Periodic points of algebraic functions related to a continued fraction of Ramanujan

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Abstract. A continued fraction $v(\tau)$ of Ramanujan is evaluated at certain arguments in the field $K = \mathbb{Q}(\sqrt{-d})$, with $-d \equiv 1 \pmod{8}$, in which the ideal $(2) = \mathfrak{p}_2 \mathfrak{p}_2'$ is a product of two prime ideals. These values of $v(\tau)$ are shown to generate the inertia field of $\mathfrak{p}_2$ or $\mathfrak{p}_2'$ in an extended ring class field over the field $K$. The conjugates over $\mathbb{Q}$ of these same values, together with $0, -1 \pm \sqrt{2}$, are shown to form the exact set of periodic points of a fixed algebraic function $\hat{F}(x)$, independent of $d$. These are analogues of similar results for the Rogers-Ramanujan continued fraction.

Contents

1. Introduction 783
2. Preliminaries. 787
3. Identities for $u(\tau)$ and $v(\tau)$ 789
4. The relation between $v(\tau)$ and $p(\tau)$. 793
5. The relation between $v(\tau)$ and $b(\tau)$. 797
6. The field generated by $v(\omega/8)$. 800
7. The diophantine equation. 806
8. Values of $v(\tau)$ as periodic points. 815
9. The periodic points of $\hat{F}(x)$ and a class number formula. 820
10. Appendix 824
References 825

1. Introduction

This paper is concerned with values of Ramanujan’s continued fraction

$$v(\tau) = \frac{q^{1/2}}{1 + q} \frac{q^2}{1 + q^2} \frac{q^4}{1 + q^4} \frac{q^6}{1 + q^6} \ldots, \quad q = e^{2\pi i \tau},$$

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sometimes referred to as the Ramanujan-Göllnitz-Gordon continued fraction, which is also given by the infinite product

\[ v(\tau) = q^{1/2} \prod_{n=1}^{\infty} \left(1 - q^n \right)^{\left(\frac{2}{n}\right)} = e^{2\pi i \tau}, \]

for \( \tau \) in the upper half-plane. Here, \( \left(\frac{2}{n}\right) \) is the Kronecker symbol. See [12], [9, p. 153], [5], [6]. The continued fraction \( v(\tau) \) is analogous to the Rogers-Ramanujan continued fraction

\[ r(\tau) = q^{1/5} \prod_{n=1}^{\infty} \left(1 - q^n \right)^{\left(\frac{2}{n}\right)} = e^{2\pi i \tau}, \]

whose properties were considered in the papers [17], [18]. In [17] it was shown that certain values of \( r(\tau) \), for \( \tau \) in the imaginary quadratic field \( K = \mathbb{Q}(\sqrt{-d}) \) with \(-d \equiv \pm 1 \pmod{5}\), are periodic points of a fixed algebraic function, independent of \( d \), and generate certain class fields \( \prod \mathbb{Q} / \mathbb{Q} \) over \( K \). Here \( \prod \mathbb{Q} \) is the ray class field of conductor \( \mathcal{I} \mathcal{O}_f \) over \( K \); and \( \mathcal{O}_f \) is the ring class field of conductor \( f \) corresponding to the order \( \mathcal{R}_{-d} \) of discriminant \(-d = b_K f^2 \) in \( K \) (\( b_K \) is the discriminant of \( K \)).

Here we will show that a similar situation holds for certain values of the continued fraction \( v(\tau) \). We consider discriminants of the form \(-d \equiv 1 \pmod{8}\) and arguments in the field \( K = \mathbb{Q}(\sqrt{-d}) \). Let \( R_K \) be the ring of integers in this field and let the prime ideal factorization of \( 2 \) in \( R_K \) be \( (2) = \mathfrak{p}_2 \mathfrak{p}_2' \). We define the algebraic integer \( w \) by

\[ w = \frac{a + \sqrt{-d}}{2}, \quad a^2 + d \equiv 0 \pmod{2^5}, \quad (N(w), f) = 1, \quad \text{(1.1)} \]

where \( \mathfrak{p}_2 = (2, w) \). Also, the positive (and odd) integer \( f \) is defined by \(-d = b_K f^2 \), where \( b_K \) is the discriminant of \( K / \mathbb{Q} \).

We will show that

\[ v(w/8) = \pm \frac{1 + \sqrt{1 + \pi^2}}{\pi}, \]

where \( \pi \) is a generator in \( \mathcal{O}_f \) of the ideal \( \mathfrak{p}_2 \) (or rather, its extension \( \mathfrak{p}_2 \mathcal{O}_f \), in \( \mathcal{O}_f \)). The algebraic integer \( \pi \) and its conjugate \( \xi \) in \( \mathcal{O}_f \) were studied in [14] and shown to satisfy

\[ \pi^4 + \xi^4 = 1, \quad (\pi) = \mathfrak{p}_2, \quad (\xi) = \mathfrak{p}_2', \quad \xi = \frac{\pi^2 + 1}{\pi^2 - 1}, \quad \text{(1.2)} \]

where \( \tau = \left( \frac{\mathcal{O}_f / \mathfrak{p}_2}{\mathfrak{p}_2} \right) \) is the Artin symbol (Frobenius automorphism) for the prime ideal \( \mathfrak{p}_2 \) and the ring class field \( \mathcal{O}_f \) over \( K \) whose conductor is \( f \). It follows from results of [14] that

\[ \pi = (-1)^c \mathfrak{p}(w), \]
where \( c \) is an integer satisfying the congruence

\[
c \equiv 1 - \frac{a^2 + d}{32} \quad (\text{mod} \ 2)
\]

and \( \wp(\tau) \) is the modular function \( \wp(\tau) = \frac{f_3(\tau/2)}{f_1(\tau/2)} \), defined in terms of the Weber-Schlöfli functions \( f_2(\tau), f(\tau) \). (See [20], [8], [19].) The above formula for \( v(w/8) \) follows from the identity

\[
\frac{2}{\wp(8\tau)} = \frac{1 - v^2(\tau)}{v(\tau)} = \frac{1}{v(\tau)} - v(\tau),
\]

for \( \tau \) in the upper half-plane, which we prove in Proposition 4.1. (Also see [7, Thm. 8.6, p. 475].)

As in [17], we consider a diophantine equation, namely

\[
\mathcal{E}_2 : X^2 + Y^2 = \sigma^2(1 + X^2Y^2), \quad \sigma = -1 + \sqrt{2}.
\]

An identity for the continued fraction \( v(\tau) \) implies that

\[
(X, Y) = (v(w/8), -1/w)
\]

is a point on \( \mathcal{E}_2 \). We prove the following theorem relating the coordinates of this point.

**Theorem A.** Let \( w \) be given by (1.1) with \( \wp_2 = (2, w) \) in \( R_K \) and \( -d = \mathfrak{d}_K f^2 \equiv 1 \) (mod 8).

(a) The field \( F_1 = \mathbb{Q}(v(w/8)) = \mathbb{Q}(v^2(w/8)) \) equals the field \( \Sigma_{\wp_2^2} \Omega_f \), where \( \Sigma_{\wp_2^2} \) is the ray class field of conductor \( \mathfrak{f} = \wp_2^3 \) and \( \Omega_f \) is the ring class field of conductor \( f \) over the field \( K \). The field \( F_1 \) is the inertia field for \( \wp_2 \) in the extended ring class field \( L_{O_{8/8}} = \Sigma_6 \Omega_f \) over \( K \), where \( O = R_{-d} \) is the order of discriminant \(-d \) in \( K \).

(b) We have \( F_2 = \mathbb{Q}(v(-1/w)) = \Sigma_{\wp_2^2} \Omega_f \), the inertia field of \( \wp_2 \) in \( L_{O_{8/8}}/K \).

(c) If \( \tau_2 \) is the Frobenius automorphism \( \tau_2 = \left( \frac{F_1/K}{\wp_2} \right) \), then

\[
v(-1/w) = \frac{v(w/8)^2 + (-1)^c \sigma}{\sigma v(w/8)^2} - (-1)^c.
\]  

(1.3)

See Theorems 6.1, 7.3 and 7.5 and their corollaries. From part (c) of this theorem we deduce the following.

**Theorem B.**

(a) If \( w \) and \( c \) are as above, then the generator \( (-1)^{1+c}v(w/8) \) of the field \( \Sigma_{\wp_2^2} \Omega_f \) over \( \mathbb{Q} \) is a periodic point of the multivalued algebraic function \( \hat{F}(x) \) given by

\[
\hat{F}(x) = \frac{x^2 - 1}{2} \pm \frac{1}{2}\sqrt{x^4 - 6x^2 + 1};
\]
and \( v^2(w/8) \) is a periodic point of the algebraic function \( \hat{T}(x) \) defined by

\[
\hat{T}(x) = \frac{1}{2}(x^2 - 4x + 1) \pm \frac{x - 1}{2}\sqrt{x^2 - 6x + 1}.
\]

(b) The minimal period of \((-1)^{1+c}v(w/8)\) (and of \(v^2(w/8)\)) is equal to the order of the automorphism \( \tau_2 \) in \( \text{Gal}(F_1/K) \).

(c) Together with the numbers \( 0, -1 \pm \sqrt{2} \), the values \((-1)^{1+c}v(w/8)\) and their conjugates over \( \mathbb{Q} \) are the only periodic points of the algebraic function \( \hat{F}(x) \) in \( \overline{\mathbb{Q}} \) or \( \mathbb{C} \). The only periodic points of \( T(x) \) in \( \overline{\mathbb{Q}} \) or \( \mathbb{C} \) are \( 0, (-1 \pm \sqrt{2})^2 \), and the conjugates of the values \( v^2(w/8) \) over \( \mathbb{Q} \).

We understand by a periodic point of the multivalued algebraic function \( \hat{F}(x) \) the following. Let \( f(x, y) = x^2y + x^2 + y^2 - y \) be the minimal polynomial of \( \hat{F}(x) \) over \( \mathbb{Q}(x) \). A periodic point of \( \hat{F}(x) \) is an algebraic number \( a \) for which there exist \( a_1, a_2, \ldots, a_{n-1} \in \overline{\mathbb{Q}} \) satisfying

\[
f(a, a_1) = f(a_1, a_2) = \cdots = f(a_{n-1}, a) = 0.
\]

A similar definition can be given over any ground field \( k \). See [15], [16]. Thus, if \( a \in \overline{\mathbb{Q}} \) is a periodic point of \( \hat{F}(x) \), so are its conjugates over \( \mathbb{Q} \), because \( f(x, y) \) has coefficients in \( \mathbb{Q} \). We show in Section 8 that \( v^2(w/8) \) is actually a periodic point in the usual sense of the single-valued 2-adic function

\[
T(x) = x^2 - 4x + 2 - (x - 1)(x - 3) \sum_{k=1}^{\infty} C_{k-1} \frac{2^k}{(x - 3)^{2k}},
\]

defined on a subset of the maximal unramified, algebraic extension \( K_2 \) of the 2-adic field \( \mathbb{Q}_2 \). (\( C_k \) is the \( k \)-th Catalan number.) This follows from the fact that

\[
v(w/8)^2 \tau_2 = T(v(w/8)^2),
\]
in the completion \( F_{1,q} \subset K_2 \) of \( F_1 = \Sigma_{p^2} \Omega_f \) with respect to a prime divisor \( q \) of \( \mathfrak{g} \) in \( F_1 \). This implies that the minimal period of \( v^2(w/8) \) with respect to the function \( T(x) \) is \( n = \text{ord}(\tau_2) \).

From Theorems A and B we conclude the following.

**Theorem C.** Let \( K = \mathbb{Q}(\sqrt{-d}) \), with \( -d \equiv 1 \mod 8 \) and \( (2) = \mathfrak{g} \mathfrak{g}' \mathfrak{g}'' \) in \( R_K \). Then every class field over \( K \) of the form \( \Sigma_{p^2} \Omega_f \) or \( \Sigma_{p^2} \Omega_f' \) (with \( f \) odd) is generated over \( \mathbb{Q} \) by an individual periodic point of the function \( \hat{F}(x) \) (or of \( \hat{T}(x) \)). Furthermore, all but three periodic points of \( \hat{F}(x) \) in \( \overline{\mathbb{Q}} \) generate a class field \( \Sigma_{p^2} \Omega_f \) in this family over some imaginary quadratic field \( K = \mathbb{Q}(\sqrt{-d}) \), for which \( -d = b_kf^2 \equiv 1 \mod 8 \).

These are all analogues of the corresponding facts for the Rogers-Ramanujan continued fraction \( r(\tau) \) which were proved in [17] and [18].

An important corollary of the fact that the conjugates of the values \( v(w/8) \) in Theorem B are, together with the three fixed points, all the periodic points of the algebraic function \( \hat{F}(x) \), is the following class number formula. In this
formula, \( h(-d) \) denotes the class number of the order \( R_{-d} \) of discriminant \(-d\) in the quadratic field \( K = K_d \), and \( \mathfrak{D}_{n,2} \) is the finite set of negative discriminants \(-d \equiv 1 \pmod{8} \) for which the Frobenius automorphism \( \tau_2 \) in Theorem A has order \( n \) in \( \text{Gal}(F_1/K_d) \), where \( F_1 = F_{1,d} \) also depends on \( d \):

\[
\sum_{-d \in \mathfrak{D}_{n,2}} h(-d) = \frac{1}{2} \sum_{k|n} \mu(n/k)2^k, \quad n > 1. \tag{1.4}
\]

(\( \mu(n) \) is the Möbius function.) See Theorem 9.2. This fact is the analogue for the prime \( p = 2 \) of Theorem 1.3 in [18] for the prime \( p = 5 \), or of Conjecture 1 of that paper for a prime \( p > 5 \).

The layout of the paper is as follows. Section 2 contains a number of \( q \)-identities (following Ramanujan) and theta function identities which we use to prove identities for various modular functions in Sections 3-5. Most of these identities are known; straightforward proofs – which use theta functions, but not the theory of modular forms or functions – are included here for the sake of completeness. In Sections 6 and 7 we prove Theorem A. The proofs of Theorem B and (1.4) are given in Sections 8 and 9.


2. Preliminaries.

As is customary, let us set

\[
(a; q)_0 := 1, \quad (a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k), \quad n \geq 1
\]

and

\[
(a; q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k), \quad |q| \leq 1.
\]

Ramanujan’s general theta function \( f(a, b) \) is defined as

\[
f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}. \tag{2.1}
\]

Three special cases are defined, in Ramanujan’s notation, as

\[
\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \tag{2.2}
\]

\[
\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2}, \tag{2.3}
\]

\[
f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2}. \tag{2.4}
\]
Jacobi’s triple product identity, in Ramanujan’s notation, takes the form
\[ f(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty. \] (2.5)

Using this, the above three functions can be written as
\[ \varphi(q) = (-q; q^2)_\infty (q^2; q^2)_\infty, \] (2.6)
\[ \psi(q) = (-q; q)_\infty (q^2; q^2)_\infty = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}, \] (2.7)
\[ f(-q) = (q; q)_\infty. \] (2.8)

The equality that relates the right hand sides of both the equations for \( f(-q) \) in (2.4) and (2.8) is Euler’s pentagonal number theorem.

Another important function that plays a prominent role is given by
\[ \chi(q) := (-q; q^2)_\infty. \] (2.9)

All the above four functions satisfy a myriad of relations, most of which are listed and proved in Berndt’s books on Ramanujan’s notebooks, and we will refer to them as needed.

Last but not least, the Dedekind-eta function is defined as
\[ \eta(\tau) = q^{1/24} f(-q), \quad q = e^{2\pi i \tau}, \quad \text{Im} \tau > 0. \] (2.10)

Most of the identities that we use later on are listed here in order, for the sake of convenience.

\[ \varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2), \] (2.11)
\[ \varphi^4(q) - \varphi^4(-q) = 16q\psi^4(q^2), \] (2.12)
\[ \varphi(q)\psi(q^2) = \psi^2(q), \] (2.13)
\[ \varphi(-q) + \varphi(q^2) = 2\frac{f^2(q^3, q^5)}{\psi(q)}, \] (2.14)
\[ \varphi(-q) - \varphi(q^2) = -2q\frac{f^2(q, q^7)}{\psi(q)}, \] (2.15)
\[ \varphi(q)\varphi(-q) = \varphi^2(-q^2), \] (2.16)
\[ \varphi(q) + \varphi(-q) = 2\varphi(q^4), \] (2.17)
\[ \varphi^2(q) - \varphi^2(-q) = 8q\psi^2(q^4). \] (2.18)

All of the above identities and their proofs can be found in [2, p. 40, Entry 25] and in [2, p. 51, Example (iv)].

For \( \tau \in \mathcal{H} \), the upper half plane, and \( q = e(\tau) = e^{2\pi i \tau} \), the theta constant with characteristic \( \left[ \begin{array}{c} \varepsilon \\ \varepsilon' \end{array} \right] \in \mathbb{R} \) is defined as
\[ \theta \left[ \begin{array}{c} \varepsilon \\ \varepsilon' \end{array} \right] (\tau) = \sum_{n \in \mathbb{Z}} e \left( \frac{1}{2} (n + \varepsilon \tau) + \frac{\varepsilon'}{2} (n + \varepsilon') \right). \] (2.19)
It satisfies the following basic properties for \( l, m, n \in \mathbb{Z} \) with \( N \) positive:

\[
\theta \left[ \frac{\varepsilon}{\varepsilon'} \right](\tau) = e\left( \mp \frac{em}{2} \right) \theta \left[ \pm \varepsilon + 2l \right] \left( \pm \varepsilon' + 2m \right) \left( \frac{N^2\tau}{\varepsilon'N} \right), \quad (2.20)
\]

\[
\theta \left[ \frac{\varepsilon}{\varepsilon'} \right](\tau) = \sum_{k=0}^{N-1} \theta \left[ \frac{\varepsilon + 2k}{N\varepsilon'} \right] \left( N^2\tau \right). \quad (2.21)
\]

We also have the transformation law, for \( \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{SL}_2(\mathbb{Z}) \):

\[
\theta \left[ \frac{\varepsilon}{\varepsilon'} \right] \left( a\tau + b \\ c\tau + d \right) = \kappa \sqrt{c\tau + d} \theta \left[ \frac{ae + ce' - ac}{bd + de' + bd} \right] \left( \frac{1}{\kappa_0} \right), \quad (2.22)
\]

where

\[
\kappa = e\left( -\frac{1}{4}(ae + ce')bd - \frac{1}{8}(ab\varepsilon^2 + cd\varepsilon'^2 + 2bce\varepsilon') \right)\kappa_0,
\]

and \( \kappa_0 \) is an eighth root of unity depending only on the matrix \( \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \).

In particular, we have:

\[
\theta \left[ \frac{\varepsilon}{\varepsilon'} \right] (\tau + 1) = e\left( -\frac{\varepsilon}{4}(1 + \varepsilon) \right) \theta \left[ \frac{\varepsilon + 1}{\varepsilon + \varepsilon'} \right] \left( \tau \right), \quad (2.23)
\]

\[
\theta \left[ \frac{\varepsilon}{\varepsilon'} \right] \left( \frac{1}{\tau} \right) = e\left( -\frac{1}{8} \right) \sqrt{\tau} e\left( \frac{\varepsilon\varepsilon'}{4} \right) \theta \left[ \frac{\varepsilon'}{-\varepsilon} \right] \left( \tau \right). \quad (2.24)
\]

We also have the product formula:

\[
\theta \left[ \frac{\varepsilon}{\varepsilon'} \right] \left( \tau \right) = e\left( \frac{\varepsilon\varepsilon'}{4} \right) q^{\frac{1}{2}} \prod_{n=1}^{\infty} \left( 1 - q^n \right) (1 + e\left( \frac{\varepsilon'}{2} \right) q^{n-\frac{1+\varepsilon}{2}}) \left( 1 + e\left( -\frac{\varepsilon'}{2} \right) q^{n-\frac{1-\varepsilon}{2}} \right),
\]

which follows from Jacobi’s triple product identity.

More information about these theta constants and the above formulas, as well as their proofs, can all be found in [10, pp. 71-81]. Also see [9, pp. 143, 158-159].

### 3. Identities for \( u(\tau) \) and \( v(\tau) \)

Let us define the functions \( u(\tau) \) and \( v(\tau) \) as

\[
u(\tau) = q^{1/2} \prod_{n=1}^{\infty} \left( 1 - q^n \right)^{\frac{1}{2}}.
\]

The functions \( u(\tau) \) and \( v(\tau) \) satisfy the following identities.
Proposition 3.1. (a) If $x = u(\tau)$ and $y = u(2\tau)$, we have
$$x^4(y^4 + 1) = 2y^2.$$ 
(b) If $x = v(\tau)$ and $y = v(2\tau)$, we have
$$x^2y + x^2 + y^2 = y.$$

Remark. The curve $E : f(x, y) = 0$ defined by
$$f(x, y) = x^2y + x^2 + y^2 - y$$
is an elliptic curve with $j(E) = 1728$, so $E$ has complex multiplication by $R = \mathbb{Z}[i]$.

Proof. (a) From (2.11), we have
$$\varphi^2(-q) = 2\varphi^2(q^2) - \varphi^2(q),$$
where
$$\varphi(q) = (q; q^2)_\infty^2(q^2; q^2)_\infty$$
and $\psi(q) = (q^2; q^2)_\infty / (q; q^2)_\infty$
are as defined in (2.6) and (2.7). Squaring both sides gives us
$$\varphi^4(-q) = 4\varphi^4(q^2) - 4\varphi^2(q)\varphi^2(q^2) + \varphi^4(q).$$
Using
$$\varphi^4(q) - \varphi^4(-q) = 16q\psi^4(q^2),$$
which is (2.12), we obtain
$$\varphi^4(q^2) + 4q\psi^4(q^2) = \varphi^2(q)\varphi^2(q^2).$$
Dividing both sides by $\varphi^4(q^2)$ and using the relation $\psi^2(q) = \varphi(q)\psi(q^2)$ from (2.13) we get
$$1 + 4\frac{\psi^4(q^2)}{\varphi^4(q^2)} = \frac{\varphi^2(q)}{\varphi^2(q^2)} = \frac{\psi^2(q^2)}{\psi^2(q^2)} \cdot \frac{\varphi^4(q)}{\varphi^4(q^2)}.$$ 
(3.1)

Since
$$u(\tau) = \sqrt{2}q^{1/8} \prod_{n=1}^{\infty} (1 + q^n)^{-1/2} = \sqrt{2}q^{1/8} \frac{(q; q^2)_\infty}{(-q^2; q^2)_\infty} = \sqrt{2}q^{1/8} \frac{\psi(q)}{\varphi(q)},$$
the result follows by substituting the last equality for $u(\tau)$ into (3.1).

(b) From [9, p. 153, (9.7)] we have the following relation between $u = u(\tau)$ and $v = v(\tau)$:
$$u^4(v^2 + 1)^2 + 4v(v^2 - 1) = 0;$$ 
(3.2)
which we rewrite as $u^4 = \frac{4v(1-v^2)}{(v^2 + 1)^2}$. (See the proof of Proposition 10.1 in the Appendix.) Substituting this expression for $u^4$ into the relation $u^4(\tau)[u^4(2\tau) + 1] = 2u^2(2\tau)$, after squaring, we obtain

$$\frac{16x^2(1-x^2)^2}{(x^2 + 1)^4} \cdot \left[ \frac{4y(1-y^2)}{(y^2 + 1)^2} + 1 \right]^2 = 4 \cdot \frac{4y(1-y^2)}{(y^2 + 1)^2},$$

where $x = v(\tau), y = v(2\tau)$. Clearing the denominators gives us

$$x^2(1-x^2)^2(y^2 - 2y - 1)^4 = y(1-y^2)(y^2 + 1)^2(x^2 + 1)^4.$$ 

Now moving everything to one side and factoring the polynomial using Maple, we finally arrive at

$$(x^2y + x^2 + y^2 - y)(x^2y^2 - x^2y + y + 1)(x^2y^2 + 2xy^2 + x^2 - 4xy + y^2 - 2x + 1)\times(x^2y^2 - 2xy^2 + x^2 + 4xy + y^2 + 2x + 1) = 0.$$

From the definitions of $x$ and $y$, it is clear that $x = O(q^{1/2})$ and $y = O(q)$ as $q$ tends to 0. Hence, the first factor above (and none of the others) vanishes for $q$ sufficiently small. By the identity theorem of complex analysis, the first factor vanishes for $|q| < 1$. This proves the result. □

**Remark.** The identity in part (b) of Proposition 3.1 can be written as

$$v^2(\tau) = v(2\tau)\frac{1 - v(2\tau)}{1 + v(2\tau)}.$$

See [5, Thm. 2.2]. This is analogous to the identity for the Rogers-Ramanujan continued fraction $r(\tau)$:

$$r^5(\tau) = r(5\tau)\frac{r^4(5\tau) - 3r^3(5\tau) + 4r^2(5\tau) - 2r(5\tau) + 1}{r^4(5\tau) + 2r^3(5\tau) + 4r^2(5\tau) + 3r(5\tau) + 1}.$$ 

Also see [4, p. 167], [3, pp. 19-20].

**Proposition 3.2.** The functions $x = v^2(\tau)$ and $y = v^2(2\tau)$ satisfy the relation

$$g(x, y) = y^2 - (x^2 - 4x + 1)y + x^2 = 0.$$ 

**Proof.** For $x = v(\tau)$ and $y = v(2\tau)$, we have the relation

$$x^2 + y^2 = y(1 - x^2).$$

Squaring both sides and moving all the terms to the left side, we obtain

$$x^4 + y^4 + 4x^2y^2 - x^4y^2 - y^2 = 0.$$ 

Hence, $x = v^2(\tau)$ and $y = v^2(2\tau)$ satisfy the relation

$$g(x, y) = x^2 + y^2 + 4xy - x^2y - y = 0.$$

□
Let $A, \tilde{A}$ denote the linear fractional mappings
\[ A(x) = \frac{\sigma x + 1}{x - \sigma}, \quad \tilde{A}(x) = \frac{-x + \sigma}{\sigma x + 1}, \quad \sigma = -1 + \sqrt{2}. \] (3.3)

**Proposition 3.3.** The following identity holds:
\[ u\left(-\frac{1}{\tau}\right) = \tilde{A}(u(\tau/4)) = \frac{\bar{\sigma} u(\tau/4) + 1}{u(\tau/4) - \bar{\sigma}} = \frac{-u(\tau/4) + \sigma}{\sigma u(\tau/4) + 1}, \]
where $\bar{\sigma} = -1 - \sqrt{2}$.

**Proof.** This follows from the formula
\[ u(\tau) = e^{-2\pi i/8} \frac{\theta[3/4][1}(8\tau)}{\theta[1/4][1}(8\tau)}, \]
using the formulas (2.20), (2.21), (2.24). (Also see [10].) Namely, we have:
\[ u\left(-\frac{1}{\tau}\right) = e^{-2\pi i/8} \frac{\theta[3/4][1}(\frac{-8}{\tau})}{\theta[1/4][1}(\frac{-8}{\tau})} = \frac{\theta[1/4][1}(\frac{\tau}{8})}{\theta[1/4][1}(\frac{\tau}{8})} = \sum_{k=0}^{3} \theta \left[ \frac{1+2k}{4} \right] (2\tau), \]
which after some simplification yields
\[ \frac{\theta[1/4][1}(\frac{-8}{\tau})}{\theta[3/4][1}(\frac{-8}{\tau})} = \frac{-1 + e^{3\pi i/8} u(\tau/4) + e^{2\pi i/8} + e^{3\pi i/2}}{e^{2\pi i/8} + e^{3\pi i/2} u(\tau/4) + [1 + e^{7\pi i/8}]}.
\]
This yields that
\[ u\left(-\frac{1}{\tau}\right) = \frac{\bar{\sigma} u(\tau/4) + 1}{u(\tau/4) - \bar{\sigma}} = \frac{-u(\tau/4) + \sigma}{\sigma u(\tau/4) + 1}. \]

\[ \square \]

The set of mappings
\[ \tilde{H} = \{ x, A(x), \tilde{A}(x), -1/x \} \]
forms a group under composition. We also have the formula
\[ (\sigma x + 1)^2(\sigma y + 1)^2 f(\tilde{A}(x), \tilde{A}(y)) = 2^3 \sigma^2 f(y, x). \]

**Proposition 3.4.** The function $u(\tau)$ satisfies the following:
\[ u^2\left(-\frac{1}{8\tau}\right) = \frac{u^2(\tau) - \sigma^2}{\sigma^2 u^2(\tau) - 1}, \quad \sigma = -1 + \sqrt{2}. \] (3.4)
Proof. Replacing \( \tau \) by \( 8 \tau \) in Proposition 3.3 and squaring gives us
\[
\nu^2 \left( \frac{-1}{8\tau} \right) = \frac{(-\nu(2\tau) + \sigma)^2}{(\sigma\nu(2\tau) + 1)^2}
\]
\[
= \frac{(-\nu + \sigma)^2}{(\sigma\nu + 1)^2}
\]
\[
= \frac{\nu^2 - 2\nu \sigma + \sigma^2}{\sigma^2 \nu^2 + 2\sigma \nu + 1},
\]
where \( \nu = \nu(2\tau) \). Then, replace \( 2\sigma \) by \( 1 - \sigma^2 \) to obtain
\[
\nu^2 \left( \frac{-1}{8\tau} \right) = \frac{\nu^2 - \nu \sigma + \sigma^2}{\sigma^2 \nu^2 + \nu - \sigma^2 \nu + 1}
\]
\[
= \frac{\sigma^2 (\nu + 1) - (\nu - \nu^2)}{(\nu + 1) - \sigma^2 (\nu - \nu^2)}.
\]
Now replace \( (\nu - \nu^2) \) by \( \nu^2 (\nu + 1) \), using Proposition 3.1(b), to get the result:
\[
\nu^2 \left( \frac{-1}{8\tau} \right) = \frac{\sigma^2 (\nu + 1) - \nu^2 (\nu + 1)}{(\nu + 1) - \sigma^2 \nu^2 (\nu + 1)}
\]
\[
= \frac{(\sigma^2 - \nu^2)(\nu + 1)}{(1 - \sigma^2 \nu^2)(\nu + 1)}
\]
\[
= \frac{\nu^2 - \sigma^2}{\sigma^2 \nu^2 - 1},
\]
where \( \nu = \nu(\tau) \). \( \square \)

For later use we denote the linear fractional map which occurs in (3.4) by \( t(x) \):
\[
t(x) = \frac{x - \sigma^2}{\sigma^2 x - 1}.
\]
(3.5)
A straightforward calculation shows that
\[
(\nu^2(\nu - 1)^2(\nu^2 \nu - 1)^2 g(t(\nu), t(\nu)) = 2^5 \nu^4 g(\nu, x).
\]
(3.6)

4. The relation between \( \nu(\tau) \) and \( p(\tau) \).

In this section and the next we shall prove several identities between \( \nu(\tau) \) and the functions \( p(\tau) \) and \( b(\tau) \) defined as follows. Let \( f_2, f_1, f \) denote the Weber-Schläfli functions (see [8, p. 233], [19, p. 148]). Then the functions \( p(\tau) \) and \( b(\tau) \) are given by

\[
p(\tau) = \frac{f_2(\tau/2)^2}{f(\tau/2)^2} = 2q^{1/16} \prod_{n=1}^{\infty} \left( \frac{1 + q^{n/2}}{1 + q^{n/2-1/4}} \right)^2,
\]
(4.1)
\[
b(\tau) = \frac{f_1(\tau/2)^2}{f(\tau/2)^2} = 2 \prod_{n=1}^{\infty} \left( \frac{1 - q^{n/2-1/4}}{1 + q^{n/2-1/4}} \right)^2.
\]
(4.2)
Note that $b(\tau)$ occurs in [14, §10, (10.3)].

**Proposition 4.1.** We have the identity

$$\frac{2}{\wp(8\tau)} = \frac{1 - v^2(\tau)}{v(\tau)} = \frac{1}{v(\tau)} - v(\tau). \quad (4.3)$$

**Proof.** (See [2, pp. 221-222].) The function $v(\tau)$ satisfies

$$v(\tau) = q^{1/2} \prod_{n \geq 1} (1 - q^n)^{\frac{(3)}{n}} = q^{1/2} \prod_{n \geq 1} \frac{(1 - q^{8n-1})(1 - q^{8n-7})}{(1 - q^{8n-3})(1 - q^{8n-5})}$$

This gives that

$$\frac{1}{v(\tau)} - v(\tau) = q^{-1/2} \frac{(q^3; q^8)_\infty (q^5; q^8)_\infty (q^7; q^8)_\infty}{(q; q^8)_\infty (q^7; q^8)_\infty} - q^{1/2} \frac{(q; q^8)_\infty (q^7; q^8)_\infty}{(q^2; q^8)_\infty (q^7; q^8)_\infty}$$

Multiplying the numerator and the denominator by $(q^8; q^8)_\infty$ and applying Jacobi's triple product identity in the form

$$f(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty,$$

with $(a, b) = (-q^3, -q^5)$ for the first term in the numerator and $(a, b) = (-q, -q^7)$ for the second, we obtain

$$\frac{1}{v(\tau)} - v(\tau) = \frac{q^{1/2} (q^3; q^2)_\infty (q^5; q^2)_\infty (q^7; q^2)_\infty - q (q; q^8)_\infty (q^7; q^8)_\infty (q^2; q^8)_\infty}{q^{1/2} (q; q^2)_\infty (q^8; q^8)_\infty}$$

Now replace $q$ by $-q$ in (2.14), (2.15) and apply this to the numerator to get

$$\frac{1}{v(\tau)} - v(\tau) = \frac{\psi(-q)[\varphi(q) + \varphi(q^2)] - \psi(-q)[\varphi(q) - \varphi(q^2)]}{2 q^{1/2} (q; q^2)_\infty (q^8; q^8)_\infty}$$

This yields that

$$\frac{1}{v(\tau)} - v(\tau) = q^{-1/2} \frac{(q^2; q^2)_\infty (-q^2; q^2)_\infty (q^4; q^4)_\infty}{(-q; q^2)_\infty (q; q^2)_\infty (q^8; q^8)_\infty}$$
\[= q^{-1/2} \frac{(-q^2; q^4)_{\infty}^2 (q^2; q^4)_{\infty} (q^4; q^4)_{\infty}}{(q^2; q^4)_{\infty} (q^8; q^8)_{\infty}}\]

\[= q^{-1/2} \frac{(-q^2; q^4)_{\infty}^2 (q^4; q^4)_{\infty}}{(q^8; q^8)_{\infty}}\]

\[= q^{-1/2} \frac{(-q^2; q^4)_{\infty}^2 (q^4; q^4)_{\infty}}{(-q^4; q^4)_{\infty}^2}.\]

Since

\[\mathfrak{p}(8\tau) = 2 q^{1/2} \prod_{n \geq 1} \left( \frac{1 + q^{4n}}{1 + q^{4n-2}} \right)^2 = 2 q^{1/2} \frac{(-q^4; q^4)_{\infty}^2}{(-q^2; q^4)_{\infty}^2},\]

we get the result by substituting into the last equality. \[\square\]

**Proposition 4.2.** The function \(\mathfrak{p}(\tau)\) satisfies the identity

\[\mathfrak{p}^2(\tau)\mathfrak{p}^2(2\tau) + \mathfrak{p}^2(\tau) - 2\mathfrak{p}(2\tau) = 0.\]

**Proof.** We use the relation between \(x = \nu(\tau)\) and \(y = \nu(2\tau)\) from Proposition 3.1(b): \(x^2 = \frac{y(1-y)}{(1+y)}\). This gives

\[\left( \frac{2x}{1 - x^2} \right)^2 = \frac{4x^2}{(1 - x^2)^2} = \frac{4 \cdot \frac{x(1-y)}{1+y}}{\left(1 - \frac{x(1-y)}{1+y}\right)^2}\]

\[= \frac{4y(1-y)(1+y)}{((1+y) - y(1-y))^2}\]

\[= \frac{4y(1-y^2)}{(1 + y^2)^2}\]

\[= \frac{4y(1-y^2)}{4y^2 + (1 - y^2)^2}.\]

Now divide both the numerator and the denominator by \((1 - y^2)^2\) to obtain

\[\left( \frac{2x}{1 - x^2} \right)^2 = \frac{\frac{4y}{1-y^2}}{\frac{4y^2}{(1-y^2)^2} + 1} = \frac{2 \cdot \left( \frac{2y}{1-y^2} \right)}{\left( \frac{2y}{1-y^2} \right)^2 + 1}. \tag{4.4}\]

From Proposition 4.1, we know that

\[\mathfrak{p}(8\tau) = \frac{2\nu(\tau)}{1 - \nu^2(\tau)} = \frac{2x}{1 - x^2},\]

and

\[\mathfrak{p}(16\tau) = \frac{2\nu(2\tau)}{1 - \nu^2(2\tau)} = \frac{2y}{1 - y^2}.\]
Thus, (4.4) becomes
\[ p^2(8\tau) = \frac{2p(16\tau)}{p^2(16\tau) + 1}. \]
Replacing \( \tau \) by \( \tau/8 \) and rearranging gives us the result. \( \Box \)

**Proposition 4.3.** a) The functions \( x = b(\tau) \) and \( y = b(2\tau) \) satisfy the relation
\[ x^2y^2 + 4y^2 - 16x = 0. \]
b) The following identity holds between \( x = b(\tau) \) and \( z = b(4\tau) \):
\[ (b(\tau) + 2)b^4(4\tau) = 2^8(b^3(\tau) + 4b(\tau)). \]

**Proof.** a) On putting \( 4\tau \) for \( \tau \) in \( x \), we have
\[ b(4\tau) = 2\prod_{n=1}^{\infty} \left( 1 - \frac{q^{2n-1}}{q^{2n-1} + 1} \right)^2 = 2 \frac{(q; q^2)_{\infty}^2}{(-q; q^2)_{\infty}^2} \frac{\varphi(-q)}{\varphi(q)}. \]
From (2.11), we have
\[ \varphi^2(-q) + \varphi^2(q) = 2\varphi^2(q^2). \]
Multiplying both sides by \( \varphi^2(-q^2) = \varphi(q)\varphi(-q) \) from (2.16), we obtain
\[ \varphi^2(-q)\varphi^2(-q^2) + \varphi^2(q)\varphi^2(-q^2) = 2\varphi(q)\varphi(-q)\varphi^2(q^2). \]
Now dividing both sides by \( \varphi^2(q)\varphi^2(q^2) \) gives us
\[ \frac{\varphi^2(-q)}{\varphi^2(q)} \cdot \frac{\varphi^2(-q^2)}{\varphi^2(q^2)} + \frac{\varphi^2(-q^2)}{\varphi^2(q^2)} = 2 \frac{\varphi(-q)}{\varphi(q)}. \]
Hence, we see that \( x = b(4\tau) \) and \( y = b(8\tau) \) satisfy the relation
\[ x^2y^2 + 4y^2 - 16x = 0. \]
Now replace \( \tau \) by \( \tau/4 \).

b) From (2.17), upon taking fourth powers, we get
\[ \left[ \varphi(-q) + \varphi(q) \right]^4 = 16 \varphi^4(q^4). \]
Multiplying both sides by \( \varphi^4(-q^4)/[\varphi^4(q)\varphi^4(q^4)] \) gives us
\[ \frac{\left[ \varphi(-q) + \varphi(q) \right]^4}{\varphi^4(q)} \cdot \frac{\varphi^4(-q^4)}{\varphi^4(q^4)} = 16 \frac{\varphi(-q)\varphi(q)}{\varphi^4(q)}. \]
Then using (2.16) twice for the right side, we obtain
\[ \frac{\left[ \varphi(-q) + \varphi(q) \right]^4}{\varphi^4(q)} \cdot \frac{\varphi^4(-q^4)}{\varphi^4(q^4)} = 16 \frac{\varphi(-q)\varphi(q)}{\varphi^4(q)} \cdot \varphi^2(q^2). \]
Now use (2.11) for the last factor on the right side to get
\[ \frac{\left[ \varphi(-q) + \varphi(q) \right]^4}{\varphi^4(q)} \cdot \frac{\varphi^4(-q^4)}{\varphi^4(q^4)} = 8 \frac{\varphi(-q)}{\varphi^3(q)} \cdot \left[ \varphi^2(-q) + \varphi^2(q) \right]. \]
This implies that
\[
\left[ \frac{\varphi(-q)}{\varphi(q)} + 1 \right]^4 \cdot \left[ \frac{\varphi(-q^4)}{\varphi(q^4)} \right]^4 = 8 \cdot \frac{\varphi(-q)}{\varphi(q)} \cdot \left( \frac{\varphi(-q)}{\varphi(q)} \right)^2 + 1.
\]

The result follows on multiplying through by \(2^8\) and substituting
\[
b(4\tau) = 2 \frac{\varphi(-q)}{\varphi(q)} \quad \text{and} \quad b(16\tau) = 2 \frac{\varphi(-q^4)}{\varphi(q^4)}
\]
into the above equation, and then replacing \(\tau\) by \(\tau/4\).

5. The relation between \(v(\tau)\) and \(b(\tau)\).

We begin this section by proving the following identity.

**Proposition 5.1.**

\[
\frac{(v^2(\tau) + 1)^2}{v^4(\tau) - 6v^2(\tau) + 1} = \frac{4}{b^2(4\tau)}.
\]

**Proof.** We prove (5.1) using the identity relating the Weber-Schläfli functions from [20, p. 86, (12)] (see also [8, p. 234, (12.18)]):

\[
f_8^8(\tau) + f_8^8(\tau) = f_8^8(\tau).
\]

From the definitions (4.1) and (4.2) of \(p(\tau)\) and \(b(\tau)\), this identity translates to

\[
b^4(4\tau) = 1 - p^4(4\tau).
\]

Using the result of Proposition 4.1, we write this equation as

\[
b^4(4\tau) = 1 - \left( \frac{2 v(\tau/2)}{1 - v^2(\tau/2)} \right)^4 = 1 - \frac{16 v^4(\tau/2)}{(1 - v^2(\tau/2))^4}.
\]

Setting \(x = v(\tau/2)\) and \(y = v(\tau)\) and using the relation between \(x\) and \(y\) from Proposition 3.1(b) in the form \(x^2 = \frac{y(1-y)}{(1+y)}\) gives that

\[
b^4(4\tau) = 1 - \frac{16 x^4}{(1 - x^2)^4} = 1 - \frac{16 \left( \frac{y(1-y)}{(1+y)} \right)^2}{\left( 1 - \frac{y(1-y)}{(1+y)} \right)^4}
\]

\[= 1 - \frac{16 y^2(1 - y^2)^2}{(1 + y^2)^4} = \frac{(y^2 + 1)^4 - 16 y^2(y^2 - 1)^2}{(y^2 + 1)^4}
\]

\[= \frac{(y^2 - 1)^2 + 4y^2}{(y^2 + 1)^4}
\]

\[= \frac{(y^2 - 1)^2 - 4y^2}{(y^2 + 1)^4}
\]
\[
\frac{(y^4 - 6y^2 + 1)^2}{(y^2 + 1)^4},
\]
which is equivalent to (5.1). (The plus sign holds on taking the square-root because \( b(i\infty) = 2, v^2(i\infty) = 0 \).) \( \square \)

Proposition 5.1 will now be used to prove the following formula for the function \( j(\tau) \) in terms of \( v(\tau) \).

**Proposition 5.2.** If \( v = v(\tau) \) and \( \tau \) lies in the upper half-plane, we have

\[
j(\tau) = \frac{(v^{16} + 232v^{14} + 732v^{12} - 1192v^{10} + 710v^8 - 1192v^6 + 732v^4 + 232v^2 + 1)^3}{v^2(v^2 - 1)^2(v^2 + 1)^4(v^4 - 6v^2 + 1)^8}.
\]

**Proof.** Let

\[
G(x) = \frac{(x^2 - 16x + 16)^3}{x(x - 16)}.
\]

Then from [14, p. 1967, (2.8)] the function

\[
\alpha(\tau) = \zeta_8^{-1} \eta(\tau/4)^2 / \eta(\tau)^2, \quad \zeta_8 = e^{2\pi i / 8}, \tag{5.2}
\]

satisfies the relation

\[
j(\tau) = \frac{(\alpha^8 - 16\alpha^4 + 16)^3}{\alpha^4(\alpha^4 - 16)} = G(\alpha^4(\tau)). \tag{5.3}
\]

Moreover, \( \alpha(\tau) \) and \( b(\tau) \) satisfy

\[
16\alpha^4(\tau) + 16b^4(\tau) = \alpha^4(\tau)b^4(\tau),
\]

so that

\[
\alpha^4(\tau) = \frac{16b^4(\tau)}{b^4(\tau) - 16}. \tag{5.4}
\]

Setting \( b = b(\tau) \), we substitute for \( \alpha = \alpha(\tau) \) in (5.3) and find that

\[
j(\tau) = G \left( \frac{16b^4}{b^4 - 16} \right) = \frac{(b^8 + 224b^4 + 256)^3}{b^4(b^4 - 16)^4}, \quad b = b(\tau).
\]

Now replace \( \tau \) by \( 4\tau \) and use (5.1) to replace \( b^4(4\tau) \) by

\[
b^4(4\tau) = \frac{16(v^4 - 6v^2 + 1)^2}{(v^2 + 1)^4},
\]

giving

\[
j(4\tau) = \frac{(v^{16} - 8v^{14} + 12v^{12} + 8v^{10} + 230v^8 + 8v^6 + 12v^4 - 8v^2 + 1)^3}{v^8(v^2 + 1)^4(v^2 - 1)^8(v^4 - 6v^2 + 1)^2}, \tag{5.5}
\]

with \( v = v(\tau) \). Replacing \( v(\tau) \) by \( \hat{A}(v(-1/4\tau)) \) from Proposition 3.3 gives that

\[
j(4\tau) = j_2(x^2),
\]
where \( x = v(-1/4\tau) \) and \( j_2(x) \) is the rational function
\[
j_2(x) = \frac{(x^8 + 232x^7 + 732x^6 - 1192x^5 + 710x^4 - 1192x^3 + 732x^2 + 232x + 1)^3}{x(x-1)^2(x+1)^4(x^2 - 6x + 1)^8}.
\] (5.6)

Finally, replace \( \tau \) by \( \tau/4 \) to give that
\[
j(\tau) = j_2(v^2(-1/\tau)),
\]
which implies that \( j_2(v^2(\tau)) = j(-1/\tau) = j(\tau) \), completing the proof. \( \square \)

We highlight the relation
\[
j(\tau) = j_2(v^2(\tau)),
\] (5.7)
which we will make use of in Section 7. Using the linear fractional map \( t(x) \) from (3.5) and the identity \( v^2(-1/8\tau) = t(v^2(\tau)) \) in (3.4) yields
\[
j\left(\frac{-1}{8\tau}\right) = j_2\left(v^2\left(\frac{-1}{8\tau}\right)\right) = j_2(t(v^2(\tau))).
\]

A calculation on Maple shows that
\[
j_{22}(x) = j_2(t(x)) = \frac{(x^8 - 8x^7 + 12x^6 + 8x^5 - 10x^4 + 8x^3 + 12x^2 - 8x + 1)^3}{x^8(x-1)^4(x+1)^2(x^2 - 6x + 1)}.
\]

Therefore,
\[
j\left(\frac{-1}{8\tau}\right) = j_{22}(v^2(\tau)).
\] (5.8)

We take this opportunity to prove the following known identity (see [9, p. 154]) from the results we have established so far.

**Proposition 5.3.**
\[
v^{-2}(\tau) + v^2(\tau) - 6 = \frac{\eta^4(\tau)\eta^2(4\tau)}{\eta^2(2\tau)\eta^4(8\tau)}.
\] (5.9)

**Proof.** We will show that (5.9) follows from (5.1). We first have that
\[
v^{-2}(\tau) + v^2(\tau) - 6 = \frac{v^4(\tau) - 6v^2(\tau) + 1}{v^2(\tau)} = \frac{8}{\left(\frac{v^2(\tau)+1}{v^4(\tau)-6v^2(\tau)+1}\right)} - 1
\] \[= \frac{8}{\left(\frac{4}{b^2(4\tau)}\right)} - 1 = \frac{8b^2(4\tau)}{4 - b^2(4\tau)}, \]
by (5.1). Using the expression \( b(4\tau) = 2\varphi(-q)/\varphi(q) \) from the proof of Proposition 4.3a) and (2.18) gives

\[
v^{-2}(\tau) + v^2(\tau) - 6 = \frac{8\left(\frac{4\varphi(-q)}{\varphi^2(q)}\right)^2}{4 - \left(\frac{4\varphi(-q)}{\varphi^2(q)}\right)^2} = \frac{8\varphi^2(-q)}{\varphi^2(q) - \varphi^2(-q)} = \frac{8\varphi^2(-q)}{8\varphi^2(q^4)}.
\]

Now putting \( \varphi(-q) = (q; q^2)_{\infty}^2(q^2; q^2)_{\infty} \) and \( \psi(q) = (q^2; q^2)_{\infty}^2/(q; q)_{\infty}^2 \),

\[
v^{-2}(\tau) + v^2(\tau) - 6 = \varphi^2(-q) \cdot \left(\frac{1}{q\psi^2(q^4)}\right)
\]

\[
= (q; q^2)_{\infty}^4(q^2; q^2)_{\infty}^2 \cdot \left(\frac{(q^4; q^8)_{\infty}^2}{q(q^8; q^8)_{\infty}^2}\right)
\]

\[
= \frac{(q; q)_{\infty}^4}{(q^2; q^2)_{\infty}^2} \cdot \frac{(q^4; q^8)_{\infty}^2}{q(q^8; q^8)_{\infty}^2}
\]

\[
= \frac{q^{1/6}(q; q)_{\infty}^4 \cdot q^{1/3}(q^4; q^4)_{\infty}^2}{q^{1/6}(q^2; q^2)_{\infty}^2 \cdot q^{1/3}(q^8; q^8)_{\infty}^2}
\]

\[
= \frac{\eta^4(\tau)\eta^2(4\tau)}{\eta^2(2\tau)\eta^4(8\tau)},
\]

using that \( \eta(\tau) = q^{1/24}(q; q)_{\infty} \). \( \square \)

6. The field generated by \( \nu(w/8) \).

As in the Introduction, let \( -d \equiv 1 \pmod{8} \) and set \( -d = \mathfrak{d}_Kf^2 \), where \( \mathfrak{d}_K \) is the discriminant of the field \( K = \mathbb{Q}(\sqrt{-d}) \). Further, let \( 2 \equiv \varphi_2 \varphi'_2 \) in the ring of integers \( \mathcal{O}_f \) of \( K \). We denote by \( \Sigma_f \) the ray class field of conductor \( f \) over \( K \) and \( \Omega_f \) the ring class field of conductor \( f \) over \( K \).

In this section we take \( \tau = w/8 \), where

\[
w = \frac{a + \sqrt{-d}}{2}, \quad \text{with } a^2 + d \equiv 0 \pmod{2^5}, \ (N(w), f) = 1. \quad (6.1)
\]

For this value of \( w \),

\[ b^4(8\tau) = b^4(w) \]

is the fourth power of the number

\[ \beta = i^{-a}b(w) \quad (6.2) \]

from [14, (10.3), Thms. 10.6, 10.7]. We also need the number \( \pi \) from [14, (10.2),(10.9)], which is given by

\[ \pi = i^t \left( \frac{\mathfrak{f}^2(w/2)^2}{\mathfrak{f}(w/2)^2} \right)^t = i^t \mathfrak{p}(w), \]
\[ \hat{c} \equiv a \left( 2 - \frac{a^2 + d}{16} \right) \pmod{4}. \]

(We have replaced \( v \) in the formulas of [14] by \( a \) and \( a \) by \( \hat{c} \).) But here the integer \( a^2 + d \) is divisible by 32, by (6.1), so \( \hat{c} \) is even. Replacing \( \hat{c} \) by the integer \( c = \hat{c}/2 \), satisfying
\[ c \equiv 1 - \frac{a^2 + d}{32} \pmod{2} \]
yields
\[ \pi = (-1)^c p(w), \quad w = \frac{a + \sqrt{-d}}{2}. \] (6.3)

It follows from the results of [14] that \( \xi = \beta/2 \) and \( \pi \) lie in the ring class field \( \Omega_f \) of the quadratic field \( K = \mathbb{Q}(\sqrt{-d}) \) (where \( -d = b_K f^2 \) and \( b_K \) is the discriminant of \( K/\mathbb{Q} \)) and \( \xi^4 + \pi^4 = 1 \). Furthermore, \( \mathbb{Q}(\pi) = \mathbb{Q}(\pi^4) = \Omega_f \). We also note that \( (\xi) = \varphi_2 \) and \( (\pi) = \varphi_2 \) in \( \Omega_f \), so that \( (\xi \pi) = (2) \).

From (4.3) and (6.3) we have that
\[ (-1)^c \frac{2}{\pi} = \frac{1}{v(w/8)} - v(w/8) = \frac{1 - v^2(w/8)}{v(w/8)}. \] (6.4)

In particular, \( v(w/8) \) satisfies a quadratic equation over \( \Omega_f \) and the map \( \rho : v(w/8) \to \frac{-1}{v(w/8)} \) leaves the right side of (6.4) invariant. On squaring (6.4), we see that \( X = v^2(w/8) \) satisfies the equation
\[ X^2 - \left(2 + \frac{4}{\pi^2}\right)X + 1 = 0, \] (6.5)
and therefore
\[ v^2(w/8) = \frac{\pi^2 + 2 \pm 2\sqrt{\pi^2 + 1}}{\pi^2} = \left(1 \pm \sqrt{1 + \frac{1}{\pi^2}}\right)^2. \]

Hence
\[ v(w/8) = \pm \frac{1 \pm \sqrt{1 + \pi^2}}{\pi}. \] (6.6)

It follows from these expressions that
\[ \Omega_f(v(w/8)) = \Omega_f(v^2(w/8)) = \Omega_f(\sqrt{1 + \pi^2}). \]

We now prove the following.

**Theorem 6.1.** If
\[ w = \frac{a + \sqrt{-d}}{2}, \quad \text{with } a^2 + d \equiv 0 \pmod{2^5}, \]
and \( \varphi_2 = (2, w) \) in \( R_K \), then the field \( \mathbb{Q}(v(w/8)) = \mathbb{Q}(\sqrt{1 + \pi^2}) \) coincides with the class field \( \Sigma_{\varphi_2} \Omega_f \) over \( K = \mathbb{Q}(\sqrt{-d}) \). The units \( v(w/8) \) and \( v^2(w/8) \) have degree \( 4h(-d) \) over \( \mathbb{Q} \).
Proof. Let $\Lambda = \mathbb{Q}(\sqrt{1 + \pi^2})$. It is clear that $\Lambda$ contains the ring class field $\Omega_f$, since $\mathbb{Q}(\pi^2) = \Omega_f$. We use the fact that $1 + \pi^2 \equiv \wp' \pmod{\Omega_f}$. From this fact it is clear that $1 + \pi^2$ is not a square in $\Omega_f$, since $\wp'$ is unramified in $\Omega_f/K$. Hence, $[\Lambda : \Omega_f] = 2$. Further, the prime divisors $q$ of $\wp'$ in $\Omega_f$ are certainly ramified in $\Lambda$. Equation (6.5) implies that $x = v^2(w/8)$ satisfies $(x - 1)^2/(4x) = 1/\pi^2$, and therefore $\mathbb{Q}(v^2(w/8)) = \mathbb{Q}(\sqrt{1 + \pi^2})$. This implies that $[\mathbb{Q}(v^2(w/8)) : \mathbb{Q}] = 4h(-d)$, since

$$[\Lambda : \mathbb{Q}] = [\Lambda : \Omega_f][\Omega_f : K][K : \mathbb{Q}] = 4h(-d).$$

Since $v^2(\tau)$ is a modular function for $\Gamma_1(8)$ ([9, p.154]), it follows from Schertz [19, Thm. 5.1.2] that $v^2(w/8) \in \Sigma_f$, the ray class field of conductor $8f$ over $K$. More precisely, $v^2(w/8) \in L_{\mathcal{O},8}$, where $L_{\mathcal{O},8} = \Sigma_f\Omega_f$ is an extended ring class field corresponding to the order $\mathcal{O} = \mathcal{R}_d$. See [8, p. 315]. Thus, $\Lambda \subset L_{\mathcal{O},8}$ is an abelian extension of $K$, whose conductor $\mathfrak{f}$ divides $8f$ in $K$. The discriminant of the polynomial $X^2 - (1 + \pi^2)$ is of course $4(1 + \pi^2) \equiv \wp^4\wp' \pmod{\mathfrak{p}}$. Since the ramification index of each $q | \wp'$ is $e_q = 2$ in $\Lambda/\Omega_f$, Dedekind’s discriminant theorem says that at least $\wp^2$ divides the discriminant $\mathfrak{b} = \mathfrak{b}_{\Lambda/\Omega_f}$, and since the power of $q$ in $\mathfrak{b}$ is odd and at most 3 ($\Omega_f/K$ is unramified over 2), it follows that $\wp^3$ exactly divides $\mathfrak{b}$. We claim now that $\wp$ is unramified in $\Lambda$.

From above $x = v^2(w/8)$ satisfies $(x - 1)^2 - 4/x^2 = 0$. Thus $x_1 = x - 1$ satisfies $h(x_1) = 0$, with

$$h(X) = X^2 - \frac{4}{\pi^2}(X + 1), \quad \text{disc}(h(X)) = \frac{16}{\pi^4} + \frac{4}{\pi^2},$$

where the ideal $\left(\frac{16}{\pi^4}\right) = \left(\frac{2}{\pi}\right)^4 = (\xi)^4 = \wp^4 \pmod{\mathfrak{p}}$ is not divisible by $\wp^2$. This shows that $\text{disc}(h(X))$ is not divisible by $\wp^2$ and therefore that $\wp$ is unramified in $\mathbb{Q}(v^2(w/8))$. Thus $\mathfrak{b} = \wp^3$.

Now $[\Sigma_f : \Sigma_1] = \frac{1}{2} \phi_K(\wp_2^3\wp_2^6) = 8$, where $\phi_K$ is the Euler function for the quadratic field $K$, and $\mathbb{Q}(\xi/\phi) \subset \Sigma_f$. Since the prime divisors of 2 do not ramify in $\Omega_f$, we have that $\Omega_f \cap \Sigma_f = \Sigma_1$ and therefore

$$[L_{\mathcal{O},8} : \Omega_f] = [\Sigma_f : \Omega_f] = [\Sigma_8 : \Sigma_1] = 8,$$

from which we obtain

$$\text{Gal}(\Sigma_8\Omega_f/\Omega_f) \cong \text{Gal}(\Sigma_8/\Sigma_1).$$

By this isomorphism the intermediate fields $L\Omega_f$ of $\Sigma_8\Omega_f/\Omega_f$ are in $1 - 1$ correspondence with the intermediate fields $L$ of $\Sigma_8/\Sigma_1$.

The ray class field $\Sigma_{\wp^2\wp^6,\wp^3}$ has degree 4 over the Hilbert class field $\Sigma_1$, and two of its quadratic subfields are $\Sigma_{\wp^2}$ and $\Sigma_{\wp^2\wp^6} = \Sigma_4 = \Sigma_1(i)$. It follows that $\text{Gal}(\Sigma_{\wp^2\wp^6,\wp^3}/\Sigma_1) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and the third quadratic subfield has conductor equal to $\mathfrak{f}' = \wp_2^5\wp_2^3$ over $K$. The other quadratic intermediate fields of
\( \Sigma_8 / \Sigma_1 \) are \( \Sigma_1(\sqrt{2}) \) and \( \Sigma_1(\sqrt{-2}) \), both of which have conductor \((8) = 2^3\mathfrak{p}_2^3\) over \( K \), the field \( \Sigma_{\varphi_2^3}^1 \), and a field whose conductor over \( K \) is \( \varphi_2^1\mathfrak{p}_2^3 \). Hence, \( L = \Sigma_{\varphi_2^3}^1 \) is the only quadratic intermediate field whose conductor is not divisible by \( \varphi_2 \). This proves that \( \mathbb{Q}(v^2(8)) = \Sigma_{\varphi_2^3}^1 \Omega_f \) and (6.6) shows that \( \mathbb{Q}(v(w/8)) = \mathbb{Q}(v^2(w/8)) = \Sigma_{\varphi_2^3}^1 \Omega_f \).

**Corollary 6.2.** The field \( \mathbb{Q}(v(w/8)) = \Sigma_{\varphi_2^3}^1 \Omega_f \) is the inertia field for the prime ideal \( \varphi_2 \) in the extension \( L_{\mathfrak{O}_8} / K = \Sigma_8 \Omega_f / K \).

**Proof.** The above proof implies that \( \text{Gal}(\Sigma_8 \Omega_f / \Omega_f) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \), since there are 7 quadratic intermediate fields. Any subfield containing \( \Omega_f \) which properly contains \( \Sigma_{\varphi_2^3}^1 \) must also contain another quadratic subfield, in which \( \varphi_2 \) must ramify.

**Corollary 6.3.** If \(-d \equiv 1 \pmod{8}\) and \( w \) is given by (6.1), then the quantity
\[
A = \frac{\eta^2(w/8)\eta(w/2)}{\eta(w/4)\eta^2(w)}
\]
generates the class field \( \Sigma_{\varphi_2^3}^1 \Omega_f \) for \( K = \mathbb{Q}(\sqrt{-d}) \) over \( \mathbb{Q} \).

**Proof.** We appeal to equation (5.9). Setting \( \eta = v(w/8) \), first use the equation preceding (6.6) to see that
\[
A^2 = \eta^{-2} + \eta^2 - 6 = \frac{\pi^2 + 2 + 2\sqrt{1 + \pi^2}}{\pi^2} + \frac{\pi^2 + 2 + 2\sqrt{1 + \pi^2}}{\pi^2} - 6 = 4 \frac{1 - \pi^2}{\pi^2}.
\]
This gives that \( A = \pm 2 \frac{\sqrt{1 - \pi^2}}{\pi} \). Since \( \sqrt{1 - \pi^2} \sqrt{1 + \pi^2} = \sqrt{1 - \pi^4} = \pm \xi^2 \in \Omega_f \) and \( Q(A^2) = \Omega_f \), we get that \( Q(A) = Q(\sqrt{1 + \pi^2}) = \Sigma_{\varphi_2^3}^1 \Omega_f \), by the result of Theorem 6.1.

The fact that \( v^2(w/8) \in L_{\mathfrak{O}_8} \) in the above proof is derived in [8, p. 317] using Shimura’s Reciprocity Law. We can give a more elementary proof of this fact by showing that \( \sqrt{1 + \pi^2} \in L_{\mathfrak{O}_8} \), as follows. We focus on the elliptic curve
\[
E_1(\alpha) : Y^2 + XY + \frac{1}{\alpha^4} Y = X^3 + \frac{1}{\alpha^4} X^2,
\]
which is the Tate normal form for a point of order 4, with
\[
\alpha^4 = \alpha(w)^4 = - \left( \frac{\eta(w/4)}{\eta(w)} \right)^8,
\]
as in (5.2). From [14, (2.10), Prop. 3.2, p. 1970], the curve \( E_1 = E_1(\alpha) \) has complex multiplication by the order \( \mathfrak{O} = R_{-d} \) of discriminant \(-d \) in \( K \). Now,
with $\beta = i^{-a}b(w)$ as in (6.2),

$$\frac{1}{\alpha^4} = \frac{\beta^4 - 16}{16\beta^4} = \frac{1}{16} - \frac{1}{\beta^4},$$

and Lynch [13] has given explicit expressions for the points of order 8 on $E_1$ in terms of $\beta$. Lynch [13, Prop. 3.3.1, p. 38] defines the following expressions:

$$b_1 = \frac{\beta\sqrt{2} + (\beta^2 + 4)^{1/2} + (\beta^2 - 4)^{1/2}}{2\beta\sqrt{2}},$$

$$b_2 = \frac{\beta\sqrt{2} + (\beta^2 + 4)^{1/2} - (\beta^2 - 4)^{1/2}}{2\beta\sqrt{2}},$$

$$b_3 = \frac{\beta\sqrt{2} - (\beta^2 + 4)^{1/2} + (\beta^2 - 4)^{1/2}}{2\beta\sqrt{2}},$$

$$b_4 = \frac{\beta\sqrt{2} - (\beta^2 + 4)^{1/2} - (\beta^2 - 4)^{1/2}}{2\beta\sqrt{2}}.$$  

With these expressions, Lynch shows [13, Thm. 3.3.1, p. 41] that the points $(X, Y) = P_1 = (b_1 b_3 b_4, -b_1 b_3 b_4)$ and $P_2 = (b_2 b_3 b_4, -b_2 b_3 b_4)$ are points of order 8 on $E_1(\alpha)$. By [11, Satz 2] or [14, Prop. 6.4] the corresponding Weber functions satisfy

$$g_2 g_3 \Delta \left( X(P_i) + \frac{4b + 1}{12} \right) \in \Sigma_8 \Omega_f, \quad b = \frac{1}{\alpha^4}.$$  

(See [14, (6.1)]. The expression inside the parentheses arises from putting the curve $E_1(\alpha)$ in standard Weierstrass form.) As in [14, p. 1976], $b, g_2, g_3, \Delta \in \Omega_f$, so that $X(P_i) = b_1 b_3 b_4 \in L_{O,8}$ for $i = 1, 2$. This implies that

$$(b_1 + b_2) b_3 b_4 = \left( \frac{\sqrt{2}\beta + (\beta^2 + 4)^{1/2}}{\sqrt{2}\beta} \right) \left( \frac{\beta^2 + 4 - \sqrt{2}\beta(\beta^2 + 4)^{1/2}}{4\beta^2} \right)$$

$$= \frac{4 - \beta^2}{4\sqrt{2}\beta^3} (\beta^2 + 4)^{1/2}$$

lies in $L_{O,8}$. But we know that $4 - \beta^2 \neq 0$. In addition, $\sqrt{2} \in Q(\xi_8) \subset \Sigma_8$ and $\beta \in \Omega_f$, so that $(\beta^2 + 4)^{1/2} = 2\sqrt{\xi_2} + 1 \in L_{O,8}$, with $\xi = \beta/2$. Now $\pi$ and $\xi$ are conjugate over $Q$, hence $\pm \sqrt{1 + \pi^2}$ is conjugate to $\sqrt{1 + \xi^2}$ over $Q$. Since $\Sigma_8 \Omega_f$ is normal over $Q$, this implies that $\sqrt{1 + \pi^2} \in L_{O,8}$, which proves the assertion.

**Proposition 6.4.** Assume $c$ in (6.3) is odd. The map $A(x) = \frac{\sigma x + 1}{x - \sigma}$ (see (3.3)) fixes the set of conjugates of $v(w/8)$. If $f_d(x)$ is the minimal polynomial of $v(w/8)$ over
Q, then

\[(x - \sigma)^{4h(-d)} f_d(A(x)) = 2^{3h(-d)} \sigma^{2h(-d)} f_d(x).\]

**Proof.** Note that (6.4) implies that the minimal polynomial of \(v(w/8)\) is

\[f_d(x) = 2^{-h(-d)(x^2 - 1)^{2h(-d)}} b_d \left( \frac{2x}{1 - x^2} \right), \quad \text{(6.7)}\]

where \(b_d(x)\) is the minimal polynomial of \(\pi\). Note that the degree of \(b_d(x)\) is \(2h(-d)\) and the constant term of \(b_d(x)\) is

\[N_{\Omega_j/\mathbb{Q}(\pi)} = N_{\Omega_j/\mathbb{Q}(\varphi_2)} = N_{K/\mathbb{Q}(\varphi_2^{h(-d)})} = 2^{h(-d)}\]

from [14]. Thus, \(\deg(f_d(x)) = 4h(-d)\), which implies by Theorem 6.1 that \(f_d(x)\) is irreducible.

We use (6.7) to prove the proposition, as follows. Setting \(h = h(-d)\) and assuming \(c\) is odd, we have that

\[(x - \sigma)^{4h} f_d(A(x)) = 2^{-h(x - \sigma)^{4h}(A(x)^2 - 1)^{2h}} b_d \left( \frac{2A(x)}{A(x)^2 - 1} \right)\]

\[= 2^{-h(x - \sigma)^{4h}} \left( \frac{-2\sigma(x^2 - 2x - 1)}{(x - \sigma)^2} \right)^{2h} b_d \left( \frac{-x^2 + 2x - 1}{x^2 - 2x - 1} \right)\]

\[= 2^h \sigma^{2h} (x^2 - 2x - 1)^{2h} b_d \left( \frac{P(x) + 1}{P(x) - 1} \right),\]

where

\[P(x) = \frac{2x}{x^2 - 1} \quad \text{and} \quad \frac{P(x) + 1}{P(x) - 1} = -\frac{x^2 + 2x - 1}{x^2 - 2x - 1} = R(x).\]

We also know from [14] that the map \(x \to \frac{x + 1}{x - 1}\) permutes the roots of \(b_d(x)\) and

\[(x - 1)^{2h} b_d \left( \frac{x + 1}{x - 1} \right) = 2^h b_d(x).\]

This gives that \(b_d \left( \frac{P(x) + 1}{P(x) - 1} \right) = (P(x) - 1)^{-2h} 2^h b_d(P(x))\) and therefore that

\[(x - \sigma)^{4h} f_d(A(x)) = 2^h \sigma^{2h} (x^2 - 2x - 1)^{2h} (P(x) - 1)^{-2h} 2^h b_d(P(x))\]

\[= 2^h \sigma^{2h} (x^2 - 2x - 1)^{2h} \left( \frac{x^2 - 1}{x^2 - 2x - 1} \right)^{2h} b_d(P(x))\]

\[= 2^{3h} \sigma^{2h} (x^2 - 1)^{2h} b_d(P(x))\]

\[= 2^{3h} \sigma^{2h} f_d(x).\]
We also check that
\[
x^4 f_d \left( \frac{-1}{x} \right) = 2^{-h} x^4 \left( \frac{1}{x^2} - 1 \right)^{2h-d} b_d(P(-1/x)) = 2^{-h}(x^2 - 1)^{2h} b_d(P(x)) = f_d(x).
\]
We conclude the following. Recall the definition of \( \tilde{A}(x) \) from (3.3).

**Proposition 6.5.** If \( c \) is odd, the mappings in the group
\[
\tilde{H}_1 = \{ x, A(x), \tilde{A}(x), -1/x \}
\]
permute the roots of \( f_d(x) \).

Now let \( c \) be even, \( \delta = 1 + \sqrt{2} \), and \( B(x) = \frac{\delta x + 1}{x - \delta} = \frac{x + \sigma}{\sigma x - 1} = -\tilde{A}(-x) \).

Then we have
\[
(x - \delta)^4 f_d(B(x)) = 2^{-h}(x - \delta)^{4h}(B^2(x) - 1)^{2h} b_d \left( \frac{2B(x)}{1 - B^2(x)} \right)^{2h} b_d \left( -\frac{x^2 - 2x - 1}{x^2 + 2x - 1} \right)
\]
\[
= 2^{h} \delta^{2h}(x^2 + 2x - 1)^{2h} b_d \left( \frac{2x + 1}{1 - x^2} \right) b_d \left( \frac{2x - 1}{1 - x^2} \right)
\]
\[
= 2^{h} \delta^{2h}(x^2 + 2x - 1)^{2h} \cdot 2^{h} \left( \frac{2x}{1 - x^2} - 1 \right)^{-2h} b_d \left( \frac{2x}{1 - x^2} \right)
\]
\[
= 2^{2h} \delta^{2h} \cdot (x^2 - 1)^{2h} b_d \left( \frac{2x}{1 - x^2} \right)
\]
\[
= 2^{3h} \delta^{2h} f_d(x).
\]

Setting \( \tilde{B}(x) = B(-1/x) = \frac{-\sigma x + 1}{x + \sigma} = -A(-x) \), we have the following.

**Proposition 6.6.** If \( c \) is even, the mappings in the group
\[
\tilde{H}_0 = \{ x, B(x), \tilde{B}(x), -1/x \}
\]
permute the roots of \( f_d(x) \).

**7. The diophantine equation.**

From (3.4) we know that \( (X, Y) = (v(w/8), v(-1/w)) \) is a solution of the diophantine equation
\[
E_2 : X^2 + Y^2 = \sigma^2(1 + X^2Y^2), \quad \sigma = -1 + \sqrt{2}.
\]
This seems to be an analogue of the equation $C_5$ in [17]. Set
\[ F_2(X, Y) = X^2 + Y^2 - \sigma^2(1 + X^2Y^2). \]

Then
\[ (\sigma Y + 1)^2F_2(X, \tilde{A}(Y)) = 4\sqrt{2}\sigma^2(X^2Y + X^2 + Y^2 - Y) = 4\sqrt{2}\sigma^2 f(X, Y). \]

Since
\[ \tilde{A}(x) = \frac{-x + \sigma}{\sigma x + 1} = \frac{-\delta x + 1}{x + \delta}, \quad \delta = \frac{1}{\sigma} = 1 + \sqrt{2}, \]
the linear fractional map $\tilde{A}(x)$ is the analogue of the map $T(x)$ in [17, p. 1199].

Considering Thm. 5.1 in [17, p. 1205] suggests the following conjecture.

**Conjecture 7.1.** Assume $c$ is odd. If $\tau_2 = \left( \frac{\Sigma_{\varphi_2^2 \Omega f/K}}{\varphi_2^2} \right)$, then
\[ -v(-1/w) = \tilde{A}(v(w/8)^{\tau_2}) = \frac{-v(w/8)^{\tau_2} + \sigma}{\sigma v(w/8)^{\tau_2} + 1}, \]
where $w$ is given by (6.1).

To prove this conjecture, we first appeal to Proposition 4.2, which implies that
\[ p(2\tau) = \frac{1 \pm \sqrt{1 - p^4(\tau)}}{p^2(\tau)}. \]

Setting $\tau = w$, (6.3) gives that
\[ p(2w) = \frac{1 \pm \sqrt{1 - \pi^4}}{\pi^2} = \frac{1 \pm \xi^2}{\pi^2}. \]

Note that
\[ \frac{1 + \xi^2}{\pi^2} - \frac{1}{\pi^4} = \frac{1 - \xi^4}{\pi^4} = 1 \]
and $\frac{1}{\pi^2} = -\pi^{\tau_2}$ from [16, p. 333]. Thus, $1 + \xi^2 = -\pi^{\tau_2}$. 

**Theorem 7.2.** If $w$ is given by (6.1) we have
\[ p(2w) = \frac{1 + \xi^2}{\pi^2} = \frac{-1}{\pi^{\tau_2}}. \]

**Proof.** We use an argument from [14, Section 10]. With the number $\beta = i^{-a}b(w)$ from (6.2) we have [14, eq. (8.0), p. 1980]
\[ j(w) = \frac{(\beta^8 + 224\beta^4 + 256)^3}{\beta^4(\beta^4 - 16)^4}. \]

(See the proof of Proposition 5.2.) Furthermore, the roots of the equation
\[ 0 = (X - 16)^3 - j(w)X = (X - 16)^3 - \frac{(\beta^8 + 224\beta^4 + 256)^3}{\beta^4(\beta^4 - 16)^4}X \]
are, on the one hand, given by the values
\[ X = f_2^{24}(w), \quad -f_1^{24}(w), \quad -f_2^{24}(w); \]
(see [8, p. 233, Th. 12.17]) and on the other, are equal to the expressions
\[ X = -\frac{(\beta^2 - 4)^4}{\beta^2(\beta^2 + 4)^2} - \frac{(\beta^2 + 4)^4}{\beta^2(\beta^2 - 4)^2} - \frac{2^{12}\beta^4}{(\beta^4 - 16)^2}. \]

See [14, p. 2000]. From [14, p. 2000] we also have (since our value \( w \) satisfies the conditions for \( w \) in [14, Prop. 3.1])
\[ f_2^{24}(w) = -\frac{(\beta^2 + 4)^4}{\beta^2(\beta^2 - 4)^2}; \] (7.1)
since \( f_2^{24}(w) \) must be a unit (from the results of [21]). There are two cases to consider.

Case 1. First assume that
\[ f_2^{24}(w) = -\frac{(\beta^2 - 4)^4}{\beta^2(\beta^2 + 4)^2}; \] (7.2)
\[ f_1^{24}(w) = \frac{2^{12}\beta^4}{(\beta^4 - 16)^2}. \]

In this case, (7.1) and (7.2) give the following formula:
\[ p_{12}(2w) = \frac{f_2(w)^{24}}{f(w)^{24}} = \frac{(\beta^2 + 4)^6}{(\beta - 2)^6(\beta + 2)^6}. \]

Now we use the following ideal factorizations in the ring class field \( \Omega_f \):
\[ (\beta^2 + 4) = \wp_2^2\wp_2', \quad (\beta - 2) = \wp_2^2\wp_2', \quad (\beta + 2) = \wp_2^3\wp_2'. \] (7.3)

See [16, Lemma 4]. These factorizations imply that
\[ p_{12}(2w) \cong \left( \frac{\wp_2^3\wp_2'}{\wp_2^5\wp_2'} \right)^6 = \frac{1}{\wp_2^{12}} \quad \text{in} \ \Omega_f, \]
which implies that
\[ p(2w) \cong \frac{1}{\wp_2}. \] (7.4)

By the remarks preceding the statement of the theorem, this shows that \( p(2w) \) is not an algebraic integer, giving that \( p(2w) = \frac{1 + \xi^2}{\pi^2} = -\pi^{-\xi_2}. \)

Case 2. The alternative to (7.2) is
\[ f_2^{24}(w) = -\frac{2^{12}\beta^4}{(\beta^4 - 16)^2}; \] (7.5)
\[ f_1^{24}(w) = \frac{(\beta^2 - 4)^4}{\beta^2(\beta^2 + 4)^2}. \]
In this case we have
\[ p^{12}(2w) = \frac{f_2(w)^{24}}{f(w)^{24}} = \left( \frac{2^2 + 4}{2^2 \beta} \right)^6 \cong \left( \frac{\wp_2 \wp_2^2 \wp_2^2}{\wp_2^2 \wp_2^2 \wp_2^2} \right)^6 = \frac{1}{\wp_2^{112}}, \]
giving that \( p(2w) \cong \frac{1}{\wp_2} \). However, this is impossible, since the above remarks show that the only prime divisors occurring in the factorization of \( p(2w) \) are prime divisors of \( \wp_2 \). This shows that Case 2 is impossible, and Case 1 proves the formula of the theorem. \( \square \)

Now we set
\[ \eta = \nu(w/8), \lambda = -\nu(-1/w), \nu = \nu(w/4). \quad (7.6) \]
We first show \( \lambda \) is a root of the minimal polynomial \( f_d(x) \) of \( \nu(w/8) \) (c odd). We have from Proposition 3.3 that
\[ \frac{2\lambda}{\lambda^2 - 1} = \frac{-2\wp(v)}{\wp^2(v) - 1} = \frac{\nu^2 + 2\nu - 1}{\nu^2 - 2\nu - 1}. \]
Proposition 4.1 and Theorem 7.2 give further that
\[ \frac{2\lambda}{\lambda^2 - 1} = \frac{\nu - 1}{\nu - 2} = \frac{-2}{\nu(2w)} + 2 = \frac{\pi^{\tau_2} + 1}{\pi^{\tau_2} - 1}. \quad (7.7) \]
Since \( \frac{\pi^{\tau_2} + 1}{\pi^{\tau_2} - 1} \) is a root of \( b_d(x) \), we have from (6.7) that
\[ f_d(\lambda) = 2^{-h(-d)}(\lambda^2 - 1)^{2h(-d)} b_d \left( \frac{2\lambda}{\lambda^2 - 1} \right) = 0. \]
Hence, \( \lambda = -\nu(-1/w) \) is a conjugate of \( \nu(w/8) \).

**Theorem 7.3.** If \( c \) is odd, we have the formula
\[ \lambda = -\nu(-1/w) = \tilde{A}(\nu(w/8)^{\tau_2}) = \frac{-\nu(w/8)^{\tau_2} + \sigma}{\sigma \nu(w/8)^{\tau_2} + 1}, \quad \sigma = -1 + \sqrt{2}, \]
where \( w \) is given by (6.1).

**Proof.** We will prove that \( \tilde{A}(\lambda) = \nu(w/8)^{\tau_2} = \eta^{\tau_2} \) by showing that
\[ \tilde{A}(\lambda) - \eta^{\tau_2} \equiv 0 \pmod{\wp_2}. \]
We have \( \eta^2 + \lambda^2 = \sigma^2(1 + \eta^2 \lambda^2) \), which implies that
\[ \tilde{A}(\lambda) - \eta^2 = \frac{-\lambda + \sigma}{\sigma \lambda + 1} - \frac{-\lambda^2 + \sigma^2}{1 - \sigma^2 \lambda^2} = \frac{-\lambda + \sigma}{\sigma \lambda + 1} + \frac{\sigma^2 - \lambda^2}{\sigma^2 \lambda^2 - 1} \]
\[ = \frac{(-\lambda + \sigma)(\sigma \lambda - 1) + \sigma^2 - \lambda^2}{\sigma^2 \lambda^2 - 1} \]
\[ = \frac{-(\sigma + 1)\lambda^2 + (\sigma^2 + 1)\lambda + \sigma^2 - \sigma}{\sigma^2 \lambda^2 - 1}. \]
\[
\frac{-\sqrt{2}\lambda^2 + (4 - 2\sqrt{2})\lambda + 4 - 3\sqrt{2}}{(\sigma\lambda + 1)(\sigma\lambda - 1)}
= \frac{-\sqrt{2}(\lambda - \sigma)^2}{\sigma^2(\lambda - \sigma)(\lambda + \sigma)}.
\]

We multiply the last expression by
\[
A(\lambda) - \frac{1}{\eta^2} = \frac{(-4 + 3\sqrt{2})(\lambda - \sigma)^2}{\lambda^2 - \sigma^2} = \frac{\sqrt{2}\sigma^2(\lambda - \sigma)^2}{\lambda^2 - \sigma^2},
\]
which is obtained from the last calculation by fixing \(\lambda\) and mapping \(\sqrt{2}\) to \(-\sqrt{2}\).

This yields the formula
\[
(\tilde{A}(\lambda) - \eta^2)(A(\lambda) - \frac{1}{\eta^2}) = -2(\lambda - \sigma)(\lambda + \sigma) = -2\frac{\lambda^2 + 2\lambda - 1}{\lambda^2 - 2\lambda - 1}.
\]

(7.8)

Now
\[
\frac{\lambda^2 + 2\lambda - 1}{\lambda^2 - 2\lambda - 1} = \frac{1 + \frac{2\lambda}{\lambda^2 - 1}}{1 - \frac{2\lambda}{\lambda^2 - 1}},
\]

(7.9)

where
\[
\frac{2\lambda}{\lambda^2 - 1} = \frac{\pi^{\tau_2} + 1}{\pi^{\tau_2} - 1},
\]

from (7.7). It follows from (7.9) that
\[
\frac{\lambda^2 + 2\lambda - 1}{\lambda^2 - 2\lambda - 1} = \frac{1 + \frac{\pi^{\tau_2 + 1}}{\pi^{\tau_2 - 1}}}{1 - \frac{\pi^{\tau_2 + 1}}{\pi^{\tau_2 - 1}}} = -\pi^{\tau_2}.
\]

Thus, (7.8) becomes
\[
(\tilde{A}(\lambda) - \eta^2)(A(\lambda) - \frac{1}{\eta^2}) = 2\pi^{\tau_2}
\]

and therefore \((\pi) = \mathfrak{q}_2\) yields that
\[
(\tilde{A}(\lambda) - \eta^2)(A(\lambda) - \frac{1}{\eta^2}) \equiv 0 \pmod{\mathfrak{q}_2^2}.
\]

It follows that
\[
\tilde{A}(\lambda) \equiv \eta^2 \text{ or } A(\lambda) \equiv \frac{1}{\eta^2} \pmod{\mathfrak{q}},
\]

(7.10)

for each prime divisor \(\mathfrak{q}\) of \(\mathfrak{q}_2\) in \(F_1 = \mathbb{Q}(\eta)\). But \(A(\lambda) = -1/\tilde{A}(\lambda)\) and \(\eta\) are units, so the second congruence in (7.10) implies the first. This proves that
\[
\tilde{A}(\lambda) \equiv \eta^2 \pmod{\mathfrak{q}_2}
\]

(7.11)

in \(F_1\). Note that \(\tilde{A}(\lambda)\) and \(\lambda = -v(-1/w)\) are roots of \(f_d(x)\) (Proposition 6.5), so \(F_2 = \mathbb{Q}(\lambda)\) is isomorphic to \(F_1 = \mathbb{Q}(\eta) = \mathbb{Q}(v(w/8))\). However, by (3.4),
\[
\lambda^2 = v^2(-1/w) = \frac{-v(w/8)^2 + \sigma^2}{1 - \sigma^2v(w/8)^2}.
\]
does not lie in $F_1$, since $\sqrt{2} \notin F_1$ (otherwise $\wp_2$ would be ramified in $F_1$; note that $v(w/8)$ is not a fourth root of unity, so the determinant of the linear fractional transformation in $\sigma^2$ is nonzero). It follows that from Theorem 6.1 that
\[ F_2 = \mathbb{Q}(\lambda) = \Sigma_{\wp_2} \Omega_f. \]

The same argument now shows that $\bar{A}(\lambda) = \frac{x + \sigma}{\sigma x + 1} \notin F_2$, so $\bar{A}(\lambda) \in F_1$. Therefore, $\psi : \eta \rightarrow \bar{A}(\lambda)$ is an automorphism of $F_1$, and since $\wp_2$ is not ramified in $F_1$ but $\wp_2'$ is, it follows that $\psi$ fixes $\wp_2$, implying that it fixes the field $K$.

Recalling the rational function $j_2(x)$ from (5.6), a computation on Maple shows that
\[ j_2\left(\left(\frac{1 - v}{1 + v}\right)^2\right) = j_2(v^2) = j_2(v^2(w/4)) = j(w/4), \]
by (5.7). Now Proposition 3.3 and the fact that $\bar{A}(x)$ has order 2 imply that $v(w/4) = \bar{A}(v(-1/w))$ and
\[
\frac{1 - v(w/4)}{1 + v(w/4)} = \frac{1 - \bar{A}(v(-1/w))}{1 + \bar{A}(v(-1/w))} = \frac{v(-1/w) + \sigma}{-\sigma v(-1/w) + 1} = \bar{A}(v(-1/w)) = \bar{A}(\lambda). \quad (7.12)
\]

This implies that
\[ j_2(\bar{A}(\lambda)^2) = j_2\left(\left(\frac{1 - v}{1 + v}\right)^2\right) = j(w/4). \]

On the other hand, equation (5.7) gives
\[ j(w/8)^\psi = j_2(\eta^{2\psi}) = j_2(\bar{A}(\lambda)^2) = j(w/4) = j(w/8)^{\tau_2}. \]

Hence $\psi|_{\Omega_f} = \tau_2|_{\Omega_f}$. It follows that $\psi = \tau_2$ or $\psi = \rho \tau_2$, where $\rho : \eta \rightarrow -1/\eta$ is the nontrivial automorphism of $F_1/\Omega_f$. If $\psi = \rho \tau_2$, then by (7.11)
\[ \eta^\psi = \bar{A}(\lambda) \equiv \eta^2 \pmod{\wp_2} \]
and $\eta^{\tau_2} \equiv \eta^2 \pmod{\wp_2}$ imply that
\[ \eta^2 \equiv \eta^{\rho \tau_2} = \frac{1}{\eta^{\tau_2}} \equiv \frac{1}{\eta^2} \pmod{\wp_2}. \]

It follows from this congruence that $\eta^4 + 1 \equiv (\eta + 1)^4 \equiv 0 \pmod{\wp_2}$ and hence $\eta \equiv 1 \pmod{\wp_2}$, since $\wp_2$ is unramified in $F_1/K$. This implies in turn that $z = \eta - \eta^{-1} \equiv 0 \pmod{\wp_2}$. But this contradicts (4.3) (with $\tau = w/8$) and (6.3), according to which $z = 2/\pi$ is relatively prime to $\wp_2$. Hence, $\psi = \tau_2$ must be the Artin symbol for $\wp_2$ in $F_1/K$. This completes the proof. \qed
Corollary 7.4. Assume $c$ is odd. If $\tau_2 = \left( \frac{\Sigma_{\psi^2} \Omega_f / K}{\varphi_2} \right)$, then

$$v(w/8)^{\tau_2} = \frac{1 - v(w/4)}{1 + v(w/4)}$$

and

$$f(v(w/8), v(w/8)^{\tau_2}) = 0.$$

**Proof.** The first formula is immediate from $\eta^\psi = \eta^{\tau_2} = \bar{A}(\lambda)$ and (7.12). The second follows from Proposition 3.1 and

$$f(v(w/8), v(w/4)) = 0 = f\left( v(w/8), \frac{1 - v(w/4)}{1 + v(w/4)} \right),$$

since

$$f\left( x, \frac{1 - y}{1 + y} \right) = \frac{2f(x, y)}{(1 + y)^2}.$$

□

Theorem 7.5. If $c$ is even, then

$$v(w/8)^{\tau_2} = \frac{v(w/4) - 1}{v(w/4) + 1}$$

and

$$v(-1/w) = B(v(w/8)^{\tau_2}) = \frac{v(w/8)^{\tau_2} + \sigma}{\sigma v(w/8)^{\tau_2} - 1}.$$

**Proof.** From Proposition 3.3, we have that

$$v(-1/w) = \bar{A}(v(w/4)) = -B(-v(w/4)),$$

where

$$B(x) = \frac{x + \sigma}{\sigma x - 1} = -\frac{(-x) + \sigma}{\sigma(-x) + 1} = -\bar{A}(-x).$$

Hence, according to (7.12), we obtain

$$v(w/8)^{\tau_2} = \frac{v(w/4) - 1}{v(w/4) + 1} = B(v(-1/w)) \iff v(-1/w) = B(v(w/8)^{\tau_2}),$$

showing that both the statements in the theorem are equivalent. We now show that Proposition 6.6 implies that $v(w/8)$ and $v(-1/w)$ are conjugate algebraic integers.

In similar fashion to (7.6), we set

$$\eta = v(w/8), \quad \bar{\lambda} = v(-1/w) = -\lambda, \quad \nu = v(w/4).$$

Then, according to (7.7), we get

$$\frac{2\bar{\lambda}}{1 - \bar{\lambda}^2} = -\frac{2 - \left( \frac{1}{\nu} - \nu \right)}{2 + \left( \frac{1}{\nu} - \nu \right)} = -\frac{2 - \frac{2}{v(2w)}}{2 + \frac{2}{v(2w)}} = -\frac{1 + \pi^{\tau_2}}{1 - \pi^{\tau_2}} = \pi^{\tau_2} + 1 = \frac{\pi^{\tau_2} + 1}{\pi^{\tau_2} - 1}. $$
Since $\pi^{r_2+1}$ is a root of $b_d(x)$, we have that

$$f_d(\lambda) = 2^{-h}\lambda^2 - 1)^{2h}b_d\left(\frac{2\lambda}{1 - \lambda^2}\right) = 0,$$

showing that $\lambda = \nu(-1/w)$ is a conjugate of $\eta = \nu(w/8)$.

Now,

$$B(\lambda) - \eta^2 = \frac{\lambda + \sigma - \frac{\lambda^2}{1 - \sigma^2\lambda^2}}{\sigma\lambda - 1} - \frac{\lambda - \sigma}{\sigma\lambda + 1} = \frac{\lambda - \sigma + \lambda^2 - \lambda^2}{\sigma\lambda^2 - 1}$$

$$= \frac{(\lambda - \sigma)(\lambda - 1) + (\lambda^2 - \lambda^2)}{(\sigma\lambda + 1)(\sigma\lambda - 1)}$$

$$= \frac{-\sqrt{2}\sigma(\lambda^2 + 2\lambda - 1)}{\sigma^2(\lambda - \sigma)(\lambda + \sigma)}$$

$$= \frac{-\sqrt{2}\sigma(\lambda - \sigma)(\lambda + \sigma)}{\sigma^2(\lambda - \sigma)(\lambda + \sigma)}$$

$$= \frac{\sqrt{2}\sigma(\lambda + \sigma)}{(\lambda - \sigma)}.$$ 

In the above calculation, mapping $\sqrt{2}$ to $-\sqrt{2}$, while fixing $\lambda$, gives us

$$B(\lambda) - \eta^2 = -\frac{2\sigma(\lambda + \sigma)}{(\lambda - \sigma)}.$$ 

Multiplying the above two expressions gives us

$$(B(\lambda) - \eta^2)(B(\lambda) - \frac{1}{\eta^2}) = 2\frac{(\lambda + \sigma)(\lambda + \sigma)}{(\lambda - \sigma)(\lambda - \sigma)}$$

$$= 2\frac{2\lambda^2 - 2\lambda - 1}{\lambda^2 + 2\lambda - 1}$$

$$= 2 + \frac{\frac{2\lambda^2}{1 - \lambda^2} - 1}{1 - \frac{2\lambda^2}{1 - \lambda^2}}$$

$$= 2 + \frac{\pi^{r_2+1}}{\pi^{r_2}-1} - \frac{\pi^{r_2+1}}{\pi^{r_2}-1} = -2\pi^{r_2}.$$ 

Now a similar argument to the end of the proof of Theorem 7.3 applies here and shows that the automorphism $\psi$ on $F_1$ taking $\eta$ to $\lambda$ is $\eta^\psi = B(\lambda)$. As before, $\psi$ coincides with $\tau_2$, giving that $\lambda = \nu(-1/w) = B(\eta^{r_2}) = B(\nu(w/8)^{r_2})$. Also see the argument below.

\[\square\]

**Corollary 7.6.** If $c$ is even, the point $(x, y) = (-\eta, -\eta^{r_2})$ lies on the curve $f(x, y) = 0$:

$$f(-\nu(w/8), -\nu(w/8)^{r_2}) = 0, \quad \tau_2 = \left(\frac{\Sigma_{\psi^{r_2}}\Omega_f/K}{\mathfrak{q}_2}\right).$$
Proof. We have
\[0 = f(v(w/8), v(w/4)) = f\left(v(w/8), -\frac{v(w/4) - 1}{v(w/4) + 1}\right)\]
\[= f(v(w/8), -v(w/8)^{\tau}) = f(-v(w/8), -v(w/8)^{\tau}).\]

Combining the arguments in the proofs of Theorems 7.3 and 7.5 for \(c\) odd and \(c\) even yields the following corollary.

**Corollary 7.7.** The field \(F_2 = \mathbb{Q}(v(-1/w)) = \Sigma\mathfrak{p}_{2}^c\Omega_f\) is the inertia field for the prime ideal \(\mathfrak{p}^c\) in the extension \(L_{\Omega,8}/K\).

We also give an alternate argument to show \(\psi = \tau_2\) in the proofs of Theorems 7.3 and 7.5. We first note that the modular function \(j(\tau)\) can be expressed in terms of \(z = v(\tau) - \frac{1}{v(\tau)}\), namely
\[j(\tau) = J(z) = \frac{(z^8 + 240z^6 + 2144z^4 + 3840z^2 + 256)^3}{z^2(z^2 + 4)(z - 2)^8(z + 2)^8},\]
using Proposition 5.2. Now set \(z = \eta - \eta^{-1} = \pm \frac{2}{\pi}\), so that \((z, \mathfrak{p}_2) = 1\). This allows us to reduce the above formula modulo \(\mathfrak{p}_2\), giving that
\[j(w/8) \equiv z^{24} \equiv z^2 (mod \mathfrak{p}_2).\]

This shows that \(j(w/8)^{\tau}\) is conjugate to \(z^5\) modulo each prime divisor \(\mathfrak{p}\) of \(\mathfrak{p}_2\) in \(\Omega_f\), for each automorphism \(\tau \in \text{Gal}(\Omega_f/K)\); and this implies that the class equation \(H_{-d}(X)\) and the minimal polynomial \(\mu_d(X)\) of \(z\) over \(K\) are congruent:
\[H_{-d}(X) \equiv \mu_d(X) (mod \mathfrak{p}_2).\]

A theorem of Deuring says that the discriminant of \(H_{-d}(X)\) is odd (since \(\left(\frac{-d}{2}\right) = +1\)), so the discriminant of \(\mu_d(X)\) is not divisible by \(\mathfrak{p}_2\). This implies that the discriminant of the minimal polynomial \(\bar{\mu}_d(X) = X^{h(-d)\mu_d(X - \frac{1}{X})}\) of \(\eta\) over \(K\) is relatively prime to \(\mathfrak{p}_2\), as well. This is because
\[\mu_d(X) = \prod_{i=1}^{h(-d)} (X - (\eta_i - \frac{1}{\eta_i}))\]
is a product over the conjugates \(z_i = \eta_i - \frac{1}{\eta_i}\) of \(z\), so that
\[X^{h(-d)\mu_d(X - \frac{1}{X})} = \prod_{i=1}^{h(-d)} (X^2 - (\eta_i - \frac{1}{\eta_i}X - 1),\]
\[= \prod_{i=1}^{h(-d)} (X^2 - z_iX - 1),\]
where \(z_i = \eta_i - \frac{1}{\eta_i}.\)
Hence,
\[
\text{disc}(\tilde{\mu}_d(X)) = \prod_{i=1}^{h(-d)} (z_i^2 + 4) \prod_{i<j} \text{Res}(X^2 - z_iX - 1, X^2 - z_jX - 1)^2
\]
\[
= \prod_{i=1}^{h(-d)} (z_i^2 + 4) \prod_{i<j} (z_i - z_j)^4
\]
\[
= \prod_{i=1}^{h(-d)} (z_i^2 + 4) (\text{disc}(\mu_d(X)))^2.
\]

Now the \(z_i\) are conjugate over \(K\), so each \(z_i\) is relatively prime to \(\wp_2\), which implies that \((z_i^2 + 4, \wp_2) = 1\) for each \(i\). This proves the claim that \((\text{disc}(\tilde{\mu}_d(X)), \wp_2) = 1\). This proves Theorem 7.8.

Let \(R_{\wp_2}\) denote the ring of elements of \(K\) which are integral for \(\wp_2\). Then the powers of \(\eta = \wp(w/8)\) form a basis over \(R_{\wp_2}\) for the ring \(\tilde{R}\) of elements of \(F_1 = \mathbb{Q}(\eta)\) which are integral for \(\wp_2\).

Given this theorem, the congruence
\[
\eta^\psi \equiv \eta^2 \pmod{\wp_2}
\]
implies that
\[
\alpha^\psi \equiv \alpha^2 \pmod{\wp_2},
\]
for all \(\alpha \in F_1\) which are integral for \(\wp_2\). Since \(F_1/K\) is abelian and \(\wp_2\) is unramified in this extension, this implies by definition of the Artin symbol that \(\psi = \tau_2\).

8. Values of \(\wp(\tau)\) as periodic points.

We now define the following algebraic functions. The roots of \(f(x, y) = y^2 + (x^2 - 1)y + x^2\) (see Proposition 3.1) as a function of \(y\) are
\[
\hat{f}(x) = -\frac{x^2 - 1}{2} \pm \frac{1}{2}\sqrt{x^4 - 6x^2 + 1}. \tag{8.1}
\]
Also, the roots of \(g(x, y) = y^2 - (x^2 - 4x + 1)y + x^2\) (see Proposition 3.2) are given by
\[
\hat{g}(x) = \frac{1}{2}(x^2 - 4x + 1) \pm \frac{1}{2}\sqrt{(x^2 - 2x + 1)(x^2 - 6x + 1)}
\]
\[
= \frac{1}{2}(x^2 - 4x + 1) \pm \frac{x - 1}{2}\sqrt{x^2 - 6x + 1}. \tag{8.2}
\]
We prove the following.

Theorem 8.1. If \(w \in R_K\) is the algebraic integer defined by
\[
w = \frac{a + \sqrt{-d}}{2}, \text{ with } a^2 + d \equiv 0 \pmod{2^5}
\]
and the integer \( c \) satisfies
\[
c \equiv 1 - \frac{a^2 + d}{32} \quad (\text{mod } 2),
\]
then the generator \((-1)^{1+c} \nu(w/8)\) of the field \( \Sigma_{\wp_2} \Omega_f \) over \( \mathbb{Q} \) is a periodic point of the algebraic function \( \hat{F}(x) \) defined by (8.1) and \( \nu^2(w/8) \) is a periodic point of the function \( \hat{T}(x) \) defined by (8.2). If \( c \) is even, then \( \nu(w/8) \) is a pre-periodic point of \( \hat{F}(x) \).

**Proof.** Setting \( \eta = (-1)^{1+c} \nu(w/8) \) and \( F_1 = \mathbb{Q}(\eta) = \mathbb{Q}(\eta^2) \), we have from the corollaries to Theorems 7.3 and 7.5 that \( f(\eta, \eta^{2^n}) = 0 \), where \( \tau_2 = \left( \frac{F_1/K}{\wp_2} \right) \) is an automorphism in \( \text{Gal}(F_1/K) \). If the order of \( \tau_2 \) is \( n \), then applying powers of \( \tau_2 \) gives that
\[
f(\eta, \eta^{2^n}) = f(\eta^{2^n}, \eta^{2^{n-1}}) = \cdots = f(\eta^{2^n-1}, \eta) = 0,
\]
which implies that \( \eta \) is a periodic point of \( \hat{F}(x) \) of period \( n \). If \( c \) is even, then from Corollary 7.6 and the fact that \( f(x, y) = f(-x, y) \) we also have that
\[
f(\nu(w/8), -\nu(w/8)^{2^n}) = 0;
\]
thus, \( \nu(w/8) \) is a pre-periodic point of \( \hat{F}(x) \), since \(-\nu(w/8)^{2^n}\) is periodic.

It is straightforward to check that
\[
\hat{F}(x)^2 = \frac{1}{2}(x^4 - 4x^2 + 1) \pm \frac{1}{2}(x^2 - 1)\sqrt{x^4 - 6x^2 + 1} = \hat{T}(x^2)
\]
and that the minimal polynomial of \( \hat{F}(x)^2 \) over \( \mathbb{Q}(x) \) is \( g(x^2, y) \). In particular, \( f(x, y) = 0 \) implies that \( g(x^2, y^2) = 0 \), since
\[
g(x^2, y^2) = (-x^2y + x^2 + y^2 + y)(x^2y + x^2 + y^2 - y) = f(x, -y)f(x, y).
\]
Hence, (8.3) implies that
\[
g(\eta^2, \eta^{2^{n-1}}) = g(\eta^{2^n}, \eta^{2^{n-1}}) = \cdots = g(\eta^{2^{n-1}}, \eta^2) = 0,
\]
which shows that \( \eta^2 = \nu(w/8)^2 \) is a periodic point of \( \hat{F}(x) \).

**Remarks.**

1. Note that if \( c \) is even, meaning that \( 2^5 \mid a^2 + d \), then \( 2^6 \mid (a + 16)^2 + d \), so that \( w + 8 = \frac{a+16+\sqrt{-d}}{2} = w' \) satisfies (6.1) with \( c \) odd. Then the infinite product formula for \( \nu(\tau) \) shows that \( \nu(w/8) = \nu(w'/8 - 1) = -\nu(w'/8) \), and \( -\nu(w/8) = \nu(w'/8) \) in Corollary 7.6.

2. Given that \( f(\nu(\tau), \nu(2\tau)) = 0 \), it is tempting to try to show that \( \nu(w/8) \) is a periodic point by considering the chain of equations
\[
f(\nu(w/8), \nu(w/4)) = f(\nu(w/4), \nu(w/2)) = \cdots = f(\nu(2^{n-1}w/8), \nu(2^nw/8)) = 0,
\]
and find an integer \( n \) for which \( 2^{n-3}w = M(w/8) = \frac{au+8b}{cw+8d} \), for some unimodular matrix \( M \) for which \( \nu(M(w/8)) = \nu(w/8) \). However, this requires
that \( M \in \Gamma_1(8) \), so that \( a \equiv 1 \pmod{8} \) and \( 8 \mid c \). This condition leads to the equation

\[
2^{n-3}cw^2 + (2^nd - a)w - 8b = 0.
\]

Moreover, \( w \) is an algebraic integer, so the fact that \( 8 \mid c \) shows that \( 2^n \) must divide the other coefficients of this quadratic. Hence, \( 2^n \mid a \), which is impossible for \( n \geq 1 \). Thus, this approach does not yield an orbit leading back to \( \nu(w/8) \).

As in the papers [15]-[18], the minimal polynomials of periodic points of \( \hat{F}(x) \) can be computed using iterated resultants involving its minimal polynomial \( f(x,y) \). We set

\[
R^{(1)}(x, x_1) = f(x, x_1) = x^2x_1 + x^2 + x_1^2 - x_1
\]

and define, inductively,

\[
R^{(n)}(x, x_n) = \text{Res}_{x_{n-1}}(R^{(n-1)}(x, x_{n-1}), f(x_{n-1}, x_n)) \quad n \geq 2.
\]

Then the roots of the polynomial

\[
R_n(x) = R^{(n)}(x, x), \quad n \geq 1,
\]

are the periodic points of \( \hat{F}(x) \) whose minimal periods divide \( n \). See [15, p. 727]. For example, we compute that

\[
R_1(x) = x(x^2 + 2x - 1),
R_2(x) = x(x^2 + 2x - 1)(x^4 - x^3 + x + 1),
R_3(x) = x(x^2 + 2x - 1)(x^{12} - 5x^{11} + 2x^{10} + 10x^9 + 5x^8 + 23x^7
- 8x^6 - 23x^5 + 5x^4 - 10x^3 + 2x^2 + 5x + 1),
R_4(x) = x(x^2 + 2x - 1)(x^4 - x^3 + x + 1)(x^8 - x^7 + x^6 - 5x^5 + 5x^3 + x^2 + x + 1)
\times (x^{16} + 5x^{15} - 18x^{14} - 75x^{13} + 137x^{12} + 105x^{11} + 38x^{10} + 185x^9
- 300x^8 - 185x^7 + 38x^6 - 105x^5 + 137x^4 + 75x^3 - 18x^2 - 5x + 1).
\]

We now set \( x = z + 3 \) in the function \( \hat{T}(x) \), so that the square-root in \( \hat{T}(x) \) has the 2-adic expansion

\[
\sqrt{x^2 - 6x + 1} = \sqrt{z^2 - 8} = z\sqrt{1 - \frac{8}{z^2}} = z\sum_{k=0}^{\infty} (-1)^k \binom{1/2}{k} \frac{8^k}{z^{2k}}.
\]

We will show that this series is 2-adically convergent for (roughly) half of the primitive periodic points of the algebraic function \( \hat{T}(x) \) of a given period \( n \) in the field \( K_2(\sqrt{2}) \), where \( K_2 \) is the maximal unramified, algebraic extension of the 2-adic field \( \mathbb{Q}_2 \).

If we set

\[
T(x) = \frac{1}{2}(x^2 - 4x + 1) + \frac{x - 1}{2}\sqrt{x^2 - 6x + 1},
\]

then
then using the above series in $T(x)$ and splitting off the $k = 0$ term, we find
\[
T(x) = x^2 - 4x + 2 + (x - 1)(x - 3) \sum_{k=1}^{\infty} (-1)^k 2^{2k-1} \left( \frac{1/2}{k} \right) \frac{2^k}{(x - 3)^{2k}},
\]
for $x - 3 \in \mathcal{O}^\times$, where $\mathcal{O}$ is the ring of integers in $K_2(\sqrt{2})$. Since
\[
(-1)^{k-1} 2^{2k-1} \left( \frac{1/2}{k} \right) = C_{k-1} \in \mathbb{Z}
\]
is the Catalan sequence, it follows that
\[
T(x) \equiv x^2 \pmod{2}, \quad x - 3 \in \mathcal{O}^\times.
\]
Hence, $T(x)$ is a lift of the Frobenius automorphism for points $x$ in the set
\[
\overline{D} = \{ x \in K_2(\sqrt{2}) : |x - 3|_2 = 1 \}.
\]
Furthermore,
\[
T(x) - 3 = (x - 3)^2 + 2(x - 3) - 4 - (x - 1)(x - 3) \sum_{k=1}^{\infty} C_{k-1} \frac{2^k}{(x - 3)^{2k}}.
\]
It follows that
\[
|T(x) - 3|_2 = |x - 3|^2_2 = 1,
\]
and $T$ maps $\overline{D}$ to itself.

We next prove

**Proposition 8.2.** We have the congruences
\[
R^{(n)}(x, x_n) \equiv (x^{2^n} + x_n)(x_n + 1)^{2^{n-1}} \pmod{2};
\]
\[
R_n(x) \equiv (x^{2^n} + x)(x + 1)^{2^{n-1}} \pmod{2}.
\]

**Proof.** We have $f(x, y) = x^2y + x^2 + y^2 - y$. So, for $n = 1$, we get
\[
R^{(1)}(x, x_1) = f(x, x_1) = x^2x_1 + x^2 + x_1^2 - x_1
\equiv x^2x_1 + x^2 + x_1^2 + x_1 \pmod{2}
\equiv (x^2 + x_1)(x_1 + 1) \pmod{2}.
\]
Hence,
\[
R_1(x) \equiv (x^2 + x)(x + 1) \pmod{2}.
\]
Now for the induction step, assume the result is true for $n - 1$. Then,
\[
R^{(n)}(x, x_n) = \text{Res}_{x_{n-1}}(R^{(n-1)}(x, x_{n-1}), f(x_{n-1}, x_n))
\equiv \text{Res}_{x_{n-1}}((x^{2^{n-1}} + x_{n-1})(x_{n-1} + 1)^{2^{n-1}-1}, (x_{n-1}^2 + x_n)(x_n + 1)) \pmod{2}.
\]
By definition, the resultant of two polynomials \( f = \sum_{i=0}^{n} a_i x^i \) and \( g = \sum_{i=0}^{m} b_i x^i \), having roots \( \alpha_1, \alpha_2, ..., \alpha_n \) and \( \beta_1, \beta_2, ..., \beta_m \), respectively, is given by

\[
\text{Res}(f, g) = a_n^m \prod_{i=1}^{n} g(\alpha_i),
\]

and

\[
\text{Res}(g, f) = (-1)^{mn} \text{Res}(f, g).
\]

The roots of \((x^{n-1} + x_n)(x_n + 1)\), as a polynomial in \( x_{n-1} \), are \( \pm \sqrt{-x_n} \). Hence,

\[
\text{Res}_{x_{n-1}}(x^{n-1} + x_{n-1})(x_n + 1) = (-1)^{2n-1}(x_n + 1)^{2n-1} \left( x^{n-1} + \sqrt{-x_n}(\sqrt{-x_n} + 1) \right)^{2n-1}
\]

\[
= (-1)^{2n}(x_n + 1)^{2n} \left( x^n + x_n \right)(x_n + 1)^{2n-1}
\]

\[
= (x^n + x_n)(x_n + 1)^{2n-1}.
\]

Hence, we obtain

\[
R^{(n)}(x, x_n) \equiv (x^{n-1} + x_n)(x_n + 1)^{2n-1} \pmod{2},
\]

\[
R_n(x) \equiv (x^{n-1} + x)(x + 1)^{2n-1} \pmod{2},
\]

completing the induction. \( \square \)

**Corollary 8.3.** The degree of \( R_n(x) \) is \( \text{deg}(R_n(x)) = 2^{n+1} - 1 \).

**Proof.** This follows from the proposition, if the leading coefficient of \( R_n(x) \) is not divisible by 2. In fact, this follows from the relation

\[
R^{(n)}(x, x_n) = A_n(x_n)x^{2n} + S_n(x, x_n),
\]

where for \( n \geq 3 \),

\[
A_n(x_n) = (x_n + 1)(x_n^2 + 1)(x_n^2 - 2x_n - 1)^2(x_n^2 + 2x_n - 1)^{2n-4}
\]

and for \( n \geq 1 \),

\[
\text{deg}(A_n(x_n)) = 2^n - 1, \quad \text{deg}(S_n(x, x_n)) \leq 2^n - 2, \quad \text{deg}_{x_n}(S_n(x, x_n)) = 2^n.
\]

We refer the reader to the lemma in [15, pp. 727-728] for a similar proof. \( \square \)

The roots of the factor \( x^n + x = x(x + 1)^{\frac{x^{n-1}+1}{x+1}} = x(x + 1)h_n(x) \) other than \( x = 0, 1 \) have degree greater than 1, and therefore satisfy \( x - 3 \not\equiv 0 \pmod{2} \). It follows from Hensel’s Lemma that \( 2^n - 1 \) of the roots of \( R_n(x) \) over \( \mathbb{Q}_2 \) have the property that \( x - 3 \in \mathcal{O}_K \), and for these roots the series for \( T(x) \) converges in \( K_2 \).
Now the argument at the end of the proof of Theorem 7.3 shows that $\eta \equiv 1 \pmod{\wp^2}$, so that the image of $\eta$ in the completion $F_{1,q} \subset K_2$ of $F_1 = \Sigma_{q/2} \Omega_f$ with respect to a prime divisor $q$ of $q/2$ in $F_1$ satisfies $\eta^2 - 3 \in O^K$. Hence, the series for $T(\eta^2)$ converges. We claim now that $\eta^{2\tau_2} = T(\eta^2)$. But $g(\eta^2, \eta^{2\tau_2}) = 0$ implies that $\eta^{2\tau_2}$ is one of the values of $T(\eta^2)$. The value different from $T(\eta^2)$ in $K_2$ is

$$T_1(\eta^2) = \eta^4 - 4\eta^2 + 1 - T(\eta^2)$$

$$\equiv \eta^4 - 4\eta^2 + 1 - \eta^4 \pmod{q}.$$ 

But we also know $\eta^{2\tau_2} - 3 = (\eta^2 - 3)^{\tau_2} \in O^K$, so that $\eta^{2\tau_2} \neq T_1(\eta^2)$. This yields the following.

**Theorem 8.4.** If $w$ satisfies (6.1), then the value $\eta = v(w/8)$ and the automorphism $\tau_2 = \left(F_1/K, q/2 \right)$ satisfy

$$\eta^{2\tau_2} = T(\eta^2),$$

in the completion $F_{1,q} \subset K_2$ of $F_1 = \Sigma_{q/2} \Omega_f$ with respect to a prime divisor $q$ of $q/2$ in $F_1$, where

$$T(x) = x^2 - 4x + 2 - (x - 1)(x - 3) \sum_{k=1}^{\infty} \frac{2^k}{(x - 3)^2k}$$

converges for $x$ in $D = \{x \in K_2(\sqrt{2}) : |x - 3|_2 = 1\}$.

Since $\tau_2$ fixes the prime divisors of $q/2$, it extends naturally to an automorphism of $F_{1,q}$, and can be applied to the individual terms of the series representing $T(x)$. Thus, we see inductively that

$$\eta^{2\tau_i} = T(\eta^{2\tau_{i-1}}) = T(T^{i-1}(\eta^2)) = T^i(\eta^2)$$

is the $i$-th iterate of $T(x)$ applied to $\eta^2$. From this and the fact that $Q(\eta^2) = F_1$ we see that the order of $\tau_2$ in $Gal(F_1/K)$ is the minimal period of the periodic point $\eta^2$, and that $\eta^2$ is a periodic point in the ordinary sense of the $2$-adic function $T(x)$. This also shows that the minimal period of $\eta$ with respect to $\hat{F}(x)$ is $n = \text{ord}(\tau_2)$, since if $\eta$ had smaller minimal period $m$, then by the proof of Theorem 8.1, $\eta^2$ would have period $m < n$ with respect to the function $T(x)$. This completes the proof of the assertions of Theorem B of the Introduction regarding minimal periods.

9. The periodic points of $\hat{F}(x)$ and a class number formula.

In this section we show that the only periodic points of $\hat{F}(x)$ are the values given in Theorem 8.1. In fact, we will prove the following.
The only periodic points of the function \( F(x) \) in \( \overline{\mathbb{Q}} \) are the fixed points 0, \( \sigma, \bar{\sigma} \) and the conjugates over \( \mathbb{Q} \) of the values \( \nu(w/8) \) in Theorem 8.1 (for odd \( c \)).

Proof. Let \( \bar{g}(x, y) = x^2y^2 + 2y + x^2 \). Note that \( \bar{g}(x, y) = g(y, x) \) for the polynomial \( g(x, y) \) in [16, Thm. 2, p. 327]. By the results of that paper the numbers \( \pi, \xi \) and their conjugates over \( \mathbb{Q} \) (as \( -d \) ranges over all discriminants \( \equiv 1 \) modulo 8) are, together with 0 and \(-1\), the only periodic points of the algebraic function \( f(z) \) defined by \( \bar{g}(z, f(z)) = 0 \). The assertion of the theorem will follow from the identity

\[
(x^2 - 1)^2(y^2 - 1)^2\bar{g}\left(\frac{2x}{x^2 - 1}, \frac{2y}{y^2 - 1}\right) = 4f(x, y)(x^2y^2 - x^2y + y + 1). \tag{9.1}
\]

Here, as in Proposition 3.1, \( f(x, y) = x^2y + x^2 + y^2 - y \). Let \( \eta \) be a periodic point of \( F(x) \) in \( \overline{\mathbb{Q}} \) which is distinct from its fixed points 0, \( \sigma, \bar{\sigma} \). Then there are \( \eta_1 = \eta, \eta_2, \ldots, \eta_n \) in \( \overline{\mathbb{Q}} \) for which

\[
f(\eta_1, \eta_2) = f(\eta_2, \eta_3) = \cdots = f(\eta_n, \eta_1) = 0. \tag{9.2}
\]

Setting \( \lambda_i = \frac{2\eta_i}{\eta_i^2 - 1} \), equations (9.1) and (9.2) give that

\[
\bar{g}(\lambda_1, \lambda_2) = \bar{g}(\lambda_2, \lambda_3) = \cdots = \bar{g}(\lambda_n, \lambda_1) = 0. \tag{9.3}
\]

Note that \( \eta_i \neq \pm 1 \) since \( \pm 1 \) are preperiodic (and not periodic) for \( f(x, y) \), since

\[
f(\pm 1, y) = y^2 + 1, \quad f(\pm i, y) = y^2 - 2y - 1, \quad f(1 \pm \sqrt{2}, y) = (y + 1 \pm \sqrt{2})^2.
\]

Equation (9.3) implies that \( \lambda_1 \) is a periodic point of the function \( f(z) \) defined above. Also, \( \lambda_i \neq 0, -1 \) since \( \eta_i \notin \{0, \sigma, \bar{\sigma}\} \). By the results of [16, Thm. 2], this shows that \( \lambda_1 \) must be a conjugate of the number \( \pi \) for some discriminant \(-d\) and is therefore a root of the polynomial \( b_d(x) \). (See Proposition 6.4.) Since \( \lambda_1 = 2\eta/(\eta^2 - 1) \), this shows that \( \eta \) is a root of the minimal polynomial \( f_d(x) \) of \( \nu(w/8) \), for \( c \) odd, by (6.7). This completes the proof. \( \square \)

Remark. We can use equation (9.1) to give an alternate proof of the corollary to Theorem 7.3, as follows. We would like to show that \( f(\eta, \eta^{\tau_2}) = 0 \), where \( \eta = \nu(w/8) \) and \( \tau_2 = \left( \frac{F_1/K}{\varphi_2} \right) \), with \( F_1 = \sum_{\varphi_2'} \Omega_f \). Since \( \tau_2 | \Omega_f = \left( \frac{\Omega_f/K}{\varphi_2} \right) \), we know that \( \bar{g}(\pi, \pi^{\tau_2}) = 0 \), by [16, pp. 332-333]. Using \( \pi = \frac{2\eta}{\eta^2 - 1} \) from (6.4), equation (9.1) implies that \( f(\eta, \eta^{\tau_2})k(\eta, \eta^{\tau_2}) = 0 \), where \( k(x, y) = x^2y^2 - x^2y + y + 1 \). But \( k(\eta, \eta^{\tau_2}) \equiv k(\eta, \eta^2) \mod \varphi_2 \) in \( F_1 \). An easy computation shows that \( k(x, x^2) \equiv (x + 1)^6 \mod 2 \), so \( k(\eta, \eta^{\tau_2}) \equiv (\eta + 1)^6 \mod \varphi_2 \). If \( \eta \equiv 1 \) modulo some prime divisor \( p \) of \( \varphi_2 \) in \( F_1 \), then the relation \( \eta^2 - 2\eta - 1 = 0 \) would give that \( \frac{2}{\pi} \equiv 0 \mod p \), which is impossible since \( \frac{2}{\pi} \equiv \varphi_2' \). Hence, \( k(\eta, \eta^{\tau_2}) \neq 0 \mod \varphi_2 \), which implies \( k(\eta, \eta^{\tau_2}) \neq 0 \) and therefore \( f(\eta, \eta^{\tau_2}) = 0 \), as claimed.
Theorem 9.1 has the following consequence. As in the last remark, let \( F_1 = \Sigma \wp_i \Omega_f \) be the field generated by \( v(w/8) \) in Theorem 6.1. Then \( [F_1 : \mathbb{Q}] = 4h(-d) \) and \( F_1 \) is the inertia field for \( \wp_2 \) in the field \( \Sigma \wp_i \Omega_f \), an extended ring class field over \( K_d = \mathbb{Q}(\sqrt{-d}) \). As in Section 7, let \( \tau_2 = \left( \frac{F_1/K_d}{\wp_2} \right) \) be the Artin symbol for \( \wp_2 \) in the extension \( F_1/K_d \). Now define the set of discriminants
\[
\mathcal{D}_{n,2} = \{-d < 0 \mid -d \equiv 1 \pmod{8} \text{ and } \operatorname{ord}(\tau_2) = n \text{ in } \operatorname{Gal}(F_1/K_d)\}. \quad (9.4)
\]

**Theorem 9.2.** If \( n \geq 2 \), we have the following relation between class numbers of discriminants in the set \( \mathcal{D}_{n,2} \):
\[
\sum_{-d \in \mathcal{D}_{n,2}} h(-d) = \frac{1}{2} \sum_{k | n} \mu(n/k) 2^k. \quad (9.5)
\]

**Proof.** This proof mirrors the arguments in [18, pp.792-793, 806]. First, define
\[
P_n(x) = \prod_{k | n} R_k(x)^{\mu(n/k)}. \quad (9.6)
\]
We show that \( P_n(x) \in \mathbb{Z}[x] \). From Proposition 8.2 it is clear that \( R_n(x) \), for \( n > 1 \), is divisible (mod 2) by the \( N \) irreducible (monic) polynomials \( h_i(x) \) of degree \( n \) over \( \mathbb{F}_2 \), where
\[
N = \frac{1}{n} \sum_{k | n} \mu(n/k) 2^k,
\]
and that these polynomials are simple factors of \( R_n(x) \) (mod 2). It follows from Hensel's Lemma that \( R_n(x) \) is divisible by distinct irreducible polynomials \( h_i(x) \) of degree \( n \) over \( \mathbb{Z}_2 \), the ring of integers in \( \mathbb{Q}_2 \), for \( 1 \leq i \leq N \), with \( h_i(x) \equiv h_i(x) \pmod{2} \). In addition, all the roots of \( h_i(x) \) are periodic of minimal period \( n \) and lie in the unramified extension \( K_2 \). Furthermore, \( n \) is the smallest index for which \( h_i(x) \mid R_n(x) \) over \( \mathbb{Q}_2 \).

Now consider the identity
\[
(\sigma x + 1)^2(\sigma y + 1)^2f(\tilde{A}(x), \tilde{A}(y)) = 2^2\sigma^2 f(y, x), \quad (9.7)
\]
where \( \tilde{A}(x) = -x + \sigma x + 1 \), as in (3.3). If the periodic point \( a \) of \( \tilde{f}(x) \), with minimal period \( n > 1 \), is a root of one of the polynomials \( h_i(x) \), then \( a \) is a unit in \( K_2 \), and for some \( a_1, ..., a_{n-1} \) we have
\[
f(a, a_1) = f(a_1, a_2) = \cdots = f(a_{n-1}, a) = 0. \quad (9.8)
\]
Furthermore, \( a \not\equiv 1 \pmod{\sqrt{2}} \), since otherwise its reduction \( a \equiv \bar{a} \equiv 1 \pmod{2} \) would have degree 1 over \( \mathbb{F}_2 \) (using that \( K_2 \) is unramified over \( \mathbb{Q}_2 \)). Hence, \( a + 1 + \sqrt{2} \) is a unit in \( K_2(\sqrt{2}) \), which gives that \( \sigma a + 1 \) is a unit, as well. All of the \( a_i \) satisfy \( a_i \not\equiv 1 \pmod{\sqrt{2}} \), since the congruence \( f(1, y) \equiv (y + 1)^2 \pmod{2} \) has
only \( y \equiv 1 \mod{\sqrt{2}} \) as a solution. Hence, if some \( a_i \equiv 1 \mod{\sqrt{2}} \), then \( a_j \equiv 1 \mod{\sqrt{2}} \) for \( j > i \), which would imply that \( a \equiv 1 \mod{\sqrt{2}} \), as well. The elements \( b_i = A(a_i) \) are distinct and lie in \( K_2(\sqrt{2}) \) and satisfy

\[
b_i - 1 \equiv \frac{-a_i + \sigma - \sigma a_i - 1}{\sigma a_i + 1} \equiv \frac{-2}{\sigma a_i + 1} \equiv 0 \mod{\sqrt{2}}.
\]

The identity (9.7) yields that

\[
f(b, b_{n-1}) = f(b_{n-1}, b_{n-2}) = \cdots = f(b_1, b) = 0 \quad (9.9)
\]

in \( K_2(\sqrt{2}) \). Hence, \( b_i \equiv 1 \mod{\sqrt{2}} \), and the orbit \( \{b, b_{n-1}, \ldots, b_1\} \) is distinct from all the orbits in (9.8).

Now the map \( \tilde{A}(x) \) has order 2, so it is clear that \( b = \tilde{A}(a) \) has minimal period \( n \) in (9.9), since otherwise \( a = \tilde{A}(b) \) would have period smaller than \( n \). It follows that there are at least \( 2N \) periodic orbits of minimal period \( n > 1 \). Noting that

\[
R_1(x) = f(x, x) = x(x^2 + 2x - 1),
\]

these distinct orbits and factors account for at least

\[
3 + \sum_{d|n, d > 1} \left(2 \sum_{k|d} \mu(d/k)2^k\right) = -1 + 2 \sum \left(\sum \mu(d/k)2^k\right) = 2 \cdot 2^n - 1
\]

roots, and therefore all the roots, of \( R_n(x) \). This shows that the roots of \( R_n(x) \) are distinct and the expressions \( P_n(x) \) are polynomials. Furthermore, over \( K_2(\sqrt{2}) \) we have the factorization

\[
P_n(x) = \pm \prod_{1 \leq i \leq N} h_i(x)\tilde{h}_i(x), \quad n > 1,
\]

(9.10)

where \( \tilde{h}_i(x) = c_i(\sigma x + 1)^n h_i(\tilde{A}(x)) \), and the constant \( c_i \) is chosen to make \( \tilde{h}_i(x) \) monic.

By the results of Section 8, for each discriminant \( -d \in \mathcal{D}_{n, 2} \) we have that \( f_d(x) \mid P_n(x) \). Furthermore, every root of \( P_n(x) \) is a root of some \( f_d(x) \), by Theorem 9.1, where \( \text{ord}(\tau_2) = n \) in order for the roots of \( f_d(x) \) to have minimal period \( n \). It follows that

\[
P_n(x) = \tilde{c}_n \prod_{-d \in \mathcal{D}_{n, 2}} f_d(x),
\]

for some constant \( \tilde{c}_n \), and taking degrees on both sides and using (9.10) gives the formula

\[
2 \sum_{k|n} \mu(n/k)2^k = \sum_{-d \in \mathcal{D}_{n, 2}} 4h(-d).
\]

The formula of the theorem follows.

The result of Theorem 9.2 is the analogue of [18, Thm.1.3] for the prime 2 in place of 5. The factor \( 1/2 \) in front is to be interpreted as \( 2/\phi(8) \), replacing the factor \( 2/\phi(5) \) in the result of [18]. Also, see Conjecture 1 in the Introduction of that paper.
Theorem 9.1 will now be used to prove the corresponding fact for the algebraic function $\hat{T}(x)$ in Theorem 8.1.

**Theorem 9.3.** The periodic points of the function $\hat{T}(x)$ of (8.2) in $\overline{\mathbb{Q}}$ (or $\mathbb{C}$) are exactly the squares of the periodic points of the function $\hat{F}(x)$, i.e., the fixed points $0, \sigma^2, \vartheta^2$ and the conjugates over $\mathbb{Q}$ of the values $\nu^2(w/8)$, where $w$ is given by (6.1).

**Proof.** As in the proof of Theorem 8.1, the polynomials $g(x, y) = y^2 - (x^2 - 4x + 1)y + x^2$ and $f(x, y) = y^2 + (x^2 - 1)y + x^2$ defining $\hat{T}$ and $\hat{F}$, respectively, satisfy the identity

$$g(x^2, y^2) = f(x, -y)f(x, y).$$

Let $\eta^2$ be a periodic point of $g(x, y)$ of period $n$. Then there exist $\eta_1^2, \eta_2^2, \ldots, \eta_{n-1}^2 \in \overline{\mathbb{Q}}$ such that

$$g(\eta_1^2, \eta_2^2) = g(\eta_2^2, \eta_3^2) = \cdots = g(\eta_{n-1}^2, \eta^2) = 0.$$ 

This means that, for every $i = 0, 1, \ldots, n-1$, either

$$f(\eta_i, \eta_{i+1}) = 0 \text{ or } f(\eta_i, -\eta_{i+1}) = 0, \text{ where } \eta_0 = \eta = \eta_n.$$ 

Now if $f(\eta_i, \eta_{i+1}) = 0$ for all $i$, then $\eta$ is a periodic point of $\hat{F}(x)$.

Otherwise, there exists an $i$ such that $f(\eta_i, \eta_{i+1}) \neq 0$, but $f(\eta_i, -\eta_{i+1}) = 0$. In this case, if $i < n-1$, replace $\eta_{i+1}$ by $-\eta_{i+1}$ in the next equation of the sequence, yielding $f(-\eta_{i+1}, \eta_{i+2}) = 0$. And if this happens for $i = n-1$, then simply replace $\eta$ by $-\eta$. This works because $f(-x, y) = f(x, y)$. In other words, in the chain of equations for $f$, whenever the second argument has a negative sign, choose the next first argument with the same negative sign. And in case the last equation has second argument $\eta$ with a negative sign, then choose the first argument of the first equation as $-\eta$ also. Hence, there is a chain of equations $f(\eta_1, \eta_{n+1}) = 0$ beginning and ending with $\pm \eta$. Hence, $\pm \eta$ is a periodic point of $\hat{F}(x)$ in either case, which implies that $\eta^2$ is the square of a periodic point of $\hat{F}(x)$. This completes the proof. \(\square\)

With this theorem, we have completely proved all the statements in Theorem B of the Introduction.

**10. Appendix**

Here we give a proof of the relation between $u(\tau)$ and $v(\tau)$ that was used in the proof of Proposition 3.1b).

**Proposition 10.1.** The following relation holds between $u(\tau)$ and $v(\tau)$:

$$u^4(v^2 + 1)^2 + 4v(v^2 - 1) = 0.$$
Proof. We have derived in the proof of Proposition 4.1 that
\[ \frac{1}{\psi(\tau)} - \psi(\tau) = q^{-1/2} \frac{(-q^2; q^4)_\infty}{(-q^4; q^4)_\infty}. \]
Proceeding in a similar way, we obtain
\[ \frac{1}{\psi(\tau)} + \psi(\tau) = \frac{\psi(-q) \cdot \varphi(q)}{q^{1/2} (q; q^2)_\infty (q^8; q^8)_\infty} \]
\[ = q^{-1/2} \frac{(q^2; q^2)_\infty}{(-q; q^2)_\infty} \cdot \frac{(-q; q^2)_\infty (q^2; q^2)_\infty}{(q; q^2)_\infty (q^8; q^8)_\infty} \]
\[ = q^{-1/2} \frac{(-q; q^2)_\infty}{(q^2; q^4)_\infty} \cdot \frac{(q^2; q^2)_\infty}{(q^8; q^8)_\infty} \]
\[ = q^{-1/2} \frac{(-q^2; q^4)_\infty (q^4; q^4)_\infty}{(-q^4; q^4)_\infty}. \]
(See [2, pp. 221-222].) Putting the above two expressions to use in \[ \frac{4(1 - \psi^2)}{(1 + \psi^2)^2} = \]
\[ \frac{4(\frac{1}{\psi} - \psi)}{\left(\frac{1}{\psi} + \psi\right)^2}, \]
we find that
\[ \frac{4(1 - \psi^2)}{(1 + \psi^2)^2} = 4q^{1/2} \frac{(-q^2; q^4)_\infty (q^4; q^4)_\infty}{(-q^4; q^4)_\infty} \]
\[ = 4q^{1/2} \frac{(-q^2; q^4)_\infty (q^4; q^4)_\infty}{(-q^2; q^4)_\infty (q^2; q^2)_\infty} \]
\[ = 4q^{1/2} \frac{(-q^2; q^2)_\infty}{(-q^2; q^2)_\infty} \]
\[ = 4q^{1/2} \frac{(-q^2; q^2)_\infty}{(-q^2; q^2)_\infty} \]
\[ = u^4(\tau), \]
completing the proof. □

References

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