Canonical components of character varieties of double twist links $J(2m + 1, 2m + 1)$

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Abstract. We show that a certain smooth projective model of the canonical component of the $\text{SL}_2(\mathbb{C})$-character variety of the double twist link $J(2m + 1, 2m + 1)$, where $m$ is a positive integer, is the conic bundle over the projective line $\mathbb{P}^1$ which is isomorphic to the surface obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by repeating a one-point blow-up $6m + 3$ times.

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1. Introduction

For a complete finite-volume hyperbolic 3-manifold with cusps, the $\text{SL}_2(\mathbb{C})$-character variety of $M$, denoted by $X(M)$, is a complex algebraic set associated to representations of $\pi_1(M)$ into $\text{SL}_2(\mathbb{C})$. Thurston [8] showed that any irreducible component of such a variety containing the character of a discrete faithful representation has complex dimension equal to the number of cusps of $M$. Such components are called canonical components and are denoted by $X_0(M)$. Character varieties have been important tools in studying the topology of $M$, and canonical components encode a lot of topological information about $M$. They contain subvarieties corresponding to Dehn fillings of $M$ and their ideal points can be used to determine essential surfaces in $M$ (see [1]).

Let $J(k, l)$ denote the double twist knot/link indicated in Figure 1, where the integers $k$ and $l$ determine the number of half twists in the boxes; positive numbers correspond to right-handed twists and negative numbers correspond...
to left-handed twists. This is the rational knot/link \( C(k, -l) \) in the Conway’s notation, which corresponds to the continued fraction \( [k, -l] = k - 1/l \). It is a knot when \( kl \) is even and a two-component link when \( kl \) is odd. These are hyperbolic exactly when \( |k| \) and \( |l| \) are greater than one; the \( J(\pm 1, l) = J(l, \pm 1) \) knot/links are torus knot/links.

Character varieties of the \( J(k, l) \) knots and links were computed and analyzed in [6] and [7] respectively. For the Whitehead link \( 5_2^5 \), which is \( J(3, 3) \), Landes [5] showed that a certain smooth projective model of the canonical component in \( \mathbb{P}^2 \times \mathbb{P}^1 \) is the conic bundle over the projective line \( \mathbb{P}^1 \) which is isomorphic to the surface obtained from \( \mathbb{P}^1 \times \mathbb{P}^1 \) by repeating a one-point blow-up nine times. Equivalently, it is isomorphic to the surface obtained from \( \mathbb{P}^2 \) by repeating a one-point blow-up ten times. Harada [2] proved similar results for the links \( 6_2^2 \) and \( 6_3^2 \) in the Rolfsen’s table. Note that a blow-up of \( \mathbb{P}^2 \) at two points is isomorphic to a blow-up of \( \mathbb{P}^1 \times \mathbb{P}^1 \) at one point, although a blow-up of \( \mathbb{P}^2 \) at one point is not isomorphic \( \mathbb{P}^1 \times \mathbb{P}^1 \) (see e.g. [3, Example 7.22]).

In [7], Petersen and the first author generalized Landes’ result to the double twist links \( J(3, 2m + 1) \) which contain the Whitehead link \( J(3, 3) \), and proved that a certain smooth projective model of the canonical component of \( J(3, 2m + 1) \) in \( \mathbb{P}^2 \times \mathbb{P}^1 \) is the conic bundle over \( \mathbb{P}^1 \) which is isomorphic to the surface obtained from \( \mathbb{P}^1 \times \mathbb{P}^1 \) by repeating a one-point blow-up \( 9m \) times if \( m \geq 1 \), and \( -(9m + 6) \) times if \( m \leq -2 \). An important step in proving this result is to show that each singular point of a certain singular projective model of the canonical component of \( J(3, 2m + 1) \) in \( \mathbb{P}^2 \times \mathbb{P}^1 \) requires only one blow-up to resolve. However, this step was assumed without proof in [7]. Note that Harada [2] proved that for the link \( 6_2^2 \), which is not a double twist link, a certain singular projective model of the canonical component in \( \mathbb{P}^2 \times \mathbb{P}^1 \) has singular points which require more than one blow-up to resolve.

In this paper, we consider the hyperbolic double twist links \( J(2m + 1, 2m + 1) \) which also contain the Whitehead link \( J(3, 3) \), and identify their canonical components topologically. Since \( J(-(2m + 1), -(2m + 1)) \) is the mirror image

![Figure 1. The double twist knot/link \( J(k, l) \).](image-url)
of \(J(2m + 1, 2m + 1)\), we only need to consider the case \(m \geq 1\). We will show the following.

**Theorem 1.** The smooth projective model of the canonical component of the \(\text{SL}_2(\mathbb{C})\)-character variety of the double twist link \(J(2m + 1, 2m + 1)\), \(m \geq 1\), is the conic bundle over the projective line \(\mathbb{P}^1\) which is isomorphic to the surface obtained from \(\mathbb{P}^1 \times \mathbb{P}^1\) by repeating a one-point blow-up \(6m + 3\) times. Equivalently, it is isomorphic to the surface obtained from \(\mathbb{P}^2\) by repeating a one-point blow-up \(6m + 4\) times.

Let us explain the meaning of the smooth projective model in Theorem 1 and sketch the proof. An affine model of the canonical component of the \(\text{SL}_2(\mathbb{C})\)-character variety of the double twist link \(J(2m + 1, 2m + 1)\) is given by the zero set of a single polynomial in three complex variables, and it is known to be an affine surface birational to \(\mathbb{C} \times \mathbb{C}\). (This fact actually holds true for all double twist links \(J(2m + 1, 2n + 1)\), by [7].) For affine complex surfaces, choosing the right projective completion is not obvious since different projective completions might result in non-isomorphic smooth projective models. In the case of the canonical component of the double twist link \(J(2m + 1, 2m + 1)\), choosing the projective completion in \(\mathbb{P}^3\) seems natural. However, this projective model has infinitely many singular points. Following [5], we will choose the projective completion in \(\mathbb{P}^2 \times \mathbb{P}^1\) which turns out to have finitely many singular points.

By compactifying the above affine model of the canonical component of \(J(2m + 1, 2m + 1)\) in \(\mathbb{P}^2 \times \mathbb{P}^1\), we obtain a projective model, denoted by \(S\), birational to \(\mathbb{P}^1 \times \mathbb{P}^1\). This projective model is not smooth; it has singular points. By resolving singular points of the surface \(S\) (using one-point blow-ups), we obtain a smooth projective model, denoted by \(\tilde{S}\). *In this paper we refer to \(\tilde{S}\) as the smooth projective model of the canonical component of the \(\text{SL}_2(\mathbb{C})\)-character variety of \(J(2m + 1, 2m + 1)\).*

The smooth projective model \(\tilde{S}\) is also birational to \(\mathbb{P}^1 \times \mathbb{P}^1\). It is known that for two birational varieties the birational equivalence between them can be written as a sequence of blow-ups and blow-downs, see e.g. [4, Chapter 5]. Since \(\mathbb{P}^1 \times \mathbb{P}^1\) is a minimal smooth projective surface (in the sense that it is not a blow-up of any smooth projective surface), we conclude that \(\tilde{S}\) is isomorphic to \(\mathbb{P}^1 \times \mathbb{P}^1\) blown up at \(N\) points. Moreover, this isomorphism (i.e. this number \(N\)) can be determined from the Euler characteristic of \(\tilde{S}\) which, in turn, depends on the Euler characteristic and singular points of \(S\).

An important part of the proof of Theorem 1 is to prove that each singular point of the singular projective model \(S\) requires only one blow-up to resolve, namely, the blow-up of \(\tilde{S}\) at each singular point is smooth everywhere except at the preimages of other singular points of \(S\). A similar proof also works for \(J(3, 2m + 1)\) and therefore fixes the gap in [7]. The remaining of the proof is in the same line as those of [5, 7].

The paper is organized as follows. In Section 2 we review Chebyshev polynomials, character varieties of double twist links, and blowing up surfaces. In Section 3, we give a proof of Theorem 1 with the assumption that each singular component
point of the projective model $S$ of the canonical component of $J(2m+1, 2m+1)$ requires only one blow-up to resolve (Proposition 3.4). Finally, we prove Proposition 3.4 in Section 4 and therefore complete the proof of Theorem 1.

2. Preliminaries

In this section, we first recall the definition of $\text{SL}_2(\mathbb{C})$-character varieties of 3-manifolds. Then, we define Chebychev polynomials of the second kind and prove some of their properties. Next, we review character varieties of two-component double twist links from [7]. Finally, we recall the definition of blowing up varieties at a point.

2.1. Character varieties. Let $M$ be a complete finite-volume hyperbolic 3-manifold with cusps. The $\text{SL}_2(\mathbb{C})$-character variety of $M$ is the set of all characters of representations $\rho : \pi_1(M) \to \text{SL}_2(\mathbb{C})$. The character associated to $\rho$ is $\chi_\rho : \pi_1(M) \to \mathbb{C}$ defined by $\chi_\rho(\gamma) = \text{tr} \rho(\gamma)$.

Let $X(M)$ denote the $\text{SL}_2(\mathbb{C})$-character variety, that is

$$X(M) = \{\chi_\rho \mid \rho : \pi_1(M) \to \text{SL}_2(\mathbb{C})\}.$$ 

The characters of reducible representations themselves form an algebraic set, which is a subset of $X(M)$. The closure of the set of characters of irreducible representations will be denoted by $X_{\text{irr}}(M)$. Any irreducible component of $X(M)$ which contains the character of a discrete faithful representation is contained in $X_{\text{irr}}(M)$ and is called a canonical component and denoted by $X_0(M)$.

Character varieties have been important tools in studying the topology of $M$, and canonical components encode a lot of topological information about $M$. They contain subvarieties corresponding to Dehn fillings of $M$ and their ideal points can be used to determine essential surfaces in $M$ (see [1]).

2.2. Chebychev polynomials. Let $S_k(z)$ be the Chebychev polynomials of the second kind defined by $S_0(z) = 1$, $S_1(z) = z$ and $S_{k+1}(z) = zS_k(z) - S_{k-1}(z)$ for all integers $k$.

It is elementary to verify the following lemma by induction.

**Lemma 2.1.** (1) With $z = a + a^{-1}$ we have

$$S_k(z) = \frac{a^{k+1} - a^{-k-1}}{a - a^{-1}}.$$ 

(2) For $k \geq 1$, the polynomial $S_k(z)$ has degree $k$ and leading term $z^k$.

The following two lemmas can be verified by using Lemma 2.1.

**Lemma 2.2.** (1) For $k \geq 1$, the polynomial $S_k(z) - S_{k-1}(z)$ has exactly $k$ distinct roots given by $z = 2 \cos \frac{(2j-1)\pi}{2k+1}$ where $1 \leq j \leq k$.

(2) For $k \geq 1$, the polynomial $S_k(z) + S_{k-1}(z)$ has exactly $k$ distinct roots given by $z = 2 \cos \frac{2j\pi}{2k+1}$ where $1 \leq j \leq k$. 
Lemma 2.3. For any integer $k$ we have
\[ S_k^2(z) + S_{k-1}^2(z) - zS_k(z)S_{k-1}(z) = 1. \]

We now prove the following two lemmas.

Lemma 2.4. For $k \geq 1$, the polynomial $2z + (z^2 - 4)S_{k-1}(z)S_k(z)$ has exactly $2k + 1$ distinct roots given by $z = 2 \cos \frac{(2j-1)\pi}{2k}$, $1 \leq j \leq k$, and $z = 2 \cos \frac{(2j-1)\pi}{2k+2}$, $1 \leq j \leq k + 1$. In particular, it is a separable polynomial in $\mathbb{C}[z]$.

Proof. Let $P(z) = 2z + (z^2 - 4)S_{k-1}(z)S_k(z)$. Consider $z = a + a^{-1}$ where $a \neq \pm 1$. Since $S_j(z) = \frac{a^{j+1} - a^{-j-1}}{a - a^{-1}}$, we have
\[ P = 2(a + a^{-1}) + (a^2 + a^{-2} - 2) \frac{a^k - a^{-k}}{a - a^{-1}} \frac{a^{k+1} - a^{-k-1}}{a - a^{-1}} = a + a^{-1} + a^{2k+1} + a^{-2k-1} = (a^k + a^{-k})(a^{k+1} + a^{-k-1}). \]

Note that $P = 0$ if $a^k = -1$ or $a^{2k+2} = -1$. Moreover, these two equations do not have any common roots. This implies that $z = 2 \cos \frac{(2j-1)\pi}{2k}$, $1 \leq j \leq k$, and $z = 2 \cos \frac{(2j-1)\pi}{2k+2}$, $1 \leq j \leq k + 1$, are distinct roots of $P$. Since the degree of $P$ is exactly $2k + 1$, these are all the roots of $P$. Therefore, $P$ is separable in $\mathbb{C}[z]$. \( \square \)

Lemma 2.5. For any integer $k$ we have
\[ \frac{dS_k(z)}{dz} = \frac{kS_{k+1}(z) - (k + 2)S_{k-1}(z)}{z^2 - 4}. \]

Proof. Write $z = a + a^{-1}$. Then $S_k(z) = \frac{a^{k+1} - a^{-k-1}}{a - a^{-1}}$ and so
\[ \frac{dS_k(z)}{dz} = \frac{dS_k(z)}{da} \int \frac{dz}{da} \frac{1}{(a - a^{-1})^2} = \frac{k(a^k + a^{-k})(a - a^{-1}) - (a^{k+1} - a^{-k-1})(1 + a^{-2})}{1 - a^{-2}} = \frac{k(a^{k+1} - a^{-k-1})}{z^2 - 4} - \frac{(k + 2)a^{k+1} - a^{-k-1}}{1 - a^{-2}}. \]
The lemma follows, since $\frac{a^{j+1} - a^{-j-1}}{a - a^{-1}} = S_j(z)$. \( \square \)

2.3. Double twist links. Recall that $J(k, l)$ is the double twist knot/link indicated in Figure 1. It is a knot when $kl$ is even and a two-component link when $kl$ is odd. The knot/link $J(k, l)$ is hyperbolic exactly when $|k|$ and $|l|$ are greater than one; the $J(\pm 1, l) = J(l, \pm 1)$ knot/links are torus knots/links. Let $X(k, l)$ denote the $SL_2(\mathbb{C})$-character variety of $S^3 \setminus J(k, l)$ and $X_0(k, l)$ its canonical component.
Character varieties of the $J(k, l)$ knots and links were computed in [6] and [7] respectively. We now review the computation for the $J(k, l)$ links with two components, so both $k$ and $l$ are odd. Suppose $k = 2m + 1$ and $l = 2n + 1$. By [6], the link group of $J(k, l)$ is $\pi_1(k, l) = \pi_1(S^3 \setminus J(k, l))$ and has presentation

$$\pi_1(k, l) = \langle a, b \mid aw^n b = w^{n+1} \rangle$$

where $w_k = (ab^{-1})^m ab(a^{-1}b)^m$. This is the Wirtinger presentation of a link diagram.

For a word $u$ in two letters $a$ and $b$, let $\bar{u}$ denote the word obtained from $u$ by writing the letters in $u$ in reversed order. By [7], the above presentation of the link group of $J(k, l)$ can be rewritten as

$$\pi_1(k, l) = \langle a, b \mid r = \bar{r} \rangle$$

where $r = w_k^m(ab^{-1})^m$.

For a representation $\rho : \pi_1(k, l) \to \text{SL}_2(\mathbb{C})$, we let $x = \text{tr}(\rho(a))$, $y = \text{tr}(\rho(b))$ and $z = \text{tr}(\rho(ab^{-1}))$. Then, by [9, Thm. 1] the algebraic set $X(k, l)$ is exactly the zero set of $\phi(x, y, z) = \text{tr}(\rho(ab)) - \text{tr}(\rho(ab^{-1})) \in \mathbb{C}[x, y, z]$. Moreover, by [7], this polynomial can be written in terms of Chebyshev polynomials as

$$\phi(x, y, z) = (xyz + 4 - x^2 - y^2 - z^2)(S_n(t)S_{m-1}(z) - S_{n-1}(t)S_m(z)), $$

where

$$t = \text{tr}(w_k) = xy - z + (xyz + 4 - x^2 - y^2 - z^2)S_m(z)S_{m-1}(z).$$

The character variety $X(k, l)$ is clearly reducible. The vanishing set of $xyz + 4 - x^2 - y^2 - z^2 \in \mathbb{C}[x, y, z]$ is the set of characters of reducible representations of $\pi_1(k, l)$ into $\text{SL}_2(\mathbb{C})$. An affine model for the algebraic set $X_{\text{irr}}(k, l)$ is the vanishing set of $S_n(t)S_{m-1}(z) - S_{n-1}(t)S_m(z) \in \mathbb{C}[x, y, z]$. Then we have the following.

**Theorem 2.6.** [7] Let $k = 2m + 1$ and $l = 2n + 1$. The algebraic set $X_{\text{irr}}(k, l)$ is birational to $C(k, l) \times \mathbb{C}$ where the curve $C(k, l)$ is given by

$$C(k, l) = \{(t, z) \in \mathbb{C}^2 \mid S_n(t)S_{m-1}(z) - S_{n-1}(t)S_m(z) = 0\}.$$

If $k \neq l$ then $C(k, l)$ is irreducible and $X_0(k, l) = X_{\text{irr}}(k, l)$ is birational to $C(k, l) \times \mathbb{C}$.

The curve $C(3, 3) = C(-3, -3)$ is given by $t = z$. If $k = l$ and $|l| > 3$ then $C(l, l)$ is the union of exactly two irreducible components: $C_0(l, l)$, given by $t = z$, and $C_1(l, l)$, the scheme-theoretic complement of $C_0(l, l)$ in $C(l, l)$. The algebraic set $X_{\text{irr}}(l, l)$ is given by the union $X_0(l, l) \cup X_1(l, l)$, where $X_0(l, l)$ is birational to $C_0(l, l) \times \mathbb{C}$ and $X_1(l, l)$ is birational to $C_1(l, l) \times \mathbb{C}$.

**2.4. One-point blow-ups.** Blowing up varieties is a standard tool for resolving singular points of surfaces. Since blowing up is a local process, it can be done in affine neighborhoods. For our purpose, understanding blowing up subvarieties of $\mathbb{A}^n$ at a point should be sufficient. For more details about blow-ups, see [3] and [4].
3. Proof of Theorem 1

Let \( m \) be a positive integer and \( l = 2m+1 \). By Theorem 2.6, an affine model of the canonical component \( X_0(l, l) \) of the \( SL_2(\mathbb{C}) \)-character variety of the double twist link \( J(l, l) \) is the zero set of the polynomial \( t - z \in \mathbb{C}[x, y, z] \), where

\[
t = xy - z + (xyz + 4 - x^2 - y^2 - z^2)S_m(z)S_{m-1}(z).
\]

Moreover, it is birational to \( C_0(l, l) \times \mathbb{C} \) where \( C_0(l, l) = \{(t, z) \in \mathbb{C}^2 \mid t = z \} \). In particular, \( X_0(l, l) \) is birational to \( \mathbb{C} \times \mathbb{C} \).

### 3.1. Projective model.

We begin by homogenizing the defining polynomial for \( X_0(l, l) \).

Let \( T_k = T_k(z, w) = w^kS_k(\frac{z}{w}) \) for \( k \geq 0 \).

**Lemma 3.1.** For \( k \geq 1 \) we have

1. \( T_k(z, 0) = z^k \),
2. \( T_k^2 + w^2T_k^2T_{k-1} - zT_kT_{k-1} = w^{2k} \),
3. \( w^{2k} + (z \pm 2w)T_kT_{k-1} = (T_k \pm w T_{k-1})^2 \).

**Proof.** (1) follows from Lemma 2.1(2).

(2) follows from Lemma 2.3.

(3) From (2), we have \( w^{2k} + z T_kT_{k-1} = T_k^2 + w^2T_{k-1}^2 \). Hence, \( w^{2k} + (z \pm 2w)T_kT_{k-1} = (T_k \pm w T_{k-1})^2 \).

The homogenization of the defining polynomial \( t - z = xy - 2z + (xyz + 4 - x^2 - y^2 - z^2)S_m(z)S_{m-1}(z) \) in \( \mathbb{P}^2 \times \mathbb{P}^1 = \{([x : y : u], [z : w]) \mid u \neq 0 \} \) is

\[
F = (xyzw - 2uw^2)w^{2m} + (xyzw + 4uw^2w^2 - x^2w^2 - y^2w^2 - u^2z^2)T_mT_{m-1}.
\]
3.2. Singular points. We now determine the singular points of the projective model of $X_0(l, l)$. To do this, we consider solutions $([x : y : u], [z : w]) \in \mathbb{P}^2 \times \mathbb{P}^1$ of $F = F_x = F_y = F_u = F_z = F_w = 0$.

First, we compute these partial derivatives by direct calculations.

**Lemma 3.2.** The first order partial derivatives of $F$ are given by

\[
F_x = (yw^{2m} + (yz - 2xw)T_mT_{m-1})w, \\
F_y = (xw^{2m} + (xz - 2yw)T_mT_{m-1})w, \\
F_u = -2u(2zw^{2m} + (z^2 - 4w^2)T_mT_{m-1}), \\
F_z = -2u^2w^{2m} + (xyw - 2u^2z)T_mT_{m-1} \\
+ (xyzw + 4u^2w^2 - x^2w^2 - y^2w^2 - u^2z^2)T_{m-1}z, \\
F_w = (2m + 1)xyw^{2m} - 4mu^2zw^{2m-1} + (xyz + 8u^2w - 2x^2w - 2y^2w)T_mT_{m-1} \\
+ (xyzw + 4u^2w^2 - x^2w^2 - y^2w^2 - u^2z^2)T_{m-1}w.
\]

We can now determine the singular points.

**Proposition 3.3.** The singular points $([x : y : u], [z : w]) \in \mathbb{P}^2 \times \mathbb{P}^1$ of $F$ are

- $s_1 = ([0 : 1 : 0], [1 : 0])$.
- $s_2 = ([1 : 0 : 0], [1 : 0])$.
- $s_3^{(k)} = ([1 : 1 : 0], [z_3^{(k)} : 1])$, where $z_3^{(k)} = 2\cos\frac{(2k-1)\pi}{2m+1}$, $1 \leq k \leq m$.
- $s_4^{(k)} = ([1 : -1 : 0], [z_4^{(k)} : 1])$, where $z_4^{(k)} = 2\cos\frac{2k\pi}{2m+1}$, $1 \leq k \leq m$.

The number of singular points is $2m + 2$.

**Proof.** Consider the equations $F = F_x = F_y = F_u = F_z = F_w = 0$. We break the analysis down into two cases: $w = 0$ and $w \neq 0$.

**Case 1:** $w = 0$. We can assume $z = 1$. Note that $T_k(1, 0) = 1$ for all $k \geq 1$. By Lemma 3.2, we have $F_x = F_y = 0$, $F_u = -u^2$ and $F_w = -2u$. Then $F = F_u = 0$ are equivalent to $u = 0$. Now we have $F_z = 0$ and $F_w = xy$. Thus $F_w = 0$ becomes $xy = 0$. In this case, there are two singular points $([0 : 1 : 0], [1 : 0])$ and $([1 : 0 : 0], [1 : 0])$.

**Case 2:** $w \neq 0$. In this case, we first solve $F_x = F_y = 0$ and then $F = F_u = 0$. Finally, we show that the equations $F_z = F_w = 0$ follow from $F = F_x = F_y = F_u = 0$.

Since $w \neq 0$, we can assume $w = 1$. We first claim that $(x, y) \neq (0, 0)$. Indeed, assuming $(x, y) = (0, 0)$ we have

\[
F = -2z + (4 - z^2)S_{m-1}(z)S_m(z).
\]

By Lemma 2.4, this polynomial is separable in $\mathbb{C}[z]$, so the equations $F = F_z = 0$ cannot occur. Hence, $(x, y) \neq (0, 0)$. 


Consider the equations $F_x = F_y = 0$. By Lemma 2.3, we have $S_m^2(z) + S_{m-1}^2(z) - zS_m(z)S_{m-1}(z) = 1$. This implies that

\[ F_x = y + (yz - 2x)S_m(z)S_{m-1}(z) = y(S_m^2(z) + S_{m-1}^2(z)) - 2xS_m(z)S_{m-1}(z), \]

\[ F_y = x + (xz - 2y)S_m(z)S_{m-1}(z) = x(S_m^2(z) + S_{m-1}^2(z)) - 2yS_m(z)S_{m-1}(z). \]

Hence,

\[ 2S_m(z)S_{m-1}(z)F_x + (S_m^2(z) + S_{m-1}^2(z))F_y = x(S_m^2(z) - S_{m-1}^2(z))^2, \]

\[ 2S_m(z)S_{m-1}(z)F_y + (S_m^2(z) + S_{m-1}^2(z))F_x = y(S_m^2(z) - S_{m-1}^2(z))^2. \]

Since $x$ and $y$ are not simultaneously equal to 0, the equations $F_x = F_y = 0$ imply that $S_m^2(z) - S_{m-1}^2(z) = 0$. We now consider the subcases $S_m(z) - S_{m-1}(z) = 0$ and $S_m(z) + S_{m-1}(z) = 0$ separately.

**Subcase 2a**: $S_m(z) - S_{m-1}(z) = 0$. By Lemma 2.2, $z = 2 \cos \frac{(2k-1)\pi}{2m+1}$ for some $1 \leq k \leq m$. From $S_m^2(z) + S_{m-1}^2(z) - zS_m(z)S_{m-1}(z) = 1$ and $S_m(z) - S_{m-1}(z) = 0$, we have $S_m^2(z) = \frac{1}{2}$. This implies that $F_x = \frac{2(y-x)}{2-2z}$ and $F_y = \frac{2(x-y)}{2-2z}$. Hence, $F_x = F_y = 0$ are equivalent to $x = y$. Since $S_m^2(z) = \frac{1}{2}$, we have $F = u^2(2-z)$ and $F_u = 2u(2-z)$. Hence, $F = F_u = 0$ are equivalent to $u = 0$. Then, by Lemma 3.2 we have

\[ F_z = \left[ S_m(z)S_{m-1}(z) + (z - 2)(S_m(z)S_{m-1}(z))^2 \right] x^2, \]

\[ F_w = \left[ (2m + 1) + (z - 4)S_m(z)S_{m-1}(z) + (z - 2)(T_mT_{m-1})_w \right] x^2. \]

We claim that $F_z = F_w = 0$. Indeed, by taking derivative of the identity $S_m^2(z) + S_{m-1}^2(z) - zS_m(z)S_{m-1}(z) = 1$ and using $S_m(z) = S_{m-1}(z)$, we get $(2 - z)(S'_m(z) + S'_{m-1}(z)) = S_m(z)$. It follows that $F_z = 0$.

Similarly, by taking partial derivative w.r.t. $w$ of the identity $T_m^2 + w^2T_{m-1}^2 - z T_mT_{m-1} = w^{2m}$ (by Lemma 3.1(2)) and using $S_m(z) = S_{m-1}(z)$, we get

\[ (2 - z)(T_m)_w + (T_{m-1})_w S_m(z) + 2S_m^2(z) = 2m. \]

It follows that

\[ (2m + 1) + (z - 4)S_m(z)S_{m-1}(z) + (z - 2)(T_mT_{m-1})_w = 1 + (z - 2)S_m^2(z) = 0. \]

Hence, $F_w = 0$.

We have proved that the singular points in this subcase are $\{(1: 1: 0), [z : 1]\}$ where $z = 2 \cos \frac{(2k-1)\pi}{2m+1}$ for some $1 \leq k \leq m$.

**Subcase 2b**: $S_m(z) + S_{m-1}(z) = 0$. Similar to the above, singular points in this subcase are $\{(1: -1: 0), [z : 1]\}$ where $z = 2 \cos \frac{2k\pi}{2m+1}$ for some $1 \leq k \leq m$. $\square$

Let $S = \mathcal{Z}(F) \subset \mathbb{P}^2 \times \mathbb{P}^1$ be the vanishing set of $F$.

**Proposition 3.4.** Each singular point $p$ of $S$ requires only one blow-up to resolve. Namely, the blow-up of $S$ at $p$ is smooth everywhere except at the preimages of other singular points $q \neq p$ of $S$. 

We will prove Proposition 3.4 in the last section.

3.3. Euler characteristic. As in [5], to compute the Euler characteristic $\chi(S)$ we observe that $F = G + u^2H$, where $G, H$ are polynomials independent of $u$. Explicitly,

$$
G = xyw^{2m+1} + (xyzw - x^2w^2 - y^2w^2)T_mT_{m-1}, \\
H = -2z w^{2m} + (4w^2 - z^2)T_mT_{m-1}.
$$

Recall that $T_k = T_k(z, w) = w^kS_k(\frac{z}{w}) \in \mathbb{C}[z, w]$. By Lemma 3.1(2), we have $T_m^2 + w^2T_{m-1}^2 - zT_mT_{m-1} = w^{2m}$. Hence, we can write

$$
G = (xT_m - ywT_{m-1})(yT_m - wxT_{m-1})w.
$$

Due to the special form of $F$ as above, we introduce the rational map

$$
\varphi : S \cong \mathbb{Z}(F) \subseteq \mathbb{P}^2 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1
$$

defined by $((x : y : u), [z : w]) \mapsto ([x : y], [z : w])$. This will play an important role in the computation of $\chi(S)$.

We first determine the domain of $\varphi$.

Lemma 3.5. The domain of $\varphi$ is the set $U = S \setminus A$, where $A$ is the set of points $([0 : 0 : 1], [z : 1])$ in $\mathbb{P}^2 \times \mathbb{P}^1$ satisfying $-2z + (4 - z^2)S_m(z)S_{m-1}(z) = 0$.

Proof. The map $\varphi$ is not defined at points of the set

$$
A = \{([0 : 0 : 1], [z : w]) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid F = 0\} \subset S.
$$

When $(x, y, u) = (0, 0, 1)$ we have $G = 0$ and so $F = H$. If $(z, w) = (1, 0)$ then $H = -T_m(1, 0)T_{m-1}(1, 0) = -1 \neq 0$. If $w = 1$ then $H = -2z + (4 - z^2)S_m(z)S_{m-1}(z)$. Hence, $A$ is equal to the set of points $([0 : 0 : 1], [z : 1])$ in $\mathbb{P}^2 \times \mathbb{P}^1$ satisfying $-2z + (4 - z^2)S_m(z)S_{m-1}(z) = 0$. \qed

Note that the set $A$ has cardinality $2m + 1$. We next determine the image $\varphi(U)$.

Lemma 3.6. We have

$$
\varphi(U) = \mathbb{P}^1 \times \mathbb{P}^1 - B,
$$

where $B$ is the set of points $([x : y], [z : 1]) \in \mathbb{P}^1 \times \mathbb{P}^1$ satisfying $-2z + (4 - z^2)S_m(z)S_{m-1}(z) = 0$ and $(xS_m(z) - yS_{m-1}(z))(yS_m(z) - xS_{m-1}(z)) \neq 0$.

Proof. Note that a point $([x : y], [z : w]) \in \mathbb{P}^1 \times \mathbb{P}^1$ is not in the image $\varphi(U)$ if and only if $F([x : y : u], [z : w]) \in \mathbb{C}[u]$ is a nonzero constant. This is equivalent to $H = 0$ and $G \neq 0$. Recall that $G = (xT_m - ywT_{m-1})(yT_m - wxT_{m-1})w$.

Since $G \neq 0$, we have $w \neq 0$. We can assume $w = 1$, so $H = -2z + (4 - z^2)S_m(z)S_{m-1}(z)$ and $G = (xS_m(z) - yS_{m-1}(z))(yS_m(z) - xS_{m-1}(z))$. The lemma then follows. \qed

Lemma 3.7. We have

$$
\chi(B) = 0.
$$
Proof. Let \( P(z) = -2z + (4 - z^2)S_m(z)S_{m-1}(z) \). By Lemma 2.4, \( P(z) \) is separable in \( \mathbb{C}[z] \). Moreover, by Lemma 2.2, \( P(z) \) and \( S_m(z) \pm S_{m-1}(z) \) do not share any common roots. Hence, if \( P(z) = 0 \) then \( S_m(z) \neq S_{m-1}(z) \). We have

\[
B = \bigsqcup_{z \in \mathbb{F}} (\mathbb{P}^1 \setminus \{(S_m(z) : S_{m-1}(z)), (S_{m-1}(z) : S_m(z)]\}) \times \{z : 1\}
\]

Since \( \mathbb{P}^1 \) with two points removed has Euler characteristic zero, we obtain \( \chi(B) = 0 \). □

Let \( C = \mathbb{Z}(G) \) be the zero set of \( G \) in \( \mathbb{P}^1 \times \mathbb{P}^1 \).

Lemma 3.8. We have

\[
\chi(C) = 4 - 2m.
\]

Proof. To compute the Euler characteristic of \( C \), we write \( C = C_1 \cup C_2 \cup C_3 \) where \( C_i \)'s are subsets of \( \mathbb{P}^1 \times \mathbb{P}^1 \) defined by

\[
\begin{align*}
C_1 &= \mathbb{Z}(w) = \mathbb{P}^1 \times \{(1 : 0)\}, \\
C_2 &= \mathbb{Z}(x T_m - y w T_{m-1}), \\
C_3 &= \mathbb{Z}(y T_m - x w T_{m-1}).
\end{align*}
\]

Note that \( C_1 \cap C_2 = \{(1 : 0), (1 : 0)\} \) and \( C_1 \cap C_3 = \{(0 : 1), (1 : 0)\} \). Moreover, \( (x : y), (z : w) \in C_2 \cap C_3 \) if and only if \( x = y \) and \( T_m = w T_{m-1}, \) or \( x = -y \) and \( T_m = -w T_{m-1} \). If \( (z, w) = (1, 0) \) then \( T_k = 1 \) and so \( T_m \neq \pm w T_{m-1} \). If \( w = 1 \) then the equation \( T_m = \pm w T_{m-1} \) is equivalent to \( S_m(z) = \pm S_{m-1}(z) \). Hence,

\[
C_2 \cap C_3 = \{(1 : 1), (1 : 1) : S_m(z) = S_{m-1}(z) = 0\}
\]

which has cardinality \( 2m \). Hence,

\[
\begin{align*}
\chi(C) &= \chi(C_1) + \chi(C_2) + \chi(C_3) - \chi(C_1 \cap C_2) - \chi(C_1 \cap C_3) - \chi(C_2 \cap C_3) \\
&= 2 + 2 + 2 - 1 - 1 - 2m + 0 = 4 - 2m.
\end{align*}
\]

Note that \( C_1 \cap C_2 \cap C_3 = \emptyset \). □

We are now ready to compute the Euler characteristic of the surface \( S = \mathbb{Z}(F) \).

Proposition 3.9. We have

\[
\chi(S) = 4m + 5.
\]

Proof. Recall that \( F = G + u^2 H \), where \( G, H \) are polynomials independent of \( u \), and \( \varphi : S \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \) is defined by \( ([x : y : u], [z : w]) \mapsto ([x : y], [z : w]) \).

Note that \( \chi(S) = \chi(U) + \chi(A) \). Since \( A \) is a finite set of cardinality \( 2m + 1 \), we have \( \chi(A) = 2m + 1 \). To compute \( \chi(U) \) we notice that a fixed point \( ([x : y], [z : w]) \in \varphi(U) = (\mathbb{P}^1 \times \mathbb{P}^1) \setminus B \) has
where \( B = \{([x : y], [z : w]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid G \neq 0, H = 0, \} \).

Recall that \( C = \{([x : y], [z : w]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid G = 0, H = 0 \} \subset \mathbb{P}^1 \times \mathbb{P}^1 \). Let \( L = \{([x : y], [z : w]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid G = 0, H \neq 0 \} \). Note that
\[
\{(x : y), (z : w) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid G \neq 0, H = 0 \} = \varphi(U) \setminus C,
\{(x : y), (z : w) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid G = 0, H \neq 0 \} = C \setminus L.
\]

Note that \( \varphi(U) \) is the disjoint union of three subsets \( \varphi(U) \setminus C, C \setminus L \) and \( L \).

Hence, \( U = \varphi^{-1}(\varphi(U)) \) can be written as the disjoint union of three subsets \( \varphi^{-1}(\varphi(U) \setminus C), \varphi^{-1}(C \setminus L) \) and \( \varphi^{-1}(L) \). Since
\[
\chi(\varphi^{-1}(\varphi(U) \setminus C)) = 2\chi(\varphi(U) \setminus C),
\chi(\varphi^{-1}(C \setminus L)) = \chi(C \setminus L),
\chi(\varphi^{-1}(L)) = \lvert L \rvert \chi(\mathbb{A}^1) = \lvert L \rvert = \chi(L).
\]
we have
\[
\chi(U) = 2\chi(\varphi(U) \setminus C) + \chi(C \setminus L) + \chi(L)
= 2\chi(\mathbb{P}^1 \times \mathbb{P}^1 \setminus (B \cup C)) + \chi(C)
= (2\chi(\mathbb{P}^1 \times \mathbb{P}^1) - 2\chi(B) - 2\chi(C)) + \chi(C)
= 2\chi(\mathbb{P}^1 \times \mathbb{P}^1) - 2\chi(B) - \chi(C)
= 8 - 0 - (4 - 2m) = 2m + 4.
\]

Finally, since \( \chi(A) = 2m + 1 \) we obtain \( \chi(S) = \chi(U) + \chi(A) = 4m + 5 \). \( \square \)

### 3.4. Proof of Theorem 1.

Recall that \( S = Z(F) \subset \mathbb{P}^2 \times \mathbb{P}^1 \) is the vanishing set of \( F \). Let \( S_{\text{sing}} \) be the set of singular points of \( S \). By Proposition 3.3, its cardinality is \( \lvert S_{\text{sing}} \rvert = 2m + 2 \).

Let \( \hat{S} \) be the smooth projective surface obtained from \( S \) by resolving all the singular points of \( S \). By Proposition 3.4, each singular point of \( S \) requires one blow-up to resolve. Moreover, from its proof in Section 4 we see that the preimage of each singular point is locally a conic and hence locally isomorphic to \( \mathbb{P}^1 \).

This implies that
\[
\chi(\hat{S}) = \chi(S \setminus S_{\text{sing}}) + \lvert S_{\text{sing}} \rvert \cdot \chi(\mathbb{P}^1) = (\chi(S) - \lvert S_{\text{sing}} \rvert) + 2\lvert S_{\text{sing}} \rvert = \chi(S) + \lvert S_{\text{sing}} \rvert.
\]

Hence,
\[
\chi(\hat{S}) = \chi(S) + \lvert S_{\text{sing}} \rvert = (4m + 5) + (2m + 2) = 6m + 7.
\]

Since \( S \) is birational to \( \mathbb{P}^1 \times \mathbb{P}^1 \), \( \hat{S} \) is a smooth projective surface birational to \( \mathbb{P}^1 \times \mathbb{P}^1 \). It is known that \( \mathbb{P}^1 \times \mathbb{P}^1 \) is a minimal smooth projective surface, namely, it is not a blow-up of any smooth projective surface (see e.g. [3] and [4]). Hence, we can blow down \( \hat{S} \) over \( \mathbb{P}^1 \) some number of times so that it becomes a fiber bundle \( \mathbb{P}^1 \times \mathbb{P}^1 \) over \( \mathbb{P}^1 \).
Let $N$ be such that $\mathcal{S}$ is obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by $N$ one-point blow-ups. Then
\[
\chi(\mathcal{S}) = (\chi(\mathbb{P}^1 \times \mathbb{P}^1) - N) + N \cdot \chi(\mathbb{P}^1) = 4 + N.
\]
Hence, $N = \chi(\mathcal{S}) - 4 = 6m + 3$. This proves Theorem 1.

4. Blow-ups at singular points

In this section, we prove Proposition 3.4 and therefore complete the proof of Theorem 1. We will show that each of the singular points $s_1$ and $s^{(k)}_3$ of the projective model $\mathcal{S}$ requires only one blow-up to resolve. Namely, the blow-up of $\mathcal{S}$ at $p = s_1$ (or $p = s^{(k)}_3$) is smooth everywhere except at the preimages of the singular points $q \neq p$ of $\mathcal{S}$. The proofs for $s_2$ and $s^{(k)}_4$ are similar.

Recall that the defining equation for $\mathcal{S}$ in $\mathbb{P}^2 \times \mathbb{P}^1 = \{(x : y : u), [z : w]\}$ is
\[
F = (xyw - 2u^2z)w^{2m} - (x^2w^2 + y^2w^2 + u^2w^2 - xyzw - 4u^2w^2)T_mT_{m-1},
\]
where $T_k = T_k(z, w) = w^kS_k(\frac{z}{w})$.

4.1. Singular point $s_1$. To perform the blow-up of $\mathcal{S}$ at $s_1 = (\{0 : 1 : 0\}, \{1 : 0\})$, we consider the affine open set $A'_1$ such that $y \neq 0$ and $z \neq 0$. Since $A'_1$ contains the singular points $s_3^{(k)}$ and $s_4^{(k)}$ where $1 \leq k \leq m$, we actually look at the blow-up of $\mathcal{S}$ at $s_1$ in the affine open set $A_1 = A'_1 \setminus \bigcup_{1 \leq k \leq m} \{s_3^{(k)}, s_4^{(k)}\}$. The local affine coordinates for $A_1 \cong \mathbb{A}^3$ are $x, u, w$. So to blow up $\mathcal{S}$ at $s_1$, we blow up $X_1 = Z(F|_{y=1, z=1})$ at the point $(x, u, w) = (0, 0, 0)$ in $A_1$. Using coordinates $a, b, c$ for $\mathbb{P}^2$, the blow-up $Y_1$ of $X_1$ at $(0, 0, 0)$ is the closed subset in $A_1 \times \mathbb{P}^2$ defined as the zero set of the following polynomials:
\[
F_1 = F|_{y=1, z=1} = (xw - 2u^2)w^{2m} - (x^2w^2 + w^2 + u^2 - xw - 4u^2w^2)T_m(1, w)T_{m-1}(1, w),
\]
\[
e_1 = xb - ua,
\]
\[
e_2 = xc - ua,
\]
\[
e_3 = wb - uc.
\]
We will determine the local model of $Y_1$ and check for smoothness by looking at $Y_1$ in the affine open sets defined by $a \neq 0, b \neq 0, \text{ and } c \neq 0$.

Let $D(w) = T_m(1, w)T_{m-1}(1, w)$. Note that $D(0) = 1$ (by Lemma 3.1(1)).

4.1.1. $a \neq 0$. First we look at $Y_1$ in the affine open set defined by $a \neq 0$ (we can assume $a = 1$). In this open set, the defining equations for $Y_1$ become
\[
F_1 = (xw - 2u^2)w^{2m} - (x^2w^2 + w^2 + u^2 - xw - 4u^2w^2)D(w),
\]
\[
e_1 = xb - u,
\]
\[
e_2 = xc - w,
\]
\[
e_3 = wb - uc.
\]
From equations \( e_1 = 0 \) and \( e_2 = 0 \), we have \( u = xb \) and \( w = xc \). By replacing \( u \) with \( xb \) and \( w \) with \( xc \) in \( F_1 \), we obtain
\[
F_1 = x^2 \left[ (c - 2b^2)(xc)^2m - (x^2c^2 + c^2 + b^2 - c - 4x^2b^2c^2)D(xc) \right].
\]
The first factor corresponds to the exceptional plane \( E_1 \) and the other factor is the defining equation for the local model of \( Y_1 \). Note that the preimage of \( s_1 \) is exactly the intersection of \( E_1 \) and \( Y_1 \) which is equal to the smooth conic \( c^2 + b^2 - c = 0 \). This local model of \( Y_1 \) is smooth in \( A_1 \times \mathbb{P}^2 \) if we can show that
\[
R(b, c, x) : = (c - 2b^2)(xc)^{2m} - (x^2c^2 + c^2 + b^2 - c - 4x^2b^2c^2)D(xc)
\]
is smooth. We now prove that the system \( R = R_b = R_c = R_x = 0 \) has no solutions.

By direct calculations, we have
\[
\begin{align*}
R_b &= -2b \left( 2x^2mc^{2m} + (1 - 4x^2c^2)D \right), \\
R_c &= (xc)^{2m} + 2m(c - 2b^2)x^{2m}c^{2m-1} - (2x^2c + 2c - 1 - 8x^2b^2c)D \\
&\quad - (x^2c^2 + c^2 + b^2 - c - 4x^2b^2c^2)xD_w, \\
R_x &= 2m(c - 2b^2)x^{2m-1}c^{2m} - (2c^2x - 8xb^2c^2)D \\
&\quad - (x^2c^2 + c^2 + b^2 - c - 4x^2b^2c^2)cD_w.
\end{align*}
\]

Note that
\[
R - bR_b/2 = c \left( x^2mc^{2m} - (x^2c + c - 1)D \right), \\
xR_x - cR_c = c \left( -x^2mc^{2m} + (2c - 1)D \right).
\]

Assume that \( R = R_b = R_c = R_x = 0 \) at some point \( (b, c, x) \). We will consider the two cases \( b = 0 \) and \( b \neq 0 \) separately.

Suppose \( b = 0 \). We claim that \( xc \neq 0 \). Indeed, if \( c = 0 \) then \( R_c = D(0) = 1 \neq 0 \). If \( c \neq 0 \) and \( x = 0 \), then \( R - bR_b/2 = 0 \) implies that \((c-1)D(0) = 1 \). So \( c = 1 \) and \( R_c =-D(0) = -1 \neq 0 \). Hence, \( xc \neq 0 \). From \( R - bR_b/2 = 0 \) and \( xR_x - cR_c = 0 \), we have \( x^{2m}c^{2m} - (x^2c + c - 1)D = 0 \) and \(-x^2mc^{2m} + (2c - 1)D = 0 \).

So \( x^2c + c - 1 = 2c - 1 \), i.e. \( x = \pm 1 \). Then \( D = \frac{x^{2m-2m}}{2c-1} = \frac{w^{2m}}{\pm 2w-1} \). Since \( D = T_m(1, w)T_{m-1}(w) = w^{2m-1}S_m(\frac{1}{w})S_{m-1}(\frac{1}{w}) \), we obtain \( S_m(\frac{1}{w})S_{m-1}(\frac{1}{w}) = \frac{w}{\pm 2w-1} \).

This is equivalent to \((\pm 2 - \frac{1}{w})S_m(\frac{1}{w})S_{m-1}(\frac{1}{w}) = 1 \), i.e. \((S_m(\frac{1}{w}) \neq S_{m-1}(\frac{1}{w}))^2 = 0 \) (by Lemma 2.3). Hence,
\[
([x : y : u], [z : w]) = ([x : 1 : u], [1 : w])
\]
\[
= ([\pm 1 : 1 : 0], [1 : w])
\]
\[
= ([1 : \pm 1 : 0], [\frac{1}{w} : 1]),
\]
which is equal to either \( s_3^{(k)} \) or \( s_4^{(k)} \). This point is not in \( A_1 \), since it has already been removed from \( A_1 \).

Suppose \( b \neq 0 \). Then \( R_b = 0 \) implies that \( 2x^2mc^{2m} + (1 - 4x^2c^2)D = 0 \). Note that \( xc \neq 0 \). (Otherwise \( 2x^2mc^{2m} + (1 - 4x^2c^2)D = D(0) = 1 \neq 0 \). From
that

$$\frac{R - bR_c/2}{2} = 0$$

and

$$xR_x - cR_x = 0$$

we also have

$$x^{2m}c^{2m} - (c^2c + c - 1)D = 0$$

and

$$−x^{2m}c^{2m} + (2c - 1)D = 0.$$  

This implies that

$$x^2c + c - 1 = 2c - 1 = \frac{1}{2}(4c^2 - 1).$$

Hence, $x^2 = 1$ and $2c - 1 = \frac{1}{2}(4c^2 - 1)$, so $c = 1/2$. But then

$$2x^{2m}c^{2m} + (1 − 4x^2c^2)D = 2x^{2m}c^{2m} \neq 0,$$

a contradiction.

4.1.2. $b \neq 0$. Now we look at $Y_1$ in the affine open set defined by $b \neq 0$ (we can assume $b = 1$). In this open set, the defining equations for $Y_1$ become

$$F_1 = (xw - 2u^2)w^{2m} - (x^2w^2 + w^2 + u^2 - xw - 4uw^2)D(u),$$

$$e_1 = x - u,$n

$$e_2 = xc - wa,$n

$$e_3 = w - uc.$$

From equations $e_1 = 0$ and $e_3 = 0$, we have $x = ua$ and $w = uc$. By replacing $x$ with $ua$ and $w$ with $uc$ in $F_1$, we obtain

$$F_1 = u^2 [(ac - 2)(uc)^{2m} - (a^2c^2u^2 + c^2 + 1 - ac - 4u^2c^2)D(uc)].$$

The first factor corresponds to the exceptional plane $E_1$ and the other factor is the defining equation for the local model of $Y_1$. Note that the preimage of $s_1$ is exactly the intersection of $E_1$ and $Y_1$ which is equal to the smooth conic $c^2 + 1 − ac = 0$. This local model of $Y_1$ is smooth in $A_1 \times \mathbb{P}^2$ if we can show that

$$R(a, c, u) := (ac - 2)(uc)^{2m} - (a^2c^2u^2 + c^2 + 1 - ac - 4u^2c^2)D(uc)$$

is smooth. We now prove that the system $R = R_a = R_c = R_u = 0$ has no solutions.

By direct calculations, we have

$$R_a = c\left(u^{2m}c^{2m} - (2au^2c - 1)D\right),$$

$$R_c = a(uc)^{2m} + 2m(ac - 2)u^{2m}c^{2m-1} - (2a^2c^2u^2 + 2c - a - 8u^2c)D$$

$$- (a^2c^2u^2 + c^2 + 1 - ac - 4u^2c^2)uDw, $$

$$R_u = 2m(ac - 2)u^{2m-1}c^{2m} - (2a^2c^2u^2 - 8uc^2)D$$

$$- (a^2c^2u^2 + c^2 + 1 - ac - 4u^2c^2)cDw.$$  

Note that

$$uR_a - cR_c = c\left(-au^{2m}c^{2m} + (2c - a)D\right).$$

Assume that $R = R_a = R_c = R_u = 0$ at some point $(a, c, u)$. If $c = 0$, then $R = −D(0) = −1 \neq 0$, a contradiction. Hence, $c \neq 0$. Then $R_a = 0$ implies that

$$u^{2m}c^{2m} - (2au^2c - 1)D = 0.$$  

Note that $u \neq 0$. (Otherwise $u^{2m}c^{2m} - (2au^2c - 1)D = D(0) = 1 \neq 0$.) Hence, $2au^2c - 1 \neq 0$ and $D = \frac{u^{2m}c^{2m}}{2au^2c - 1}$. From

$$uR_a - cR_c = 0,$$

we get

$$-au^{2m}c^{2m} + (2c - a)\frac{u^{2m}c^{2m}}{2au^2c - 1} = 0.$$  

This implies that

$$- a + \frac{2c - a}{2au^2c - 1} = 0,$$

i.e., $a^2u^2 = 1$.

Similarly, from $R = (ac - 2)(uc)^{2m} - (a^2c^2u^2 + c^2 + 1 - ac - 4u^2c^2)\frac{u^{2m}c^{2m}}{2au^2c - 1} = 0$

we have

$$ac - 2 - \frac{a^2c^2u^2 + c^2 + 1 - ac - 4u^2c^2}{2au^2c - 1} = 0.$$  

Since $u^2 = 1/a^2$, we obtain $ac - 2 -$
\[ \frac{2c^2+1-4c^2/a^2}{2c/a-1} = 0. \] This is equivalent to \( \left( \frac{2c}{a} - 1 \right)^2 = 0 \), i.e. \( 2c = a \). But then \( 2au^2c - 1 = a^2u^2 - 1 = 0 \), a contradiction.

4.1.3. \( c \neq 0 \). Finally we look at \( Y_1 \) in the affine open set defined by \( c \neq 0 \) (we can assume \( c = 1 \)). In this open set, the defining equations for \( Y_1 \) become

\[
F_1 = (xw - 2u^2)w^2m - (x^2w^2 + w^2 + xw - 4u^2w^2)D(w),
\]

\[
e_1 = xb - ua,
\]

\[
e_2 = x - wa,
\]

\[
e_3 = wb - u.
\]

From equations \( e_2 = 0 \) and \( e_3 = 0 \), we have \( x = wa \) and \( u = wb \). By replacing \( x \) with \( wa \) and \( u \) with \( wb \) in \( F_1 \), we obtain

\[
F_1 = w^2 [(a - 2b^2)w^2m - (a^2w^2 + 1 + b^2 - a - 4b^2w^2)D(w)].
\]

The first factor corresponds to the exceptional plane \( E_1 \) and the other factor is the defining equation for the local model of \( Y_1 \). Note that the preimage of \( s_1 \) is exactly the intersection of \( E_1 \) and \( Y_1 \) which is equal to the smooth conic \( 1 + b^2 - a = 0 \). This local model of \( Y_1 \) is smooth in \( A_1 \times \mathbb{P}^2 \) if we can show that

\[
R(a, b, w): = (a - 2b^2)w^2m - (a^2w^2 + 1 + b^2 - a - 4b^2w^2)D(w),
\]

is smooth. We now prove that the system \( R = R_a = R_b = R_w = 0 \) has no solutions.

By direct calculations, we have

\[
R_a = w^2m - (2aw^2 - 1)D,
\]

\[
R_b = -2b (2w^2m + (1 - 4w^2)D),
\]

\[
R_w = 2m(a - 2b^2)w^{2m-1} - (2a^2w - 8b^2w)D
\]

\[-(a^2w^2 + 1 + b^2 - a - 4b^2w^2)D_w.
\]

Note that

\[
R - (a - 2b^2)R_a = (a^2w^2 - 1 + b^2 + 4b^2w^2 - 4ab^2w^2)D.
\]

Assume that \( R = R_a = R_b = R_w = 0 \) at some point \((a, b, w)\). We will consider the two cases \( b = 0 \) and \( b \neq 0 \) separately.

Suppose \( b = 0 \). Then \( R - (a - 2b^2)R_a = 0 \) implies that \( (a^2w^2 - 1)D = 0 \). If \( D = 0 \), then from \( R_a = 0 \) we have \( w = 0 \). This implies that \( D = D(0) = 1 \neq 0 \), a contradiction. Hence, \( a^2w^2 - 1 = 0 \), i.e. \( a = \pm 1/w \). From \( R_a = 0 \), we have \( D = \frac{w^2m}{\pm 2w-1} \). This is equivalent to \( (S_m(\frac{1}{w}) \not\equiv S_{m-1}(\frac{1}{w}))^2 = 0 \). Hence,

\[
([x : y : u], [z : w]) = ([aw : 1 : bw], [1 : w])
\]

\[= ([\pm 1 : 1: 0], [1 : w])
\]

\[= ([1 : 1 : 0], [\frac{1}{w} : 1]),
\]
which corresponds to either \( s_3^{(k)} \) or \( s_4^{(k)} \). This point is not in \( A_1 \), since it has already been removed from \( A_1 \).

Suppose \( b \neq 0 \). From \( R_b = 0 \), we have \( 2w^{2m} + (1 - 4w^2)D = 0 \). This implies that \( w \neq 0 \) (otherwise \( 2w^{2m} + (1 - 4w^2)D = D(0) = 1 \neq 0 \), so \( 4w^2 - 1 \neq 0 \) and \( D = \frac{2w^{2m}}{4w^2 - 1} \neq 0 \). Then \( R_a = 0 \) becomes \( 1 - \frac{2(2aw^2 - 1)}{4w^2 - 1} = 0 \), which means that \( a = 1 + \frac{1}{4w^2} \). From \( R - (a - 2b^2)R_a = 0 \) and \( D \neq 0 \), we have \( a^2w^2 - 1 + b^2 + 4b^2w^2 - 4ab^2w^2 = 0 \). But \( b^2 + 4b^2w^2 - 4ab^2w^2 = b^2(1 + 4w^2 - 4aw^2) = 0 \), so \( a^2w^2 - 1 = 0 \). Hence, \( a = 1 + \frac{1}{4w^2} = 1 + \frac{a^2}{4} \), i.e. \( a = 2 \). This implies that \( 4w^2 - 1 = 0 \), which contradicts \( 4w^2 - 1 \neq 0 \).

**4.1.4. Conclusion.** From the cases \( a \neq 0 \), \( b \neq 0 \), and \( c \neq 0 \) considered above, we conclude that the singular point \( s_1 \) requires only one blow-up to resolve.

**4.2. Singular points \( s_3^{(k)} \).** To perform the blow-up of \( S \) at

\[
s_3^{(k)} = (1 : 1 : 0, z_3^{(k)} : 1),
\]

we consider the affine open set \( A'_3 \) such that \( x \neq 0 \) and \( z \neq 0 \). Since \( A'_3 \) contains all other singularities except \( s_1 \), we actually look at the blow-up of \( S \) at \( s_1 \) in the affine open set \( A_3 = A'_3 \setminus (S_{\text{sing}} \setminus \{s_1, s_3^{(k)}\}) \). The local affine coordinates for \( A_3 \cong \mathbb{A}^3 \) are \( y, u, w \). So to blow up \( S \) at \( s_3^{(k)} \), we blow up \( X_3 = \mathbb{Z}(F|_{x=1, z=z_3^{(k)}}) \) at the point \((y, u, w) = (1, 0, 1)\) in \( A_3 \). For short, we write \( z_0 \) for \( z_3^{(k)} \). Note that \( S_m(z_0) - S_{m-1}(z_0) = 0 \). Using coordinates \( a, b, c \) for \( \mathbb{P}^2 \), the blow-up \( Y_3 \) of \( X_3 \) at \((1, 0, 1)\) is the closed subset in \( A_3 \times \mathbb{P}^2 \) defined as the zero set of the following polynomials:

\[
F_3 = F|_{x=1, z=z_0} = (yw - 2u^2z_0)w^{2m} + (yz_0w + 4u^2w^2 - w^2 - y^2w^2 - u^2z_0^2)P(w),
\]

\[
e_1 = ua - (y - 1)b,
\]

\[
e_2 = (w - 1)a - (y - 1)c,
\]

\[
e_3 = (w - 1)b - uc,
\]

where \( P(w) = T_m(z_0, w)T_{m-1}(z_0, w) \). Note that \( P(0) = z_0^{2m-1} \) (by Lemma 3.1(1)).

We will determine the local model of \( Y_3 \) and check for smoothness by looking at \( Y_3 \) in the affine open sets defined by \( a \neq 0 \), \( b \neq 0 \), and \( c \neq 0 \).

By Lemma 3.1(3), we have \( w^{2m} + (z - 2w)T_mT_{m-1} = (T_m - wT_{m-1})^2 \). Hence,

\[
F_3 = yw(w^{2m} + (z_0 - 2w)P) - 2u^2z_0w^{2m} + (4u^2w^2 - (y - 1)^2w^2 - u^2z_0^2)P = yw(T_m(z_0, w) - T_{m-1}(z_0, w))^2 - 2u^2z_0w^{2m} + (4u^2w^2 - (y - 1)^2w^2 - u^2z_0^2)P.
\]

Let

\[
Q = Q(w) = \frac{T_m(z_0, w) - wT_{m-1}(z_0, w)}{w - 1}.
\]
Lemma 4.1. We have \( S_m^2(z_0) = \frac{1}{2-z_0} \) and
\[
Q(1) = -\frac{(2m + 1)z_0}{z_0 + 2} S_m(z_0).
\]

Proof. Since \( S_m^2(z_0) + S_{m-1}^2(z_0) - z_0 S_m(z_0) S_{m-1}(z_0) = 1 \) (by Lemma 2.3) and \( S_m(z_0) - S_{m-1}(z_0) = 0 \), we get \( S_m^2 = \frac{1}{2-z_0} \). By L'Hospital rule, we have
\[
Q(1) = \frac{S_m(z_0) - S_{m-1}(z_0)}{w-1} |_{w=1}
\]
\[
= -\frac{z_0}{w^2} (S_m'(z_0) - S_{m-1}'(z_0)) |_{w=1}
\]
\[
= -z_0 (S_m'(z_0) - S_{m-1}'(z_0)).
\]

Since \( S_m(z_0) = S_{m-1}(z_0) \), we have \( S_{m+1}(z) = (z_0 - 1)S_m(z_0) \) and \( S_{m-2}(z) = (z_0 - 1)S_m(z_0) \). Lemma 2.5 then implies that
\[
S_m'(z_0) = \frac{mS_{m+1}(z_0) - (m + 2)S_{m-1}(z_0)}{z_0^2 - 4} = \frac{m(z_0 - 1) - (m + 2)}{z_0^2 - 4} S_m(z_0),
\]
\[
S_{m-1}'(z_0) = \frac{(m - 1)S_m(z_0) - (m + 1)S_{m-2}(z_0)}{z_0^2 - 4} = \frac{m - 1 - (m + 1)(z_0 - 1)}{z_0^2 - 4} S_m(z_0).
\]

Hence, \( Q(1) = -z_0 (S_m'(z_0) - S_{m-1}'(z_0)) = -\frac{(2m+1)z_0}{z_0 + 2} S_m(z_0) \). \( \square \)

4.2.1. \( a \neq 0 \). First we look at \( Y_3 \) in the affine open set defined by \( a \neq 0 \) (we can assume \( a = 1 \)). In this open set, the defining equations for \( Y_3 \) become
\[
F_3 = (yw - 2u^2z_0)w^{2m} + (yz_0w + 4u^2w^2 - w^2 - y^2w^2 - u^2z_0^2)P(w),
\]
\[
e_1 = u - (y-1)b,
\]
\[
e_2 = (w-1) - (y-1)c,
\]
\[
e_3 = (w-1)b - uc.
\]

From equations \( e_1 = 0 \) and \( e_2 = 0 \), we have \( u = (y-1)b \) and \( w = (y-1)c + 1 \). By replacing \( u \) with \( (y-1)b \) and \( w \) with \( (y-1)c + 1 \) in \( F_3 \), we obtain
\[
F_3 = (yw - 1)^2Q^2 - 2u^2z_0w^{2m} + (4u^2w^2 - (y-1)^2w^2 - u^2z_0^2)P
\]
\[
= (y-1)^2 [yw c^2Q^2 - 2b^2z_0w^{2m} + (4b^2w^2 - w^2 - b^2z_0^2)P] .
\]
Let
\[ R(b, c, y) = ywc^2Q^2 - 2b^2z_0w^{2m} + (4b^2w^2 - w^2 - b^2z_0^2)P, \]
where \( w = (y - 1)c + 1 \). Then
\[ R|_{y=1} = c^2Q^2(1) - 2b^2z_0 + (4b^2 - 1 - b^2z_0^2)P(1) \]
\[ = c^2 \left( \frac{(2m+1)^2z_0^2}{(z_0 + 2)^2}S_m(z_0) - 2b^2z_0 + (4b^2 - 1 - b^2z_0^2)S_m(z_0)S_{m-1}(z_0) \right) \]
\[ = \frac{1}{2 - z_0} \left( c^2 \left( \frac{(2m+1)^2z_0^2}{(z_0 + 2)^2} - 2b^2z_0(2 - z_0) + (4b^2 - 1 - b^2z_0^2) \right) \right) \]
\[ = \frac{1}{2 - z_0} \left( c^2 \left( \frac{(2m+1)^2z_0^2}{(z_0 + 2)^2} + b^2(z_0 - 2)^2 - 1 \right) \right). \]

We have \( F_3 = (y - 1)^2R \). The first factor corresponds to the exceptional plane \( E_3 \) and the other factor is the defining equation for the local model of \( Y_3 \). Note that the preimage of \( S_j^{(k)} \) is exactly the intersection of \( E_3 \) and \( Y_3 \) which is equal to the smooth conic \( c^2(2m+1)^2z_0^2 + b^2(z_0 - 2)^2 - 1 = 0 \). This local model of \( Y_3 \) is smooth in \( A_3 \times \mathbb{P}^2 \) if we can show that \( R(b, c, y) \) is smooth.

We now prove that the system \( R = R_b = R_c = R_y = 0 \) has no solutions. By direct calculations, we have
\[ R_b = 2b \left( -2z_0w^{2m} + (4w^2 - z_0^2)P \right), \]
\[ R_c = y(y - 1)c^2Q^2 + 2ywQ^2 + ywc^2(y - 1)(Q^2)_w - 4mb^2z_0(y - 1)w^{2m-1} \]
\[ + (8b^2w - 2w)(y - 1)P + (4b^2w^2 - w^2 - b^2z_0^2)(y - 1)P_w, \]
\[ R_y = wc^2Q^2 + yc^2Q^2 + ywc^3(Q^2)_w - 4mb^2z_0cw^{2m-1} \]
\[ + (8b^2w - 2w)cP + (4b^2w^2 - w^2 - b^2z_0^2)cP_w. \]

Note that
\[ R - bR_b/2 = w(yc^2Q^2 - wP), \]
\[ cR_c - (y - 1)R_y = (y + 1)wc^2Q^2. \]

Assume that \( R = R_b = R_c = R_y = 0 \) at some point \((b, c, y)\). We first claim that \( w \neq 0 \). Indeed, if \( w = 0 \) then \( R = 0 \) implies that \(-b^2z_0^2P(0) = 0 \). Since \( P(0) = z_0^{2m-1} \neq 0 \), we get \( b = 0 \). Then \( R_y = 0 \) implies that \( yc^2Q^2(0) = 0 \). Note that \( c \neq 0 \) (since \( w = (y - 1)c + 1 = 0 \) and \( Q(0) = T_m(z_0, 0) = z_0^{2m} \neq 0 \)). Hence, \( y = 0 \). Then \((x : y : u), [z : w]) = ([1 : 0 : 0], [z_0 : 0]) = s_2 \) which has been removed from \( A_3 \). This proves that \( w \neq 0 \).

Now \( cR_c - (y - 1)R_y = 0 \) implies \( y = -1 \) or \( c^2Q^2 = 0 \). If \( c^2Q^2 = 0 \) then \( w^{2m} + (z_0 - 2w)P = (y - 1)^2c^2Q^2 = 0 \), which implies that \( P \neq 0 \). Then \( R - bR_b/2 = -w^2P \neq 0 \), a contradiction. Hence, \( y = -1 \).

Since \( w^{2m} + (z_0 - 2w)P = (w - 1)^2Q^2 = (y - 1)^2c^2Q^2 = 4c^2Q^2 \), we have \( c^2Q^2 = \frac{w^{2m} + (z_0 - 2w)P}{4} \). From \( R - bR_b/2 = 0 \), we get \(-w^{2m} + (z_0 - 2w)P - wP = 0 \),
which implies that \( w^{2m} + (z_0 + 2w)P = 0 \). By Lemma 3.1(3), this is equivalent to \( T_m(z_0, w) + wT_{m-1}(z_0, w) = 0 \), i.e. \( S_m(z_0) + S_{m-1}(z_0) = 0 \). So

\[
([x : y : u], [z : w]) = ([1 : -1 : 0], [z_0 : w]) = ([1 : -1 : 0], [\frac{z_0}{w} : 1]) = s_4^{(k)}
\]

which has been removed from \( A_3 \).

### 4.2.2. \( b \neq 0 \)

Now we look at \( Y_3 \) in the affine open set defined by \( b \neq 0 \) (we can assume \( b = 1 \)). In this open set, the defining equations for \( Y_3 \) become

\[
F_3 = (yw - 2uz_0)w^{2m} + (yz_0w + 4u^2w^2 - w^2 - y^2w^2 - u^2z_0^2)P(w),
\]

\[
e_1 = ua - (y - 1),
\]

\[
e_2 = (w - 1)a - (y - 1)c,
\]

\[
e_3 = (w - 1) - uc.
\]

From equations \( e_1 = 0 \) and \( e_3 = 0 \), we have \( y = au + 1 \) and \( w = uc + 1 \). By replacing \( y \) with \( au + 1 \) and \( w \) with \( uc + 1 \) in \( F_3 \), we obtain

\[
F_3 = yw(w - 1)^2Q^2 - 2u^2z_0w^{2m} + (4u^2w^2 - (y - 1)^2w^2 - u^2z_0^2)P
\]

\[
= u^2[(au + 1)wc^2Q^2 - 2z_0w^{2m} + (4w^2 - a^2w^2 - z_0^2)P].
\]

Let

\[
R(a, c, u) = (au + 1)wc^2Q^2 - 2z_0w^{2m} + (4w^2 - a^2w^2 - z_0^2)P(w),
\]

where \( w = uc + 1 \). Then

\[
R\mid_{u=0} = c^2Q^2(1) - 2z_0 + (4 - a^2 - z_0^2)P(1),
\]

\[
= c^2\frac{(2m + 1)^2z_0^2}{(z_0 + 2)^2}S_m(z_0) - 2z_0 + (4 - a^2 - z_0^2)S_m(z_0)S_{m-1}(z_0)
\]

\[
= \frac{1}{2 - z_0}\left(c^2\frac{(2m + 1)^2z_0^2}{(z_0 + 2)^2} - 2z_0(2 - z_0) + (4 - a^2 - z_0^2)\right)
\]

\[
= \frac{1}{2 - z_0}\left(c^2\frac{(2m + 1)^2z_0^2}{(z_0 + 2)^2} - a^2 + (z_0 - 2)^2\right).
\]

We have \( F_3 = u^2R \). The first factor corresponds to the exceptional plane \( E_3 \) and the other factor is the defining equation for the local model of \( Y_3 \). Note that the preimage of \( s_3^{(k)} \) is exactly the intersection of \( E_3 \) and \( Y_3 \) which is equal to the smooth conic \( c^2\frac{(2m + 1)^2z_0^2}{(z_0 + 2)^2} - a^2 + (z_0 - 2)^2 = 0 \). This local model of \( Y_3 \) is smooth in \( A_3 \times \mathbb{P}^2 \) if we can show that \( R(a, c, u) \) is smooth.
We now prove that the system $R = R_a = R_c = R_u = 0$ has no solutions. By direct calculations, we have

\[
R_a = w(uw^2Q^2 - 2awP),
\]

\[
R_c = (au + 1)uw^2Q^2 + (au + 1)uwQ^2 + (au + 1)uw^3Q^2 - 4mz_0uw^{2m-1} + (4u - a^2)uwP + (4u^2 - a^2w^2 - z_0^2)uwP_w,
\]

\[
R_u = aw^2Q^2 + (au + 1)uw^2Q^2 + (au + 1)uw^3Q^2 - 4mz_0uw^{2m-1} + (4u - a^2)uwP + (4u^2 - a^2w^2 - z_0^2)uwP_w.
\]

Note that

\[
R - aR_a/2 = (au/2 + 1)uw^2Q^2 - 2z_0uw^{2m} + (4u^2 - z_0^2)P,
\]

\[
cR_c - uR_u = (au + 2)uw^2Q^2.
\]

We first claim that $w \neq 0$. Indeed, if $w = 0$ then $R = 0$ implies that $-z_0^2P(0) = 0$. But $P(0) = z_0^{2m-1} \neq 0$, a contradiction. Hence, $w \neq 0$.

From $cR_c - uR_u = 0$ and $R - aR_a/2 = 0$, we have $(au + 2)uw^2Q^2 = 0$ and $-2z_0uw^{2m} + (4u^2 - z_0^2)P = 0$. Since $z_0uw^{2m} \neq 0$, we get $4u^2 - z_0^2 \neq 0$ and

\[
P = \frac{2z_0uw^{2m}}{4u^2 - z_0^2}.
\]

If $c^2Q^2 = 0$, then $w^{2m} + (z_0 - 2w)P = (w - 1)^2Q^2 = u^2c^2Q^2 = 0$. This implies that $2w - z_0 \neq 0$ and $P = \frac{w^{2m}}{2w - z_0}$. Together with $P = \frac{2z_0uw^{2m}}{4u^2 - z_0^2}$, we get $\frac{2z_0}{2w - z_0} = 1$. So $z_0 = 2w$, which contradicts $2w - z_0 \neq 0$.

If $au + 2 = 0$, then $a = -2/u$. From $R_a = 0$, we have $u^2c^2Q + 4wP = 0$, i.e. $(w - 1)^2Q^2 + 4wP = 0$. This is equivalent to $w^{2m} + (z_0 - 2w)P + 4wP = 0$. So $2w + z_0 \neq 0$ and $P = -\frac{u^{2m}}{2w + z_0}$. Together with $P = \frac{2z_0uw^{2m}}{4w^2 - z_0^2}$, we get $\frac{2z_0}{2w + z_0} = -1$. So $z_0 = -2w$, which contradicts $2w + z_0 \neq 0$.

4.2.3. $c \neq 0$. Finally we look at $Y_3$ in the affine open set defined by $c \neq 0$ (we can assume $b = 1$). In this open set, the defining equations for $Y_3$ become

\[
F_3 = (yw - 2u^2z_0)uw^{2m} + (yz_0w + 4u^2w^2 - w^2 - y^2w^2 - u^2z_0^2)P(w),
\]

\[
e_1 = ua - (y - 1)b,
\]

\[
e_2 = (w - 1)a - (y - 1),
\]

\[
e_3 = (w - 1)b - u.
\]

From equations $e_2 = 0$ and $e_3 = 0$, we have $y = a(w - 1) + 1$ and $u = b(w - 1)$. By replacing $y$ with $a(w - 1) + 1$ and $u$ with $b(w - 1)$ in $F_3$, we obtain

\[
F_3 = yu(w - 1)^2Q^2 - 2u^2z_0uw^{2m} + (4u^2w^2 - (y - 1)^2w^2 - u^2z_0^2)P(w)
\]

\[
= (w - 1)^2 \left[(a(w - 1) + 1)wQ^2 - 2b^2z_0uw^{2m} + (4b^2w^2 - a^2w^2 - b^2z_0^2)P(w)\right].
\]

Let

\[
R(a, b, w) = (a(w - 1) + 1)wQ^2(w) - 2b^2z_0uw^{2m} + (4b^2w^2 - a^2w^2 - b^2z_0^2)P(w).
\]
Then
\[ R_{|w=1} = Q(1) - 2b^2z_0 + (4b^2 - a^2 - b^2z_0^2)P(1), \]
\[ = \frac{(2m + 1)^2z_0^2}{(z_0 + 2)^2} - 2b^2z_0 + (4b^2 - a^2 - b^2z_0^2)S_m(z_0)S_{m-1}(z_0) \]
\[ = \frac{1}{2 - z_0} \left( \frac{(2m + 1)^2z_0^2}{(z_0 + 2)^2} - 2b^2z_0(2 - z_0) + (4b^2 - a^2 - b^2z_0^2) \right) \]
\[ = \frac{1}{2 - z_0} \left( \frac{(2m + 1)^2z_0^2}{(z_0 + 2)^2} - a^2 + b^2(z_0 - 2)^2 \right). \]

We have \( F_3 = (w - 1)^2R. \) The first factor corresponds to the exceptional plane \( E_3 \) and the other factor is the defining equation for the local model of \( Y_3. \) Note that the preimage of \( s_3^{(k)} \) is exactly the intersection of \( E_3 \) and \( Y_3 \) which is equal to the smooth conic \( \frac{(2m + 1)^2z_0^2}{(z_0 + 2)^2} - a^2 + b^2(z_0 - 2)^2 = 0. \) This local model of \( Y_3 \) is smooth in \( A_3 \times \mathbb{P}^2 \) if we can show that \( R(a, b, w) \) is smooth.

We now prove that the system \( R = R_a = R_b = R_w = 0 \) has no solutions. By direct calculations, we have
\[ R_a = (w - 1)wQ^2 - 2aw^2P, \]
\[ R_b = 2b(-2z_0w^{2m} + (4w^2 - z_0^2)P), \]
\[ R_w = awQ^2 + (a(w - 1) + 1)Q^2 + (a(w - 1) + 1)w(Q^2)_w - 4mb^2z_0w^{2m-1} + 2(4b^2 - a^2)wP + (4b^2w^2 - a^2w^2 - b^2z_0^2)P_w. \]

Note that
\[ 2R - bR_b - aR_a = (a(w - 1) + 2)wQ^2. \]

We first claim that \( w \neq 0. \) Indeed, if \( w = 0 \) then \( R = 0 \) implies that \( b^2z_0^2P(0) = 0. \) Since \( z_0 \neq 0 \) and \( P(0) = 1, \) we have \( b = 0. \) Then \( R_w = 0 \) becomes \( (a(w - 1) + 1)Q^2 = 0. \) Note that \( Q(0) = z_0^{2m} \neq 0, \) hence \( a(w - 1) + 1 = 0. \) Then \( ([x : y : u], [z : w]) = ([1 : 0 : 0], [z_0 : 0]) = s_2 \) which has been removed from \( A_3. \) Hence, \( w \neq 0. \)

From \( 2R - bR_b - aR_a = 0, \) we have \( a(w - 1) + 2 \) or \( Q = 0. \) Similarly, \( R_b = 0 \) implies that \( b = 0 \) or \( -2z_0w^{2m} + (4w^2 - z_0^2)P = 0. \) There are four cases to consider.

Case 1: Suppose \( b = 0 \) and \( Q = 0. \) Then \( R_a = 0 \) implies that \( aP = 0. \) Note that \( P \neq 0, \) since \( w^{2m} + (z_0 - 2w)P = (w - 1)^2Q^2 = 0. \) Hence, \( a = 0. \) From \( Q = 0, \) we have \( T_m(z_0, w) = wT_m(z_0, w) = 0, \) which is equivalent to \( S_m(z_0) - S_{m-1}(z_0) = 0, \) so \( z_0 \neq z_0^{(l)} \) for some \( l. \) Note that \( Q(1) = \frac{1}{2 - z_0} (2m + 1^2z_0^2) \neq 0, \) so \( w \neq 1. \) This implies that \( z_0^{(l)} = z_0 \neq z_0^{(k)}. \) Since \( ([x : y : u], [z : w]) = ([1 : 1 : 0], [z_0^{(l)} : 1]) = s_3^{(l)} \) has been removed from \( A_3, \) we obtain a contradiction.
Case 2: Suppose $b = 0$ and $a(w - 1) + 2 = 0$. Then $a = -2/(w - 1)$ and $y = a(w - 1) + 1 = -1$. From $R = 0$, we have $(w - 1)^2Q^2 + 4wP = 0$, i.e. $w^{2m} + (z_0 - 2w)P + 4wP = 0$. By Lemma 3.1(3), this is equivalent to $S_m(\frac{z_0}{w}) + S_{m-1}(\frac{z_0}{w}) = 0$, so $z_0 = z^{(l)}_4$ for some $l$. Then $([x : y : u], [z : w]) = ([1 : -1 : 0], [z^{(l)}_4 : 1]) = s^{(l)}_4$ which has been removed from $A_3$.

Case 3: Suppose $-2z_0w^{2m} + (4w^2 - z_0^2)P = 0$ and $Q = 0$. Then $4w^2 - z_0^2 \neq 0$ and $P = \frac{2z_0w^{2m}}{4w^2 - z_0^2}$. From $Q = 0$, we have $w^{2m} + (z_0 - 2w)P = (w - 1)^2Q^2 = 0$.

Hence, $1 + (z_0 - 2w)\frac{2z_0}{4w^2 - z_0^2} = 0$, i.e. $1 - \frac{2z_0}{z_0 + 2w} = 0$. This implies that $z_0 = 2w$, which contradicts $4w^2 - z_0^2 \neq 0$.

Case 4: Suppose $-2z_0w^{2m} + (4w^2 - z_0^2)P = 0$ and $a(w - 1) + 2 = 0$. From $R_a = 0$, we have $(w - 1)^2Q^2 + 4wP = 0$, which is equivalent to $w^{2m} + (z_0 - 2w)P + 4wP = 0$. So $1 + (z_0 + 2w)\frac{2z_0}{4w^2 - z_0^2} = 0$, i.e. $1 - \frac{2z_0}{z_0 - 2w} = 0$. This implies that $z_0 = -2w$, which contradicts $4w^2 - z_0^2 \neq 0$.

4.2.4. Conclusion. From the cases $a \neq 0$, $b \neq 0$, and $c \neq 0$ considered above, we conclude that the singular point $s^{(k)}_3$ requires only one blow-up to resolve.

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