(D_h A_\varphi f)(z) = h\,(A_\varphi f)(hz) = \frac{1}{2\pi h z^2}\int_\Pi f(w)\varphi\Big(\frac{hz}{\overline{w}}\Big)d\mu(w).$$

and

$$\begin{aligned}(A_\varphi D_h f)(z) &= \frac{1}{2\pi z^2}\int_\Pi (D_h f)(w)\varphi\Big(\frac{z}{\overline{w}}\Big)d\mu(w)\\ &= \frac{h}{2\pi z^2}\int_\Pi f(hw)\varphi\Big(\frac{z}{\overline{w}}\Big)d\mu(w).\end{aligned}$$

Using the change of variable $w \mapsto \dfrac{w}{h}$, we get

$$(A_\varphi D_h f)(z) = \frac{1}{2\pi h z^2}\int_\Pi f(w)\varphi\Big(\frac{hz}{\overline{w}}\Big)d\mu(w) = (D_h A_\varphi f)(z),\ \forall z \in \Pi.$$

Therefore, $D_h A_\varphi = A_\varphi D_h$ for all $h \in \mathbb{R}_+$. That is, the operator A_φ is angular. Hence, by Theorem 3.6, there exists $\sigma \in L^\infty(\mathbb{R})$ such that $A_\varphi = R^* M_\sigma R$. □

Lemma 3.8. *Let $\varphi_1, \varphi_2 \in \mathcal{G}$ such that the operators $A_{\varphi_1}, A_{\varphi_2} \in \mathcal{B}(\mathcal{A}^2(\Pi))$. Then $A_{\varphi_1} = A_{\varphi_2}$ if and only if $\varphi_1 = \varphi_2$.*

Proof. We are given that $\varphi_1, \varphi_2 \in \mathcal{G}$ such that the operators $A_{\varphi_1}, A_{\varphi_2} \in \mathcal{B}(\mathcal{A}^2(\Pi))$. If $\varphi_1 = \varphi_2$ then $A_{\varphi_1} = A_{\varphi_2}$. Conversely, suppose $A_{\varphi_1} = A_{\varphi_2}$. Let

$$K_{\varphi_1}(z,\overline{w}) = \frac{1}{2\pi z^2}\,\varphi_1\Big(\frac{z}{\overline{w}}\Big),\ K_{\varphi_2}(z,\overline{w}) = \frac{1}{2\pi z^2}\,\varphi_2\Big(\frac{z}{\overline{w}}\Big),\ \ z,w \in \Pi.$$

Then for all $f \in \mathcal{A}^2(\Pi)$, we have

$$\begin{aligned}(A_{\varphi_1} f)(z) &= \int_\Pi f(w)K_{\varphi_1}(z,\overline{w})d\mu(w)\\ &= \int_\Pi f(w)K_{\varphi_2}(z,\overline{w})d\mu(w) = (A_{\varphi_2} f)(z),\ \ z \in \Pi.\end{aligned}$$

That is,

$$\int_{\Pi} f(w)(K_{\varphi_1} - K_{\varphi_2})(z,\overline{w})d\mu(w) = 0$$
$$\implies \int_{\Pi} f(w)\overline{\overline{(K_{\varphi_1} - K_{\varphi_2})}(z,\overline{w})}d\mu(w) = 0.$$

For $z \in \Pi$, we define $\Phi_z(w) := \overline{(K_{\varphi_1} - K_{\varphi_2})(z,\overline{w})}$ for all $w \in \Pi$. Clearly, $\Phi_z \in \mathcal{A}^2(\Pi)$. Therefore, we have $\langle f, \Phi_z \rangle_{\mathcal{A}^2} = 0$ for all $f \in \mathcal{A}^2(\Pi)$. This gives $\Phi_z \equiv 0$. Since $z \in \Pi$ is arbitrary, we get $\Phi_z(w) = 0$ for all $z, w \in \Pi$. That is, $\overline{(K_{\varphi_1} - K_{\varphi_2})(z,\overline{w})} = 0$ for all $z, w \in \Pi$. This implies

$$\frac{1}{2\pi z^2}\,\varphi_1\Big(\frac{z}{\overline{w}}\Big) = \frac{1}{2\pi z^2}\,\varphi_2\Big(\frac{z}{\overline{w}}\Big),\ \forall z, w \in \Pi.$$

Hence $\varphi_1\Big(\frac{z}{\overline{w}}\Big) = \varphi_2\Big(\frac{z}{\overline{w}}\Big)$ for all $z, w \in \Pi$. That is, $\varphi_1 = \varphi_2$. □

Now we are ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $\varphi \in \mathcal{G}$ such that A_φ given by (1.2) is bounded on $\mathcal{A}^2(\Pi)$. By Lemma 3.7, there exists $\sigma \in L^\infty(\mathbb{R})$ such that $A_\varphi = R^* M_\sigma R$. But Lemma 3.4 implies that $R^* M_\sigma R = A_\psi$, where

$$\psi(z) = \int_{\mathbb{R}} \sigma(t)\Big(\frac{2t}{1 - e^{-2t\pi}}\Big) z^{1+it} dt,\ z \in \mathbb{C}_-.$$

By Lemma 3.3, we get $\psi \in \mathcal{G}$. As $A_\varphi = A_\psi$ with $\varphi, \psi \in \mathcal{G}$, by Lemma 3.8, it follows that $\varphi = \psi$. That is,

$$\varphi(z) = \int_{\mathbb{R}} \sigma(t)\Big(\frac{2t}{1 - e^{-2t\pi}}\Big) z^{1+it} dt,\ z \in \mathbb{C}_-.$$

Conversely, suppose $\sigma \in L^\infty(\mathbb{R})$ and φ is given by (1.3). Then by Lemma 3.4, it follows that $A_\varphi = R^* M_\sigma R$. Since M_σ is bounded operator on $L^2(\mathbb{R})$, we get $A_\varphi \in \mathcal{B}(\mathcal{A}^2(\Pi))$. This completes the proof of the theorem. □

As a consequence of Theorem 1.1, we have that the every angular operator T is of the form A_φ for some $\varphi \in \mathcal{G}$ and vice-versa. Let $\mathfrak{A}$ be the collection of all angular operators on $\mathcal{A}^2(\Pi)$, then we have $\mathfrak{A} = \{A_\varphi \in \mathcal{B}(\mathcal{A}^2(\Pi)) : \varphi \in \mathcal{G}\}$. That is

$$\mathfrak{A} = \left\{ A_\varphi \in \mathcal{B}(\mathcal{A}^2(\Pi)) \,\middle|\, \begin{array}{l} \varphi(z) = \int_{\mathbb{R}} \sigma(t)\Big(\frac{2t}{1 - e^{-2t\pi}}\Big) z^{1+it} dt, z \in \mathbb{C}_- \text{ for} \\ \text{some } \sigma \in L^\infty(\mathbb{R}) \end{array} \right\}.$$

3.1. Operator theoretic properties of angular operators. In this subsection, we study various operator theoretic properties for the operator $A_\varphi \in \mathcal{B}(\mathcal{A}^2(\Pi))$ in terms of the symbol φ.

Using Theorems 2.1, 1.1 and 3.6, one can easily prove the following results. The proofs are left to the reader.

Theorem 3.9 (Adjoint of A_φ). *Let φ be a function on $\mathbb{C}_-$ and $\sigma \in L^\infty(\mathbb{R})$ such that they satisfy (1.3). Then $A_\varphi^* = A_{\widetilde{\varphi}}$, where $\widetilde{\varphi} \in \mathcal{G}$ and it is given by*

$$\widetilde{\varphi}(z) = \int_{\mathbb{R}} \overline{\sigma(t)}\Big(\frac{2t}{1-e^{-2t\pi}}\Big) z^{1+it}\,dt,\ z \in \mathbb{C}_-.$$

Theorem 3.10. *Let φ be a function on $\mathbb{C}_-$ and $\sigma \in L^\infty(\mathbb{R})$ such that they satisfy (1.3). Then we have the following:*

(1) *A_φ is normal;*
(2) *A_φ is compact if and only if $\varphi \equiv 0$;*
(3) *The collection $\mathfrak{A}$ is a maximal commutative C^*-subalgebra of $\mathcal{B}(\mathcal{A}^2(\Pi))$.*

Theorem 3.11 (Spectrum of A_φ). *Let φ be a function on $\mathbb{C}_-$ and $m \in L^\infty(\mathbb{R})$ such that they satisfy (1.3), with m instead of σ. Then we have the following:*

(1) *$\sigma(A_\varphi) = \sigma_a(A_\varphi) = \sigma_e(A_\varphi) = \mathrm{ess}(m)$;*
(2) *$\lambda \in \sigma_p(A_\varphi)$ if and only if the Lebesgue measure of $\{x : m(x) = \lambda\}$ is positive.*

Now, we give the structure of common reducing subspaces of operators in the collection $\mathfrak{A}$. Before that, we recall some basic definitions and results.

Definitions 3.12. [9, Definition 4.41] *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. A closed subspace $\mathcal{M}$ of $\mathcal{H}$ is an invariant subspace of T if $T(\mathcal{M}) \subseteq \mathcal{M}$ and $\mathcal{M}$ is said to be a reducing subspace of T if it is invariant under both T and T^*.*

Lemma 3.13. [9, Proposition 4.42] *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Then $\mathcal{M}$ is an invariant subspace of T if and only if $P_{\mathcal{M}}TP_{\mathcal{M}} = TP_{\mathcal{M}}$ and it is a reducing subspace of T if and only if $TP_{\mathcal{M}} = P_{\mathcal{M}}T$, where $P_{\mathcal{M}}$ is an orthogonal projection associated to $\mathcal{M}$.*

Theorem 3.14 (Common reducing subspace). *Let $\mathcal{M}$ be a closed subspace of $\mathcal{A}^2(\Pi)$. Then $\mathcal{M}$ is a reducing subspace of all the operators in $\mathfrak{A}$ if and only if $\mathcal{M} = A_{\varphi_0}(\mathcal{A}^2(\Pi))$, where*

$$\varphi_0(z) = \int_{\mathbb{R}} \chi_E(t)\Big(\frac{2t}{1-e^{-2t\pi}}\Big) z^{1+it}\,dt,\ \ z \in \mathbb{C}_-$$

and χ_E is a characteristic function associated to a measurable set E.

Proof. Let $\mathcal{M}$ be a closed subspace of $\mathcal{A}^2(\Pi)$. By Lemma 3.13 and Theorem 1.1, $\mathcal{M}$ is a reducing subspace of operators in $\mathfrak{A} \iff A_\varphi P_{\mathcal{M}} = P_{\mathcal{M}} A_\varphi$ for all $A_\varphi \in \mathfrak{A} \iff M_m(RP_{\mathcal{M}}R^*) = (RP_{\mathcal{M}}R^*)M_m$ for all $m \in L^\infty(\mathbb{R})$. Since $\mathfrak{A}$ is a maximal commutative C^*-algebra, we get $(RP_{\mathcal{M}}R^*) = M_\sigma$ for some $\sigma \in L^\infty(\mathbb{R})$.

Since $M_\sigma(= RP_{\mathcal{M}}R^*)$ is an orthogonal projection on $L^2(\mathbb{R})$, there exists a Lebesgue measurable set $E \subseteq \mathbb{R}$ such that $\sigma = \chi_E$ almost everywhere on $\mathbb{R}$ and $M_\sigma = M_{\chi_E}$. Hence $P_{\mathcal{M}} = RM_{\chi_E}R^*$. By Theorem 1.1, we get $P_{\mathcal{M}} = A_{\varphi_0}$, where

$$\varphi_0(z) = \int_{\mathbb{R}} \chi_E(t)\Big(\frac{2t}{1-e^{-2t\pi}}\Big)z^{1+it}dt, \;\; z \in \mathbb{C}_-.$$

This proves the theorem. □

4. Angular Toeplitz operators

Let P be the orthogonal projection on $L^2(\Pi)$ with range $\mathcal{A}^2(\Pi)$ and let $\mathbf{a} \in L^\infty(\Pi)$. Then the Toeplitz operator $T_{\mathbf{a}} : L^2(\Pi) \to L^2(\Pi)$ is defined by $T_{\mathbf{a}}f = P\mathbf{a}f$. Let $\mathbf{a} \in L^\infty(\Pi)$. Then $\mathbf{a}$ is said to an angular function if $\mathbf{a}(z) = \mathbf{a}(\arg z)$ almost everywhere on Π. For a Toeplitz operator $T_{\mathbf{a}}$, $\mathbf{a} \in L^\infty(\Pi)$, we have the following results.

Theorem 4.1. [10, Proposition 3.1] *Let $\mathbf{a} \in L^\infty(\Pi)$, then the Toeplitz operator $T_{\mathbf{a}}$ is angular if and only if $\mathbf{a}$ is an angular function.*

Theorem 4.2. [10] *Let $\mathbf{a} \in L^\infty(\Pi)$ be an angular function. Then $T_{\mathbf{a}} = R^*M_{\gamma_{\mathbf{a}}}R$, where $\gamma_{\mathbf{a}} \in L^\infty(\mathbb{R})$ and it is given by*

$$\gamma_{\mathbf{a}}(t) = \frac{2t}{1-e^{-2t\pi}} \int_0^\pi \mathbf{a}(x)e^{-2xt}dx, \;\; t \in \mathbb{R}. \tag{4.1}$$

Let $\mathbf{a} \in L^\infty(\Pi)$ be an angular function. By Theorem 1.1 and Theorem 4.2, we have $A_{\varphi_{\mathbf{a}}} = R^*M_{\gamma_{\mathbf{a}}}R = T_{\mathbf{a}}$, where $\varphi_{\mathbf{a}} \in \mathcal{G}$ and it is given by

$$\varphi_{\mathbf{a}}(z) = \int_{\mathbb{R}} \gamma_{\mathbf{a}}(t)\Big(\frac{2t}{1-e^{-2t\pi}}\Big)z^{1+it}dt, \;\; z \in \mathbb{C}_- \tag{4.2}$$

and $\gamma_{\mathbf{a}}$ is given by (4.1). Let $\mathcal{A}_{top} = \big\{T_{\mathbf{a}} : \mathbf{a} \in L^\infty(\Pi) \text{ is angular}\big\}$. Then from above, it is clear that

$$\mathcal{A}_{top} = \big\{A_{\varphi_{\mathbf{a}}} : \mathbf{a} \in L^\infty(\Pi) \text{ is angular and } \varphi_{\mathbf{a}} \text{ is given by (4.2)}\big\}.$$

Let $\Gamma = \{\gamma_{\mathbf{a}} : \mathbf{a} \in L^\infty(\Pi) \text{ is angular and } \gamma_{\mathbf{a}} \text{ is given by (4.1)}\}$. Then the map $\eta : \Gamma \to \mathcal{A}_{top}; \gamma_{\mathbf{a}} \mapsto A_{\varphi_{\mathbf{a}}}$ is a $*$-isometric isomorphism.

Let $\mathcal{AT}$ be the C^*-algebra generated by $\mathcal{A}_{top}$. Let VSO($\mathbb{R}$) be the collection of all bounded **very slowly oscillating** functions on $\mathbb{R}$, that is the functions which are uniformly continuous with respect to the "arcsinh-metric" $\rho(x, y) = |\operatorname{arcsinh}(x) - \operatorname{arcsinh}(y)|$. From [10], we have that VSO($\mathbb{R}$) is a closed C^*-algebra subalgebra of $L^\infty(\mathbb{R})$ and it is equal to the C^*-algebra generated by Γ. Let

$$\widetilde{\mathcal{G}} = \left\{ \varphi \in \mathcal{G} \;\middle|\; \begin{array}{l} \varphi(z) = \int_{\mathbb{R}} \sigma(t)\Big(\frac{2t}{1-e^{-2t\pi}}\Big)z^{1+it}dt, z \in \mathbb{C}_- \text{ for} \\ \text{some } \sigma \in \text{VSO}(\mathbb{R}) \end{array} \right\}.$$

Then it is easy to prove the following result.

Theorem 4.3. *The C^*-algebra $\mathcal{AT}$ generated by $\mathcal{A}_{top}$ is given by*

$$\mathcal{AT} = \{A_\varphi : \varphi \in \widetilde{\mathcal{G}}\}.$$

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References

[1] BAIS, S.R.; VENKU NAIDU, D. Study of twisted Bargmann transform via Bargmann transform. *Forum Math.* **33** (2021), 1659-1670. MR4333996, Zbl 1486.30142. 46

[2] BAIS, S.R.; VENKU NAIDU, D. $\mathcal{L}$-invariant and radial singular integral operators on the Fock space. *J. Pseudo-Differ. Oper. Appl.* **14**, No. 1 (2023), paper no. 11. MR4540871, Zbl 1514.42013. 43, 46

[3] BAIS, S.R.; VENKU NAIDU, D. A note on C^*-algebra of Toeplitz operators with $\mathcal{L}$-invariant symbols on the Fock space. *Complex Anal. Oper. Theory* **17**, No. 6 (2023), paper no. 99. MR4627306, Zbl 07736014. 43, 46

[4] BAIS, S.R.; VENKU NAIDU, D.; MOHAN, P. Integral representation of vertical operators on the Bergman space over the upper half-plane. *C. R. Math. Acad. Sci. Paris* **361** (2023), 1593-1604. 43, 45, 46

[5] BARGMANN, V. On a Hilbert space of analytic functions and an associated integral transform. *Comm. Pure Appl. Math.* **14** (1961), 187-214. MR0157250 (28 #486), Zbl 0107.09102. 46

[6] BHATIA, R. Notes on functional analysis. Texts and Readings in Mathematics, 50. *Hindustan Book Agency, New Delhi,* 2009. MR2477477 (2010a:46001), Zbl 1175.46001. 44, 45

[7] CAO, G.F.; LI, J.; SHEN, M.; WICK, B.D.; YAN, L. A boundedness criterion for singular integral operators of convolution type on the Fock space. *Adv. Math.* **363** (2020), 107001, 33 pp. MR4053517, Zbl 1437.32001. 43

[8] CONWAY, J.B. A course in operator theory. Graduate Studies in Mathematics, 21. *American Mathematical Society, Providence, RI,* 2000. MR1721402, Zbl 0936.47001. 45

[9] DOUGLAS, R.G. Banach algebra techniques in operator theory. Pure and Applied Mathematics, Vol. 49. *Academic Press, New York,* 1972. MR0361893 (50 #14335), Zbl 0247.47001. 44, 54

[10] ESMERAL, K; MAXIMENKO, E.A. C^*-algebra of angular Toeplitz operators on Bergman spaces over the upper half-plane. *Commun. Math. Anal.* **17** (2014), no. 2, 151-162. MR3292965, Zbl 1327.30063. 44, 52, 55

[11] ESMERAL, K; MAXIMENKO, E.A.; VASILEVSKI, N. C^*-algebra generated by angular Toeplitz operators on the weighted Bergman spaces over the upper half-plane. *Integral Equations Operator Theory* **83** (2015), no. 3, 413-428. MR3413929, Zbl 1329.47078. 44, 45

[12] ESMERAL, K; VASILEVSKI, N. C^*-algebra generated by horizontal Toeplitz operators on the Fock space. *Bol. Soc. Mat. Mex. (3)* **22** (2016), no. 2, 567-582. MR3544154, Zbl 1369.47033. 44

[13] FOLLAND, G.B. Fourier Analysis and Its Applications. *AMS, Providence, RI, 1992.* MR1145236 (93f:42001), Zbl 0786.42001. 46

[14] GRUDSKY, S.; QUIROGA-BARRANCO, R.; VASILEVSKI, N. Commutative C^*-algebras of Toeplitz operators and quantization on the unit disk. *J. Funct. Anal.* **234** (2006), no. 1, 1-44. MR2214138 (2006k:47058), Zbl 1100.47023. 44

[15] GRUDSKY, S.; VASILEVSKI, N. Bergman-Toeplitz operators: radial component influence. *Integral Equations Operator Theory* **40** (2001), no. 1, 16-33. MR1829512 (2002g:47053), Zbl 0993.47023. 44
[16] GRUDSKY, S. M.; VASILEVSKI, N. L. Toeplitz operators on the Fock space: radial component effects. *Integral Equations Operator Theory* **44** (2002), no. 1, 10-37. MR1913421 (2003g:47049), Zbl 1039.47015. 44
[17] HERRERA YAÑEZ, C.; HUTNÍK, O.; MAXIMENKO, E. A. Vertical symbols, Toeplitz operators on weighted Bergman spaces over the upper half-plane and very slowly oscillating functions. *C. R. Math. Acad. Sci. Paris* **352** (2014), no. 2, 129-132. MR3151880, Zbl 1293.47030. 44
[18] HERRERA YAÑEZ, C.; MAXIMENKO, E. A.; VASILEVSKI, N. Vertical Toeplitz operators on the upper half-plane and very slowly oscillating functions. *Integral Equations Operator Theory* **77** (2013), no. 2, 149-166. MR3101779, Zbl 1293.47029. 44
[19] STEIN, E.M. Singular integrals and differentiability properties of functions. Princeton Mathematical Series, No. 30. *Princeton Univ. Press, Princeton, NJ,* 1970. MR0290095 (44 #7280), Zbl 0207.13501. 51
[20] VASILEVSKI, N.L. Commutative algebras of Toeplitz operators on the Bergman space. Operator Theory: Advances and Applications, 185. *Birkhäuser Verlag, Basel,* 2008. MR2441227 (2009m:47071), Zbl 1168.47057. 44, 46
[21] ZHU, K. Singular integral operators on the Fock space. *Integral Equations Operator Theory* **81** (2015), no. 4, 451-454. MR3323562, Zbl 1332.30083. 43
[22] ZHU, K. Towards a dictionary for the Bargmann transform, *Handbook of analytic operator theory*. CRC Press/Chapman Hall Handb. Math. Ser. *CRC Press, Boca Raton, FL,* 2019, pp. 319–349. MR3966623, Zbl 1429.30053. 46
[23] ZHU, K. Operator Theory in Function Spaces. Mathematical Surveys and Monographs, 138. *American Mathematical Society, Providence,* 2007. MR2311536, Zbl 1123.47001. 44

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