New York Journal of Mathematics

New York J. Math. **30** (2024) 783–827.

Periodic points of algebraic functions related to a continued fraction of Ramanujan

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ABSTRACT. A continued fraction $v(\tau)$ of Ramanujan is evaluated at certain arguments in the field $K = \mathbb{Q}(\sqrt{-d})$, with $-d \equiv 1 \pmod 8$, in which the ideal $(2) = \wp_2\wp_2'$ is a product of two prime ideals. These values of $v(\tau)$ are shown to generate the inertia field of $\wp_2$ or $\wp_2'$ in an extended ring class field over the field K. The conjugates over $\mathbb{Q}$ of these same values, together with $0, -1\pm\sqrt{2}$, are shown to form the exact set of periodic points of a fixed algebraic function $\hat{F}(x)$, independent of d. These are analogues of similar results for the Rogers-Ramanujan continued fraction.

CONTENTS

1. Introduction

This paper is concerned with values of Ramanujan's continued fraction

$$v(\tau) = \frac{q^{1/2}}{1+q+}\,\frac{q^2}{1+q^3+}\,\frac{q^4}{1+q^5+}\,\frac{q^6}{1+q^7+}\cdots,\quad q = e^{2\pi i\tau},$$

Received October 18, 2022.

2020 *Mathematics Subject Classification.* 14H05, 37F05, 11R37, 11D88, 11R29.

Key words and phrases. Periodic points, algebraic function, 2-adic field, extended ring class fields, Ramanujan continued fraction.

ISSN 1076-9803/2024

sometimes referred to as the Ramanujan-Göllnitz-Gordon continued fraction, which is also given by the infinite product

$$v(\tau) = q^{1/2} \prod_{n=1}^{\infty} (1-q^n)^{\left(\frac{2}{n}\right)}, \ \ q = e^{2\pi i \tau},$$

for τ in the upper half-plane. Here, $\left(\frac{2}{n}\right)$ is the Kronecker symbol. See [12], [9, p. 153], [5], [6]. The continued fraction $v(\tau)$ is analogous to the Rogers-Ramanujan continued fraction

$$r(\tau) = q^{1/5} \prod_{n=1}^{\infty} (1-q^n)^{\left(\frac{5}{n}\right)} \ \ q = e^{2\pi i \tau},$$

whose properties were considered in the papers [17], [18]. In [17] it was shown that certain values of $r(\tau)$, for τ in the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$ with $-d \equiv \pm 1 \pmod 5$, are periodic points of a fixed algebraic function, independent of d, and generate certain class fields $\Sigma_{\mathfrak{f}}\Omega_f$ over K. Here $\Sigma_{\mathfrak{f}}$ is the ray class field of conductor $\mathfrak{f} = \wp_5$ or $\wp_5'$ over K, where $(5) = \wp_5\wp_5'$ in the ring of integers R_K of K; and Ω_f is the ring class field of conductor f corresponding to the order R_{-d} of discriminant $-d = \mathfrak{d}_K f^2$ in K ($\mathfrak{d}_K$ is the discriminant of K).

Here we will show that a similar situation holds for certain values of the continued fraction $v(\tau)$. We consider discriminants of the form $-d \equiv 1 \pmod 8$ and arguments in the field $K = \mathbb{Q}(\sqrt{-d})$. Let R_K be the ring of integers in this field and let the prime ideal factorization of (2) in R_K be $(2) = \wp_2\wp_2'$. We define the algebraic integer w by

$$w = \frac{a+\sqrt{-d}}{2}, \ \ a^2 + d \equiv 0 \pmod{2^5}, \ \ (N(w), f) = 1, \tag{1.1}$$

where $\wp_2 = (2, w)$. Also, the positive (and odd) integer f is defined by $-d = \mathfrak{d}_K f^2$, where $\mathfrak{d}_K$ is the discriminant of $K/\mathbb{Q}$.

We will show that

$$v(w/8) = \pm \frac{1 \pm \sqrt{1+\pi^2}}{\pi},$$

where π is a generator in Ω_f of the ideal $\wp_2$ (or rather, its extension $\wp_2 R_{\Omega_f}$ in Ω_f). The algebraic integer π and its conjugate ξ in Ω_f were studied in [14] and shown to satisfy

$$\pi^4 + \xi^4 = 1, \ \ (\pi) = \wp_2, \ (\xi) = \wp_2', \ \ \xi = \frac{\pi^{\tau^2}+1}{\pi^{\tau^2}-1}, \tag{1.2}$$

where $\tau = \left(\frac{\Omega_f/K}{\wp_2}\right)$ is the Artin symbol (Frobenius automorphism) for the prime ideal $\wp_2$ and the ring class field Ω_f over K whose conductor is f. It follows from results of [14] that

$$\pi = (-1)^c \mathfrak{p}(w),$$

where c is an integer satisfying the congruence

$$c \equiv 1 - \frac{a^2 + d}{32} \pmod 2$$

and $\mathfrak{p}(\tau)$ is the modular function $\mathfrak{p}(\tau) = \frac{\mathfrak{f}_2^2(\tau/2)}{\mathfrak{f}^2(\tau/2)}$, defined in terms of the Weber-Schläfli functions $\mathfrak{f}_2(\tau), \mathfrak{f}(\tau)$. (See [20], [8], [19].) The above formula for $v(w/8)$ follows from the identity

$$\frac{2}{\mathfrak{p}(8\tau)} = \frac{1 - v^2(\tau)}{v(\tau)} = \frac{1}{v(\tau)} - v(\tau),$$

for τ in the upper half-plane, which we prove in Proposition 4.1. (Also see [7, Thm. 8.6, p. 475].)

As in [17], we consider a diophantine equation, namely

$$\mathcal{C}_2 : X^2 + Y^2 = \sigma^2(1 + X^2Y^2), \ \ \sigma = -1 + \sqrt{2}.$$

An identity for the continued fraction $v(\tau)$ implies that

$$(X, Y) = (v(w/8), v(-1/w))$$

is a point on $\mathcal{C}_2$. We prove the following theorem relating the coordinates of this point.

Theorem A. *Let w be given by (1.1) with $\wp_2 = (2, w)$ in R_K and $-d = \mathfrak{d}_K f^2 \equiv 1$ (mod 8).*

(a) *The field $F_1 = \mathbb{Q}(v(w/8)) = \mathbb{Q}(v^2(w/8))$ equals the field $\Sigma_{\wp_2'^3}\Omega_f$, where $\Sigma_{\wp_2'^3}$ is the ray class field of conductor $\mathfrak{f} = \wp_2'^3$ and Ω_f is the ring class field of conductor f over the field K. The field F_1 is the inertia field for $\wp_2$ in the extended ring class field $L_{\mathcal{O},8} = \Sigma_8\Omega_f$ over K, where $\mathcal{O} = R_{-d}$ is the order of discriminant $-d$ in K.*

(b) *We have $F_2 = \mathbb{Q}(v(-1/w)) = \Sigma_{\wp_2^3}\Omega_f$, the inertia field of $\wp_2'$ in $L_{\mathcal{O},8}/K$.*

(c) *If τ_2 is the Frobenius automorphism $\tau_2 = \left(\frac{F_1/K}{\wp_2}\right)$, then*

$$v(-1/w) = \frac{v(w/8)^{\tau_2} + (-1)^c\sigma}{\sigma v(w/8)^{\tau_2} - (-1)^c}. \tag{1.3}$$

See Theorems 6.1, 7.3 and 7.5 and their corollaries. From part (c) of this theorem we deduce the following.

Theorem B.

(a) *If w and c are as above, then the generator $(-1)^{1+c}v(w/8)$ of the field $\Sigma_{\wp_2'^3}\Omega_f$ over $\mathbb{Q}$ is a periodic point of the multivalued algebraic function $\hat{F}(x)$ given by*

$$\hat{F}(x) = -\frac{x^2 - 1}{2} \pm \frac{1}{2}\sqrt{x^4 - 6x^2 + 1};$$

and $v^2(w/8)$ is a periodic point of the algebraic function $\hat{T}(x)$ defined by

$$\hat{T}(x) = \frac{1}{2}(x^2 - 4x + 1) \pm \frac{x-1}{2}\sqrt{x^2 - 6x + 1}.$$

(b) *The minimal period of $(-1)^{1+c}v(w/8)$ (and of $v^2(w/8)$) is equal to the order of the automorphism τ_2 in $Gal(F_1/K)$.*

(c) *Together with the numbers $0, -1 \pm \sqrt{2}$, the values $(-1)^{1+c}v(w/8)$ and their conjugates over $\mathbb{Q}$ are the only periodic points of the algebraic function $\hat{F}(x)$ in $\overline{\mathbb{Q}}$ or $\mathbb{C}$. The only periodic points of $\hat{T}(x)$ in $\overline{\mathbb{Q}}$ or $\mathbb{C}$ are $0, (-1 \pm \sqrt{2})^2$, and the conjugates of the values $v^2(w/8)$ over $\mathbb{Q}$.*

We understand by a periodic point of the multivalued algebraic function $\hat{F}(x)$ the following. Let $f(x,y) = x^2y + x^2 + y^2 - y$ be the minimal polynomial of $\hat{F}(x)$ over $\mathbb{Q}(x)$. A periodic point of $\hat{F}(x)$ is an algebraic number a for which there exist $a_1, a_2, \ldots, a_{n-1} \in \overline{\mathbb{Q}}$ satisfying

$$f(a, a_1) = f(a_1, a_2) = \cdots = f(a_{n-1}, a) = 0.$$

A similar definition can be given over any ground field k. See [15], [16]. Thus, if $a \in \overline{\mathbb{Q}}$ is a periodic point of $\hat{F}(x)$, so are its conjugates over $\mathbb{Q}$, because $f(x,y)$ has coefficients in $\mathbb{Q}$. We show in Section 8 that $v^2(w/8)$ is actually a periodic point in the usual sense of the single-valued 2-adic function

$$T(x) = x^2 - 4x + 2 - (x-1)(x-3)\sum_{k=1}^{\infty} C_{k-1}\frac{2^k}{(x-3)^{2k}},$$

defined on a subset of the maximal unramified, algebraic extension K_2 of the 2-adic field $\mathbb{Q}_2$. (C_k is the k-th Catalan number.) This follows from the fact that

$$v(w/8)^{2\tau_2} = T(v(w/8)^2),$$

in the completion $F_{1,\mathfrak{q}} \subset \mathsf{K}_2$ of $F_1 = \Sigma_{\wp_2'^3}\Omega_f$ with respect to a prime divisor $\mathfrak{q}$ of $\wp_2$ in F_1. This implies that the minimal period of $v^2(w/8)$ with respect to the function $T(x)$ is $n = \mathrm{ord}(\tau_2)$.

From Theorems A and B we conclude the following.

Theorem C. *Let $K = \mathbb{Q}(\sqrt{-d})$, with $-d \equiv 1 \bmod 8$ and $(2) = \wp_2\wp_2'$ in R_K. Then every class field over K of the form $\Sigma_{\wp_2^3}\Omega_f$ or $\Sigma_{\wp_2'^3}\Omega_f$ (with f odd) is generated over $\mathbb{Q}$ by an individual periodic point of the function $\hat{F}(x)$ (or of $\hat{T}(x)$). Furthermore, all but three periodic points of $\hat{F}(x)$ in $\overline{\mathbb{Q}}$ generate a class field $\Sigma_{\wp_2^3}\Omega_f$ in this family over some imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$, for which $-d = \mathfrak{d}_K f^2 \equiv 1$ (mod 8).*

These are all analogues of the corresponding facts for the Rogers-Ramanujan continued fraction $r(\tau)$ which were proved in [17] and [18].

An important corollary of the fact that the conjugates of the values $v(w/8)$ in Theorem B are, together with the three fixed points, all the periodic points of the algebraic function $\hat{F}(x)$, is the following class number formula. In this formula, $h(-d)$ denotes the class number of the order R_{-d} of discriminant $-d$ in the quadratic field $K = K_d$, and $\mathfrak{D}_{n,2}$ is the finite set of negative discriminants $-d \equiv 1 \pmod 8$ for which the Frobenius automorphism τ_2 in Theorem A has order n in $\mathrm{Gal}(F_1/K_d)$, where $F_1 = F_{1,d}$ also depends on d:

$$\sum_{-d \in \mathfrak{D}_{n,2}} h(-d) = \frac{1}{2} \sum_{k|n} \mu(n/k) 2^k, \quad n > 1. \tag{1.4}$$

($\mu(n)$ is the Möbius function.) See Theorem 9.2. This fact is the analogue for the prime $p = 2$ of Theorem 1.3 in [18] for the prime $p = 5$, or of Conjecture 1 of that paper for a prime $p > 5$.

The layout of the paper is as follows. Section 2 contains a number of q-identities (following Ramanujan) and theta function identities which we use to prove identities for various modular functions in Sections 3-5. Most of these identities are known; straightforward proofs – which use theta functions, but not the theory of modular forms or functions – are included here for the sake of completeness. In Sections 6 and 7 we prove Theorem A. The proofs of Theorem B and (1.4) are given in Sections 8 and 9.

Sections 2-5 and portions of Sections 6-9 also appear in the Ph.D. dissertation [1] of the first author.

2. Preliminaries.

As is customary, let us set

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=0}^{n-1} (1 - aq^k), \quad n \geq 1$$

and

$$(a;q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k), \quad |q| \leq 1.$$

Ramanujan's general theta function $f(a,b)$ is defined as

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}\, b^{n(n-1)/2}. \tag{2.1}$$

Three special cases are defined, in Ramanujan's notation, as

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \tag{2.2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2}, \tag{2.3}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n\, q^{n(3n-1)/2}. \tag{2.4}$$

Jacobi's triple product identity, in Ramanujan's notation, takes the form

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.5}$$

Using this, the above three functions can be written as

$$\varphi(q) = (-q;q^2)^2_\infty(q^2;q^2)_\infty, \tag{2.6}$$

$$\psi(q) = (-q;q)_\infty(q^2;q^2)_\infty = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{2.7}$$

$$f(-q) = (q;q)_\infty. \tag{2.8}$$

The equality that relates the right hand sides of both the equations for $f(-q)$ in (2.4) and (2.8) is Euler's pentagonal number theorem.

Another important function that plays a prominent role is given by

$$\chi(q) := (-q;q^2)_\infty. \tag{2.9}$$

All the above four functions satisfy a myriad of relations, most of which are listed and proved in Berndt's books on Ramanujan's notebooks, and we will refer to them as needed.

Last but not least, the Dedekind-eta function is defined as

$$\eta(\tau) = q^{1/24} f(-q), \quad q = e^{2\pi i\tau}, \quad \operatorname{Im}\tau > 0. \tag{2.10}$$

Most of the identities that we use later on are listed here in order, for the sake of convenience.

$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2), \tag{2.11}$$

$$\varphi^4(q) - \varphi^4(-q) = 16q\psi^4(q^2), \tag{2.12}$$

$$\varphi(q)\psi(q^2) = \psi^2(q), \tag{2.13}$$

$$\varphi(-q) + \varphi(q^2) = 2\frac{f^2(q^3,q^5)}{\psi(q)}, \tag{2.14}$$

$$\varphi(-q) - \varphi(q^2) = -2q\frac{f^2(q,q^7)}{\psi(q)}, \tag{2.15}$$

$$\varphi(q)\varphi(-q) = \varphi^2(-q^2), \tag{2.16}$$

$$\varphi(q) + \varphi(-q) = 2\varphi(q^4), \tag{2.17}$$

$$\varphi^2(q) - \varphi^2(-q) = 8q\psi^2(q^4). \tag{2.18}$$

All of the above identities and their proofs can be found in [2, p. 40, Entry 25] and in [2, p. 51, Example (iv)].

For $\tau \in \mathcal{H}$, the upper half plane, and $q = e(\tau) = e^{2\pi i\tau}$, the theta constant with characteristic $\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right] \in \mathbb{R}$ is defined as

$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\tau) = \sum_{n\in\mathbb{Z}} e\Big(\frac{1}{2}\Big(n+\frac{\epsilon}{2}\Big)^2\tau + \frac{\epsilon'}{2}\Big(n+\frac{\epsilon}{2}\Big)\Big). \tag{2.19}$$

It satisfies the following basic properties for $l, m, n \in \mathbb{Z}$ with N positive:

$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\tau) = e\big(\mp\frac{\epsilon m}{2}\big)\,\theta\begin{bmatrix}\pm\epsilon + 2l\\ \pm\epsilon' + 2m\end{bmatrix}(\tau), \tag{2.20}$$

$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\tau) = \sum_{k=0}^{N-1}\theta\begin{bmatrix}\frac{\epsilon+2k}{N}\\ N\epsilon'\end{bmatrix}(N^2\tau). \tag{2.21}$$

We also have the transformation law, for $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$:

$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}\big(\frac{a\tau+b}{c\tau+d}\big) = \kappa\sqrt{c\tau+d}\,\theta\begin{bmatrix}a\epsilon + c\epsilon' - ac\\ b\epsilon + d\epsilon' + bd\end{bmatrix}(\tau), \tag{2.22}$$

where

$$\kappa = e\big(-\tfrac{1}{4}(a\epsilon + c\epsilon')bd - \tfrac{1}{8}(ab\epsilon^2 + cd\epsilon'^2 + 2bc\epsilon\epsilon')\big)\kappa_0,$$

and κ_0 is an eighth root of unity depending only on the matrix $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$.

In particular, we have:

$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\tau+1) = e\big(-\tfrac{\epsilon}{4}(1+\tfrac{\epsilon}{2})\big)\,\theta\begin{bmatrix}\epsilon\\ \epsilon+\epsilon'+1\end{bmatrix}(\tau), \tag{2.23}$$

$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}\big(\tfrac{-1}{\tau}\big) = e\big(-\tfrac{1}{8}\big)\sqrt{\tau}\,e\big(\tfrac{\epsilon\epsilon'}{4}\big)\,\theta\begin{bmatrix}\epsilon'\\ -\epsilon\end{bmatrix}(\tau). \tag{2.24}$$

We also have the product formula:

$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\tau) = e\big(\frac{\epsilon\epsilon'}{4}\big)q^{\frac{\epsilon^2}{8}}\prod_{n\geq 1}(1-q^n)\big(1+e\Big(\frac{\epsilon'}{2}\Big)q^{n-\frac{1+\epsilon}{2}}\big)\big(1+e\Big(\frac{-\epsilon'}{2}\Big)q^{n-\frac{1-\epsilon}{2}}\big), \tag{2.25}$$

which follows from Jacobi's triple product identity.

More information about these theta constants and the above formulas, as well as their proofs, can all be found in [10, pp. 71-81]. Also see [9, pp. 143, 158-159].

3. Identities for $u(\tau)$ and $v(\tau)$

Let us define the functions $u(\tau)$ and $v(\tau)$ as

$$u(\tau) = \sqrt{2}\,q^{1/8}\prod_{n=1}^{\infty}\big(1+q^n\big)^{(-1)^n},$$

$$v(\tau) = q^{1/2}\prod_{n=1}^{\infty}\big(1-q^n\big)^{\left(\frac{2}{n}\right)}.$$

The functions $u(\tau)$ and $v(\tau)$ satisfy the following identities.

Proposition 3.1. *(a) If $x = u(\tau)$ and $y = u(2\tau)$, we have*

$$x^4(y^4+1) = 2y^2.$$

(b) If $x = v(\tau)$ and $y = v(2\tau)$, we have

$$x^2y + x^2 + y^2 = y.$$

Remark. The curve $E : f(x,y) = 0$ defined by

$$f(x,y) = x^2y + x^2 + y^2 - y$$

is an elliptic curve with $j(E) = 1728$, so E has complex multiplication by $\mathsf{R} = \mathbb{Z}[i]$.

Proof. (a) From (2.11), we have

$$\varphi^2(-q) = 2\varphi^2(q^2) - \varphi^2(q),$$

where

$$\varphi(q) = (-q;q^2)_\infty^2(q^2;q^2)_\infty \text{ and } \psi(q) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}$$

are as defined in (2.6) and (2.7). Squaring both sides gives us

$$\varphi^4(-q) = 4\varphi^4(q^2) - 4\varphi^2(q)\varphi^2(q^2) + \varphi^4(q).$$

Using

$$\varphi^4(q) - \varphi^4(-q) = 16q\psi^4(q^2),$$

which is (2.12), we obtain

$$\varphi^4(q^2) + 4q\,\psi^4(q^2) = \varphi^2(q)\,\varphi^2(q^2).$$

Dividing both sides by $\varphi^4(q^2)$ and using the relation $\psi^2(q) = \varphi(q)\,\psi(q^2)$ from (2.13) we get

$$1 + 4q\frac{\psi^4(q^2)}{\varphi^4(q^2)} = \frac{\varphi^2(q)}{\varphi^2(q^2)} = \frac{\psi^2(q^2)}{\varphi^2(q^2)} \cdot \frac{\varphi^4(q)}{\psi^4(q)}. \tag{3.1}$$

Since

$$u(\tau) = \sqrt{2}q^{1/8}\prod_{n=1}^{\infty}(1+q^n)^{(-1)^n} = \sqrt{2}q^{1/8}\frac{(-q^2;q^2)_\infty}{(-q;q^2)_\infty} = \sqrt{2}q^{1/8}\frac{\psi(q)}{\varphi(q)},$$

the result follows by substituting the last equality for $u(\tau)$ into (3.1).

(b) From [9, p. 153, (9.7)] we have the following relation between $u = u(\tau)$ and $v = v(\tau)$:

$$u^4(v^2+1)^2 + 4v(v^2-1) = 0; \tag{3.2}$$

which we rewrite as $u^4 = \dfrac{4v(1-v^2)}{(v^2+1)^2}$. (See the proof of Proposition 10.1 in the Appendix.) Substituting this expression for u^4 into the relation $u^4(\tau)\big[u^4(2\tau)+1\big] = 2u^2(2\tau)$, after squaring, we obtain

$$\frac{16x^2(1-x^2)^2}{(x^2+1)^4}\cdot\left[\frac{4y(1-y^2)}{(y^2+1)^2}+1\right]^2 = 4\cdot\frac{4y(1-y^2)}{(y^2+1)^2},$$

where $x = v(\tau), y = v(2\tau)$. Clearing the denominators gives us

$$x^2(1-x^2)^2(y^2-2y-1)^4 = y(1-y^2)(y^2+1)^2(x^2+1)^4.$$

Now moving everything to one side and factoring the polynomial using Maple, we finally arrive at

$$(x^2y+x^2+y^2-y)(x^2y^2-x^2y+y+1)(x^2y^2+2xy^2+x^2-4xy+y^2-2x+1) \\ \times(x^2y^2-2xy^2+x^2+4xy+y^2+2x+1)=0.$$

From the definitions of x and y, it is clear that $x = O(q^{1/2})$ and $y = O(q)$ as q tends to 0. Hence, the first factor above (and none of the others) vanishes for q sufficiently small. By the identity theorem of complex analysis, the first factor vanishes for $|q| < 1$. This proves the result. □

Remark. The identity in part (b) of Proposition 3.1 can be written as

$$v^2(\tau) = v(2\tau)\frac{1-v(2\tau)}{1+v(2\tau)}.$$

See [5, Thm. 2.2]. This is analogous to the identity for the Rogers-Ramanujan continued fraction $r(\tau)$:

$$r^5(\tau) = r(5\tau)\frac{r^4(5\tau)-3r^3(5\tau)+4r^2(5\tau)-2r(5\tau)+1}{r^4(5\tau)+2r^3(5\tau)+4r^2(5\tau)+3r(5\tau)+1}.$$

Also see [4, p. 167], [3, pp. 19-20].

Proposition 3.2. *The functions $x = v^2(\tau)$ and $y = v^2(2\tau)$ satisfy the relation*

$$g(x,y) = y^2-(x^2-4x+1)y+x^2 = 0.$$

Proof. For $x = v(\tau)$ and $y = v(2\tau)$, we have the relation

$$x^2+y^2 = y(1-x^2).$$

Squaring both sides and moving all the terms to the left side, we obtain

$$x^4+y^4+4x^2y^2-x^4y^2-y^2 = 0.$$

Hence, $x = v^2(\tau)$ and $y = v^2(2\tau)$ satisfy the relation

$$g(x,y) = x^2+y^2+4xy-x^2y-y = 0.$$

□

Let $A, \bar{A}$ denote the linear fractional mappings

$$A(x) = \frac{\sigma x + 1}{x - \sigma}, \quad \bar{A}(x) = \frac{-x + \sigma}{\sigma x + 1}, \quad \sigma = -1 + \sqrt{2}. \tag{3.3}$$

Proposition 3.3. *The following identity holds:*

$$v\left(\frac{-1}{\tau}\right) = \bar{A}(v(\tau/4)) = \frac{\bar{\sigma} v(\tau/4) + 1}{v(\tau/4) - \bar{\sigma}} = \frac{-v(\tau/4) + \sigma}{\sigma v(\tau/4) + 1},$$

where $\bar{\sigma} = -1 - \sqrt{2}$.

Proof. This follows from the formula

$$v(\tau) = e^{-2\pi i/8} \frac{\theta\left[{3/4 \atop 1}\right](8\tau)}{\theta\left[{1/4 \atop 1}\right](8\tau)},$$

using the formulas (2.20), (2.21), (2.24). (Also see [10].) Namely, we have:

$$v\left(\frac{-1}{\tau}\right) = e^{-2\pi i/8} \frac{\theta\left[{3/4 \atop 1}\right]\left(\frac{-8}{\tau}\right)}{\theta\left[{1/4 \atop 1}\right]\left(\frac{-8}{\tau}\right)} = \frac{\theta\left[{1 \atop 3/4}\right]\left(\frac{\tau}{8}\right)}{\theta\left[{1 \atop 1/4}\right]\left(\frac{\tau}{8}\right)} = \frac{\sum_{k=0}^{3} \theta\left[{\frac{1+2k}{4} \atop 3}\right](2\tau)}{\sum_{k=0}^{3} \theta\left[{\frac{1+2k}{4} \atop 1}\right](2\tau)},$$

which after some simplification yields

$$v\left(\frac{-1}{\tau}\right) = \frac{\left[-1 + e^{3\pi i/8}\right] v(\tau/4) + \left[e^{2\pi i/8} + e^{3\pi i/2}\right]}{\left[e^{2\pi i/8} + e^{3\pi i/2}\right] v(\tau/4) + \left[1 + e^{7\pi i/8}\right]}.$$

This yields that

$$v\left(\frac{-1}{\tau}\right) = \frac{\bar{\sigma} v(\tau/4) + 1}{v(\tau/4) - \bar{\sigma}} = \frac{-v(\tau/4) + \sigma}{\sigma v(\tau/4) + 1}.$$

□

The set of mappings

$$\tilde{H} = \{x, A(x), \bar{A}(x), -1/x\}$$

forms a group under composition. We also have the formula

$$(\sigma x + 1)^2 (\sigma y + 1)^2 f(\bar{A}(x), \bar{A}(y)) = 2^3 \sigma^2 f(y, x).$$

Proposition 3.4. *The function* $v(\tau)$ *satisfies the following:*

$$v^2\left(\frac{-1}{8\tau}\right) = \frac{v^2(\tau) - \sigma^2}{\sigma^2 v^2(\tau) - 1}, \quad \sigma = -1 + \sqrt{2}. \tag{3.4}$$

Proof. Replacing τ by 8τ in Proposition 3.3 and squaring gives us

$$\begin{aligned} v^2\left(\frac{-1}{8\tau}\right) &= \frac{(-v(2\tau)+\sigma)^2}{(\sigma v(2\tau)+1)^2} \\ &= \frac{(-y+\sigma)^2}{(\sigma y+1)^2} \\ &= \frac{y^2-2\sigma y+\sigma^2}{\sigma^2 y^2+2\sigma y+1}, \end{aligned}$$

where $y = v(2\tau)$. Then, replace 2σ by $1-\sigma^2$ to obtain

$$\begin{aligned} v^2\left(\frac{-1}{8\tau}\right) &= \frac{y^2-y+\sigma^2 y+\sigma^2}{\sigma^2 y^2+y-\sigma^2 y+1} \\ &= \frac{\sigma^2(y+1)-(y-y^2)}{(y+1)-\sigma^2(y-y^2)}. \end{aligned}$$

Now replace $(y-y^2)$ by $x^2(y+1)$, using Proposition 3.1(b), to get the result:

$$\begin{aligned} v^2\left(\frac{-1}{8\tau}\right) &= \frac{\sigma^2(y+1)-x^2(y+1)}{(y+1)-\sigma^2 x^2(y+1)} \\ &= \frac{(\sigma^2-x^2)(y+1)}{(1-\sigma^2 x^2)(y+1)} \\ &= \frac{x^2-\sigma^2}{\sigma^2 x^2-1}, \end{aligned}$$

where $x = v(\tau)$. □

For later use we denote the linear fractional map which occurs in (3.4) by $t(x)$:

$$t(x) = \frac{x-\sigma^2}{\sigma^2 x-1}. \tag{3.5}$$

A straightforward calculation shows that

$$(\sigma^2 x-1)^2(\sigma^2 y-1)^2 g(t(x),t(y)) = 2^5\sigma^4 g(y,x). \tag{3.6}$$

4. The relation between $v(\tau)$ and $\mathfrak{p}(\tau)$.

In this section and the next we shall prove several identities between $v(\tau)$ and the functions $\mathfrak{p}(\tau)$ and $\mathfrak{b}(\tau)$ defined as follows. Let $\mathfrak{f}, \mathfrak{f}_1, \mathfrak{f}_2$ denote the Weber-Schläfli functions (see [8, p. 233], [19, p. 148]). Then the functions $\mathfrak{p}(\tau)$ and $\mathfrak{b}(\tau)$ are given by

$$\mathfrak{p}(\tau) = \frac{\mathfrak{f}_2(\tau/2)^2}{\mathfrak{f}(\tau/2)^2} = 2q^{1/16}\prod_{n=1}^{\infty}\left(\frac{1+q^{n/2}}{1+q^{n/2-1/4}}\right)^2, \tag{4.1}$$

$$\mathfrak{b}(\tau) = 2\frac{\mathfrak{f}_1(\tau/2)^2}{\mathfrak{f}(\tau/2)^2} = 2\prod_{n=1}^{\infty}\left(\frac{1-q^{n/2-1/4}}{1+q^{n/2-1/4}}\right)^2. \tag{4.2}$$

Note that $\mathfrak{b}(\tau)$ occurs in [14, §10, (10.3)].

Proposition 4.1. *We have the identity*

$$\frac{2}{\mathfrak{p}(8\tau)} = \frac{1-v^2(\tau)}{v(\tau)} = \frac{1}{v(\tau)} - v(\tau). \tag{4.3}$$

Proof. (See [2, pp. 221-222].) The function $v(\tau)$ satisfies

$$\begin{aligned} v(\tau) &= q^{1/2}\prod_{n\geq 1}(1-q^n)^{\left(\frac{8}{n}\right)} = q^{1/2}\prod_{n\geq 1}\frac{(1-q^{8n-1})(1-q^{8n-7})}{(1-q^{8n-3})(1-q^{8n-5})} \\ &= q^{1/2}\frac{(q;q^8)_\infty(q^7;q^8)_\infty}{(q^3;q^8)_\infty(q^5;q^8)_\infty}. \end{aligned}$$

This gives that

$$\begin{aligned} \frac{1}{v(\tau)} - v(\tau) &= q^{-1/2}\frac{(q^3;q^8)_\infty(q^5;q^8)_\infty}{(q;q^8)_\infty(q^7;q^8)_\infty} - q^{1/2}\frac{(q;q^8)_\infty(q^7;q^8)_\infty}{(q^3;q^8)_\infty(q^5;q^8)_\infty} \\ &= \frac{(q^3;q^8)^2_\infty(q^5;q^8)^2_\infty - q\,(q;q^8)^2_\infty(q^7;q^8)^2_\infty}{q^{1/2}\,(q;q^8)_\infty(q^3;q^8)_\infty(q^5;q^8)_\infty(q^7;q^8)_\infty} \\ &= \frac{(q^3;q^8)^2_\infty(q^5;q^8)^2_\infty - q\,(q;q^8)^2_\infty(q^7;q^8)^2_\infty}{q^{1/2}\,(q;q^2)_\infty}. \end{aligned}$$

Multiplying the numerator and the denominator by $(q^8;q^8)^2_\infty$ and applying Jacobi's triple product identity in the form

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty,$$

with $(a,b) = (-q^3,-q^5)$ for the first term in the numerator and $(a,b) = (-q,-q^7)$ for the second, we obtain

$$\begin{aligned} \frac{1}{v(\tau)} - v(\tau) &= \frac{(q^3;q^8)^2_\infty(q^5;q^8)^2_\infty(q^8;q^8)^2_\infty - q\,(q;q^8)^2_\infty(q^7;q^8)^2_\infty(q^8;q^8)^2_\infty}{q^{1/2}\,(q;q^2)_\infty(q^8;q^8)^2_\infty} \\ &= \frac{f^2(-q^3,-q^5) - q\,f^2(-q,-q^7)}{q^{1/2}\,(q;q^2)_\infty(q^8;q^8)^2_\infty}. \end{aligned}$$

Now replace q by $-q$ in (2.14), (2.15) and apply this to the numerator to get

$$\begin{aligned} \frac{1}{v(\tau)} - v(\tau) &= \frac{\psi(-q)\big[\varphi(q)+\varphi(q^2)\big] - \psi(-q)\big[\varphi(q)-\varphi(q^2)\big]}{2\,q^{1/2}\,(q;q^2)_\infty(q^8;q^8)^2_\infty} \\ &= \frac{\psi(-q)\times\varphi(q^2)}{q^{1/2}\,(q;q^2)_\infty(q^8;q^8)^2_\infty}. \end{aligned}$$

This yields that

$$\frac{1}{v(\tau)} - v(\tau) = q^{-1/2}\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}\times\frac{(-q^2;q^4)^2_\infty(q^4;q^4)_\infty}{(q;q^2)_\infty(q^8;q^8)^2_\infty}$$

$$= q^{-1/2}\,\frac{(-q^2;q^4)_\infty^2(q^2;q^2)_\infty(q^4;q^4)_\infty}{(q^2;q^4)_\infty(q^8;q^8)_\infty^2}$$
$$= q^{-1/2}\,\frac{(-q^2;q^4)_\infty^2(q^4;q^4)_\infty^2}{(q^8;q^8)_\infty^2}$$
$$= q^{-1/2}\,(-q^2;q^4)_\infty^2(q^4;q^8)_\infty^2$$
$$= q^{-1/2}\,\frac{(-q^2;q^4)_\infty^2}{(-q^4;q^4)_\infty^2}.$$

Since

$$\mathfrak{p}(8\tau) = 2\,q^{1/2}\prod_{n\geq 1}\left(\frac{1+q^{4n}}{1+q^{4n-2}}\right)^2 = 2\,q^{1/2}\,\frac{(-q^4;q^4)_\infty^2}{(-q^2;q^4)_\infty^2},$$

we get the result by substituting into the last equality. □

Proposition 4.2. *The function $\mathfrak{p}(\tau)$ satisfies the identity*

$$\mathfrak{p}^2(\tau)\mathfrak{p}^2(2\tau) + \mathfrak{p}^2(\tau) - 2\mathfrak{p}(2\tau) = 0.$$

Proof. We use the relation between $x = \upsilon(\tau)$ and $y = \upsilon(2\tau)$ from Proposition 3.1(b): $x^2 = \frac{y(1-y)}{(1+y)}$. This gives

$$\left(\frac{2x}{1-x^2}\right)^2 = \frac{4x^2}{(1-x^2)^2} = \frac{4\cdot\frac{y(1-y)}{(1+y)}}{\left(1-\frac{y(1-y)}{(1+y)}\right)^2}$$
$$= \frac{4y(1-y)(1+y)}{\big((1+y)-y(1-y)\big)^2}$$
$$= \frac{4y(1-y^2)}{(1+y^2)^2}$$
$$= \frac{4y(1-y^2)}{4y^2+(1-y^2)^2}.$$

Now divide both the numerator and the denominator by $(1-y^2)^2$ to obtain

$$\left(\frac{2x}{1-x^2}\right)^2 = \frac{\frac{4y}{1-y^2}}{\frac{4y^2}{(1-y^2)^2}+1} = \frac{2\cdot\left(\frac{2y}{1-y^2}\right)}{\left(\frac{2y}{1-y^2}\right)^2+1}. \tag{4.4}$$

From Proposition 4.1, we know that

$$\mathfrak{p}(8\tau) = \frac{2\upsilon(\tau)}{1-\upsilon^2(\tau)} = \frac{2x}{1-x^2},$$

and

$$\mathfrak{p}(16\tau) = \frac{2\upsilon(2\tau)}{1-\upsilon^2(2\tau)} = \frac{2y}{1-y^2}.$$

Thus, (4.4) becomes

$$\mathfrak{p}^2(8\tau) = \frac{2\mathfrak{p}(16\tau)}{\mathfrak{p}^2(16\tau)+1}.$$

Replacing τ by $\tau/8$ and rearranging gives us the result. □

Proposition 4.3. *a) The functions $x = \mathfrak{b}(\tau)$ and $y = \mathfrak{b}(2\tau)$ satisfy the relation*

$$x^2y^2 + 4y^2 - 16x = 0.$$

b) The following identity holds between $x = \mathfrak{b}(\tau)$ and $z = \mathfrak{b}(4\tau)$:

$$(\mathfrak{b}(\tau)+2)^4\mathfrak{b}^4(4\tau) = 2^8(\mathfrak{b}^3(\tau)+4\mathfrak{b}(\tau)).$$

Proof. a) On putting 4τ for τ in x, we have

$$\mathfrak{b}(4\tau) = 2\prod_{n=1}^{\infty}\left(\frac{1-q^{2n-1}}{1+q^{2n-1}}\right)^2 = 2\,\frac{(q;q^2)_\infty^2}{(-q;q^2)_\infty^2} = 2\,\frac{\varphi(-q)}{\varphi(q)}.$$

From (2.11), we have

$$\varphi^2(-q) + \varphi^2(q) = 2\varphi^2(q^2).$$

Multiplying both sides by $\varphi^2(-q^2) = \varphi(q)\varphi(-q)$ from (2.16), we obtain

$$\varphi^2(-q)\varphi^2(-q^2) + \varphi^2(q)\varphi^2(-q^2) = 2\varphi(q)\varphi(-q)\varphi^2(q^2).$$

Now dividing both sides by $\varphi^2(q)\varphi^2(q^2)$ gives us

$$\frac{\varphi^2(-q)}{\varphi^2(q)}\cdot\frac{\varphi^2(-q^2)}{\varphi^2(q^2)} + \frac{\varphi^2(-q^2)}{\varphi^2(q^2)} = 2\,\frac{\varphi(-q)}{\varphi(q)}.$$

Hence, we see that $x = \mathfrak{b}(4\tau)$ and $y = \mathfrak{b}(8\tau)$ satisfy the relation

$$x^2y^2 + 4y^2 - 16x = 0.$$

Now replace τ by $\tau/4$.

b) From (2.17), upon taking fourth powers, we get

$$\left[\varphi(-q)+\varphi(q)\right]^4 = 16\,\varphi^4(q^4).$$

Multiplying both sides by $\varphi^4(-q^4)/\left[\varphi^4(q)\varphi^4(q^4)\right]$ gives us

$$\frac{\left[\varphi(-q)+\varphi(q)\right]^4}{\varphi^4(q)}\cdot\frac{\varphi^4(-q^4)}{\varphi^4(q^4)} = 16\,\frac{\varphi^4(-q^4)}{\varphi^4(q)}.$$

Then using (2.16) twice for the right side, we obtain

$$\frac{\left[\varphi(-q)+\varphi(q)\right]^4}{\varphi^4(q)}\cdot\frac{\varphi^4(-q^4)}{\varphi^4(q^4)} = 16\,\frac{\varphi(-q)\varphi(q)}{\varphi^4(q)}\cdot\varphi^2(q^2).$$

Now use (2.11) for the last factor on the right side to get

$$\frac{\left[\varphi(-q)+\varphi(q)\right]^4}{\varphi^4(q)}\cdot\frac{\varphi^4(-q^4)}{\varphi^4(q^4)} = 8\,\frac{\varphi(-q)}{\varphi^3(q)}\cdot\left[\varphi^2(-q)+\varphi^2(q)\right].$$

This implies that

$$\left[\frac{\varphi(-q)}{\varphi(q)}+1\right]^4\cdot\left[\frac{\varphi(-q^4)}{\varphi(q^4)}\right]^4=8\cdot\frac{\varphi(-q)}{\varphi(q)}\cdot\left[\left(\frac{\varphi(-q)}{\varphi(q)}\right)^2+1\right].$$

The result follows on multiplying through by 2^8 and substituting

$$\mathfrak{b}(4\tau)=2\frac{\varphi(-q)}{\varphi(q)}\quad\text{and}\quad\mathfrak{b}(16\tau)=2\frac{\varphi(-q^4)}{\varphi(q^4)}$$

into the above equation, and then replacing τ by $\tau/4$. □

5. The relation between $v(\tau)$ and $\mathfrak{b}(\tau)$.

We begin this section by proving the following identity.

Proposition 5.1.

$$\frac{(v^2(\tau)+1)^2}{v^4(\tau)-6v^2(\tau)+1}=\frac{4}{\mathfrak{b}^2(4\tau)}. \tag{5.1}$$

Proof. We prove (5.1) using the identity relating the Weber-Schläfli functions from [20, p. 86, (12)] (see also [8, p. 234, (12.18)]):

$$\mathfrak{f}_1^8(\tau)+\mathfrak{f}_2^8(\tau)=\mathfrak{f}^8(\tau).$$

From the definitions (4.1) and (4.2) of $\mathfrak{p}(\tau)$ and $\mathfrak{b}(\tau)$, this identity translates to

$$\frac{\mathfrak{b}^4(4\tau)}{16}=1-\mathfrak{p}^4(4\tau).$$

Using the result of Proposition 4.1, we write this equation as

$$\frac{\mathfrak{b}^4(4\tau)}{16}=1-\left(\frac{2\,v(\tau/2)}{1-v^2(\tau/2)}\right)^4=1-\frac{16\,v^4(\tau/2)}{\left(1-v^2(\tau/2)\right)^4}.$$

Setting $x=v(\tau/2)$ and $y=v(\tau)$ and using the relation between x and y from Proposition 3.1(b) in the form $x^2=\frac{y(1-y)}{(1+y)}$ gives that

$$\begin{aligned}\frac{\mathfrak{b}^4(4\tau)}{16}&=1-\frac{16\,x^4}{(1-x^2)^4}=1-\frac{16\left(\frac{y(1-y)}{(1+y)}\right)^2}{\left(1-\frac{y(1-y)}{(1+y)}\right)^4}\\&=1-\frac{16\,y^2(1-y^2)^2}{(1+y^2)^4}=\frac{(y^2+1)^4-16\,y^2(y^2-1)^2}{(y^2+1)^4}\\&=\frac{\left((y^2-1)^2+4y^2\right)^2-16\,y^2(y^2-1)^2}{(y^2+1)^4}\\&=\frac{\left((y^2-1)^2-4y^2\right)^2}{(y^2+1)^4}\end{aligned}$$

$$= \frac{(y^4 - 6y^2 + 1)^2}{(y^2 + 1)^4},$$

which is equivalent to (5.1). (The plus sign holds on taking the square-root because $\mathfrak{b}(i\infty) = 2, v^2(i\infty) = 0$.) □

Proposition 5.1 will now be used to prove the following formula for the function $j(\tau)$ in terms of $v(\tau)$.

Proposition 5.2. *If $v = v(\tau)$ and τ lies in the upper half-plane, we have*

$$j(\tau) = \frac{(v^{16} + 232v^{14} + 732v^{12} - 1192v^{10} + 710v^8 - 1192v^6 + 732v^4 + 232v^2 + 1)^3}{v^2(v^2 - 1)^2(v^2 + 1)^4(v^4 - 6v^2 + 1)^8}.$$

Proof. Let

$$G(x) = \frac{(x^2 - 16x + 16)^3}{x(x - 16)}.$$

Then from [14, p. 1967, (2.8)] the function

$$\alpha(\tau) = \zeta_8^{-1} \frac{\eta(\tau/4)^2}{\eta(\tau)^2}, \;\; \zeta_8 = e^{2\pi i/8}, \tag{5.2}$$

satisfies the relation

$$j(\tau) = \frac{(\alpha^8 - 16\alpha^4 + 16)^3}{\alpha^4(\alpha^4 - 16)} = G(\alpha^4(\tau)). \tag{5.3}$$

Moreover, $\alpha(\tau)$ and $\mathfrak{b}(\tau)$ satisfy

$$16\alpha^4(\tau) + 16\mathfrak{b}^4(\tau) = \alpha^4(\tau)\mathfrak{b}^4(\tau),$$

so that

$$\alpha^4(\tau) = \frac{16\mathfrak{b}^4(\tau)}{\mathfrak{b}^4(\tau) - 16}. \tag{5.4}$$

Setting $b = \mathfrak{b}(\tau)$, we substitute for $\alpha = \alpha(\tau)$ in (5.3) and find that

$$j(\tau) = G\left(\frac{16b^4}{b^4 - 16}\right) = \frac{(b^8 + 224b^4 + 256)^3}{b^4(b^4 - 16)^4}, \;\; b = \mathfrak{b}(\tau).$$

Now replace τ by 4τ and use (5.1) to replace $\mathfrak{b}^4(4\tau)$ by

$$\mathfrak{b}^4(4\tau) = \frac{16(v^4 - 6v^2 + 1)^2}{(v^2 + 1)^4},$$

giving

$$j(4\tau) = \frac{(v^{16} - 8v^{14} + 12v^{12} + 8v^{10} + 230v^8 + 8v^6 + 12v^4 - 8v^2 + 1)^3}{v^8(v^2 + 1)^4(v^2 - 1)^8(v^4 - 6v^2 + 1)^2}, \tag{5.5}$$

with $v = v(\tau)$. Replacing $v(\tau)$ by $\bar{A}(v(-1/4\tau))$ from Proposition 3.3 gives that

$$j(4\tau) = j_2(x^2),$$

where $x = \upsilon(-1/4\tau)$ and $j_2(x)$ is the rational function

$$j_2(x) = \frac{(x^8 + 232x^7 + 732x^6 - 1192x^5 + 710x^4 - 1192x^3 + 732x^2 + 232x + 1)^3}{x(x-1)^2(x+1)^4(x^2-6x+1)^8}. \tag{5.6}$$

Finally, replace τ by $\tau/4$ to give that

$$j(\tau) = j_2(\upsilon^2(-1/\tau)),$$

which implies that $j_2(\upsilon^2(\tau)) = j(-1/\tau) = j(\tau)$, completing the proof. □

We highlight the relation

$$j(\tau) = j_2(\upsilon^2(\tau)), \tag{5.7}$$

which we will make use of in Section 7. Using the linear fractional map $t(x)$ from (3.5) and the identity $\upsilon^2(-1/8\tau) = t(\upsilon^2(\tau))$ in (3.4) yields

$$j\left(\frac{-1}{8\tau}\right) = j_2\left(\upsilon^2\left(\frac{-1}{8\tau}\right)\right) = j_2(t(\upsilon^2(\tau))).$$

A calculation on Maple shows that

$$j_{22}(x) = j_2(t(x)) = \frac{(x^8 - 8x^7 + 12x^6 + 8x^5 - 10x^4 + 8x^3 + 12x^2 - 8x + 1)^3}{x^8(x-1)^4(x+1)^2(x^2-6x+1)}.$$

Therefore,

$$j\left(\frac{-1}{8\tau}\right) = j_{22}(\upsilon^2(\tau)). \tag{5.8}$$

We take this opportunity to prove the following known identity (see [9, p. 154]) from the results we have established so far.

Proposition 5.3.

$$\upsilon^{-2}(\tau) + \upsilon^2(\tau) - 6 = \frac{\eta^4(\tau)\eta^2(4\tau)}{\eta^2(2\tau)\eta^4(8\tau)}. \tag{5.9}$$

Proof. We will show that (5.9) follows from (5.1). We first have that

$$\begin{aligned}\upsilon^{-2}(\tau) + \upsilon^2(\tau) - 6 &= \frac{\upsilon^4(\tau) - 6\upsilon^2(\tau) + 1}{\upsilon^2(\tau)} \\ &= \frac{8}{\left(\frac{(\upsilon^2(\tau)+1)^2}{\upsilon^4(\tau)-6\upsilon^2(\tau)+1}\right) - 1} \\ &= \frac{8}{\left(\frac{4}{\mathfrak{b}^2(4\tau)}\right) - 1} = \frac{8\mathfrak{b}^2(4\tau)}{4 - \mathfrak{b}^2(4\tau)},\end{aligned}$$

by (5.1). Using the expression $\mathfrak{b}(4\tau) = 2\varphi(-q)/\varphi(q)$ from the proof of Proposition 4.3a) and (2.18) gives

$$v^{-2}(\tau) + v^2(\tau) - 6 = \frac{8\left(\frac{4\varphi^2(-q)}{\varphi^2(q)}\right)}{4 - \left(\frac{4\varphi^2(-q)}{\varphi^2(q)}\right)} = \frac{8\varphi^2(-q)}{\varphi^2(q) - \varphi^2(-q)} = \frac{8\varphi^2(-q)}{8q\psi^2(q^4)}.$$

Now putting $\varphi(-q) = (q;q^2)^2_\infty (q^2;q^2)_\infty = \frac{(q;q)^2_\infty}{(q^2;q^2)_\infty}$ and $\psi(q) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{(q^2;q^2)^2_\infty}{(q;q)_\infty}$ yields

$$\begin{aligned} v^{-2}(\tau) + v^2(\tau) - 6 &= \varphi^2(-q) \cdot \left(\frac{1}{q\psi^2(q^4)}\right) \\ &= (q;q^2)^4_\infty (q^2;q^2)^2_\infty \cdot \left(\frac{(q^4;q^8)^2_\infty}{q(q^8;q^8)^2_\infty}\right) \\ &= \left(\frac{(q;q)^4_\infty}{(q^2;q^2)^2_\infty}\right) \cdot \left(\frac{(q^4;q^4)^2_\infty}{q(q^8;q^8)^4_\infty}\right) \\ &= \frac{q^{1/6}(q;q)^4_\infty \cdot q^{1/3}(q^4;q^4)^2_\infty}{q^{1/6}(q^2;q^2)^2_\infty \cdot q^{4/3}(q^8;q^8)^4_\infty} \\ &= \frac{\eta^4(\tau)\eta^2(4\tau)}{\eta^2(2\tau)\eta^4(8\tau)}, \end{aligned}$$

using that $\eta(\tau) = q^{1/24}(q;q)_\infty$. □

6. The field generated by $v(w/8)$.

As in the Introduction, let $-d \equiv 1 \pmod 8$ and set $-d = \mathfrak{d}_K f^2$, where $\mathfrak{d}_K$ is the discriminant of the field $K = \mathbb{Q}(\sqrt{-d})$. Further, let $2 \cong \wp_2\wp_2'$ in the ring of integers R_K of K. We denote by $\Sigma_{\mathfrak{f}}$ the ray class field of conductor $\mathfrak{f}$ over K and Ω_f the ring class field of conductor f over K.

In this section we take $\tau = w/8$, where

$$w = \frac{a + \sqrt{-d}}{2}, \quad \text{with } a^2 + d \equiv 0 \pmod{2^5}, \ (N(w), f) = 1. \tag{6.1}$$

For this value of w,

$$\mathfrak{b}^4(8\tau) = \mathfrak{b}^4(w)$$

is the fourth power of the number

$$\beta = i^{-a}\mathfrak{b}(w) \tag{6.2}$$

from [14, (10.3), Thms. 10.6, 10.7]. We also need the number π from [14, (10.2),(10.9)], which is given by

$$\pi = i^{\bar{c}} \frac{\mathfrak{f}_2(w/2)^2}{\mathfrak{f}(w/2)^2} = i^{\bar{c}}\mathfrak{p}(w),$$

$$\bar{c} \equiv a\left(2 - \frac{a^2+d}{16}\right) \pmod 4.$$

(We have replaced v in the formulas of [14] by a and a by $\bar{c}$.) But here the integer $a^2 + d$ is divisible by 32, by (6.1), so $\bar{c}$ is even. Replacing $\bar{c}$ by the integer $c = \bar{c}/2$, satisfying

$$c \equiv 1 - \frac{a^2+d}{32} \pmod 2$$

yields

$$\pi = (-1)^c \mathfrak{p}(w), \quad w = \frac{a+\sqrt{-d}}{2}. \tag{6.3}$$

It follows from the results of [14] that $\xi = \beta/2$ and π lie in the ring class field Ω_f of the quadratic field $K = \mathbb{Q}(\sqrt{-d})$ (where $-d = \mathfrak{d}_K f^2$ and $\mathfrak{d}_K$ is the discriminant of $K/\mathbb{Q}$) and $\xi^4 + \pi^4 = 1$. Furthermore, $\mathbb{Q}(\pi) = \mathbb{Q}(\pi^4) = \Omega_f$. We also note that $(\xi) = \wp_2'$ and $(\pi) = \wp_2$ in Ω_f, so that $(\xi\pi) = (2)$.

From (4.3) and (6.3) we have that

$$(-1)^c \frac{2}{\pi} = \frac{1}{v(w/8)} - v(w/8) = \frac{1 - v^2(w/8)}{v(w/8)}. \tag{6.4}$$

In particular, $v(w/8)$ satisfies a quadratic equation over Ω_f and the map $\rho : v(w/8) \to \frac{-1}{v(w/8)}$ leaves the right side of (6.4) invariant. On squaring (6.4), we see that $X = v^2(w/8)$ satisfies the equation

$$X^2 - \left(2 + \frac{4}{\pi^2}\right)X + 1 = 0, \tag{6.5}$$

and therefore

$$v^2(w/8) = \frac{\pi^2 + 2 \pm 2\sqrt{\pi^2+1}}{\pi^2} = \left(\frac{1 \pm \sqrt{1+\pi^2}}{\pi}\right)^2.$$

Hence

$$v(w/8) = \pm\frac{1 \pm \sqrt{1+\pi^2}}{\pi}. \tag{6.6}$$

It follows from these expressions that

$$\Omega_f(v(w/8)) = \Omega_f(v^2(w/8)) = \Omega_f(\sqrt{1+\pi^2}).$$

We now prove the following.

Theorem 6.1. *If*

$$w = \frac{a+\sqrt{-d}}{2}, \quad \textit{with } a^2 + d \equiv 0 \ (\textit{mod } 2^5),$$

and $\wp_2 = (2, w)$ *in* R_K, *then the field* $\mathbb{Q}(v(w/8)) = \mathbb{Q}(\sqrt{1+\pi^2})$ *coincides with the class field* $\Sigma_{\wp_2'^3}\Omega_f$ *over* $K = \mathbb{Q}(\sqrt{-d})$. *The units* $v(w/8)$ *and* $v^2(w/8)$ *have degree* $4h(-d)$ *over* $\mathbb{Q}$.

Proof. Let $\Lambda = \mathbb{Q}(\sqrt{1+\pi^2})$. It is clear that Λ contains the ring class field Ω_f, since $\mathbb{Q}(\pi^4) = \Omega_f$. We use the fact that $1+\pi^2 \cong \wp_2'$ from [16, Lemma 5]. From this fact it is clear that $1+\pi^2$ is not a square in Ω_f, since $\wp_2'$ is unramified in Ω_f/K. Hence, $[\Lambda : \Omega_f] = 2$. Further, the prime divisors $\mathfrak{q}$ of $\wp_2'$ in Ω_f are certainly ramified in Λ. Equation (6.5) implies that $x = v^2(w/8)$ satisfies $(x-1)^2/(4x) = 1/\pi^2$, and therefore $\mathbb{Q}(v^2(w/8)) = \mathbb{Q}(\sqrt{1+\pi^2})$. This implies that $[\mathbb{Q}(v^2(w/8)) : \mathbb{Q}] = 4h(-d)$, since

$$[\Lambda : \mathbb{Q}] = [\Lambda : \Omega_f][\Omega_f : K][K : \mathbb{Q}] = 4h(-d).$$

Since $v^2(\tau)$ is a modular function for $\Gamma_1(8)$ ([9, p.154]), it follows from Schertz [19, Thm. 5.1.2] that $v^2(w/8) \in \Sigma_{8f}$, the ray class field of conductor $8f$ over K. More precisely, $v^2(w/8) \in L_{\mathcal{O},8}$, where $L_{\mathcal{O},8} = \Sigma_8\Omega_f$ is an *extended ring class field* corresponding to the order $\mathcal{O} = \mathsf{R}_{-d}$. See [8, p. 315]. Thus, $\Lambda \subset L_{\mathcal{O},8}$ is an abelian extension of K, whose conductor $\mathfrak{f}$ divides $8f$ in K. The discriminant of the polynomial $X^2 - (1+\pi^2)$ is of course $4(1+\pi^2) \cong \wp_2^2\wp_2'^3$. Since the ramification index of each $\mathfrak{q} \mid \wp_2'$ is $e_{\mathfrak{q}} = 2$ in Λ/Ω_f, Dedekind's discriminant theorem says that at least $\wp_2'^2$ divides the discriminant $\mathfrak{d} = \mathfrak{d}_{\Lambda/\Omega_f}$, and since the power of $\mathfrak{q}$ in $\mathfrak{d}$ is odd and at most 3 (Ω_f/K is unramified over 2), it follows that $\wp_2'^3$ exactly divides $\mathfrak{d}$. We claim now that $\wp_2$ is unramified in Λ.

From above $x = v^2(w/8)$ satisfies $(x-1)^2 - \frac{4}{\pi^2}x = 0$. Thus $x_1 = x - 1$ satisfies $h(x_1) = 0$, with

$$h(X) = X^2 - \frac{4}{\pi^2}(X+1), \quad \mathrm{disc}(h(X)) = \frac{16}{\pi^4} + 4\frac{4}{\pi^2},$$

where the ideal $(\frac{16}{\pi^4}) = \left(\frac{2}{\pi}\right)^4 = (\xi)^4 = \wp_2'^4$ is not divisible by $\wp_2$. This shows that $\mathrm{disc}(h(X))$ is not divisible by $\wp_2$ and therefore that $\wp_2$ is unramified in $\mathbb{Q}(v^2(w/8))$. Thus $\mathfrak{d} = \wp_2'^3$.

Now $[\Sigma_8 : \Sigma_1] = \frac{1}{2}\phi_K(\wp_2^3\wp_2'^3) = 8$, where ϕ_K is the Euler function for the quadratic field K, and $\mathbb{Q}(\zeta_8) \subset \Sigma_8$. Since the prime divisors of 2 do not ramify in Ω_f, we have that $\Omega_f \cap \Sigma_8 = \Sigma_1$ and therefore

$$[L_{\mathcal{O},8} : \Omega_f] = [\Sigma_8\Omega_f : \Omega_f] = [\Sigma_8 : \Sigma_1] = 8,$$

from which we obtain

$$\mathrm{Gal}(\Sigma_8\Omega_f/\Omega_f) \cong \mathrm{Gal}(\Sigma_8/\Sigma_1).$$

By this isomorphism the intermediate fields $L\Omega_f$ of $\Sigma_8\Omega_f/\Omega_f$ are in $1-1$ correspondence with the intermediate fields L of Σ_8/Σ_1.

The ray class field $\Sigma_{\wp_2^2\wp_2'^3}$ has degree 4 over the Hilbert class field Σ_1, and two of its quadratic subfields are $\Sigma_{\wp_2'^3}$ and $\Sigma_{\wp_2^2\wp_2'^2} = \Sigma_4 = \Sigma_1(i)$. It follows that $\mathrm{Gal}(\Sigma_{\wp_2^2\wp_2'^3}/\Sigma_1) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and the third quadratic subfield has conductor equal to $\mathfrak{f}' = \wp_2^2\wp_2'^3$ over K. The other quadratic intermediate fields of

Σ_8/Σ_1 are $\Sigma_1(\sqrt{2})$ and $\Sigma_1(\sqrt{-2})$, both of which have conductor $(8) = \wp_2^3\wp_2'^3$ over K, the field $\Sigma_{\wp_2^3}$, and a field whose conductor over K is $\wp_2'^2\wp_2^3$. Hence, $L = \Sigma_{\wp_2'^3}$ is the only quadratic intermediate field whose conductor is not divisible by $\wp_2$. This proves that $\mathbb{Q}(v^2(w/8)) = \Sigma_{\wp_2'^3}\Omega_f$ and (6.6) shows that $\mathbb{Q}(v(w/8)) = \mathbb{Q}(v^2(w/8)) = \Sigma_{\wp_2'^3}\Omega_f$. □

Corollary 6.2. *The field $\mathbb{Q}(v(w/8)) = \Sigma_{\wp_2'^3}\Omega_f$ is the inertia field for the prime ideal $\wp_2$ in the extension $L_{\mathcal{O},8}/K = \Sigma_8\Omega_f/K$.*

Proof. The above proof implies that $\mathrm{Gal}(\Sigma_8\Omega_f/\Omega_f) \cong \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$, since there are 7 quadratic intermediate fields. Any subfield containing Ω_f which properly contains $\Sigma_{\wp_2'^3}$ must also contain another quadratic subfield, in which $\wp_2$ must ramify. □

Corollary 6.3. *If $-d \equiv 1$ (mod 8) and w is given by (6.1), then the quantity*

$$A = \frac{\eta^2(w/8)\eta(w/2)}{\eta(w/4)\eta^2(w)}$$

generates the class field $\Sigma_{\wp_2'^3}\Omega_f$ for $K = \mathbb{Q}(\sqrt{-d})$ over $\mathbb{Q}$.

Proof. We appeal to equation (5.9). Setting $\eta = v(w/8)$, first use the equation preceding (6.6) to see that

$$\begin{aligned} \mathsf{A}^2 = \eta^{-2} + \eta^2 - 6 &= \frac{\pi^2 + 2 \mp 2\sqrt{1+\pi^2}}{\pi^2} + \frac{\pi^2 + 2 \pm 2\sqrt{1+\pi^2}}{\pi^2} - 6 \\ &= 4\frac{1-\pi^2}{\pi^2}. \end{aligned}$$

This gives that $\mathsf{A} = \pm\frac{2}{\pi}\sqrt{1-\pi^2}$. Since $\sqrt{1-\pi^2}\sqrt{1+\pi^2} = \sqrt{1-\pi^4} = \pm\xi^2 \in \Omega_f$ and $\mathbb{Q}(\mathsf{A}^2) = \Omega_f$, we get that $\mathbb{Q}(\mathsf{A}) = \mathbb{Q}(\sqrt{1+\pi^2}) = \Sigma_{\wp_2'^3}\Omega_f$, by the result of Theorem 6.1. □

The fact that $v^2(w/8) \in L_{\mathcal{O},8}$ in the above proof is derived in [8, p. 317] using Shimura's Reciprocity Law. We can give a more elementary proof of this fact by showing that $\sqrt{1+\pi^2} \in L_{\mathcal{O},8}$, as follows. We focus on the elliptic curve

$$E_1(\alpha):\ Y^2 + XY + \frac{1}{\alpha^4}Y = X^3 + \frac{1}{\alpha^4}X^2,$$

which is the Tate normal form for a point of order 4, with

$$\alpha^4 = \alpha(w)^4 = -\left(\frac{\eta(w/4)}{\eta(w)}\right)^8,$$

as in (5.2). From [14, (2.10), Prop. 3.2, p. 1970], the curve $E_1 = E_1(\alpha)$ has complex multiplication by the order $\mathcal{O} = \mathsf{R}_{-d}$ of discriminant $-d$ in K. Now, with $\beta = i^{-a}\mathfrak{b}(w)$ as in (6.2),

$$\frac{1}{\alpha^4} = \frac{\beta^4 - 16}{16\beta^4} = \frac{1}{16} - \frac{1}{\beta^4},$$

and Lynch [13] has given explicit expressions for the points of order 8 on E_1 in terms of β. Lynch [13, Prop. 3.3.1, p. 38] defines the following expressions:

$$\begin{aligned}
b_1 &= \frac{\beta\sqrt{2} + (\beta^2+4)^{1/2} + (\beta^2-4)^{1/2}}{2\beta\sqrt{2}},\\
b_2 &= \frac{\beta\sqrt{2} + (\beta^2+4)^{1/2} - (\beta^2-4)^{1/2}}{2\beta\sqrt{2}},\\
b_3 &= \frac{\beta\sqrt{2} - (\beta^2+4)^{1/2} + (\beta^2-4)^{1/2}}{2\beta\sqrt{2}},\\
b_4 &= \frac{\beta\sqrt{2} - (\beta^2+4)^{1/2} - (\beta^2-4)^{1/2}}{2\beta\sqrt{2}}.
\end{aligned}$$

With these expressions, Lynch shows [13, Thm. 3.3.1, p. 41] that the points

$$(X, Y) = P_1 = (b_1 b_3 b_4, -b_1 b_3^3 b_4) \text{ and } P_2 = (b_2 b_3 b_4, -b_2 b_3 b_4^3)$$

are points of order 8 on $E_1(\alpha)$. By [11, Satz 2] or [14, Prop. 6.4] the corresponding Weber functions satisfy

$$\frac{g_2 g_3}{\Delta}\left(X(P_i) + \frac{4b+1}{12}\right) \in \Sigma_8\Omega_f, \quad b = \frac{1}{\alpha^4}.$$

(See [14, (6.1)]. The expression inside the parentheses arises from putting the curve $E_1(\alpha)$ in standard Weierstrass form.) As in [14, p. 1976], $b, g_2, g_3, \Delta \in \Omega_f$, so that $X(P_i) = b_i b_3 b_4 \in L_{\mathcal{O},8}$ for $i = 1, 2$. This implies that

$$\begin{aligned}
(b_1 + b_2) b_3 b_4 &= \left(\frac{\sqrt{2}\beta + (\beta^2+4)^{1/2}}{\sqrt{2}\beta}\right)\left(\frac{\beta^2 + 4 - \sqrt{2}\beta(\beta^2+4)^{1/2}}{4\beta^2}\right)\\
&= \frac{4-\beta^2}{4\sqrt{2}\beta^3}(\beta^2+4)^{1/2}
\end{aligned}$$

lies in $L_{\mathcal{O},8}$. But we know that $4 - \beta^2 \neq 0$. In addition, $\sqrt{2} \in \mathbb{Q}(\zeta_8) \subset \Sigma_8$ and $\beta \in \Omega_f$, so that $(\beta^2+4)^{1/2} = 2\sqrt{\xi^2+1} \in L_{\mathcal{O},8}$, with $\xi = \beta/2$. Now π and ξ are conjugate over $\mathbb{Q}$, hence $\pm\sqrt{1+\pi^2}$ is conjugate to $\sqrt{1+\xi^2}$ over $\mathbb{Q}$. Since $\Sigma_8\Omega_f$ is normal over $\mathbb{Q}$, this implies that $\sqrt{1+\pi^2} \in L_{\mathcal{O},8}$, which proves the assertion.

Proposition 6.4. *Assume c in (6.3) is odd. The map $A(x) = \frac{\sigma x+1}{x-\sigma}$ (see (3.3)) fixes the set of conjugates of $\upsilon(w/8)$. If $f_d(x)$ is the minimal polynomial of $\upsilon(w/8)$ over*

$\mathbb{Q}$, *then*

$$(x-\sigma)^{4h(-d)} f_d(A(x)) = 2^{3h(-d)}\sigma^{2h(-d)} f_d(x).$$

Proof. Note that (6.4) implies that the minimal polynomial of $v(w/8)$ is

$$f_d(x) = 2^{-h(-d)}(x^2-1)^{2h(-d)} b_d\left((-1)^c \frac{2x}{1-x^2}\right), \tag{6.7}$$

where $b_d(x)$ is the minimal polynomial of π. Note that the degree of $b_d(x)$ is $2h(-d)$ and the constant term of $b_d(x)$ is

$$N_{\Omega_f/\mathbb{Q}}(\pi) = N_{\Omega_f/\mathbb{Q}}(\wp_2) = N_{K/\mathbb{Q}}(\wp_2^{h(-d)}) = 2^{h(-d)}$$

from [14]. Thus, $\deg(f_d(x)) = 4h(-d)$, which implies by Theorem 6.1 that $f_d(x)$ is irreducible.

We use (6.7) to prove the proposition, as follows. Setting $h = h(-d)$ and assuming c is odd, we have that

$$\begin{aligned}(x-\sigma)^{4h} f_d(A(x)) &= 2^{-h}(x-\sigma)^{4h}(A(x)^2-1)^{2h} b_d\left(\frac{2A(x)}{A(x)^2-1}\right)\\ &= 2^{-h}(x-\sigma)^{4h}\left(\frac{-2\sigma(x^2-2x-1)}{(x-\sigma)^2}\right)^{2h} b_d\left(-\frac{x^2+2x-1}{x^2-2x-1}\right)\\ &= 2^h\sigma^{2h}(x^2-2x-1)^{2h} b_d\left(\frac{P(x)+1}{P(x)-1}\right),\end{aligned}$$

where

$$P(x) = \frac{2x}{x^2-1} \quad\text{and}\quad \frac{P(x)+1}{P(x)-1} = -\frac{x^2+2x-1}{x^2-2x-1} = R(x).$$

We also know from [14] that the map $x \to \frac{x+1}{x-1}$ permutes the roots of $b_d(x)$ and

$$(x-1)^{2h} b_d\left(\frac{x+1}{x-1}\right) = 2^h b_d(x).$$

This gives that $b_d\left(\frac{P(x)+1}{P(x)-1}\right) = (P(x)-1)^{-2h} 2^h b_d(P(x))$ and therefore that

$$\begin{aligned}(x-\sigma)^{4h} f_d(A(x)) &= 2^h\sigma^{2h}(x^2-2x-1)^{2h}(P(x)-1)^{-2h}2^h b_d(P(x))\\ &= 2^{2h}\sigma^{2h}(x^2-2x-1)^{2h}\left(\frac{x^2-1}{x^2-2x-1}\right)^{2h} b_d(P(x))\\ &= 2^{3h}\sigma^{2h}2^{-h}(x^2-1)^{2h} b_d(P(x))\\ &= 2^{3h}\sigma^{2h} f_d(x).\end{aligned}$$

□

We also check that

$$x^{4h} f_d\left(\frac{-1}{x}\right) = 2^{-h} x^{4h} \left(\frac{1}{x^2} - 1\right)^{2h(-d)} b_d(P(-1/x))$$
$$= 2^{-h}(x^2-1)^{2h} b_d(P(x)) = f_d(x).$$

We conclude the following. Recall the definition of $\bar{A}(x)$ from (3.3).

Proposition 6.5. *If c is odd, the mappings in the group*

$$\tilde{H}_1 = \{x, A(x), \bar{A}(x), -1/x\}$$

permute the roots of $f_d(x)$.

Now let c be even, $\delta = 1 + \sqrt{2}$, and $B(x) = \dfrac{\delta x + 1}{x - \delta} = \dfrac{x + \sigma}{\sigma x - 1} = -\bar{A}(-x)$. Then we have

$$\begin{aligned}
(x-\delta)^{4h} f_d(B(x)) &= 2^{-h}(x-\delta)^{4h}(B^2(x)-1)^{2h} b_d\left(\frac{2B(x)}{1-B^2(x)}\right)\\
&= 2^{-h}(x-\delta)^{4h}\left(\frac{2\delta(x^2+2x-1)}{(x-\delta)^2}\right)^{2h} b_d\left(-\frac{x^2-2x-1}{x^2+2x-1}\right)\\
&= 2^h\delta^{2h}(x^2+2x-1)^{2h}\ b_d\left(\frac{\frac{2x}{1-x^2}+1}{\frac{2x}{1-x^2}-1}\right)\\
&= 2^h\delta^{2h}(x^2+2x-1)^{2h}\cdot 2^h\left(\frac{2x}{1-x^2}-1\right)^{-2h} b_d\left(\frac{2x}{1-x^2}\right)\\
&= 2^{2h}\delta^{2h}(x^2+2x-1)^{2h}\cdot\left(\frac{1-x^2}{x^2+2x-1}\right)^{2h} b_d\left(\frac{2x}{1-x^2}\right)\\
&= 2^{2h}\delta^{2h}\cdot(x^2-1)^{2h} b_d\left(\frac{2x}{1-x^2}\right)\\
&= 2^{2h}\delta^{2h}\cdot 2^h f_d(x)\\
&= 2^{3h}\delta^{2h} f_d(x).
\end{aligned}$$

Setting $\bar{B}(x) = B(-1/x) = \dfrac{-\sigma x + 1}{x + \sigma} = -A(-x)$, we have the following.

Proposition 6.6. *If c is even, the mappings in the group*

$$\tilde{H}_0 = \{x, B(x), \bar{B}(x), -1/x\}$$

permute the roots of $f_d(x)$.

7. The diophantine equation.

From (3.4) we know that $(X, Y) = (\upsilon(w/8), \upsilon(-1/w))$ is a solution of the diophantine equation

$$\mathcal{C}_2 : X^2 + Y^2 = \sigma^2(1 + X^2Y^2), \quad \sigma = -1 + \sqrt{2}.$$

This seems to be an analogue of the equation $\mathcal{C}_5$ in [17]. Set

$$F_2(X,Y) = X^2 + Y^2 - \sigma^2(1 + X^2Y^2).$$

Then

$$(\sigma Y + 1)^2 F_2(X, \bar{A}(Y)) = 4\sqrt{2}\sigma^2(X^2Y + X^2 + Y^2 - Y) = 4\sqrt{2}\sigma^2 f(X,Y).$$

Since

$$\bar{A}(x) = \frac{-x+\sigma}{\sigma x + 1} = \frac{-\delta x + 1}{x + \delta}, \quad \delta = \frac{1}{\sigma} = 1 + \sqrt{2},$$

the linear fractional map $\bar{A}(x)$ is the analogue of the map $T(x)$ in [17, p. 1199]. Considering Thm. 5.1 in [17, p. 1205] suggests the following conjecture.

Conjecture 7.1. *Assume c is odd. If* $\tau_2 = \left(\frac{\Sigma_{\wp_2'^3}\Omega_f/K}{\wp_2}\right)$*, then*

$$-v(-1/w) = \bar{A}(v(w/8)^{\tau_2}) = \frac{-v(w/8)^{\tau_2} + \sigma}{\sigma v(w/8)^{\tau_2} + 1},$$

where w is given by (6.1)*.*

To prove this conjecture, we first appeal to Proposition 4.2, which implies that

$$\mathfrak{p}(2\tau) = \frac{1 \pm \sqrt{1 - \mathfrak{p}^4(\tau)}}{\mathfrak{p}^2(\tau)}.$$

Setting $\tau = w$, (6.3) gives that

$$\mathfrak{p}(2w) = \frac{1 \pm \sqrt{1-\pi^4}}{\pi^2} = \frac{1 \pm \xi^2}{\pi^2}.$$

Note that

$$\frac{1+\xi^2}{\pi^2}\frac{1-\xi^2}{\pi^2} = \frac{1-\xi^4}{\pi^4} = 1$$

and $\dfrac{1-\xi^2}{\pi^2} = -\pi^{\tau_2}$ from [16, p. 333]. Thus, $\dfrac{1+\xi^2}{\pi^2} = -\pi^{-\tau_2}$.

Theorem 7.2. *If w is given by* (6.1) *we have*

$$\mathfrak{p}(2w) = \frac{1+\xi^2}{\pi^2} = \frac{-1}{\pi^{\tau_2}}.$$

Proof. We use an argument from [14, Section 10]. With the number $\beta = i^{-a}\mathfrak{b}(w)$ from (6.2) we have [14, eq. (8.0), p. 1980]

$$j(w) = \frac{(\beta^8 + 224\beta^4 + 256)^3}{\beta^4(\beta^4 - 16)^4}.$$

(See the proof of Proposition 5.2.) Furthermore, the roots of the equation

$$0 = (X-16)^3 - j(w)X = (X-16)^3 - \frac{(\beta^8 + 224\beta^4 + 256)^3}{\beta^4(\beta^4-16)^4}X$$

are, on the one hand, given by the values

$$X = \mathfrak{f}^{24}(w), \ \ -\mathfrak{f}_1^{24}(w), \ \ -\mathfrak{f}_2^{24}(w);$$

(see [8, p. 233, Th. 12.17]) and on the other, are equal to the expressions

$$X = -\frac{(\beta^2-4)^4}{\beta^2(\beta^2+4)^2}, \ \ \frac{(\beta^2+4)^4}{\beta^2(\beta^2-4)^2}, \ \ -\frac{2^{12}\beta^4}{(\beta^4-16)^2}.$$

See [14, p. 2000]. From [14, p. 2000] we also have (since our value w satisfies the conditions for w in [14, Prop. 3.1])

$$\mathfrak{f}_2^{24}(w) = -\frac{(\beta^2+4)^4}{\beta^2(\beta^2-4)^2}, \tag{7.1}$$

since $\mathfrak{f}_2^{24}(w)$ must be a unit (from the results of [21]). There are two cases to consider.

Case 1. First assume that

$$\mathfrak{f}^{24}(w) = -\frac{(\beta^2-4)^4}{\beta^2(\beta^2+4)^2}, \tag{7.2}$$
$$\mathfrak{f}_1^{24}(w) = \frac{2^{12}\beta^4}{(\beta^4-16)^2}.$$

In this case, (7.1) and (7.2) give the following formula:

$$\mathfrak{p}^{12}(2w) = \frac{\mathfrak{f}_2(w)^{24}}{\mathfrak{f}(w)^{24}} = \frac{(\beta^2+4)^6}{(\beta-2)^6(\beta+2)^6}.$$

Now we use the following ideal factorizations in the ring class field Ω_f:

$$(\beta^2+4) = \wp_2^3\wp_2'^2, \ \ (\beta-2) = \wp_2^2\wp_2', \ \ (\beta+2) = \wp_2^3\wp_2'. \tag{7.3}$$

See [16, Lemma 4]. These factorizations imply that

$$\mathfrak{p}^{12}(2w) \cong \left(\frac{\wp_2^3\wp_2'^2}{\wp_2^5\wp_2'^2}\right)^6 = \frac{1}{\wp_2^{12}} \ \text{ in } \Omega_f,$$

which implies that

$$\mathfrak{p}(2w) \cong \frac{1}{\wp_2}. \tag{7.4}$$

By the remarks preceding the statement of the theorem, this shows that $\mathfrak{p}(2w)$ is not an algebraic integer, giving that $\mathfrak{p}(2w) = \dfrac{1+\xi^2}{\pi^2} = -\pi^{-\tau_2}$.

Case 2. The alternative to (7.2) is

$$\mathfrak{f}^{24}(w) = -\frac{2^{12}\beta^4}{(\beta^4-16)^2}, \tag{7.5}$$
$$\mathfrak{f}_1^{24}(w) = \frac{(\beta^2-4)^4}{\beta^2(\beta^2+4)^2}.$$

In this case we have

$$\mathfrak{p}^{12}(2w) = \frac{\mathfrak{f}_2(w)^{24}}{\mathfrak{f}(w)^{24}} = \left(\frac{\beta^2+4}{2^2\beta}\right)^6 \cong \left(\frac{\wp_2^3\wp_2'^2}{\wp_2^2\wp_2'^2\wp_2\wp_2'^2}\right)^6 = \frac{1}{\wp_2'^{12}},$$

giving that $\mathfrak{p}(2w) \cong \frac{1}{\wp_2'}$. However, this is impossible, since the above remarks show that the only prime divisors occuring in the factorization of $\mathfrak{p}(2w)$ are prime divisors of $\wp_2$. This shows that Case 2 is impossible, and Case 1 proves the formula of the theorem. □

Now we set

$$\eta = v(w/8), \ \ \lambda = -v(-1/w), \ \ \nu = v(w/4). \tag{7.6}$$

We first show λ is a root of the minimal polynomial $f_d(x)$ of $v(w/8)$ (c odd). We have from Proposition 3.3 that

$$\frac{2\lambda}{\lambda^2-1} = \frac{-2\bar{A}(\nu)}{\bar{A}^2(\nu)-1} = \frac{\nu^2+2\nu-1}{\nu^2-2\nu-1}.$$

Proposition 4.1 and Theorem 7.2 give further that

$$\frac{2\lambda}{\lambda^2-1} = \frac{\nu-\frac{1}{\nu}+2}{\nu-\frac{1}{\nu}-2} = \frac{\frac{-2}{\mathfrak{p}(2w)}+2}{\frac{-2}{\mathfrak{p}(2w)}-2} = \frac{\pi^{\tau_2}+1}{\pi^{\tau_2}-1}. \tag{7.7}$$

Since $\frac{\pi^{\tau_2}+1}{\pi^{\tau_2}-1}$ is a root of $b_d(x)$, we have from (6.7) that

$$f_d(\lambda) = 2^{-h(-d)}(\lambda^2-1)^{2h(-d)}b_d\left(\frac{2\lambda}{\lambda^2-1}\right) = 0.$$

Hence, $\lambda = -v(-1/w)$ is a conjugate of $v(w/8)$.

Theorem 7.3. *If c is odd, we have the formula*

$$\lambda = -v(-1/w) = \bar{A}(v(w/8)^{\tau_2}) = \frac{-v(w/8)^{\tau_2}+\sigma}{\sigma v(w/8)^{\tau_2}+1}, \ \ \sigma = -1+\sqrt{2},$$

where w is given by (6.1)*.*

Proof. We will prove that $\bar{A}(\lambda) = v(w/8)^{\tau_2} = \eta^{\tau_2}$ by showing that

$$\bar{A}(\lambda) - \eta^2 \equiv 0 \pmod{\wp_2}.$$

We have $\eta^2+\lambda^2 = \sigma^2(1+\eta^2\lambda^2)$, which implies that

$$\begin{aligned}\bar{A}(\lambda)-\eta^2 &= \frac{-\lambda+\sigma}{\sigma\lambda+1} - \frac{-\lambda^2+\sigma^2}{1-\sigma^2\lambda^2} = \frac{-\lambda+\sigma}{\sigma\lambda+1} + \frac{\sigma^2-\lambda^2}{\sigma^2\lambda^2-1} \\ &= \frac{(-\lambda+\sigma)(\sigma\lambda-1)+\sigma^2-\lambda^2}{\sigma^2\lambda^2-1} \\ &= \frac{-(\sigma+1)\lambda^2+(\sigma^2+1)\lambda+\sigma^2-\sigma}{\sigma^2\lambda^2-1}\end{aligned}$$

$$= \frac{-\sqrt{2}\lambda^2 + (4 - 2\sqrt{2})\lambda + 4 - 3\sqrt{2}}{(\sigma\lambda + 1)(\sigma\lambda - 1)}$$
$$= \frac{-\sqrt{2}(\lambda - \sigma)^2}{\sigma^2(\lambda - \bar{\sigma})(\lambda + \bar{\sigma})}.$$

We multiply the last expression by

$$A(\lambda) - \frac{1}{\eta^2} = \frac{(-4 + 3\sqrt{2})(\lambda - \bar{\sigma})^2}{\lambda^2 - \sigma^2} = \frac{\sqrt{2}\sigma^2(\lambda - \bar{\sigma})^2}{\lambda^2 - \sigma^2},$$

which is obtained from the last calculation by fixing λ and mapping $\sqrt{2}$ to $-\sqrt{2}$. This yields the formula

$$\left(\bar{A}(\lambda) - \eta^2\right)\left(A(\lambda) - \frac{1}{\eta^2}\right) = \frac{-2(\lambda - \sigma)(\lambda - \bar{\sigma})}{(\lambda + \sigma)(\lambda + \bar{\sigma})} = -2\frac{\lambda^2 + 2\lambda - 1}{\lambda^2 - 2\lambda - 1}. \tag{7.8}$$

Now

$$\frac{\lambda^2 + 2\lambda - 1}{\lambda^2 - 2\lambda - 1} = \frac{1 + \frac{2\lambda}{\lambda^2 - 1}}{1 - \frac{2\lambda}{\lambda^2 - 1}}, \tag{7.9}$$

where

$$\frac{2\lambda}{\lambda^2 - 1} = \frac{\pi^{\tau_2} + 1}{\pi^{\tau_2} - 1}$$

from (7.7). It follows from (7.9) that

$$\frac{\lambda^2 + 2\lambda - 1}{\lambda^2 - 2\lambda - 1} = \frac{1 + \frac{\pi^{\tau_2} + 1}{\pi^{\tau_2} - 1}}{1 - \frac{\pi^{\tau_2} + 1}{\pi^{\tau_2} - 1}} = -\pi^{\tau_2}.$$

Thus, (7.8) becomes

$$\left(\bar{A}(\lambda) - \eta^2\right)\left(A(\lambda) - \frac{1}{\eta^2}\right) = 2\pi^{\tau_2}$$

and therefore $(\pi^{\tau_2}) = (\pi) = \wp_2$ yields that

$$\left(\bar{A}(\lambda) - \eta^2\right)\left(A(\lambda) - \frac{1}{\eta^2}\right) \equiv 0 \pmod{\wp_2^2}.$$

It follows that

$$\bar{A}(\lambda) \equiv \eta^2 \text{ or } A(\lambda) \equiv \frac{1}{\eta^2} \pmod{\mathfrak{q}}, \tag{7.10}$$

for each prime divisor $\mathfrak{q}$ of $\wp_2$ in $F_1 = \mathbb{Q}(\eta)$. But $A(\lambda) = -1/\bar{A}(\lambda)$ and η are units, so the second congruence in (7.10) implies the first. This proves that

$$\bar{A}(\lambda) \equiv \eta^2 \pmod{\wp_2} \tag{7.11}$$

in F_1. Note that $\bar{A}(\lambda)$ and $\lambda = -v(-1/w)$ are roots of $f_d(x)$ (Proposition 6.5), so $F_2 = \mathbb{Q}(\lambda)$ is isomorphic to $F_1 = \mathbb{Q}(\eta) = \mathbb{Q}(v(w/8))$. However, by (3.4),

$$\lambda^2 = v^2(-1/w) = \frac{-v(w/8)^2 + \sigma^2}{1 - \sigma^2 v(w/8)^2}$$

does not lie in F_1, since $\sqrt{2} \notin F_1$ (otherwise $\wp_2$ would be ramified in F_1; note that $v(w/8)$ is not a fourth root of unity, so the determinant of the linear fractional transformation in σ^2 is nonzero). It follows that from Theorem 6.1 that

$$F_2 = \mathbb{Q}(\lambda) = \Sigma_{\wp_2^3}\Omega_f.$$

The same argument now shows that $\bar{A}(\lambda) = \frac{-\lambda+\sigma}{\sigma\lambda+1} \notin F_2$, so $\bar{A}(\lambda) \in F_1$. Therefore, $\psi : \eta \to \bar{A}(\lambda)$ is an automorphism of F_1, and since $\wp_2$ is not ramified in F_1 but $\wp_2'$ is, it follows that ψ fixes $\wp_2$, implying that it fixes the field K.

Recalling the rational function $j_2(x)$ from (5.6), a computation on Maple shows that

$$j_2\left(\left(\frac{1-\nu}{1+\nu}\right)^2\right) = j_2(\nu^2) = j_2(v^2(w/4)) = j(w/4),$$

by (5.7). Now Proposition 3.3 and the fact that $\bar{A}(x)$ has order 2 imply that $v(w/4) = \bar{A}(v(-1/w))$ and

$$\begin{aligned}\frac{1-v(w/4)}{1+v(w/4)} &= \frac{1-\bar{A}(v(-1/w))}{1+\bar{A}(v(-1/w))}\\ &= \frac{v(-1/w)+\sigma}{-\sigma v(-1/w)+1}\\ &= \bar{A}(-v(-1/w)) = \bar{A}(\lambda).\end{aligned} \tag{7.12}$$

This implies that

$$j_2(\bar{A}(\lambda)^2) = j_2\left(\left(\frac{1-\nu}{1+\nu}\right)^2\right) = j(w/4).$$

On the other hand, equation (5.7) gives

$$j(w/8)^{\psi} = j_2(\eta^{2\psi}) = j_2(\bar{A}(\lambda)^2) = j(w/4) = j(w/8)^{\tau_2}.$$

Hence $\psi|_{\Omega_f} = \tau_2|_{\Omega_f}$. It follows that $\psi = \tau_2$ or $\psi = \rho\tau_2$, where $\rho : \eta \to -1/\eta$ is the nontrivial automorphism of F_1/Ω_f. If $\psi = \rho\tau_2$, then by (7.11)

$$\eta^{\psi} = \bar{A}(\lambda) \equiv \eta^2 \pmod{\wp_2}$$

and $\eta^{\tau_2} \equiv \eta^2 \pmod{\wp_2}$ imply that

$$\eta^2 \equiv \eta^{\rho\tau_2} = \frac{-1}{\eta^{\tau_2}} \equiv \frac{1}{\eta^2} \pmod{\wp_2}.$$

It follows from this congruence that $\eta^4 + 1 \equiv (\eta+1)^4 \equiv 0 \bmod \wp_2$ and hence $\eta \equiv 1 \pmod{\wp_2}$, since $\wp_2$ is unramified in F_1/K. This implies in turn that $z = \eta - \eta^{-1} \equiv 0 \pmod{\wp_2}$. But this contradicts (4.3) (with $\tau = w/8$) and (6.3), according to which $z = 2/\pi$ is relatively prime to $\wp_2$. Hence, $\psi = \tau_2$ must be the Artin symbol for $\wp_2$ in F_1/K. This completes the proof. □

Corollary 7.4. *Assume c is odd. If* $\tau_2 = \left(\frac{\Sigma_{\wp_2'^3}\Omega_f/K}{\wp_2}\right)$, *then*

$$v(w/8)^{\tau_2} = \frac{1 - v(w/4)}{1 + v(w/4)}$$

and

$$f(v(w/8), v(w/8)^{\tau_2}) = 0.$$

Proof. The first formula is immediate from $\eta^{\psi} = \eta^{\tau_2} = \bar{A}(\lambda)$ and (7.12). The second follows from Proposition 3.1 and

$$f(v(w/8), v(w/4)) = 0 = f\left(v(w/8), \frac{1 - v(w/4)}{1 + v(w/4)}\right),$$

since

$$f\left(x, \frac{1-y}{1+y}\right) = \frac{2f(x,y)}{(1+y)^2}.$$

□

Theorem 7.5. *If c is even, then*

$$v(w/8)^{\tau_2} = \frac{v(w/4) - 1}{v(w/4) + 1}$$

and

$$v(-1/w) = B(v(w/8)^{\tau_2}) = \frac{v(w/8)^{\tau_2} + \sigma}{\sigma v(w/8)^{\tau_2} - 1}.$$

Proof. From Proposition 3.3, we have that

$$v(-1/w) = \bar{A}(v(w/4)) = -B(-v(w/4)),$$

where

$$B(x) = \frac{x + \sigma}{\sigma x - 1} = -\frac{-(-x) + \sigma}{\sigma(-x) + 1} = -\bar{A}(-x).$$

Hence, according to (7.12), we obtain

$$v(w/8)^{\tau_2} = \frac{v(w/4) - 1}{v(w/4) + 1} = B(v(-1/w)) \iff v(-1/w) = B(v(w/8)^{\tau_2}),$$

showing that both the statements in the theorem are equivalent. We now show that Proposition 6.6 implies that $v(w/8)$ and $v(-1/w)$ are conjugate algebraic integers.

In similar fashion to (7.6), we set

$$\eta = v(w/8), \ \ \tilde{\lambda} = v(-1/w) = -\lambda, \ \ \nu = v(w/4).$$

Then, according to (7.7), we get

$$\frac{2\tilde{\lambda}}{1 - \tilde{\lambda}^2} = -\frac{2 - (\frac{1}{\nu} - \nu)}{2 + (\frac{1}{\nu} - \nu)} = -\frac{2 - \frac{2}{\mathfrak{p}(2w)}}{2 + \frac{2}{\mathfrak{p}(2w)}} = -\frac{1 + \pi^{\tau_2}}{1 - \pi^{\tau_2}} = \frac{\pi^{\tau_2} + 1}{\pi^{\tau_2} - 1}.$$

Since $\frac{\pi^{\tau_2}+1}{\pi^{\tau_2}-1}$ is a root of $b_d(x)$, we have that

$$f_d(\tilde{\lambda}) = 2^{-h}(\tilde{\lambda}^2-1)^{2h} b_d\left(\frac{2\tilde{\lambda}}{1-\tilde{\lambda}^2}\right) = 0,$$

showing that $\tilde{\lambda} = v(-1/w)$ is a conjugate of $\eta = v(w/8)$.

Now,

$$\begin{aligned}
B(\tilde{\lambda}) - \eta^2 &= \frac{\tilde{\lambda}+\sigma}{\sigma\tilde{\lambda}-1} - \frac{\sigma^2-\tilde{\lambda}^2}{1-\sigma^2\tilde{\lambda}^2} = \frac{\lambda-\sigma}{\sigma\lambda+1} - \frac{\sigma^2-\lambda^2}{1-\sigma^2\lambda^2} \\
&= \frac{(\lambda-\sigma)(\sigma\lambda-1)+(\sigma^2-\lambda^2)}{\sigma^2\lambda^2-1} \\
&= \frac{(\sigma-1)\lambda^2-(\sigma^2+1)\lambda+(\sigma^2+\sigma)}{(\sigma\lambda+1)(\sigma\lambda-1)} \\
&= \frac{-\sqrt{2}\,\sigma(\lambda^2+2\lambda-1)}{\sigma^2(\lambda-\bar{\sigma})(\lambda+\bar{\sigma})} \\
&= \frac{-\sqrt{2}\,\sigma(\lambda-\sigma)(\lambda-\bar{\sigma})}{\sigma^2(\lambda-\bar{\sigma})(\lambda+\bar{\sigma})} \\
&= \frac{\sqrt{2}\,\bar{\sigma}(\tilde{\lambda}+\sigma)}{(\tilde{\lambda}-\bar{\sigma})}.
\end{aligned}$$

In the above calculation, mapping $\sqrt{2}$ to $-\sqrt{2}$, while fixing $\tilde{\lambda}$, gives us

$$\bar{B}(\tilde{\lambda}) - \frac{1}{\eta^2} = -\frac{\sqrt{2}\,\sigma(\tilde{\lambda}+\bar{\sigma})}{(\tilde{\lambda}-\sigma)}.$$

Multiplying the above two expressions gives us

$$\begin{aligned}
\left(B(\tilde{\lambda})-\eta^2\right)\left(\bar{B}(\tilde{\lambda})-\frac{1}{\eta^2}\right) &= 2\,\frac{(\tilde{\lambda}+\sigma)(\tilde{\lambda}+\bar{\sigma})}{(\tilde{\lambda}-\sigma)(\tilde{\lambda}-\bar{\sigma})} = 2\frac{\tilde{\lambda}^2-2\tilde{\lambda}-1}{\tilde{\lambda}^2+2\tilde{\lambda}-1} \\
&= 2\frac{1+\left(\frac{2\tilde{\lambda}}{1-\tilde{\lambda}^2}\right)}{1-\left(\frac{2\tilde{\lambda}}{1-\tilde{\lambda}^2}\right)} = 2\frac{1+\left(\frac{\pi^{\tau_2}+1}{\pi^{\tau_2}-1}\right)}{1-\left(\frac{\pi^{\tau_2}+1}{\pi^{\tau_2}-1}\right)} = -2\pi^{\tau_2}.
\end{aligned}$$

Now a similar argument to the end of the proof of Theorem 7.3 applies here and shows that the automorphism ψ on F_1 taking η to $\tilde{\lambda}$ is $\eta^\psi = B(\tilde{\lambda})$. As before, ψ coincides with τ_2, giving that $\tilde{\lambda} = v(-1/w) = B(\eta^{\tau_2}) = B(v(w/8)^{\tau_2})$. Also see the argument below. □

Corollary 7.6. *If c is even, the point $(x, y) = (-\eta, -\eta^{\tau_2})$ lies on the curve $f(x, y) = 0$:*

$$f(-v(w/8), -v(w/8)^{\tau_2}) = 0, \quad \tau_2 = \left(\frac{\Sigma_{\wp_2'^3}\Omega_f/K}{\wp_2}\right).$$

Proof. We have

$$0 = f(v(w/8), v(w/4)) = f\left(v(w/8), -\frac{v(w/4)-1}{v(w/4)+1}\right)$$
$$= f(v(w/8), -v(w/8)^{\tau_2}) = f(-v(w/8), -v(w/8)^{\tau_2}).$$

□

Combining the arguments in the proofs of Theorems 7.3 and 7.5 for c odd and c even yields the following corollary.

Corollary 7.7. *The field* $F_2 = \mathbb{Q}(v(-1/w)) = \Sigma_{\wp_2^3}\Omega_f$ *is the inertia field for the prime ideal* $\wp_2'$ *in the extension* $L_{\mathcal{O},8}/K$.

We also give an alternate argument to show $\psi = \tau_2$ in the proofs of Theorems 7.3 and 7.5. We first note that the modular function $j(\tau)$ can be expressed in terms of $z = v(\tau) - \frac{1}{v(\tau)}$, namely

$$j(\tau) = J(z) = \frac{(z^8 + 240z^6 + 2144z^4 + 3840z^2 + 256)^3}{z^2(z^2+4)^2(z-2)^8(z+2)^8},$$

using Proposition 5.2. Now set $z = \eta - \frac{1}{\eta} = \pm\frac{2}{\pi}$, so that $(z, \wp_2) = 1$. This allows us to reduce the above formula modulo $\wp_2$, giving that

$$j(w/8) \equiv \frac{z^{24}}{z^{22}} \equiv z^2 \pmod{\wp_2}.$$

This shows that $j(w/8)^\tau$ is conjugate to z^τ modulo each prime divisor $\mathfrak{p}$ of $\wp_2$ in Ω_f, for each automorphism $\tau \in \mathrm{Gal}(\Omega_f/K)$; and this implies that the class equation $H_{-d}(X)$ and the minimal polynomial $\mu_d(X)$ of z over K are congruent:

$$H_{-d}(X) \equiv \mu_d(X) \pmod{\wp_2}.$$

A theorem of Deuring says that the discriminant of $H_{-d}(X)$ is odd (since $\left(\frac{-d}{2}\right) = +1$), so the discriminant of $\mu_d(X)$ is not divisible by $\wp_2$. This implies that the discriminant of the minimal polynomial $\tilde{\mu}_d(X) = X^{h(-d)}\mu_d\left(X - \frac{1}{X}\right)$ of η over K is relatively prime to $\wp_2$, as well. This is because

$$\mu_d(X) = \prod_{i=1}^{h(-d)} (X - (\eta_i - \frac{1}{\eta_i}))$$

is a product over the conjugates $z_i = \eta_i - \frac{1}{\eta_i}$ of z, so that

$$X^{h(-d)}\mu_d\left(X - \frac{1}{X}\right) = \prod_{i=1}^{h(-d)} (X^2 - (\eta_i - \frac{1}{\eta_i})X - 1),$$
$$= \prod_{i=1}^{h(-d)} (X^2 - z_iX - 1), \quad z_i = \eta_i - \frac{1}{\eta_i}.$$

Hence,

$$\begin{aligned}\operatorname{disc}(\tilde{\mu}_d(X)) &= \prod_{i=1}^{h(-d)} (z_i^2+4) \prod_{i<j} \operatorname{Res}(X^2 - z_iX - 1, X^2 - z_jX - 1)^2 \\ &= \prod_{i=1}^{h(-d)} (z_i^2+4) \prod_{i<j} (z_i - z_j)^4 \\ &= \prod_{i=1}^{h(-d)} (z_i^2+4)\, (\operatorname{disc}(\mu_d(X)))^2.\end{aligned}$$

Now the z_i are conjugate over K, so each z_i is relatively prime to $\wp_2$, which implies that $(z_i^2+4, \wp_2) = 1$ for each i. This proves the claim that $(\operatorname{disc}(\tilde{\mu}_d(X)), \wp_2) = 1$. This proves

Theorem 7.8. *Let $R_{\wp_2}$ denote the ring of elements of K which are integral for $\wp_2$. Then the powers of $\eta = \upsilon(w/8)$ form a basis over $R_{\wp_2}$ for the ring $\overline{R}$ of elements of $F_1 = \mathbb{Q}(\eta)$ which are integral for $\wp_2$.*

Given this theorem, the congruence

$$\eta^{\psi} \equiv \eta^2 \pmod{\wp_2}$$

implies that

$$\alpha^{\psi} \equiv \alpha^2 \pmod{\wp_2},$$

for all $\alpha \in F_1$ which are integral for $\wp_2$. Since F_1/K is abelian and $\wp_2$ is unramified in this extension, this implies by definition of the Artin symbol that $\psi = \tau_2$.

8. Values of $\upsilon(\tau)$ as periodic points.

We now define the following algebraic functions. The roots of $f(x,y) = y^2 + (x^2-1)y + x^2$ (see Proposition 3.1) as a function of y are

$$\hat{F}(x) = -\frac{x^2-1}{2} \pm \frac{1}{2}\sqrt{x^4 - 6x^2 + 1}. \tag{8.1}$$

Also, the roots of $g(x,y) = y^2 - (x^2 - 4x + 1)y + x^2$ (see Proposition 3.2) are given by

$$\begin{aligned}\hat{T}(x) &= \frac{1}{2}(x^2 - 4x + 1) \pm \frac{1}{2}\sqrt{(x^2 - 2x + 1)(x^2 - 6x + 1)} \\ &= \frac{1}{2}(x^2 - 4x + 1) \pm \frac{x-1}{2}\sqrt{x^2 - 6x + 1}. \end{aligned} \tag{8.2}$$

We prove the following.

Theorem 8.1. *If $w \in R_K$ is the algebraic integer defined by*

$$w = \frac{a + \sqrt{-d}}{2}, \quad \text{with } a^2 + d \equiv 0 \ (\text{mod } 2^5)$$

and the integer c satisfies

$$c \equiv 1 - \frac{a^2+d}{32} \pmod 2,$$

then the generator $(-1)^{1+c}\mathfrak{v}(w/8)$ *of the field* $\Sigma_{\wp_2'^3}\Omega_f$ *over* $\mathbb{Q}$ *is a periodic point of the algebraic function* $\hat{F}(x)$ *defined by* (8.1) *and* $\mathfrak{v}^2(w/8)$ *is a periodic point of the function* $\hat{T}(x)$ *defined by* (8.2)*. If c is even, then* $\mathfrak{v}(w/8)$ *is a pre-periodic point of* $\hat{F}(x)$*.*

Proof. Setting $\eta = (-1)^{1+c}\mathfrak{v}(w/8)$ and $F_1 = \mathbb{Q}(\eta) = \mathbb{Q}(\eta^2)$, we have from the corollaries to Theorems 7.3 and 7.5 that $f(\eta, \eta^{\tau_2}) = 0$, where $\tau_2 = \left(\frac{F_1/K}{\wp_2}\right)$ is an automorphism in $\mathrm{Gal}(F_1/K)$. If the order of τ_2 is n, then applying powers of τ_2 gives that

$$f(\eta, \eta^{\tau_2}) = f(\eta^{\tau_2}, \eta^{\tau_2^2}) = \cdots = f(\eta^{\tau_2^{n-1}}, \eta) = 0, \tag{8.3}$$

which implies that η is a periodic point of $\hat{F}(x)$ of period n. If c is even, then from Corollary 7.6 and the fact that $f(x, y) = f(-x, y)$ we also have that

$$f(\mathfrak{v}(w/8), -\mathfrak{v}(w/8)^{\tau_2}) = 0;$$

thus, $\mathfrak{v}(w/8)$ is a pre-periodic point of $\hat{F}(x)$, since $-\mathfrak{v}(w/8)^{\tau_2}$ is periodic.

It is straightforward to check that

$$\hat{F}(x)^2 = \frac{1}{2}(x^4 - 4x^2 + 1) \pm \frac{1}{2}(x^2-1)\sqrt{x^4 - 6x^2 + 1} = \hat{T}(x^2) \tag{8.4}$$

and that the minimal polynomial of $\hat{F}(x)^2$ over $\mathbb{Q}(x)$ is $g(x^2, y)$. In particular, $f(x, y) = 0$ implies that $g(x^2, y^2) = 0$, since

$$g(x^2, y^2) = (-x^2y + x^2 + y^2 + y)(x^2y + x^2 + y^2 - y) = f(x, -y)f(x, y).$$

Hence, (8.3) implies that

$$g(\eta^2, \eta^{2\tau_2}) = g(\eta^{2\tau_2}, \eta^{2\tau_2^2}) = \cdots = g(\eta^{2\tau_2^{n-1}}, \eta^2) = 0, \tag{8.5}$$

which shows that $\eta^2 = \mathfrak{v}(w/8)^2$ is a periodic point of $\hat{T}(x)$. □

Remarks.

1. Note that if c is even, meaning that $2^5 \mid\mid a^2 + d$, then $2^6 \mid (a+16)^2 + d$, so that $w + 8 = \frac{a+16+\sqrt{-d}}{2} = w'$ satisfies (6.1) with c odd. Then the infinite product formula for $\mathfrak{v}(\tau)$ shows that $\mathfrak{v}(w/8) = \mathfrak{v}(w'/8 - 1) = -\mathfrak{v}(w'/8)$, and $-\mathfrak{v}(w/8) = \mathfrak{v}(w'/8)$ in Corollary 7.6.
2. Given that $f(\mathfrak{v}(\tau), \mathfrak{v}(2\tau)) = 0$, it is tempting to try to show that $\mathfrak{v}(w/8)$ is a periodic point by considering the chain of equations

$$f(\mathfrak{v}(w/8), \mathfrak{v}(w/4)) = f(\mathfrak{v}(w/4), \mathfrak{v}(w/2)) = \cdots = f(\mathfrak{v}(2^{n-1}w/8), \mathfrak{v}(2^n w/8)) = 0,$$

and find an integer n for which $2^{n-3}w = M(w/8) = \frac{aw+8b}{cw+8d}$, for some unimodular matrix M for which $\mathfrak{v}(M(w/8)) = \mathfrak{v}(w/8)$. However, this requires that $M \in \Gamma_1(8)$, so that $a \equiv 1 \pmod 8$ and $8 \mid c$. This condition leads to the equation

$$2^{n-3}cw^2 + (2^n d - a)w - 8b = 0.$$

Moreover, w is an algebraic integer, so the fact that $8 \mid c$ shows that 2^n must divide the other coefficients of this quadratic. Hence, $2^n \mid a$, which is impossible for $n \geq 1$. Thus, this approach does not yield an orbit leading back to $v(w/8)$.

As in the papers [15]-[18], the minimal polynomials of periodic points of $\hat{F}(x)$ can be computed using iterated resultants involving its minimal polynomial $f(x, y)$. We set

$$R^{(1)}(x, x_1) = f(x, x_1) = x^2 x_1 + x^2 + x_1^2 - x_1$$

and define, inductively,

$$R^{(n)}(x, x_n) = \mathrm{Res}_{x_{n-1}}(R^{(n-1)}(x, x_{n-1}), f(x_{n-1}, x_n)) \;\; n \geq 2.$$

Then the roots of the polynomial

$$R_n(x) = R^{(n)}(x, x), \;\; n \geq 1,$$

are the periodic points of $\hat{F}(x)$ whose minimal periods divide n. See [15, p. 727]. For example, we compute that

$$\begin{aligned}
R_1(x) &= x(x^2 + 2x - 1),\\
R_2(x) &= x(x^2 + 2x - 1)(x^4 - x^3 + x + 1),\\
R_3(x) &= x(x^2 + 2x - 1)(x^{12} - 5x^{11} + 2x^{10} + 10x^9 + 5x^8 + 23x^7\\
&\qquad - 8x^6 - 23x^5 + 5x^4 - 10x^3 + 2x^2 + 5x + 1),\\
R_4(x) &= x(x^2 + 2x - 1)(x^4 - x^3 + x + 1)(x^8 - x^7 + x^6 - 5x^5 + 5x^3 + x^2 + x + 1)\\
&\qquad \times (x^{16} + 5x^{15} - 18x^{14} - 75x^{13} + 137x^{12} + 105x^{11} + 38x^{10} + 185x^9\\
&\qquad - 300x^8 - 185x^7 + 38x^6 - 105x^5 + 137x^4 + 75x^3 - 18x^2 - 5x + 1).
\end{aligned}$$

We now set $x = z + 3$ in the function $\hat{T}(x)$, so that the square-root in $\hat{T}(x)$ has the 2-adic expansion

$$\sqrt{x^2 - 6x + 1} = \sqrt{z^2 - 8} = z\sqrt{1 - \frac{8}{z^2}} = z \sum_{k=0}^{\infty} (-1)^k \binom{1/2}{k} \frac{8^k}{z^{2k}}.$$

We will show that this series is 2-adically convergent for (roughly) half of the primitive periodic points of the algebraic function $\hat{T}(x)$ of a given period n in the field $\mathsf{K}_2(\sqrt{2})$, where K_2 is the maximal unramified, algebraic extension of the 2-adic field $\mathbb{Q}_2$.

If we set

$$T(x) = \frac{1}{2}(x^2 - 4x + 1) + \frac{x - 1}{2}\sqrt{x^2 - 6x + 1},$$

then using the above series in $T(x)$ and splitting off the $k=0$ term, we find

$$T(x) = x^2 - 4x + 2 + (x-1)(x-3)\sum_{k=1}^{\infty}(-1)^k 2^{2k-1}\binom{1/2}{k}\frac{2^k}{(x-3)^{2k}},$$

for $x - 3 \in \mathcal{O}^{\times}$, where $\mathcal{O}$ is the ring of integers in $\mathsf{K}_2(\sqrt{2})$. Since

$$(-1)^{k-1}2^{2k-1}\binom{1/2}{k} = C_{k-1} \in \mathbb{Z}$$

is the Catalan sequence, it follows that

$$T(x) \equiv x^2 \pmod 2, \quad x - 3 \in \mathcal{O}^{\times}.$$

Hence, $T(x)$ is a lift of the Frobenius automorphism for points x in the set

$$\overline{\mathsf{D}} = \{x \in \mathsf{K}_2(\sqrt{2}) : |x-3|_2 = 1\}.$$

Furthermore,

$$T(x) - 3 = (x-3)^2 + 2(x-3) - 4 - (x-1)(x-3)\sum_{k=1}^{\infty} C_{k-1}\frac{2^k}{(x-3)^{2k}}.$$

It follows that

$$|T(x) - 3|_2 = |x-3|_2^2 = 1, \tag{8.6}$$

and T maps $\overline{\mathsf{D}}$ to itself.

We next prove

Proposition 8.2. *We have the congruences*

$$R^{(n)}(x, x_n) \equiv (x^{2^n} + x_n)(x_n + 1)^{2^n - 1} \ (mod\ 2);$$
$$R_n(x) \equiv (x^{2^n} + x)(x+1)^{2^n - 1} \ (mod\ 2).$$

Proof. We have $f(x,y) = x^2 y + x^2 + y^2 - y$. So, for $n = 1$, we get

$$\begin{aligned} R^{(1)}(x, x_1) = f(x, x_1) &= x^2 x_1 + x^2 + x_1^2 - x_1 \\ &\equiv x^2 x_1 + x^2 + x_1^2 + x_1 \pmod 2 \\ &\equiv (x^2 + x_1)(x_1 + 1) \pmod 2. \end{aligned}$$

Hence,

$$R_1(x) \equiv (x^2 + x)(x+1) \pmod 2.$$

Now for the induction step, assume the result is true for $n-1$. Then,

$$\begin{aligned} R^{(n)}(x, x_n) &= \mathrm{Res}_{x_{n-1}}(R^{(n-1)}(x, x_{n-1}), f(x_{n-1}, x_n)) \\ &\equiv \mathrm{Res}_{x_{n-1}}((x^{2^{n-1}} + x_{n-1})(x_{n-1}+1)^{2^{n-1}-1}, (x_{n-1}^2 + x_n)(x_n + 1)) \pmod 2. \end{aligned}$$

By definition, the resultant of two polynomials $f = \sum_{i=0}^{n} a_i x^i$ and $g = \sum_{i=0}^{m} b_i x^i$, having roots $\alpha_1, \alpha_2, \dots, \alpha_n$ and $\beta_1, \beta_2, \dots, \beta_m$, respectively, is given by

$$\mathrm{Res}(f, g) = a_n^m \prod_{i=1}^{n} g(\alpha_i),$$

and

$$\mathrm{Res}(g, f) = (-1)^{mn} \mathrm{Res}(f, g).$$

The roots of $(x_{n-1}^2 + x_n)(x_n + 1)$, as a polynomial in x_{n-1}, are $\pm\sqrt{-x_n}$. Hence,

$$\begin{aligned}
&\mathrm{Res}_{x_{n-1}}((x^{2^{n-1}} + x_{n-1})(x_{n-1} + 1)^{2^{n-1}-1}, (x_{n-1}^2 + x_n)(x_n + 1)) \\
&\quad = (-1)^{2^{n-1}\cdot 2}(x_n + 1)^{2^{n-1}} \left(x^{2^{n-1}} + \sqrt{-x_n}\right)\left(\sqrt{-x_n} + 1\right)^{2^{n-1}-1} \\
&\quad\times \left(x^{2^{n-1}} - \sqrt{-x_n}\right)\left(-\sqrt{-x_n} + 1\right)^{2^{n-1}-1} \\
&\quad = (-1)^{2^n}(x_n + 1)^{2^{n-1}} \left(x^{2^n} + x_n\right)(x_n + 1)^{2^{n-1}-1} \\
&\quad = \left(x^{2^n} + x_n\right)(x_n + 1)^{2^n-1}.
\end{aligned}$$

Hence, we obtain

$$R^{(n)}(x, x_n) \equiv (x^{2^n} + x_n)(x_n + 1)^{2^n-1} \pmod 2,$$
$$R_n(x) \equiv (x^{2^n} + x)(x + 1)^{2^n-1} \pmod 2,$$

completing the induction. □

Corollary 8.3. *The degree of $R_n(x)$ is $\deg(R_n(x)) = 2^{n+1} - 1$.*

Proof. This follows from the proposition, if the leading coefficient of $R_n(x)$ is not divisible by 2. In fact, this follows from the relation

$$R^{(n)}(x, x_n) = A_n(x_n)x^{2^n} + S_n(x, x_n),$$

where for $n \geq 3$,

$$A_n(x_n) = (x_n + 1)(x_n^2 + 1)(x_n^2 - 2x_n - 1)^2(x_n^2 + 2x_n - 1)^{2^{n-1}-4}$$

and for $n \geq 1$,

$$\deg(A_n(x_n)) = 2^n - 1, \quad \deg_x(S_n(x, x_n)) \leq 2^n - 2, \quad \deg_{x_n}(S_n(x, x_n)) = 2^n.$$

We refer the reader to the lemma in [15, pp. 727-728] for a similar proof. □

The roots of the factor $x^{2^n} + x = x(x+1)\frac{x^{2^n-1}+1}{x+1} = x(x+1)h_n(x)$ other than $x = 0, 1$ have degree greater than 1, and therefore satisfy $x - 3 \not\equiv 0 \pmod 2$. It follows from Hensel's Lemma that $2^n - 1$ of the roots of $R_n(x)$ over $\mathbb{Q}_2$ have the property that $x - 3 \in \mathcal{O}^\times$, and for these roots the series for $T(x)$ converges in K_2.

Now the argument at the end of the proof of Theorem 7.3 shows that $\eta \not\equiv 1 \pmod{\wp_2}$, so that the image of η in the completion $F_{1,\mathfrak{q}} \subset \mathsf{K}_2$ of $F_1 = \Sigma_{\wp_2'^3}\Omega_f$ with respect to a prime divisor $\mathfrak{q}$ of $\wp_2$ in F_1 satisfies $\eta^2 - 3 \in \mathcal{O}^\times$. Hence, the series for $T(\eta^2)$ converges. We claim now that $\eta^{2\tau_2} = T(\eta^2)$. But $g(\eta^2, \eta^{2\tau_2}) = 0$ implies that $\eta^{2\tau_2}$ is one of the values of $\hat{T}(\eta^2)$. The value different from $T(\eta^2)$ in K_2 is

$$\begin{aligned} T_1(\eta^2) &= \eta^4 - 4\eta^2 + 1 - T(\eta^2) \\ &\equiv \eta^4 - 4\eta^2 + 1 - \eta^4 \pmod{\mathfrak{q}} \\ &\equiv 1 \pmod{\mathfrak{q}}. \end{aligned}$$

But we also know $\eta^{2\tau_2} - 3 = (\eta^2 - 3)^{\tau_2} \in \mathcal{O}^\times$, so that $\eta^{2\tau_2} \neq T_1(\eta^2)$. This yields the following.

Theorem 8.4. *If w satisfies (6.1), then the value $\eta = v(w/8)$ and the automorphism $\tau_2 = \left(\frac{F_1/K}{\wp_2}\right)$ satisfy*

$$\eta^{2\tau_2} = T(\eta^2),$$

in the completion $F_{1,\mathfrak{q}} \subset \mathsf{K}_2$ of $F_1 = \Sigma_{\wp_2'^3}\Omega_f$ with respect to a prime divisor $\mathfrak{q}$ of $\wp_2$ in F_1, where

$$T(x) = x^2 - 4x + 2 - (x-1)(x-3)\sum_{k=1}^{\infty} C_{k-1}\frac{2^k}{(x-3)^{2k}}$$

converges for x in $\overline{D} = \{x \in \mathsf{K}_2(\sqrt{2}) : |x-3|_2 = 1\}$.

Since τ_2 fixes the prime divisors of $\wp_2$, it extends naturally to an automorphism of $F_{1,\mathfrak{q}}$, and can be applied to the individual terms of the series representing $T(x)$. Thus, we see inductively that

$$\eta^{2\tau_2^i} = T(\eta^{2\tau_2^{i-1}}) = T(T^{i-1}(\eta^2)) = T^i(\eta^2)$$

is the i-th iterate of $T(x)$ applied to η^2. From this and the fact that $\mathbb{Q}(\eta^2) = F_1$ we see that the order of τ_2 in $\mathrm{Gal}(F_1/K)$ is the minimal period of the periodic point η^2, and that η^2 *is a periodic point in the ordinary sense of the* 2*-adic function* $T(x)$. This also shows that the minimal period of η with respect to $\hat{F}(x)$ is $n = \mathrm{ord}(\tau_2)$, since if η had smaller minimal period m, then by the proof of Theorem 8.1, η^2 would have period $m < n$ with respect to the function $T(x)$. This completes the proof of the assertions of Theorem B of the Introduction regarding minimal periods.

9. The periodic points of $\hat{F}(x)$ and a class number formula.

In this section we show that the only periodic points of $\hat{F}(x)$ are the values given in Theorem 8.1. In fact, we will prove the following.

Theorem 9.1. *The only periodic points of the function $\hat{F}(x)$ in $\overline{\mathbb{Q}}$ are the fixed points $0, \sigma, \bar{\sigma}$ and the conjugates over $\mathbb{Q}$ of the values $v(w/8)$ in Theorem 8.1 (for odd c).*

Proof. Let $\tilde{g}(x,y) = x^2y^2 + 2y + x^2$. Note that $\tilde{g}(x,y) = g(y,x)$ for the polynomial $g(x,y)$ in [16, Thm. 2, p. 327]. By the results of that paper the numbers π, ξ and their conjugates over $\mathbb{Q}$ (as $-d$ ranges over all discriminants $\equiv 1$ modulo 8) are, together with 0 and -1, the only periodic points of the algebraic function $\mathfrak{f}(z)$ defined by $\tilde{g}(z, \mathfrak{f}(z)) = 0$. The assertion of the theorem will follow from the identity

$$(x^2-1)^2(y^2-1)^2\tilde{g}\left(\frac{2x}{x^2-1}, \frac{2y}{y^2-1}\right) = 4f(x,y)(x^2y^2 - x^2y + y + 1). \quad (9.1)$$

Here, as in Proposition 3.1, $f(x,y) = x^2y + x^2 + y^2 - y$. Let η be a periodic point of $\hat{F}(x)$ in $\overline{\mathbb{Q}}$ which is distinct from its fixed points $0, \sigma, \bar{\sigma}$. Then there are $\eta_1 = \eta, \eta_2, \dots, \eta_n$ in $\overline{\mathbb{Q}}$ for which

$$f(\eta_1, \eta_2) = f(\eta_2, \eta_3) = \cdots = f(\eta_n, \eta_1) = 0. \quad (9.2)$$

Setting $\lambda_i = \frac{2\eta_i}{\eta_i^2 - 1}$, equations (9.1) and (9.2) give that

$$\tilde{g}(\lambda_1, \lambda_2) = \tilde{g}(\lambda_2, \lambda_3) = \cdots = \tilde{g}(\lambda_n, \lambda_1) = 0. \quad (9.3)$$

Note that $\eta_i \neq \pm 1$ since ± 1 are preperiodic (and not periodic) for $f(x,y)$, since

$$f(\pm 1, y) = y^2 + 1, \quad f(\pm i, y) = y^2 - 2y - 1, \quad f(1 \pm \sqrt{2}, y) = (y + 1 \pm \sqrt{2})^2.$$

Equation (9.3) implies that λ_1 is a periodic point of the function $\mathfrak{f}(z)$ defined above. Also, $\lambda_i \neq 0, -1$ since $\eta_i \notin \{0, \sigma, \bar{\sigma}\}$. By the results of [16, Thm. 2], this shows that λ_1 must be a conjugate of the number π for some discriminant $-d$ and is therefore a root of the polynomial $b_d(x)$. (See Proposition 6.4.) Since $\lambda_1 = 2\eta/(\eta^2 - 1)$, this shows that η is a root of the minimal polynomial $f_d(x)$ of $v(w/8)$, for c odd, by (6.7). This completes the proof. $\square$

Remark. We can use equation (9.1) to give an alternate proof of the Corollary to Theorem 7.3, as follows. We would like to show that $f(\eta, \eta^{\tau_2}) = 0$, where $\eta = v(w/8)$ and $\tau_2 = \left(\frac{F_1/K}{\wp_2}\right)$, with $F_1 = \Sigma_{\wp_2'^3}\Omega_f$. Since $\tau_2|_{\Omega_f} = \left(\frac{\Omega_f/K}{\wp_2}\right)$, we know that $\tilde{g}(\pi, \pi^{\tau_2}) = 0$, by [16, pp. 332-333]. Using $\pi = \frac{2\eta}{\eta^2-1}$ from (6.4), equation (9.1) implies that $f(\eta, \eta^{\tau_2})k(\eta, \eta^{\tau_2}) = 0$, where $k(x,y) = x^2y^2 - x^2y + y + 1$. But $k(\eta, \eta^{\tau_2}) \equiv k(\eta, \eta^2) \bmod \wp_2$ in F_1. An easy computation shows that $k(x, x^2) \equiv (x+1)^6 \pmod 2$, so $k(\eta, \eta^{\tau_2}) \equiv (\eta + 1)^6 \bmod \wp_2$. If $\eta \equiv 1$ modulo some prime divisor $\mathfrak{p}$ of $\wp_2$ in F_1, then the relation $\eta^2 - \frac{2}{\pi}\eta - 1 = 0$ would give that $\frac{2}{\pi} \equiv 0 \pmod{\mathfrak{p}}$, which is impossible since $\frac{2}{\pi} \cong \wp_2'$. Hence, $k(\eta, \eta^{\tau_2}) \not\equiv 0 \bmod \wp_2$, which implies $k(\eta, \eta^{\tau_2}) \neq 0$ and therefore $f(\eta, \eta^{\tau_2}) = 0$, as claimed.

Theorem 9.1 has the following consequence. As in the last remark, let $F_1 = \Sigma_{\wp_2'^3}\Omega_f$ be the field generated by $\upsilon(w/8)$ in Theorem 6.1. Then $[F_1 : \mathbb{Q}] = 4h(-d)$ and F_1 is the inertia field for $\wp_2$ in the field $\Sigma_8\Omega_f$, an extended ring class field over $K_d = \mathbb{Q}(\sqrt{-d})$. As in Section 7, let $\tau_2 = \left(\frac{F_1/K_d}{\wp_2}\right)$ be the Artin symbol for $\wp_2$ in the extension F_1/K_d. Now define the set of discriminants

$$\mathfrak{D}_{n,2} = \{-d < 0 \mid -d \equiv 1 \ (\text{mod } 8) \text{ and } \text{ord}(\tau_2) = n \text{ in } \text{Gal}(F_1/K_d)\}. \quad (9.4)$$

Theorem 9.2. *If $n \geq 2$, we have the following relation between class numbers of discriminants in the set $\mathfrak{D}_{n,2}$:*

$$\sum_{-d \in \mathfrak{D}_{n,2}} h(-d) = \frac{1}{2}\sum_{k|n} \mu(n/k)2^k. \quad (9.5)$$

Proof. This proof mirrors the arguments in [18, pp.792-793, 806]. First, define

$$\mathsf{P}_n(x) = \prod_{k|n} R_k(x)^{\mu(n/k)}. \quad (9.6)$$

We show that $\mathsf{P}_n(x) \in \mathbb{Z}[x]$. From Proposition 8.2 it is clear that $R_n(x)$, for $n > 1$, is divisible (mod 2) by the N irreducible (monic) polynomials $\bar{h}_i(x)$ of degree n over $\mathbb{F}_2$, where

$$N = \frac{1}{n}\sum_{k|n} \mu(n/k)2^k,$$

and that these polynomials are simple factors of $R_n(x)$ (mod 2). It follows from Hensel's Lemma that $R_n(x)$ is divisible by distinct irreducible polynomials $h_i(x)$ of degree n over $\mathbb{Z}_2$, the ring of integers in $\mathbb{Q}_2$, for $1 \leq i \leq N$, with $h_i(x) \equiv \bar{h}_i(x)$ (mod 2). In addition, all the roots of $h_i(x)$ are periodic of minimal period n and lie in the unramified extension K_2. Furthermore, n is the smallest index for which $h_i(x) \mid R_n(x)$ over $\mathbb{Q}_2$.

Now consider the identity

$$(\sigma x + 1)^2(\sigma y + 1)^2 f(\bar{A}(x), \bar{A}(y)) = 2^3\sigma^2 f(y, x), \quad (9.7)$$

where $\bar{A}(x) = \dfrac{-x + \sigma}{\sigma x + 1}$, as in (3.3). If the periodic point a of $\hat{F}(x)$, with minimal period $n > 1$, is a root of one of the polynomials $h_i(x)$, then a is a unit in K_2, and for some $a_1, \dots, a_{n-1}$ we have

$$f(a, a_1) = f(a_1, a_2) = \cdots = f(a_{n-1}, a) = 0. \quad (9.8)$$

Furthermore, $a \not\equiv 1$ (mod $\sqrt{2}$), since otherwise its reduction $a \equiv \bar{a} \equiv 1$ (mod 2) would have degree 1 over $\mathbb{F}_2$ (using that K_2 is unramified over $\mathbb{Q}_2$). Hence, $a+1+\sqrt{2}$ is a unit in $\mathsf{K}_2(\sqrt{2})$, which gives that $\sigma a+1$ is a unit, as well. All of the a_i satisfy $a_i \not\equiv 1$ (mod $\sqrt{2}$), since the congruence $f(1, y) \equiv (y + 1)^2$ (mod 2) has only $y \equiv 1$ as a solution. Hence, if some $a_i \equiv 1 \pmod{\sqrt{2}}$, then $a_j \equiv 1$ for $j > i$, which would imply that $a \equiv 1 \pmod{\sqrt{2}}$, as well. The elements $b_i = \bar{A}(a_i)$ are distinct and lie in $\mathsf{K}_2(\sqrt{2})$ and satisfy

$$b_i - 1 \equiv \frac{-a_i + \sigma - \sigma a_i - 1}{\sigma a_i + 1} \equiv \frac{-2}{\sigma a_i + 1} \equiv 0 \pmod{\sqrt{2}}.$$

The identity (9.7) yields that

$$f(b, b_{n-1}) = f(b_{n-1}, b_{n-2}) = \cdots = f(b_1, b) = 0 \tag{9.9}$$

in $\mathsf{K}_2(\sqrt{2})$. Hence, $b_i \equiv 1 \pmod{\sqrt{2}}$, and the orbit $\{b, b_{n-1}, \ldots, b_1\}$ is distinct from all the orbits in (9.8).

Now the map $\bar{A}(x)$ has order 2, so it is clear that $b = \bar{A}(a)$ has minimal period n in (9.9), since otherwise $a = \bar{A}(b)$ would have period smaller than n. It follows that there are at least $2N$ periodic orbits of minimal period $n > 1$. Noting that

$$R_1(x) = f(x, x) = x(x^2 + 2x - 1),$$

these distinct orbits and factors account for at least

$$3 + \sum_{d|n, d>1} (2 \sum_{k|d} \mu(d/k) 2^k) = -1 + 2 \sum_{d|n} (\sum_{k|d} \mu(d/k) 2^k) = 2 \cdot 2^n - 1$$

roots, and therefore all the roots, of $R_n(x)$. This shows that the roots of $R_n(x)$ are distinct and the expressions $\mathsf{P}_n(x)$ are polynomials. Furthermore, over $\mathsf{K}_2(\sqrt{2})$ we have the factorization

$$\mathsf{P}_n(x) = \pm \prod_{1 \le i \le N} h_i(x) \tilde{h}_i(x), \quad n > 1, \tag{9.10}$$

where $\tilde{h}_i(x) = c_i(\sigma x + 1)^n h_i(\bar{A}(x))$, and the constant c_i is chosen to make $\tilde{h}_i(x)$ monic.

By the results of Section 8, for each discriminant $-d \in \mathfrak{D}_{n,2}$ we have that $f_d(x) \mid \mathsf{P}_n(x)$. Furthermore, every root of $\mathsf{P}_n(x)$ is a root of some $f_d(x)$, by Theorem 9.1, where $ord(\tau_2) = n$ in order for the roots of $f_d(x)$ to have minimal period n. It follows that

$$\mathsf{P}_n(x) = \tilde{c}_n \prod_{-d \in \mathfrak{D}_{n,2}} f_d(x),$$

for some constant $\tilde{c}_n$, and taking degrees on both sides and using (9.10) gives the formula

$$2 \sum_{k|n} \mu(n/k) 2^k = \sum_{-d \in \mathfrak{D}_{n,2}} 4h(-d).$$

The formula of the theorem follows. □

The result of Theorem 9.2 is the analogue of [18, Thm.1.3] for the prime 2 in place of 5. The factor $1/2$ in front is to be interpreted as $2/\phi(8)$, replacing the factor $2/\phi(5)$ in the result of [18]. Also, see Conjecture 1 in the Introduction of that paper.

Theorem 9.1 will now be used to prove the corresponding fact for the algebraic function $\hat{T}(x)$ in Theorem 8.1.

Theorem 9.3. *The periodic points of the function $\hat{T}(x)$ of (8.2) in $\overline{\mathbb{Q}}$ (or $\mathbb{C}$) are exactly the squares of the periodic points of the function $\hat{F}(x)$, i.e., the fixed points $0, \sigma^2, \bar{\sigma}^2$ and the conjugates over $\mathbb{Q}$ of the values $v^2(w/8)$, where w is given by (6.1).*

Proof. As in the proof of Theorem 8.1, the polynomials $g(x,y) = y^2 - (x^2 - 4x+1)y + x^2$ and $f(x,y) = y^2 + (x^2-1)y + x^2$ defining $\hat{T}$ and $\hat{F}$, respectively, satisfy the identity

$$g(x^2, y^2) = f(x,-y)f(x,y).$$

Let η^2 be a periodic point of $g(x,y)$ of period n. Then there exist $\eta_1^2, \eta_2^2, \dots, \eta_{n-1}^2 \in \overline{\mathbb{Q}}$ such that

$$g(\eta^2, \eta_1^2) = g(\eta_1^2, \eta_2^2) = \cdots = g(\eta_{n-1}^2, \eta^2) = 0.$$

This means that, for every $i = 0, 1, \dots, n-1$, either

$$f(\eta_i, \eta_{i+1}) = 0 \text{ or } f(\eta_i, -\eta_{i+1}) = 0, \text{ where } \eta_0 = \eta = \eta_n.$$

Now if $f(\eta_i, \eta_{i+1}) = 0$ for all i, then η is a periodic point of $\hat{F}(x)$.

Otherwise, there exists an i such that $f(\eta_i, \eta_{i+1}) \neq 0$, but $f(\eta_i, -\eta_{i+1}) = 0$. In this case, if $i < n-1$, replace η_{i+1} by $-\eta_{i+1}$ in the next equation of the sequence, yielding $f(-\eta_{i+1}, \eta_{i+2}) = 0$. And if this happens for $i = n-1$, then simply replace η by $-\eta$. This works because $f(-x,y) = f(x,y)$. In other words, in the chain of equations for f, whenever the second argument has a negative sign, choose the next first argument with the same negative sign. And in case the last equation has second argument η with a negative sign, then choose the first argument of the first equation as $-\eta$ also. Hence, there is a chain of equations $f(\eta_i, \eta_{i+1}) = 0$ beginning and ending with $\pm\eta$. Hence, $\pm\eta$ is a periodic point of $\hat{F}(x)$ in either case, which implies that η^2 is the square of a periodic point of $\hat{F}(x)$. This completes the proof. □

With this theorem, we have completely proved all the statements in Theorem B of the Introduction.

10. Appendix

Here we give a proof of the relation between $u(\tau)$ and $v(\tau)$ that was used in the proof of Proposition 3.1b).

Proposition 10.1. *The following relation holds between $u(\tau)$ and $v(\tau)$:*

$$u^4(v^2+1)^2 + 4v(v^2-1) = 0.$$

Proof. We have derived in the proof of Proposition 4.1 that

$$\frac{1}{v(\tau)} - v(\tau) = q^{-1/2}\,\frac{(-q^2;q^4)_\infty^2}{(-q^4;q^4)_\infty^2}.$$

Proceeding in a similar way, we obtain

$$\begin{aligned}
\frac{1}{v(\tau)} + v(\tau) &= \frac{\psi(-q)\cdot\varphi(q)}{q^{1/2}\,(q;q^2)_\infty(q^8;q^8)_\infty^2}\\
&= q^{-1/2}\,\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}\cdot\frac{(-q;q^2)_\infty^2(q^2;q^2)_\infty}{(q;q^2)_\infty(q^8;q^8)_\infty^2}\\
&= q^{-1/2}\,\frac{(-q;q^2)_\infty}{(q;q^2)_\infty}\cdot\frac{(q^2;q^2)_\infty^2}{(q^8;q^8)_\infty^2}\\
&= q^{-1/2}\,\frac{(-q;q^2)_\infty^2}{(q^2;q^4)_\infty}\cdot(q^2;q^4)_\infty^2(q^4;q^8)_\infty^2\\
&= q^{-1/2}\,\frac{(-q;q^2)_\infty^2(q^2;q^4)_\infty}{(-q^4;q^4)_\infty^2}.
\end{aligned}$$

(See [2, pp. 221-222].) Putting the above two expressions to use in $\frac{4v(1-v^2)}{(1+v^2)^2} = \frac{4\left(\frac{1}{v}-v\right)}{\left(\frac{1}{v}+v\right)^2}$, we find that

$$\begin{aligned}
\frac{4v(1-v^2)}{(1+v^2)^2} &= 4q^{1/2}\,\frac{(-q^2;q^4)_\infty^2}{(-q^4;q^4)_\infty^2}\cdot\frac{(-q^4;q^4)_\infty^4}{(-q;q^2)_\infty^4(q^2;q^4)_\infty^2}\\
&= 4q^{1/2}\,\frac{(-q^2;q^4)_\infty^2(-q^4;q^4)_\infty^2}{(-q;q^2)_\infty^4(q^2;q^4)_\infty^2}\\
&= 4q^{1/2}\frac{(-q^2;q^2)_\infty^2}{(-q;q^2)_\infty^4(q^2;q^4)_\infty^2}\\
&= 4q^{1/2}\frac{(-q^2;q^2)_\infty^4}{(-q;q^2)_\infty^4}\\
&= u^4(\tau),
\end{aligned}$$

completing the proof. □

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This paper is available via http://nyjm.albany.edu/j/2024/30-36.html.