Small angle limits of negatively curved Kähler–Einstein metrics with crossing edge singularities

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Abstract. Let \((X, D)\) be a log smooth log canonical pair such that \(K_X + D\) is ample. Extending a theorem of Guenancia and building on his techniques, we show that negatively curved Kähler–Einstein crossing edge metrics converge to Kähler–Einstein mixed cusp and edge metrics smoothly away from the divisor when some of the cone angles converge to 0. We further show that near the divisor such normalized Kähler–Einstein crossing edge metrics converge to a mixed cylinder and edge metric in the pointed Gromov–Hausdorff sense when some of the cone angles converge to 0 at (possibly) different speeds.

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1. Introduction

1.1. The small angle world. Let \(X\) be a compact Kähler manifold of dimension \(n\) and \(D \subset X\) be a smooth hypersurface. A Kähler edge metric on \(X\) with angle \(2\pi \beta\) (\(0 < \beta \leq 1\)) along \(D\) is a Kähler metric on \(X \setminus D\) that is quasi-isometric to the model edge metric at \(D\):

\[
\omega_{\text{cone}} = \beta^2 \frac{\sqrt{-1} dz_1 \wedge d\bar{z}_1}{|z_1|^{2(1-\beta)}} + \sum_{i=2}^{n} \sqrt{-1} dz_i \wedge d\bar{z}_i,
\]

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where \( z_1, \ldots, z_n \) are holomorphic coordinates and \( D \) is locally given by \( \{ z_1 = 0 \} \). Tian generalized Calabi’s conjecture to Kähler–Einstein edge metrics and studied the applications of negatively curved Kähler–Einstein edge metric to algebraic geometry by letting the cone angle tend to \( 2\pi \) [17]. Donaldson proposed using Kähler edge metrics to study the existence problem of smooth Kähler–Einstein metrics of positive curvature on \( X \) by deforming the cone angle to \( 2\pi \) [4]. Since then much research has gone into understanding the large angle limits (when \( \beta \to 1 \)) of Kähler–Einstein edge metrics in relation to the Yau–Tian–Donaldson conjecture. Cheltsov–Rubinstein initiated the program of studying Kähler–Einstein edge metrics in another extreme where the cone angle goes to zero [3]. One topic of their program is to understand the limit, when such exists, of Kähler–Einstein edge metrics as the cone angle tends to 0. This paper is following that program. We prove that on a log smooth log canonical pair \((X, D)\), i.e., \( X \) is a compact Kähler manifold and \( D = \sum \limits_{i=1}^{r} (1 - \beta_i)D_i \) is a divisor with simple normal crossing support such that \( \beta_i \in [0, 1) \) for all \( i \), assuming that \( K_X + \sum \limits_{i=1}^{r} D_i \) is ample, then the negatively curved Kähler–Einstein crossing edge metrics converge to the Kähler–Einstein mixed cusp and edge metric when some of the cone angles tend to 0. We further study the asymptotic behavior of the Kähler–Einstein crossing edge metrics near the divisor and show the rescaled Kähler–Einstein crossing edge metrics converge to mixed cylinder and edge metrics on \((\mathbb{C}^*)^m \times \mathbb{C}^{n-m}\) when some of the cone angles tend to 0. A beautiful theorem of Guenancia related the Kähler–Einstein edge metric to the Kähler–Einstein cusp metric in the smooth case [8], which confirmed a conjecture made by Mazzeo [12]. Our paper is a generalization of Guenancia’s results to the snc case. An added interesting feature of our work is the possibility that multiple angles converge to zero at (possibly) different rates. Such kind of small angle limits, where some angles tend to zero but others do not could be interesting when studying moduli spaces of Kähler–Einstein metrics since these limits could correspond to approaching various strata in the moduli space.

1.2. Guenancia’s convergence result. Let \( \mathbb{D}^* \) be the punctured unit disc in \( \mathbb{C} \). The first observation is that

\[
\omega_{\eta, \mathbb{D}^*} := \frac{\eta^2 \sqrt{-1} dz \wedge \bar{dz}}{|z|^{2(1-\eta)} (1 - |z|^{2\eta})^2}, \quad z \in \mathbb{D}^*, \eta \in (0, 1),
\]

is a Kähler edge metric with cone angle \( 2\pi \eta \) at 0 and it has constant Ricci curvature \( -2 \). When \( \eta \) tends to 0, \( \omega_{\eta, \mathbb{D}^*} \) converges pointwise (see (5) for the detail) to the following cusp metric (also called a Poincaré metric) on \( \mathbb{D}^* \):

\[
\omega_{P, \mathbb{D}^*} := \frac{\sqrt{-1} dz \wedge \bar{dz}}{|z|^2 (\log |z|)^2}.
\]

In higher dimensions, we consider the pair \((X, D)\) where \( X \) is a compact Kähler manifold of dimension \( n \) and \( D \) is a smooth divisor such that \( K_X + D \) is ample. By Kobayashi [11, Theorem 1] or Tian–Yau [18, Theorem 2.1] with complements
by Wu [19], there exists a unique complete Kähler–Einstein metric $\omega_0$ on $X \setminus D$ with cusp singularity along $D$ such that $\text{Ric } \omega_0 = -\omega_0$.

**Definition 1.1.** $\omega_0$ is said to have cusp singularities along $D$ if whenever $D$ is locally given by $\{z_1 = 0\}$, there exists a constant $C > 0$ such that

$$C^{-1}\omega_{\text{cusp}} \leq \omega_0 \leq C\omega_{\text{cusp}},$$

where $\omega_{\text{cusp}}$ is the model cusp metric:

$$\omega_{\text{cusp}} := \frac{\sqrt{-1}dz_1 \wedge d\bar{z}_1}{|z_1|^2 \log^2|z_1|^2} + \sum_{i=2}^{n} \sqrt{-1}dz_i \wedge d\bar{z}_i.$$ 

Since ampleness is an open condition, there exists some $\beta_0$ such that for $0 < \beta < \beta_0$, $K_X + (1 - \beta)D$ is also ample. Thus, by Campana–Guenancia–Păun [9, Theorem A] and Jeffres–Mazzio–Rubinstein [10, Theorem 2], there exists a unique negatively curved Kähler–Einstein edge metric $\omega_{\beta}$ for each such small $\beta \in (0, \beta_0]$. The family of metrics $\{\omega_{\beta}\}_{0 < \beta < \beta_0}$ can be seen as currents on $X$ satisfying the twisted Kähler–Einstein equation:

$$\text{Ric } \omega_{\beta} = -\omega_{\beta} + (1 - \beta)[D], \quad 0 \leq \beta < \beta_0.$$

As a generalization of the observation discussed in the beginning of this section, Guenancia related these two metrics as follows:

**Theorem 1.2.** [8, Theorem A and B] Let $\omega_0$ be defined as in Definition 1.1. $\{\omega_{\beta}\}_{0 < \beta < \beta_0}$ converge to $\omega_0$ in both the weak topology of currents and the $C^\infty_{\text{loc}}(X \setminus D)$-topology as $\beta \to 0$. Moreover, for $\beta \in (0, 1/2]$, there exists a constant $C > 0$ independent of $\beta$ such that on any coordinate chart $U$ where $D$ is given by $\{z_1 = 0\}$, the Kähler–Einstein edge metric $\omega_{\beta}$ satisfies

$$C^{-1}\omega_{\beta, \text{mod}} \leq \omega_{\beta} \leq C\omega_{\beta, \text{mod}},$$

where

$$\omega_{\beta, \text{mod}} := \frac{\beta^2\sqrt{-1}dz_1 \wedge d\bar{z}_1}{|z_1|^{2(1-\beta)}(1 - |z_1|^{2\beta})^2} + \sum_{i=2}^{n} \sqrt{-1}dz_i \wedge d\bar{z}_i.$$ 

The relation between the convergence result in Theorem 1.2 and (2) is that the weak convergence from $(\omega_{\beta})_{0 < \beta < \beta_0}$ to $\omega_0$ can be recovered from (2) by using Lebesgue’s Dominated Convergence Theorem.

As an application of Theorem 1.2, Guenancia studied the asymptotic behavior of $\omega_{\beta}$ near $D$ as $\beta \to 0$. Fix a point $p \in D$, let $U_{\beta}$ denote the punctured metric ball $B_{\omega_{\beta}}(p, 1)$ of radius 1 centered at $p$ with respect to the metric $\omega_{\beta}$. Then after renormalization by $\beta^{-2}$, there exists a subsequence of the metric spaces $(U_{\beta}, \frac{\beta^2}{\beta^2} \omega_{\beta})$ converging to $(C^* \times C^{n-1}, \omega_{\text{cyl}})$ in the pointed Gromov–Hausdorff sense, where $\omega_{\text{cyl}}$ is a so-called cylindrical metric:

**Definition 1.3.** Let $\pi : \mathbb{C}^n \to C^* \times C^{n-1}$ be the universal cover of $C^* \times C^{n-1}$ given by $\pi(z_1, ..., z_n) = (e^{z_1}, z_2, ..., z_n)$. A Kähler metric $\omega_{\text{cyl}}$ on $C^* \times C^{n-1}$ is
called cylindrical if $\pi^* \omega$ is isometric to the usual Euclidean metric on $\mathbb{C}^n$ up to a complex linear transformation.

**Theorem 1.4.** [8, Theorem C] Let $(\beta_k)_{k \in \mathbb{N}}$ be a sequence of positive numbers converging to 0. Then, up to extracting a subsequence, there exists a cylindrical metric $\omega_{\text{cyl}}$ on $\mathbb{C}^* \times \mathbb{C}^{n-1}$ such that the metric spaces $(U_{\beta_k}, \beta_k^{-2} \omega_{\beta_k})$ converge in pointed Gromov-Hausdorff topology to $(\mathbb{C}^* \times \mathbb{C}^{n-1}, \omega_{\text{cyl}})$ when $k$ tends to $+\infty$.

### 1.3. The main results.

A natural problem is to generalize Theorems 1.2 and 1.4 to the snc case when all or some of the cone angles tend to 0. This possibility is mentioned in [8] but there is no detailed proof given. In this paper, we generalize Theorems 1.2 and 1.4 to the snc setting.

From now on, let $(X, \omega)$ be an $n$-dimensional compact Kähler manifold with a smooth Kähler metric $\omega$. Fix a divisor $D_\beta := \sum_{i=1}^r (1 - \beta_i)D_i$, where $\beta_i \in (0, 1)$ for $i = 1, \ldots, r$. Assume each $D_i$ is smooth and irreducible. We further assume $D_\beta$ has simple normal crossing support, i.e., for any $p \in \text{supp}(D_\beta)$ lying in the intersection of exactly $m$ components $D_1, \ldots, D_m$, there exists a coordinate chart $(U, \{z_i\}_{i=1}^n)$ containing $p$ such that $D_j \mid_U = \{z_j = 0\}$ for $j = 1, \ldots, m$, $m \leq n$.

Suppose $K_X + \sum D_i$ is ample. Let $s_i$ denote a defining section of $D_i$ and $h_i = |\cdot|_{h_i}$ be a smooth hermitian metric on $L_{D_i}$, which is the line bundle induced by $D_i$. We normalize $h_i$ such that $\log |s_i|_{h_i}^2 + 1 < 0$ for each $i$. Denote $\beta := (\beta_1, \ldots, \beta_r) \in (0, 1)^r$.

The following result is well known (see [14, §4] for a survey).

**Theorem 1.5.** [9, 10, 13] (Solution of the Calabi–Tian conjecture in the negative regime) There exists a unique Kähler–Einstein crossing edge metric with negative curvature, denoted by $\omega_{\phi_\beta} = \omega + \sqrt{-1} \partial \bar{\partial} \phi_\beta$ on $X$ with cone angle $2\pi \beta_i$ along $D_i$ for each $i$. In another word, $\omega_{\phi_\beta}$ satisfies the Kähler–Einstein edge equation

$$\text{Ric } \omega_{\phi_\beta} - [D_\beta] = -\omega_{\phi_\beta}. $$

Analogously to [8], let us introduce a reference metric,

$$\Omega_\beta := \omega - \sum_{i=1}^r \sqrt{-1} \partial \bar{\partial} \log \left[ \frac{1 - |s_i|_{h_i}^{2\beta_i}}{\beta_i} \right]. \quad (3)$$

Our first result is as follows.

**Theorem 1.6.** Let $\omega_{\phi_\beta}$ be given by Theorem 1.5. Let $\Omega_\beta$ be given by (3). There exists a uniform constant $C > 0$, independent of $\beta \in (0, \frac{1}{2})^r$, such that

$$C^{-1} \Omega_\beta \leq \omega_{\phi_\beta} \leq C \Omega_\beta.$$
The key point of Theorem 1.6 is that the constant $C$ is uniform with respect to small $\beta_i, i = 1, \ldots, r$. According to Theorem 1.6 and Lebesgue’s Dominated Convergence Theorem, we obtain the weak convergence from $\omega_{\phi_\beta}$ to the Kähler–Einstein mixed cusp and edge metric $\omega_0$ constructed in [7] as some of the cone angles tend to 0. In particular, when $\beta \to 0 \in (0, 1)^r$, the limiting metric of such $\omega_{\phi_\beta}$ is the unique Kähler–Einstein cusp metric on $(X, \sum_{i=1}^{r} D_i)$ constructed in [11, 18, 19]. More precisely, the following result is shown in section 2.3.

**Theorem 1.7.** The Kähler–Einstein crossing edge metric $\omega_{\phi_\beta}$ converges to a Kähler–Einstein mixed cusp and edge metric on $(X, D_\beta)$ globally in a weak sense and locally in a strong sense when some of the cone angles tend to 0. In particular, $\omega_{\phi_\beta}$ converges to the Kähler–Einstein cusp metric on $(X, \sum_{i=1}^{r} D_i)$ in the above sense when $\beta \to 0 \in (0, 1)^r$.

**Remark 1.8.** In Theorem 1.7, we assume $K_X + \sum_{i=1}^{r} D_i$ to be ample to ensure the existence of a limiting Kähler–Einstein metric by the work of Kobayashi [11] and Tian–Yau–Wu [18, 19]. An interesting open problem is to study the convergence of $\omega_{\phi_\beta}$ when we only assume the ampleness of $K_X + D_\beta$ for $0 < \beta_i \ll 1, i = 1, \ldots, r$.

Theorem 1.6 and Theorem 1.7 generalize Guenancia’s Theorem 1.2 from the smooth case to the nc case.

As an application of Theorem 1.6, we study the asymptotic behavior of the Kähler–Einstein crossing edge metric $\omega_{\phi_\beta}$ near $D_\beta$ when the smallest cone angle approaches 0, with possibly other cone angles also converging to 0.

To state the result, without loss of generality, we assume for $\beta = (\beta_1, \ldots, \beta_r)$ there holds $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_r$. Fix a point $p \in D_\beta$. Choose a coordinate chart $(U, \{z_i\}_{i=1}^{n})$ containing $p$ such that $D_j|_U = \{z_j = 0\}$ for $j = 1, \ldots, m, m \leq n$. Consider a small neighborhood $U_\beta$ about $p$ defined by

$$\left\{ z \in (\mathbb{C}^*)^m \times \mathbb{C}^{n-m} : |z_1| < e^{-\frac{1}{\beta_1}}, |z_j| < (\frac{\beta_1}{\beta_j})^{\frac{1}{n}}, j = 2, \ldots, m, |z_\ell| < 1, \ell = m + 1, \ldots, n \right\}.$$  

We show that after normalization by factor $\beta_1^{-2}$, a subsequence of metrics $\omega_{\phi_\beta}$ converges to a mixed cylinder and edge metric on $(\mathbb{C}^*)^m \times \mathbb{C}^{n-m}$ (see Definition 3.1 for more details. We note that our Definition 3.1 is in a weaker sense of quasi-isometry comparing to Definition 1.3.) as $\beta_1$ tends to 0. The limiting metric has cylindrical part along the component $D_1$ where the cone angle $\beta_1$ approaches 0 while has conical singularities along other components. More precisely, the third result of this paper is as follows:
Theorem 1.9. Let \( \{\beta_{1,k}\}_{k \in \mathbb{N}} \) be a sequence of positive numbers converging to 0. Assume further that \( \{\beta_{i,k}\}_{k \in \mathbb{N}} \) does not converge to 0 for each \( i = 2, ..., r \) and all \( \beta_{i,k} \in (0, \frac{1}{2}] \). Let \( \omega_{\phi_{\beta_k}} \) be the (negatively curved) Kähler–Einstein crossing edge metric on \( (X, D_k = \sum_{i=1}^r (1 - \beta_{i,k})D_i) \). Then there exists a subsequence of the metric spaces \( (U_{\beta_k}, \frac{1}{\beta_{1,k}^2} \omega_{\phi_{\beta_k}}) \) converging in pointed Gromov-Hausdorff topology to \( ((\mathbb{C}^*)^m \times \mathbb{C}^{n-m}, \omega_{\infty}) \) where \( \omega_{\infty} \) is a mixed cylinder and edge metric.

Theorem 1.9 is a generalization of [8, Theorem C] which shows the convergence of Kähler–Einstein edge metrics to a cylindrical metric in the smooth case. Regarding complex dimension 1, i.e., in the Riemann surface case, but in the positive curvature regime, Rubinstein–Zhang showed that the (American) football equipped with the Ricci soliton metric converges to the cone-cigar soliton on \( \mathbb{R}_+ \) as two cone angles converge to 0 at a different speed and to a flat cylindrical metric as two cone angles converge to 0 at a comparable speed [16, Theorem 1.1-1.3]. In [16], the \( S^1 \)-symmetry of the metric plays an important role in the proof. In higher dimensions, we generalize Theorem 1.9 to allow more than one cone angles to tend to 0 and study the limit behavior of metrics under this joint degeneration of cone angles. The result is as follows.

Theorem 1.10. Let \( \{\beta_{1,k}\}_{k \in \mathbb{N}} \) be a sequence of positive numbers converging to 0. Assume further that for any \( i \in \{2, ..., r\} \) such that \( \{\beta_{i,k}\}_{k \in \mathbb{N}} \) also converges to 0, there holds \( \lim_{k \to \infty} \frac{\beta_{i,k}}{\beta_{1,k}} \in [0, 1] \) and all \( \beta_{i,k} \in (0, \frac{1}{2}] \). Let \( \omega_{\phi_{\beta_k}} \) be the (negatively curved) Kähler–Einstein crossing edge metric on \( (X, D_k = \sum_{i=1}^r (1 - \beta_{i,k})D_i) \). Then there exists a subsequence of the metric spaces \( (U_{\beta_k}, \frac{1}{\beta_{1,k}^2} \omega_{\phi_{\beta_k}}) \) converging in pointed Gromov-Hausdorff topology to \( ((\mathbb{C}^*)^m \times \mathbb{C}^{n-m}, \omega_{\infty}) \) where \( \omega_{\infty} \) is a mixed cylinder and edge metric with cylindrical part along components whose cone angles converge to 0 and conical part along other components.

In the language of [16], [16, Theorem 1.1-1.3] completely describe, in a geometric sense, the boundary behavior of the body of ample angles [15] of the pair \( (S^2, N + S) \), where \( N \) and \( S \) denote the north and south poles of the Riemann sphere respectively. In higher dimensions, given a pair \( (X, \widetilde{D} = \sum_{i=1}^r D_i) \), Theorem 1.10 is still not a satisfactory description of the boundary of the body of ample angles of \( (X, \widetilde{D}) \) in the negative curvature regime. Part of the reason is that different subsequences may converge to different mixed cylinder and edge metrics. A complete characterization of the moduli space of such \( (X, \widetilde{D}) \) endowed with Kähler–Einstein crossing edge metrics in the sense of [14] is still open. However, Theorem 1.10 is interesting in its own right from an analytical point of view.

1.4. Main ingredients of the proofs. We first recall the key ingredient in the proof of Theorem 1.2 is the boundedness of the holomorphic bisectional curvature of the model metric \( \omega_{\rho, \text{mod}} \), which makes it possible to use the Chern–Liu
inequality to obtain the Laplacian estimates, cf. [8, Theorem 3.2]. Therefore, one way to prove corresponding results of Theorem 1.2 in the snc setting is to first extend [8, Theorem 3.2] to the snc setting, i.e., prove boundedness of holomorphic bisectional curvature of the model metric $\Omega_\beta$ (see (3) for details). This is done in Lemma 2.4 by adapting arguments in [10, Lemma 2.3] and making use of the fact that we only need to deal with small $\beta_i$’s. Then the proof of Theorem 1.6 uses a modified maximum principle argument and Chern–Lu’s inequality to give respectively the $C^0$ and Laplacian estimates. A useful fact in the proof is the observation that $\Omega_\beta$ shares the same crossing edge singularities as $\omega_{\phi,\beta}$ (see Claim 2.8 for the detail).

An important observation is that the reference metric $\Omega_\beta$ has the property of converging to a Kähler metric with mixed cusp and edge singularities when some of the cone angles tend to 0. This observation, combined with the content of Theorem 1.6, give us the result of Theorem 1.7 as a corollary. As another consequence of Theorem 1.6, Theorem 1.9 and Theorem 1.10 treat the limit behavior of the Kähler–Einstein crossing edge metric $\omega_{\phi,\beta}$ near the divisor $D_\beta$ when some of the cone angles approach 0. After fixing a point in the divisor $D_\beta$, we first rescale the reference metric to obtain its convergence to a mixed cylinder and edge metric (see Definition 3.1) as the smallest cone angle tends to 0 in a small neighborhood of $D_\beta$. To obtain the pointed Gromov–Hausdorff convergence of the rescaled Kähler–Einstein crossing edge metric $\omega_{\phi,\beta}$ near the divisor, we actually show a stronger local smooth convergence result. We use Theorem 1.6 and the limit behavior of $\Omega_\beta$ mentioned above to obtain $C^0$-estimates of rescaled $\omega_{\phi,\beta}$. By a standard use of Evans–Krylov theory and Arzelà–Ascoli Theorem, we obtain the $C^\infty_{\text{loc}}$-convergence of the rescaled $\omega_{\phi,\beta}$ as some of the cone angles tend to 0.

2. Small angle limits of the Kähler–Einstein crossing edge metrics

Let $D^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$ be the punctured unit disc in $\mathbb{C}$. The first observation is, for $\eta \in (0, 1)$, the following Kähler metric

$$\omega_{\eta,D^*} = -\sqrt{-1} \delta \log (1 - |z|^{2\eta}) = \frac{\sqrt{-1} \eta^2 |z|^{2n-2}}{(1 - |z|^{2\eta})^2} dz \wedge d\bar{z}$$

has negative constant curvature and cone singularity with cone angle $2\pi \eta$ at $0 \in \mathbb{C}$. Indeed, direct calculation using the Poincaré–Lelong formula [6] yields

$$\text{Ric} \omega_{\eta,D^*} = 2\pi(1 - \eta) \delta_0 - 2\omega_{\eta,D^*},$$

(4)

where $\delta_0$ denotes the Dirac measure at 0.

For a fixed $z \in D^*$,

$$\lim_{\eta \to 0} \frac{\eta^2 |z|^{2\eta-2}}{(1 - |z|^{2\eta})^2} = \frac{1}{|z|^2 (\log |z|^2)^2}.$$

(5)
Thus, $\omega_{p,\mathcal{D}^*}$ converges uniformly to the Poincaré metric $\omega_{p,\mathcal{D}^*}$ defined in (1) for any compact $K \Subset \mathcal{D}^*$ when $\eta$ tends to 0. Note that

$$\text{Ric} \omega_{p,\mathcal{D}^*} = 2\pi \delta_0 - 2\omega_{p,\mathcal{D}^*}.$$ 

Thus, $\omega_{p,\mathcal{D}^*}$ is a Kähler–Einstein cusp metric on $\mathcal{D}^*$ with cusp singularity at 0.

Next we introduce a reference metric that generalizes $\omega_{p,\mathcal{D}^*}$ to higher dimensional manifolds.

### 2.1. The reference metric.

From now on, let $(X, D_\beta)$ be an $n$-dimensional compact Kähler manifold with an $\mathbb{R}$-divisor $D_\beta = \sum_{i=1}^r (1 - \beta_i)D_i$ such that $K_X + \sum_{i=1}^r D_i$ is ample, where $\beta_i \in (0, 1)$ for $i = 1, \ldots, r$. Given this assumption, $K_X + D_\beta$ is ample for small $\beta_1, \ldots, \beta_r$ since ampleness is an open condition. Assume each $D_i$ is smooth and irreducible. We further assume $D_\beta$ has simple normal crossing support, i.e., for any $p \in \text{supp}(D_\beta)$ lying in the intersection of exactly $m$ divisors $D_1, \ldots, D_m$, $m \leq n$, there exists a coordinate chart $(U, \{z_j\}_{j=1}^n)$ containing $p$ such that $D_j|_U = \{z_j = 0\}$ for $j = 1, \ldots, m$. Let $s_i$ denote a defining holomorphic section of $D_i$ and $h_i = | \cdot |_{h_i}$ be a smooth hermitian metric on $L_{D_i}$, which is the line bundle induced by $D_i$. Let $\partial_i$ denote the curvature form of each $(L_{D_i}, h_i)$. We normalize $h_i$ such that $\log |s_i|_{h_i}^2 + 1 < 0$ for each $i$. Let $\omega$

be a fixed smooth Kähler metric with $[\omega] = c_1 \left( K_X + \sum_{i=1}^r D_i \right)$. Below we denote $D := \sum_{i=1}^r D_i$ and $\beta := (\beta_1, \ldots, \beta_r)$.

Define the reference metric:

$$\Omega_\beta := \omega - \sum_{i=1}^r \sqrt{-1} \partial \bar{\partial} \log \left( 1 - \frac{|s_i|_{h_i}^{2\beta_i}}{\beta_i} \right)^2.$$  (6)

**Remark 2.1.** The appearance of $\beta_i$ in the denominator of the log term in the potential function does not affect the definition of the reference metric. We use this convention, following [8], since the potential function in (6) defined in such a way will be shown to converge weakly to a potential function for some Kähler cusp metric. See Lemma 2.10 for details.

$\Omega_\beta$ can be seen as a generalization of $\omega_{p,\mathcal{D}^*}$ to higher dimensional manifolds. First, let us recall [8, Lemma 3.1].

**Lemma 2.2.** $\Omega_\beta$ is a Kähler edge form with cone angle $2\pi \beta_i$ along $D_i$ for $i = 1, \ldots, r$. More precisely,

$$\Omega_\beta = \omega + 2 \cdot \sum_{i=1}^r \left( \sqrt{-1} \frac{\beta_i^2}{|s_i|_{h_i}^{2-2\beta_i}(1 - |s_i|_{h_i}^{2\beta_i})^2} \langle D^{1,0}s_i, D^{1,0}s_i \rangle - \frac{\beta_i}{1 - |s_i|_{h_i}^{2\beta_i}} \partial_i \right),$$  (7)
where $D^{1,0}$ is the $(1, 0)$-part of the Chern connection of $(L_D, h_i)$ for each $i$. Up to rescaling $\{h_i\}_{i=1,...,r}$, there holds $\Omega_\beta \geq \frac{1}{2} \omega$.

**Proof.** A concise proof for the case $r = 1$ is given in [8, Lemma 3.1]. For the reader’s convenience, we give a detailed proof here. In fact, it suffices to show (7) when $r = 1$. Hence, below we suppose $r = 1$ and drop the subscript $i$ for simplicity.

If we set

$$f(x) = -\log \left( \frac{1 - x^\beta}{\beta} \right)^2,$$

$$\phi = |s|^2_h = h \cdot |s|^2,$$

then $\Omega_\beta = \omega + \sqrt{-1} \partial \bar{\partial} f \circ \phi$. Recall there holds

$$\sqrt{-1} \partial \bar{\partial} f \circ \phi = \sqrt{-1}(f''(\phi) \partial \phi \land \bar{\partial} \phi + f'(\phi) \partial \bar{\partial} \phi).$$

We calculate

$$f' = \frac{2\beta x^{\beta - 1}}{1 - x^\beta},$$

$$f'' = \frac{-2\beta x^{\beta - 2} + 2\beta^2 x^{\beta - 2}}{(1 - x^\beta)^2}.$$

Then

$$\Omega_\beta = \omega + \sqrt{-1} \cdot \frac{2\beta |s|^2_h^{2\beta - 2}}{1 - |s|^2_h^{2\beta}} \partial \bar{\partial} |s|^2_h + \sqrt{-1} \left( \frac{2\beta^2 |s|^2_h^{2\beta - 4}}{(1 - |s|^2_h^{2\beta})^2} - \frac{2\beta |s|^2_h^{2\beta - 4}}{1 - |s|^2_h^{2\beta}} \right) \partial |s|^2_h \land \bar{\partial} |s|^2_h.$$

Note

$$\partial \bar{\partial} |s|^2_h = \partial \bar{s} \land \bar{\partial} h + |s|^2 \partial \bar{\partial} h + h \partial s \land \bar{\partial} s + s \partial h \land \bar{s},$$

$$\partial |s|^2_h \land \bar{\partial} |s|^2_h = |s|^4 \partial h \land \bar{\partial} h + sh |s|^2 \partial h \land \bar{\partial} s + sh |s|^2 \partial s \land \bar{\partial} h + |s|^2 h^2 \partial s \land \bar{s},$$

and the fact

$$\theta = -\sqrt{-1} \partial \bar{\partial} \log h = \sqrt{-1} \left( \frac{\partial h \land \bar{\partial} h}{h^2} - \frac{\bar{\partial} \partial h}{h} \right)$$

$$\langle D^{1,0} s, D^{1,0} s \rangle = \langle \partial s \cdot e + \frac{\partial h}{h} s \cdot e, \partial s \cdot e + \frac{\partial h}{h} s \cdot e \rangle$$

$$= h \partial s \land \bar{\partial} s + s \partial h \land \bar{s} + s \partial h \land \bar{\partial} s + \frac{|s|^2}{h} \partial h \land \bar{\partial} h.$$
We calculate
\[
\Omega_\beta = \omega + \sqrt{-1} \frac{2\beta |s|_h^{2\beta - 2}}{1 - |s|_h^{2\beta}} \partial \bar{\partial} |s|_h^2 + \sqrt{-1} \left( \frac{2\beta^2 |s|_h^{2\beta - 4}}{(1 - |s|_h^{2\beta})^2} - \frac{2\beta |s|_h^{2\beta - 4}}{1 - |s|_h^{2\beta}} \right) \partial |s|_h^2 \wedge \bar{\partial} |s|_h^2
\]
\[
= \omega + \left( \sqrt{-1} \frac{2\beta |s|_h^{2\beta - 2}}{1 - |s|_h^{2\beta}} |s|^2 \partial \bar{\partial} h - \sqrt{-1} \frac{2\beta |s|_h^{2\beta - 4}}{1 - |s|_h^{2\beta}} |s|^4 \partial h \wedge \bar{\partial} h \right)
\]
\[
+ \sqrt{-1} \frac{2\beta^2 |s|_h^{2\beta - 4}}{(1 - |s|_h^{2\beta})^2} (s \partial s \wedge \partial h + h \partial s \wedge \partial \bar{s} + s \partial h \wedge \bar{s})
\]
\[
+ \sqrt{-1} \frac{2\beta^2 |s|_h^{2\beta - 4}}{(1 - |s|_h^{2\beta})^2} (s \partial s \wedge \partial h + h \partial s \wedge \partial \bar{s} + s \partial h \wedge \bar{s})
\]
\[
= \omega - \frac{2\beta |s|_h^{2\beta}}{1 - |s|_h^{2\beta}} \sqrt{-1} \left( \frac{h \partial \bar{\partial} h + \partial \bar{\partial} h}{h^2} - \frac{h}{h} \right)
\]
\[
+ \sqrt{-1} \frac{2\beta^2 |s|_h^{2\beta - 2}}{(1 - |s|_h^{2\beta})^2} (h \partial h \wedge \bar{s} \partial \bar{s} + h \partial \bar{s} \wedge \bar{\partial} h + s \partial h \wedge \bar{s} + |s|^2 \partial h \wedge \bar{s})
\]
\[
= \omega + 2 \cdot \sqrt{-1} \frac{\beta^2 |s|_h^{2\beta - 2}}{(1 - |s|_h^{2\beta})^2} (\partial^{1,0} s, \partial^{1,0} \bar{s}) - 2 \cdot \frac{\beta |s|_h^{2\beta}}{1 - |s|_h^{2\beta}} \partial,
\]
which is what we need. Since
\[
\sqrt{-1} \frac{\beta_i^2}{|s|_{h_i}^{2 - 2\beta_i} (1 - |s|_{h_i}^{2\beta_i})^2} (\partial^{1,0} s_i, \partial^{1,0} \bar{s}_i)
\]
contributes as a non-negative \((1, 1)\)-form for each \(i\), we will show that up to rescaling \(h_i, \frac{\beta_i |s|_{h_i}^{2\beta_i}}{1 - |s|_{h_i}^{2\beta_i}}\) can be made arbitrarily small to conclude that \(\Omega_\beta \geq \frac{1}{2} \omega\). To see this, consider the function \(f_\beta(t) := \frac{\beta_i t^{\beta_i}}{1 - t^{\beta_i}}\). The function \(f_\beta(t)\) is increasing in \((0, 1)\) and satisfies \(f_\beta(0) = 0\). Hence for any \(\delta > 0, \exists t_\delta \in (0, 1),\) such that for \(t \in (0, t_\delta), f_\beta(t) \leq \delta\) for each \(i = 1, \ldots, r\). Now take \(\delta = \frac{1}{4r \cdot \sup_{X, i} ||\partial_i|_\omega}\) and rescale each \(h_i\) such that \(|s|_{h_i}^{2\beta_i} \leq t_\delta\). Then
\[
2 \cdot \sum_{i=1}^r \frac{\beta_i |s|_{h_i}^{2\beta_i}}{1 - |s|_{h_i}^{2\beta_i}} \geq - \frac{1}{2} \omega
\]
and therefore $\Omega_\beta \geq \frac{1}{2}\omega$. \hfill \Box

When $r = 1$ and $\beta = \beta_1 \in (0, \frac{1}{2}]$, the following result of Guenancia states that the reference metric $\Omega_\beta$ has uniformly bounded holomorphic bisectional curvatures on $X \setminus D_\beta$.

**Lemma 2.3.** [8, Theorem 3.2] *When $r = 1$, there exists a constant $C > 0$ depending only on $X$ such that for all $\beta \in (0, \frac{1}{2}]$, the holomorphic bisectional curvature of $\Omega_\beta$ is bounded by $C$.***

We generalize this result to the SNC case by adapting arguments from [10, Lemma 2.3].

**Lemma 2.4.** *There exists a constant $C > 0$ depending only on $X$ such that for all $\beta$ where $\beta_i \in (0, \frac{1}{2}]$ for each $i = 1, \ldots, r$, the holomorphic bisectional curvature of $\Omega_\beta$ is uniformly bounded by $C$ on $X \setminus D$.***

**Proof.** We prove the lemma following [10, Lemma 2.3]. When $r = 1$, this gives another proof for our Lemma 2.3. To deal with the more complicated $r > 1$ case, we need to treat non-diagonal terms in the metric tensor of $\Omega_\beta$ carefully. Since the idea of the proof is the same for general $r > 1$, we assume $r = 2$ for simplicity.

**Step 1: Estimate the metric tensor.**

Fix a point $p \in X \setminus D$. We can find local holomorphic coordinates such that $s_1 = z_1, s_2 = z_2$ and the hermitian metric $h_k$ on $L_{D_h}$ is given by $h_k = e^{-\phi_k}$ with $\phi_k(p) = 0$ and $d\phi_k(p) = 0$ for $k = 1, 2$. In these local coordinates, write

$$\omega = \sqrt{-1}g_{ij}dz^i \wedge d\bar{z}^j,$$

$$\theta^k = \sqrt{-1}\theta_{ij}^k dz^i \wedge d\bar{z}^j,$$

where $g_{ij}$ and $\theta_{ij}^k$ are smooth functions of the coordinate $z$ and $k = 1, 2$. Moreover, for $k = 1, 2$, we have

$$\langle D^{1,0}s_k, D^{1,0}s_k \rangle = \langle dz_k \cdot e_k - z_k\partial \phi_k \cdot e_k, dz_k \cdot e_k - z_k\partial \phi_k \cdot e_k \rangle$$

$$= e^{-\phi_k} \left(1 - z_k \frac{\partial \phi_k}{\partial z_k} - z_k \frac{\partial \phi_k}{\partial \bar{z}_k} + |z_k|^2 \frac{\partial \phi_k}{\partial z_k} \frac{\partial \phi_k}{\partial \bar{z}_k} \right) dz^k \wedge d\bar{z}^k$$

$$+ \sum_{i \neq k} e^{-\phi_k} \left(-\bar{z}_k \frac{\partial \phi_k}{\partial z_i} + |z_k|^2 \frac{\partial \phi_k}{\partial z_k} \frac{\partial \phi_k}{\partial \bar{z}_i} \right) dz^i \wedge d\bar{z}^k$$

$$+ \sum_{j \neq k} e^{-\phi_k} \left(-z_k \frac{\partial \phi_k}{\partial \bar{z}_j} + |z_k|^2 \frac{\partial \phi_k}{\partial z_j} \frac{\partial \phi_k}{\partial \bar{z}_k} \right) dz^j \wedge d\bar{z}^k$$

$$+ \sum_{i,j \neq k} e^{-\phi_k} |z_k|^2 \frac{\partial \phi_k}{\partial z_i} \frac{\partial \phi_k}{\partial \bar{z}_j} dz^i \wedge d\bar{z}^j.$$
For the sake of brevity, we introduce the following notations for $k = 1, 2$:

$$a_k = -\bar{z}_k \frac{\partial \phi_k}{\partial \bar{z}_k} - z_k \frac{\partial \phi_k}{\partial z_k} + |z_k|^2 \frac{\partial \phi_k}{\partial z_k} \frac{\partial \phi_k}{\partial \bar{z}_k},$$

$$b^k_j = \frac{\partial \phi_k}{\partial z^j} + \bar{z}_k \frac{\partial \phi_k}{\partial z_j} \frac{\partial \phi_k}{\partial \bar{z}_k}, \quad j \neq k,$$

$$c^k_{ij} = \frac{\partial \phi_k}{\partial z_i} \frac{\partial \phi_k}{\partial \bar{z}_j}, \quad i, j \neq k.$$

Then $a_k, b^k_j$ and $c^k_{ij}$ are smooth and vanish at $p$. Writing $\Omega_{\beta} = \sqrt{-1} g_{ij} dz^i \wedge d\bar{z}^j$, by (7) we have

$$g = \hat{g}_{ij} - 2 \cdot \sum_{k=1}^{2} \frac{\beta_k |s_k|^2 h_k}{1 - |s_k|^2 h_k} \delta^k_{ij} + 2 \cdot \sum_{k=1}^{2} \frac{\beta_k^2 |s_k|^2 h_k - 2}{(1 - |s_k|^2 h_k)^2} \langle D^{1,0} s_k, D^{1,0} s_k \rangle$$

$$= \hat{g}_{ij} - 2 \cdot \sum_{k=1}^{2} \frac{\beta_k e^{-\beta_k \phi_k} |z_k|^2 h_k}{1 - |z_k|^2 h_k} \delta^k_{ij} + 2 \cdot \sum_{k=1}^{2} \frac{\beta_k^2 e^{-(\beta_k - 1) \phi_k} |z_k|^2 h_k}{(1 - |z_k|^2 h_k)^2} (1 + a_1)$$

For each component, we have

$$g_{11} = \hat{g}_{11} - 2 \cdot \sum_{k=1}^{2} \frac{\beta_k e^{-\beta_k \phi_k} |z_k|^2 h_k}{1 - |z_k|^2 h_k} \delta^k_{11} + 2 \cdot \frac{\beta_k^2 e^{-(\beta_k - 1) \phi_k} |z_k|^2 h_k}{(1 - |z_k|^2 h_k)^2} e^{-\phi_k} (1 + a_1)$$

$$g_{12} = \hat{g}_{12} - 2 \cdot \sum_{k=1}^{2} \frac{\beta_k e^{-\beta_k \phi_k} |z_k|^2 h_k}{1 - |z_k|^2 h_k} \delta^k_{12} + 2 \cdot \frac{\beta_k^2 e^{-(\beta_k - 1) \phi_k} |z_k|^2 h_k}{(1 - |z_k|^2 h_k)^2} e^{-\phi_k} z_1 \delta^k_{12}$$

$$g_{ij} = \hat{g}_{ij} - 2 \cdot \sum_{k=1}^{2} \frac{\beta_k e^{-\beta_k \phi_k} |z_k|^2 h_k}{1 - |z_k|^2 h_k} \delta^k_{ij} + 2 \cdot \frac{\beta_k^2 e^{-(\beta_k - 1) \phi_k} |z_k|^2 h_k}{(1 - |z_k|^2 h_k)^2} e^{-\phi_k} z_i \delta^k_{ij}, \quad i, j \geq 3.$$
the Riemann surface associated to \( z_k \to z_k^{\beta_k} \). More specifically, whenever we work with \( \xi_k \), denoting the polar coordinates of \( \xi_k \) by \((r_k, \theta_k)\), we always assume \( \theta_k \in (0, 2\pi \beta_k) \). Under this assumption, \( \xi_k \) always take values in the space \( \{(r, \theta) \in \mathbb{C} : \theta \in [0, \pi)\} \) for \( k = 1, 2 \) for varying \( \beta_k \) as we have \( \beta_k \in (0, 1/2] \).
Moreover, when we consider the inverse map of the change of coordinates \( z_k = (\beta_k \xi_k)^{1/\beta_k} \), we assume \( \xi_k \in \{(r, \theta) \in \mathbb{C} : \theta \in [0, 2\pi \beta_k)\} \) and hence \( z_k \in \{(r, \theta) \in \mathbb{C} : \theta \in [0, 2\pi)\} \) for varying \( \beta_k \).
In the new coordinates, we have
\[
dz_1 \wedge dz_1 = |\beta_1 \xi_1|^2_1 d\xi_1 \wedge d\bar{\xi}_1,
\]
\[
dz_2 \wedge dz_2 = |\beta_2 \xi_2|^2_2 d\xi_2 \wedge d\bar{\xi}_2.
\]
From now on, we make the change of coordinates summarized as above:

\[
\xi_k = \frac{z_k^{\beta_k}}{\beta_k}, \quad k = 1, 2
\]
\[
\xi_{\ell} = z_{\ell}, \quad \ell = 3, \ldots, n.
\]
We record the components of \( \Omega_\beta \), denoted by \( \Omega_\beta = \sqrt{-1}h_{ij}d\xi^i \wedge d\xi^j \), in the new coordinates:
\[
h_{11} = |\beta_1 \xi_1|^2_1 \bar{g}_{11} - 2 \cdot \frac{1}{1 - e^{-\beta_1 \phi_1}} e^{-\beta_1 \phi_1} |\xi_1|^2_1 \bar{g}_1^1 - 2 \cdot \frac{\beta_2^3 e^{-\beta_2 \phi_2} |\xi_2|^2_2 \bar{g}_2^1}{1 - e^{-\beta_2 \phi_2}} - 2 \cdot \frac{\beta_2^3 e^{-\beta_2 \phi_2} |\xi_2|^2_2 \bar{g}_2^2}{1 - e^{-\beta_2 \phi_2}}
\]
\[
h_{12} = \left( \frac{1}{\beta_1} \frac{1}{\beta_2^2} \frac{1}{|\xi_1|} \frac{1}{|\xi_2|} \bar{g}_{12} - 2 \cdot \frac{1}{1 - e^{-\beta_1 \phi_1}} e^{-\beta_1 \phi_1} |\xi_1|^2_1 \bar{g}_1^1 - 2 \cdot \frac{\beta_2^3 e^{-\beta_2 \phi_2} |\xi_2|^2_2 \bar{g}_2^1}{1 - e^{-\beta_2 \phi_2}} \right) + 2 \cdot \frac{\beta_2^3 e^{-\beta_2 \phi_2} |\xi_2|^2_2 \bar{g}_2^2}{1 - e^{-\beta_2 \phi_2}}
\]
\[
h_{1j} = \frac{1}{\beta_1} \frac{1}{\xi_1^2} \bar{g}_{1j} - 2 \cdot \frac{1}{1 - e^{-\beta_1 \phi_1}} e^{-\beta_1 \phi_1} |\xi_1|^2_1 \bar{g}_1^j - 2 \cdot \frac{1}{1 - e^{-\beta_2 \phi_2}} e^{-\beta_2 \phi_2} |\xi_2|^2_2 \bar{g}_2^j + 2 \cdot \frac{\beta_2^3 e^{-\beta_2 \phi_2} |\xi_2|^2_2 \bar{g}_2^j}{1 - e^{-\beta_2 \phi_2}}
\]
\[
h_{ij} = \frac{1}{\beta_1} \frac{1}{\xi_1^2} \bar{g}_{1j} - 2 \cdot \frac{1}{1 - e^{-\beta_1 \phi_1}} e^{-\beta_1 \phi_1} |\xi_1|^2_1 \bar{g}_1^j - 2 \cdot \frac{1}{1 - e^{-\beta_2 \phi_2}} e^{-\beta_2 \phi_2} |\xi_2|^2_2 \bar{g}_2^j + 2 \cdot \frac{\beta_2^3 e^{-\beta_2 \phi_2} |\xi_2|^2_2 \bar{g}_2^j}{1 - e^{-\beta_2 \phi_2}}
\]
\[
(13)
\]
Note that $h_{2i}$ can be treated similarly as $h_{1j}$. A first observation is that since $\beta_k \in (0, 1/2]$, both $\beta_k^{k-2}$ and $|\xi_k|^{-2}$ are uniformly bounded. Regarding the metric tensor of $\Omega_\beta$, terms that involve $\tilde{g}_{ij}$ or $\theta_{ij}$ are also uniformly bounded. Moreover, after the change of coordinates, there does not exist singular term in each component $h_{ij}$. In conclusion, $h_{ij}$ is uniformly bounded for any $i, j$, i.e., by (13), in coordinates $(\xi_1, \xi_2, \xi_3 = z_3, \ldots, \xi_n = z_n)$, there holds

$$h_{ij} = O(1), \quad \text{for } i, j = 1, \ldots, n.$$  

In other words, we have shown that the metric tensor $h_{ij}$ is bounded from above for all $i, j$. However, the metric tensor may degenerate as $\beta_1$ or $\beta_2$ tends to 0.

According to (13), we have for rather small $\beta_1$ and $\beta_2$ the following asymptotic behaviors of each metric tensor:

$$h_{11} \sim \beta_1^2, \quad h_{22} \sim \beta_2^2,$$
$$h_{12}, h_{21} \sim \beta_1^2 \beta_2^2 |\xi_1| |\xi_2|,$$
$$h_{1j} \sim |\xi_1|, \quad j = 3, \ldots, n,$$
$$h_{ij} = O(1), \quad i, j = 3, \ldots, n. \quad (14)$$

**Step 2: Estimate the inverse of the metric tensor**

Recall the curvature tensor is given by

$$R_{ijkl} = -h_{ij,k\ell} + h^{il}h_{kj,\ell}. \quad (15)$$

To estimate the holomorphic bisectional curvature, we take two unit vectors (w.r.t. the metric $h_{ij}$) $u^i \frac{\partial}{\partial \xi_i}$ and $v^j \frac{\partial}{\partial \xi_j}$. Then by (14) there holds

$$u^1, v^1 = O(\beta_1^{-1}), \quad u^2, v^2 = O(\beta_2^{-1}),$$
$$u^i, v^i = O(1), \quad i = 3, \ldots, n. \quad (16)$$

To finish the proof we need to bound $R_{ijkl} u^i \tilde{u}^j v^k \tilde{v}^\ell$. We first consider $R_{ij,k\ell}$. We need to analyze $h^{ij}, h_{ij,k}$ and $h_{ij,k\ell}$.

We first treat $h^{ij}$. By (13) and (14), one finds that $\det\{h_{ij}\}$ is uniformly bounded and tends to 0 as $\beta_1$ or $\beta_2$ tends to 0. More precisely, by (14) there holds

$$\det\{h_{ij}\} \sim \beta_1^2 \beta_2^2. \quad (17)$$

Since $h^{ij} = C_{ji} / \det\{h_{ij}\}$, where $C_{ji}$ is the $ji$-th cofactor of the matrix $\{h_{ij}\}$, we deduce from (14) and (17) that

$$h^{11}, h^{11} \sim \beta_1^{-2}, \quad i, j \neq 2,$$
$$h^{12} \sim \beta_1^{-2} \beta_2^{-2},$$
$$h^{ij} = O(1), \quad i, j \neq 1, 2. \quad (18)$$
while $h^{2i}$ and $h^{1j}$ can be treated in the same way. Roughly speaking, $h^{ij}$ are bounded for fixed $\beta$ near $p$ but may tend to infinity with respect to $\beta$ as described above. However, by (13) and (14), for any $j$, $h_{ij}$ (respectively $h_{2j}$) degenerate at the rate of at least $\beta_1^2$ (respectively $\beta_2^2$), when $\beta_1$ (respectively $\beta_2$) is small, and taking derivatives may only cause the terms $h_{ij,k}$ and $h_{ij,k,l}$ to converge to 0 at a faster speed in $\beta$. Thus, the singularity in $h^{ij}$ does not cause a problem when we consider the curvature tensor (15), where the $h^{ij}$ terms are multiplied by corresponding $h_{i,l,k}$ or $h_{i,j,l}$. We explain this in detail later in Step 4.

**Step 3: Estimate derivatives of the metric tensor**

Now we turn to show that $h_{ij,k}$ and $h_{ij,k,l}$ are uniformly bounded and find their dependence on $\beta_1$ and $\beta_2$. For $k, \ell \in \{3, \ldots, n\}$, as taking derivative w.r.t. $\partial / \partial z_3 = \partial / \partial \xi_3, \ldots, \partial / \partial z_n = \partial / \partial \xi_n$ will not contribute extra singular terms, the uniform boundedness of $h_{ij}$ implies that this also holds for such $h_{ij,k}$ and $h_{ij,k,l}$. It remains to deal with $h_{ij,k}$ and $h_{ij,k,l}$ for $k, \ell \in \{1, 2\}$.

By the exact formula of each $h_{ij}$ in (13), we find that the exponent of the terms $\xi_1, \xi_1, \xi_2, \xi_2$ is at least 1 when $\beta_1$ and $\beta_2$ are rather small, and indeed both the term $h_{ij}$ and their first or second order derivatives are smooth in $\xi_1, \xi_1, \xi_2, \xi_2$. In summary, taking derivatives of the metric tensor does not cause singularities in $\xi_1, \xi_1, \xi_2, \xi_2$. We only need to derive the asymptotic behavior of the derivatives with respect to $\beta$. To achieve this, it is enough to deal with the following term $T$ from (13). The reason is that any other terms in (13) have higher order dependence in $\beta_1$ and $\beta_2$, before and after taking the derivatives. So we consider taking derivatives of the following term

$$\frac{\beta_1^2 e^{-\beta_1 \phi_1}}{(1 - \beta_1^2 |\xi_1|^2 e^{-\beta_1 \phi_1})^2} =: T$$

w.r.t. $\partial / \partial \xi_1$ and $\partial / \partial \xi_2$, since this is the most singular term appearing in the metric tensor. And by symmetry of indices we only need to consider $\partial / \partial \xi_1(T)$, $\partial^2 / \partial \xi_1 \partial \xi_2(T)$ and $\partial^2 / \partial \xi_1 \partial \xi_2(T)$. In the coordinates $(\xi_1, \xi_2, \xi_3 = z_3, \ldots, \xi_n = z_n)$,

$$\begin{align*}
\frac{\partial}{\partial \xi_1} &= \frac{\partial z_1}{\partial \xi_1} \frac{\partial}{\partial z_1} = \beta_1^{1/\beta_1 - 1} \xi_1^{1/\beta_1 - 1} \frac{\partial}{\partial z_1}, \\
\frac{\partial^2}{\partial \xi_1 \partial \xi_2} &= \frac{\partial z_2}{\partial \xi_1} \frac{\partial z_1}{\partial \xi_2} \frac{\partial^2}{\partial z_1 \partial z_2}, \\
\frac{\partial^2}{\partial \xi_1 \partial \xi_2} &= \frac{\partial z_1}{\partial \xi_1} \frac{\partial z_1}{\partial \xi_2} \frac{\partial^2}{\partial z_1 \partial z_2}, \\
\frac{\partial^2}{\partial \xi_1 \partial \xi_2} &= \frac{\partial z_2}{\partial \xi_1} \frac{\partial z_1}{\partial \xi_2} \frac{\partial^2}{\partial z_1 \partial z_2}.
\end{align*}$$

Thus, we have

$$\frac{\partial}{\partial \xi_1}(T) = \frac{\partial}{\partial \xi_1} \frac{\beta_1^2 e^{-\beta_1 \phi_1}}{(1 - \beta_1^2 |\xi_1|^2 e^{-\beta_1 \phi_1})^2}$$
we find that the derivatives of the metric tensor degenerate (w.r.t. \( \beta \)). Similarly, we get

\[
\frac{\partial^2}{\partial \xi_1 \partial \xi_2} (T) = O(\beta^4 \beta_2^{1/2} |\xi_1||\xi_2|^{1/2}) = O(1),
\]

since we only need to address \( \partial / \partial \xi_2 (e^{-\beta \phi_1}) \) when taking derivative of \( \partial / \partial \xi_1 (T) \) w.r.t. \( \partial / \partial \xi_2 \). Finally, direct calculations yield

\[
\frac{\partial^2}{\partial \xi_1 \partial \xi_2} T = \beta_1^{1/2} \beta_1^{1/2} |\xi_1|^{2/2} e^{-\beta \phi_1} \frac{\beta^2 e^{-\beta \phi_1}}{\partial \xi_1 \partial \xi_2} + 2\beta_1^{1/2} |\xi_1|^{2/2} e^{-\beta \phi_1} \frac{\partial e^{-\beta \phi_1}}{\partial \xi_1} + 6\beta_1^{1/2} |\xi_1|^{2/2} e^{-\beta \phi_1} \frac{\partial}{\partial \xi_1} \left( |\xi_1|^2 e^{-\beta \phi_1} \right)
\]

In conclusion, we have shown

\[
h_{i,j,k} = O(1),
\]

\[
h_{i,j,k,\ell} = O(1),
\]

for fixed \( \beta \) and any \( i, j, k, \ell = 1, ..., n \). In other words, the derivative of the metric tensor is bounded for fixed \( \beta \). Moreover, by taking derivatives of (13), we find that the derivatives of the metric tensor degenerate (w.r.t. \( \beta_1 \) and \( \beta_2 \)) at
the rate shown below:

\[
\begin{align*}
    h_{11,1} &= O(\beta_1^4), \quad \ell = 1, \ldots, n \\
    h_{11,2} &= O(\beta_1^3 \beta_2^3), \quad i = 1, \ldots, n, \\
    h_{11,k\ell} &= O(\beta_1^3), \quad k \neq 1, 2, \ell = 1, \ldots, n \\
    h_{12,1} &= O(\beta_1^3 \beta_2^3), \\
    h_{12,k} &= O(\beta_1^3 \beta_2^3), \quad k \neq 1, \\
    h_{12,1\ell} &= O(\beta_1^3 \beta_2^3), \quad \ell = 1, \ldots, n, \\
    h_{12,k\ell} &= O(\beta_1^3 \beta_2^3), \quad k \neq 1, \ell = 1, \ldots, n \\
    h_{1j,k} &= O(\beta_1^3), \quad j \neq 1, 2, k \neq 2, \\
    h_{1j,2} &= O(\beta_1^3 \beta_2^3), \quad j \neq 1, 2 \\
    h_{1j,k\ell} &= O(\beta_1^3), \quad j \neq 1, 2, k \neq 2, \quad \ell \neq 2, \\
    h_{1j,2\ell} &= O(\beta_1^3 \beta_2^3), \quad j \neq 1, 2, k = 2 \quad \text{or} \quad \ell = 2, \\
    h_{ij,1} &= O(\beta_1^{1/\beta_1-1}), \quad i, j \neq 1, 2 \\
    h_{ij,2} &= O(\beta_1^{1/\beta_2-1}), \quad i, j \neq 1, 2 \\
    h_{ij,11} &= O(\beta_1^{2/\beta_1-2}), \quad i, j \neq 1, 2 \\
    h_{ij,22} &= O(\beta_2^{2/\beta_2-2}), \quad i, j \neq 1, 2 \\
    h_{ij,12}, h_{ij,21} &= O(\beta_1^{1/\beta_1-1} \beta_2^{1/\beta_2-1}), \quad i, j \neq 1, 2 \\
    h_{ij,k}, h_{ij,k\ell} &= O(1), \quad i, j, k, \ell \neq 1, 2.
\end{align*}
\]

\textbf{Step 4: Estimate the sum} \( R_{ijk\ell} u^{i} \bar{u}^{j} v^{k} \bar{v}^{\ell} \).

We have shown the derivatives of \( h_{ij} \) are bounded. To show that \( R_{ijk\ell} u^{i} \bar{u}^{j} v^{k} \bar{v}^{\ell} \)
is bounded, we consider

\[
R_{ijk\ell} u^{i} \bar{u}^{j} v^{k} \bar{v}^{\ell} = (-h_{ij,k\ell} + h^{ii} h_{ij,k} h_{ij,\ell}) u^{i} \bar{u}^{j} v^{k} \bar{v}^{\ell}.
\]

We consider three different cases.

\textbf{When none of} \( i, j, k, \ell, s, t \) \textbf{is 1 or 2}: then the sum is uniformly bounded because none of the term blow up w.r.t. \( \beta_1 \) or \( \beta_2 \).

\textbf{When} \( s, t \neq 1, 2 \) \textbf{and} \( i, j, k, \ell \) \textbf{may take values from 1 or 2}: then we found from (20) that the common factors of powers of \( \beta_1 \) or \( \beta_2 \) in \( h_{ij,k} \) and \( h_{ij,k\ell} \) compensate for the degeneracy of \( u^1 \) and \( u^2 \) and \( v^1 \) and \( v^2 \).

\textbf{When} \( s \) \textbf{or} \( t = 1, 2 \): then in the worst case, where \( s = 1, t = 2 \), we find from (20) that all the derivatives that have 1 or 2 in the subscript have at least a degeneracy rate of \( \beta_1^3 \) or \( \beta_2^3 \). However, \( h^{ii} \) blows up at the rate of \( \beta_1^{-2} \beta_2^{-2} \). Combining this fact with (16) we find that the common factors of \( \beta_1 \) and \( \beta_2 \) in the derivatives can still compensate for the degeneracy of \( h^{ii} \) and \( u, v \).

Then we conclude that when \( \beta \) satisfies that \( \beta_k \in (0,1/2] \) for each \( k \), the holomorphic bisectional curvature of \( \Omega_{\beta} \) is uniformly bounded in \( \beta \). \( \square \)
Remark 2.5. As pointed out to the author by H. Guenancia, J. Sturm’s trick (see [14, p. 62]) can also be applied to simplify the proof of the curvature bounds in Lemma 2.3. Our proof of Lemma 2.4 deals with the general case where \( r > 1 \).

2.2. A priori estimates. By Theorem 1.5, there is a unique Kähler–Einstein crossing edge metric on \( X \) with cone angle \( 2\pi \beta_i \) along each \( D_i \), denoted by \( \omega_{\phi_{\beta}} = \omega - \sum_{i=1}^{r} \beta_i \partial_i + \sqrt{-1} \partial \bar{\partial} \phi_{\beta} \), such that

\[
\begin{cases}
\omega_{\phi_{\beta}} = \frac{e^{f+\phi_{\beta}} \omega^n}{\prod_{i=1}^{r} |s_i|_{\hat{h}_i}^{2(1-\beta_i)}}, \\
\omega_{\phi_{\beta}} = \omega - \sum_{i=1}^{r} \beta_i \partial_i + \sqrt{-1} \partial \bar{\partial} \phi_{\beta} > 0,
\end{cases}
\]

where \( f \in C^\infty(X) \). In this section, we establish a Laplacian estimate for \( \omega_{\phi_{\beta}} \) with respect to the reference metric \( \Omega_{\beta} \) by proving the following result.

**Theorem 2.6.** For \( \beta = (\beta_1, \ldots, \beta_r) \in (0, \frac{1}{2}]^r \), there exists a constant \( C > 0 \) (independent of \( \beta_1, \ldots, \beta_r \)) such that

\[
C^{-1} \Omega_{\beta} \leq \omega_{\phi_{\beta}} \leq C \Omega_{\beta},
\]

on \( X \setminus \text{supp}(D) \).

Define

\[
\psi_{\beta} := -\sum_{i=1}^{r} \log \left( \frac{1 - |s_i|_{\tilde{h}_i}^{2\beta_i}}{\beta_i} \right)^2,
\]

then

\[
\Omega_{\beta} = \omega + \sqrt{-1} \partial \bar{\partial} \psi_{\beta}.
\]

**Proof of Theorem 2.6.** We divide the proof into two steps. First, we deduce the \( C^0 \)-estimate for potential functions \( \phi_{\beta} \) and \( \psi_{\beta} \) by using a modified maximum principle. Then we derive the Laplacian estimate by applying Chern–Lu’s inequality to the identity map \( (X, \omega_{\phi_{\beta}}) \to (X, \Omega_{\beta}) \).

**Remark 2.7.** When \( r = 1 \), the proof of (22) is already given in [8, Proposition 4.2]. The main difference for the case \( r > 1 \) is that \( \omega_{\phi_{\beta}} \) and \( \Omega_{\beta} \) admit crossing edge singularities. However, thanks to Lemma 2.4, we are able to follow the arguments in [8] and treat all the components at once.

**Step 1: \( C^0 \)-estimate: Comparing \( \phi_{\beta} \) with \( \psi_{\beta} \).**

We first compare the potential functions of \( \omega_{\phi_{\beta}} \) and \( \Omega_{\beta} \). Let \( \tilde{\phi}_{\beta} := \phi_{\beta} - \psi_{\beta} \), then we get

\[
\omega_{\tilde{\phi}_{\beta}} = \frac{e^{\tilde{\phi}_{\beta}+f} \omega^n}{\prod_{i=1}^{r} |s_i|_{\hat{h}_i}^{2(1-\beta_i)}},
\]

(23)
\[
\Rightarrow (\Omega - \sum_{i=1}^{r} \beta_i \partial_i + \sqrt{-1} \partial \bar{\partial} \phi) = e^{\delta \phi + F \cdot \Omega^{n}}
\]

where

\[
F = \psi + f + \log \left( \prod_{i=1}^{r} \left( \frac{\omega^{n}}{|s_i|^{2(1-\beta_i)}} \cdot \Omega^{n}_{\beta} \right) \right).
\]

Then we claim that

**Claim 2.8.** For some uniform constant \( C > 0 \),

\[
||F||_{C^{0}(\Sigma \setminus D)} \leq C.
\]

**Proof.** First note that \( f \) is smooth on \( X \) by construction, hence it is bounded as \( X \) is compact. Therefore, it suffices to show

\[
F - f = \psi + \log \left( \prod_{i=1}^{r} \left( \frac{\omega^{n}}{|s_i|^{2(1-\beta_i)}} \cdot \Omega^{n}_{\beta} \right) \right)
\]

is bounded. To prove the claim, it is equivalent to showing

\[
\Omega^{n}_{\beta} = \frac{\prod_{i=1}^{r} \beta_i^{2}}{\prod_{i=1}^{r} |s_i|^{2(1-\beta_i)}(1 - |s_i|^{2\beta_i})^{2}} e^{O(1)} \omega^{n}
\]

near the divisor. This amounts to saying that \( \Omega^{n}_{\beta} \) has a pole of order \( \prod_{i=1}^{r} |s_i|^{2(1-\beta_i)} \). Without loss of generality, we can assume \( r = 1 \) and thus below we drop the \( i \) in the subscript for simplicity. Let \( p \in M \setminus D \) near \( D \). Let \( e \) be a local holomorphic frame for \( L_{D} \), and \( (z_1, \ldots, z_n) \) be a local holomorphic coordinate chart such that \( s = z_1 e \). Let \( h = e^{-\psi} \) be a smooth hermitian metric on \( \mathcal{O}_{X}(D) \) and \( \theta \) the curvature form of \( (L_{D}, h) \). Denote

\[
\omega = \sqrt{-1} g_{ij} dz_i \wedge d\bar{z}_j,
\]

\[
\theta = \sqrt{-1} \phi_{ij} dz_i \wedge d\bar{z}_j.
\]

Recall the expression (7) of \( \Omega_{\beta} \). We calculate

\[
\langle D^{1,0}, s^{1,0} \rangle = e^{-\phi}(dz_1 + z_1 \frac{\partial \phi}{\partial z_1}) \wedge (d\bar{z}_1 + \bar{z}_1 \frac{\partial \phi}{\partial \bar{z}_1})
\]

\[
= e^{-\phi}((1 + \bar{z}_1 \frac{\partial \phi}{\partial z_1}) + |z_1|^2 \frac{\partial \phi}{\partial z_1} \frac{\partial \phi}{\partial \bar{z}_1}) dz_1 \wedge d\bar{z}_1
\]

\[
+ \sum_{k=2}^{n} (\bar{z}_1 \frac{\partial \phi}{\partial z_k} + |z_1|^2 \frac{\partial \phi}{\partial z_1} \frac{\partial \phi}{\partial \bar{z}_k}) dz_1 \wedge d\bar{z}_k
\]

\[
+ \sum_{k=2}^{n} (z_1 \frac{\partial \phi}{\partial \bar{z}_k} + |z_1|^2 \frac{\partial \phi}{\partial \bar{z}_1} \frac{\partial \phi}{\partial z_k}) dz_1 \wedge d\bar{z}_k.
\]
Recall the formula for the determinant of block matrices as in (26),

\[ + \sum_{k,l=2}^n \left| z_1 \right|^2 \frac{\partial \phi}{\partial z_k} \frac{\partial \phi}{\partial z_l} dz_k \wedge d\bar{z}_l. \]

Hence,

\[ \frac{\beta^2}{|s|_h^{2(1-\beta)}(1 - |s|_h^{2\beta})^2} (D^{1,0} s, D^{1,0} s) \]

\[ = \frac{\beta^2}{|s|_h^{2(1-\beta)}(1 - |s|_h^{2\beta})^2} + O(1) dz_1 \wedge d\bar{z}_1 + \sum_{k=2}^n \left( \frac{\beta^2 |s|_h^{2\beta}}{z_1(1 - |s|_h^{2\beta})^2} + O(1) dz_1 \wedge d\bar{z}_k \right) + \sum_{k,l=2}^n \left( \frac{\beta^2 |s|_h^{2\beta}}{z_1(1 - |s|_h^{2\beta})^2} + O(1) dz_1 \wedge d\bar{z}_k \right) + \sum_{k,l=2}^n O(1) dz_k \wedge d\bar{z}_l. \]

Let

\[ (A_{ij})_{i,j=1}^n = (g_{ij})_{i,j=1}^n - \frac{\beta |s|_h^{2\beta}}{1 - |s|_h^{2\beta}} (\phi_{ij})_{i,j=1}^n + \frac{\beta^2}{|s|_h^{2(1-\beta)}(1 - |s|_h^{2\beta})^2} (D^{1,0} s, D^{1,0} s). \]

Write \((A_{ij})_{i,j=1}^n\) as a block matrix

\[ (A_{ij})_{i,j=1}^n = \begin{bmatrix} A_{11} & \tilde{A}_r \\ \bar{A}_c & (A_{ij})_{i,j=2}^n \end{bmatrix}, \tag{26} \]

then

\[ A_{11} = g_{11} - \frac{\beta |s|_h^{2\beta}}{1 - |s|_h^{2\beta}} \phi_{11} + \frac{\beta^2}{|s|_h^{2(1-\beta)}(1 - |s|_h^{2\beta})^2} + O(|s|_h^{2\beta}) + O(1), \]

\[ A_{1j} = g_{1j} - \frac{\beta |s|_h^{2\beta}}{1 - |s|_h^{2\beta}} \phi_{1j} + O(|s|_h^{2\beta-1}) + O(|s|_h^{2\beta}) + O(1), \quad j = 2, \ldots, n, \]

\[ A_{i1} = g_{i1} - \frac{\beta |s|_h^{2\beta}}{1 - |s|_h^{2\beta}} \phi_{i1} + O(|s|_h^{2\beta-1}) + O(|s|_h^{2\beta}) + O(1), \quad i = 2, \ldots, n, \]

\[ A_{kl} = O(|s|_h^{2\beta}) + O(1), \quad k,l = 2, \ldots, n. \]

Recall the formula for the determinant of block matrices as in (26),

\[ \det((A_{ij})_{i,j=1}^n) = \det((A_{ij})_{i,j=2}^n) \cdot (A_{11} - \tilde{A}_r ((A_{ij})_{i,j=2}^n)^{-1} \bar{A}_c). \tag{27} \]
Using (27),
\[
\frac{\Omega^n_\beta}{\omega^n} = \frac{\det(g_{ij} - \frac{\beta|s_h|^{2\beta}}{1-|s_h|^{2\beta}} \, \phi_{ij} + \frac{\beta^2}{|s_h|^{2\beta}(1-|s_h|^{2\beta})^2} (D^{1,0} s, D^{1,0} s))}{\det(g_{ij})}
\]
\[
= \frac{\det(A_{ij})_{i,j=1}^n}{\det(g_{ij})}
\]
\[
= e^{O(1)} \cdot \det(A_{ij})_{i,j=2}^n \cdot (A_{11} - \tilde{A}_c ((A_{ij})_{i,j=2}^n)^{-1} \tilde{A}_c)
\]
\[
= e^{O(1)} \cdot \left( \frac{\beta^2}{|s_h|^{2(1-\beta)}(1-|s_h|^{2\beta})^2} + O(|s_h|^{4\beta-2}) + O(|s_h|^{2\beta-1}) + O(|s_h|^{4\beta-1})
\right.
\]
\[
+ O(|s_h|^{2\beta}) + O(|s_h|^{4\beta}) + O(1) \right).
\]

Thus, one finds that the dominant term is \(\frac{\beta^2}{|s_h|^{2(1-\beta)}(1-|s_h|^{2\beta})^2}\). In another word, we have shown \(\Omega^n_\beta = e^{O(1)} \frac{\beta^2}{|s_h|^{2(1-\beta)}(1-|s_h|^{2\beta})^2} \omega^n\), which is exactly what we need.

\[\Box\]

**Lemma 2.9.** There exists a uniform constant \(C > 0\) in \(\beta\) such that,
\[
\sup_{X \setminus D} |\tilde{\phi}_{\beta}| \leq C,
\]
when \(\beta_i\) are small enough for every \(i\).

**Proof.** First note that for a fixed \(\beta\), \(\tilde{\phi}_{\beta}\) is bounded according to [10, 9]. We aim to derive a uniform bound for \(\tilde{\phi}_{\beta}\) in \(\beta\). Let \(\chi_{\beta, \varepsilon} = \tilde{\phi}_{\beta} + \varepsilon \sum_{i=1}^r \log |s_i|_{h_i}^2\) for small \(\varepsilon > 0\). Since \(\chi_{\beta, \varepsilon}(p)\) approaches \(-\infty\) when \(p \to D\), \(\chi_{\beta, \varepsilon}\) obtains its maximum on \(X \setminus D\), at say \(p_{\text{max}}\). Then
\[
0 \geq -1 \delta \bar{\delta} \chi_{\beta, \varepsilon}(p_{\text{max}}) - \varepsilon \sum_{i=1}^r \bar{\partial}_i(p_{\text{max}}),
\]
where \(\bar{\partial}_i\) is the curvature of the Chern connection on \((L_{D_i}, h_i)\). Then at \(p_{\text{max}},\)
\[
(\Omega_{\beta} - \sum_{i=1}^r \beta_i \bar{\partial}_i + \sqrt{-1} \delta \bar{\delta} \tilde{\phi}_{\beta})^n \leq (\Omega_{\beta} + \sum_{i=1}^r (\varepsilon - \beta_i) \bar{\partial}_i)^n \leq 2^n \Omega^n_{\beta},
\]
by the fact that \(\Omega_{\beta} \geq r(\varepsilon - \beta_i)\bar{\partial}_i\) for small enough \(\varepsilon\) and \(\beta_i\), as shown in Lemma 2.2. Combining (24) and (29), at \(p_{\text{max}},\)
\[
e^{\tilde{\phi}_{\beta} + F_{\beta}}(p_{\text{max}}) \leq 2^n
\[
\Rightarrow \tilde{\phi}_\beta(p_{\text{max}}) \leq n \log 2 - F_\beta(p_{\text{max}}) \\
\leq n \log 2 - \inf_{X \setminus D} F_\beta.
\]

For any \( p \in X \setminus D, \)
\[
\tilde{\phi}_\beta(p) = \chi_{\tilde{\phi}_\beta}(p) - \varepsilon \sum_{i=1}^{r} \log |s_i|_{h_i}(p) \\
\leq \chi_{\tilde{\phi}_\beta}(p_{\text{max}}) - \varepsilon \sum_{i=1}^{r} \log |s_i|_{h_i}(p) \\
\leq n \log 2 - \inf_{X \setminus D} F_\beta + \varepsilon \sum_{i=1}^{r} \log |s_i|_{h_i}(p_{\text{max}}) - \varepsilon \sum_{i=1}^{r} \log |s_i|_{h_i}(p) \\
\leq C
\]
for some constant \( C > 0, \) when letting \( \varepsilon \to 0 \) and using (25). Similarly by considering \( \tilde{\chi}_{\tilde{\phi}_\beta} := \tilde{\phi}_\beta - \varepsilon \sum_{i=1}^{r} \log |s_i|_{h_i} \), achieving its minimum on \( X \setminus D, \) we can show a lower bound for \( \tilde{\phi}_\beta \) on \( X \setminus D. \]

Step 2: The Laplacian estimates for \( \omega_{\tilde{\phi}_\beta} \) and \( \Omega_{\beta}. \)

In this section, we use Chern–Lu’s inequality to deduce the Laplacian estimate of \( \omega_{\tilde{\phi}_\beta} \) with respect to \( \Omega_{\beta}. \)

Consider the identity map
\[
\text{id} : (X \setminus D, \omega_{\tilde{\phi}_\beta}) \to (X \setminus D, \Omega_{\beta}).
\]
By definitions, \( \text{Ric}_{\omega_{\tilde{\phi}_\beta}} = -\omega_{\tilde{\phi}_\beta}. \) From Lemma 2.4, \( |\text{Bisec}_{\Omega_{\beta}}| \leq C_3 \) for some constant \( C_3 > 0 \) when \( \beta_i \in (0, \frac{1}{2}] \) for every \( i. \) Then by Chern–Lu’s inequality [10, Proposition 7.1] (see also [14, Proposition 7.2]),
\[
\Delta_{\omega_{\tilde{\phi}_\beta}}(\log \text{tr}_{\omega_{\tilde{\phi}_\beta}} \Omega_{\beta}) \geq -1 - 2C_3 \text{tr}_{\omega_{\tilde{\phi}_\beta}} \Omega_{\beta}.
\] (30)
Set for \( 0 < \varepsilon \ll 1, \)
\[
H_{\beta,\varepsilon} = \log \text{tr}_{\omega_{\tilde{\phi}_\beta}} \Omega_{\beta} - 4(C_3 + 1)\tilde{\phi}_\beta + \varepsilon \sum_{i=1}^{r} \log |s_i|_{h_i},
\]
then
\[
\Delta_{\omega_{\tilde{\phi}_\beta}} H_{\beta,\varepsilon} = \Delta_{\omega_{\tilde{\phi}_\beta}}(\log \text{tr}_{\omega_{\tilde{\phi}_\beta}} \Omega_{\beta} - 4(C_3 + 1)\tilde{\phi}_\beta) - \varepsilon \sum_{i=1}^{r} \text{tr}_{\omega_{\tilde{\phi}_\beta}} \theta_i
\] (31)
\[
\geq \Delta_{\omega_{\tilde{\phi}_\beta}}(\log \text{tr}_{\omega_{\tilde{\phi}_\beta}} \Omega_{\beta}) + 4(C_3 + 1)\left(\frac{1}{2} \text{tr}_{\omega_{\tilde{\phi}_\beta}} \Omega_{\beta} - n\right) - \text{tr}_{\omega_{\tilde{\phi}_\beta}} \Omega_{\beta},
\] (32)
where the last inequality is true by noting that \( \theta_i \leq M\Omega_{\beta} \) for some uniform constant \( M > 0 \) and assuming \( \varepsilon < \frac{1}{2rM} \) and \( \sum_{i=1}^{r} \beta_i < \frac{1}{2M}. \)
Combine (30) and (32),
\[
\Delta_{\omega_{\beta}} H_{\beta,\varepsilon} \geq \text{tr}_{\omega_{\beta}} \Omega_{\beta} - C
\]
for some uniform constant \( C > 0 \). \( H_{\beta,\varepsilon} \) achieves its maximum on \( X \setminus D \), at, say \( q_{\text{max}} \). Then by (33),
\[
\text{tr}_{\omega_{\beta}} \Omega_{\beta}(q_{\text{max}}) \leq C.
\]
For any \( q \in X \setminus D \),
\[
\log \text{tr}_{\omega_{\beta}} \Omega_{\beta}(q) = H_{\beta,\varepsilon}(q) + 4(C_3 + 1)\phi_{\beta}(q) - \varepsilon \sum_{i=1}^{r} \log |s_i|_{h_i}^2(q)
\]

\[
\leq H_{\beta,\varepsilon}(q_{\text{max}}) + 4(C_3 + 1)\phi_{\beta}(q_{\text{max}}) - \varepsilon \sum_{i=1}^{r} \log |s_i|_{h_i}^2(q_{\text{max}})
\]

\[
\leq C - 4(C_3 + 1)\phi_{\beta}(q_{\text{max}}) + \varepsilon \sum_{i=1}^{r} \log |s_i|_{h_i}^2(q_{\text{max}})
\]

\[
+ 4(C_3 + 1)\phi_{i,\varepsilon}(q) - \varepsilon \sum_{i=1}^{r} \log |s_i|_{h_i}^2(q)
\]

\[
\leq \text{some uniform constant } C,
\]
where the last inequality is true by Lemma 2.9 and letting \( \varepsilon \to 0 \) for fixed \( q \). Hence we have shown
\[
\omega_{\beta} \geq C \cdot \Omega_{\beta},
\]
on \( X \setminus \text{supp}(D) \) as desired. Since \( \omega_{\beta} \) and \( \Omega_{\beta} \) are equivalent on \( X \), we obtain the estimate (22) on \( X \setminus D \).

\[\square\]

2.3. Global convergence of \( \omega_{\phi_{\beta}} \). A smooth Kähler metric \( \Omega_{PC} \) on \( X \setminus D \) is said to have mixed cusp and edge singularities along a divisor \( D \) if whenever \( D \) is locally given by \( D = \sum_{i=1}^{t} \{ z_i = 0 \} + \sum_{j=t+1}^{m} (1 - \beta_j) \{ z_j = 0 \} \) with \( t < m \leq n \), \( \Omega_{PC} \) is quasi-isometric to the following metric:
\[
\omega_{PC} := \sum_{i=1}^{t} \sqrt{-1}dz_i \wedge d\bar{z}_i \sum_{j=t+1}^{m} \frac{\beta_j^2 \sqrt{-1}dz_j \wedge d\bar{z}_j}{|z_j|^{2(1-\beta_j)}} + \sum_{\ell=m+1}^{n} \sqrt{-1}dz_\ell \wedge d\bar{z}_\ell .
\]

In particular, when \( t = m \), \( \omega_{PC} \) has merely cusp singularities along \( D \).

In the case \( t = m \), it is well known [11, 18] that if \( K_X + D \) is ample, there exists a unique Kähler–Einstein metric on \( X \setminus D \) with cusp singularities along \( D \).

In general, it is shown that if \( K_X + D \) is ample, there exists a unique Kähler–Einstein metric on \( X \setminus D \) with mixed cusp and cone singularities along \( D [7, \]

[11, 18]

[7, \]
Theorem A. As a corollary of Theorem 2.6, we study the global weak convergence and local smooth convergence of the Kähler–Einstein crossing edge metrics $\omega_{\phi_\beta}$ to a Kähler–Einstein mixed cusp and edge metric on $(X, D_\beta)$ as some of the cone angles tend to 0. The first observation is the following lemma.

**Lemma 2.10.** Assume $\beta_i \to 0$, for $i = 1, \ldots, t < r$, and $\beta_j \to d_j \in (0, 1)$ for $j = t + 1, \ldots, r$, then $\Omega_\beta$ weakly converges to some Kähler mixed cusp and edge metric $\Omega_{PC}$. Moreover, $\Omega_\beta$ converges to $\Omega_{PC}$ in $C^\infty_{loc}(X \setminus \text{supp}(D_\beta))$.

**Proof.** Recall the definition of $\Omega_\beta$. Note that $\log \left[ (1 - |s_i|_{h_i}^{2\beta_i})/\beta_i \right]$ converges to $\log \log |s_i|^2_{h_i}$ in $L^1(X, \omega)$ and $C^\infty_{loc}(X \setminus \text{supp}(D_\beta))$ as $\beta_i \to 0$ for each $i = 1, \ldots, t$. Thus, $\Omega_\beta$ converges to

$$\Omega_{PC} := \omega - \sum_{i=1}^t \sqrt{-1} \delta \delta \log \log |s_i|^2_{h_i} - \sum_{j=t+1}^r \sqrt{-1} \delta \delta \log \left[ \frac{1 - |s_j|_{h_j}^{2d_j}}{d_j} \right]$$

in $C^\infty_{loc}(X \setminus \text{supp}(D_\beta))$ sense and weakly in the sense of currents. It remains to show that $\Omega_{PC}$ has mixed cusp and edge singularities along $D_\beta$. To see this, recall we denote by $\partial_i$ the Chern curvature form of $(L_{D_i}, h_i)$ for each $i$, then by (8) and (10), we calculate that

$$\sum_{i=1}^t \sqrt{-1} \delta \delta \log \log |s_i|^2_{h_i} = \sum_{i=1}^t 2 \sqrt{-1} \cdot \frac{(\delta \delta |s_i|^2_{h_i})(\log |s_i|^2_{h_i})(\log |s_i|^2_{h_i})}{(\log |s_i|^2_{h_i})^2} + \frac{2}{\log |s_i|^2_{h_i}} \partial_i.$$  

Thus, $\Omega_{PC}$ has cusp singularities along $D_i$ for $i = 1, \ldots, t$. The result follows. \hfill \Box

**Theorem 2.11.** The Kähler–Einstein crossing edge metric $\omega_{\phi_\beta}$ converges to the Kähler–Einstein mixed cusp and edge metric on $(X, D_\beta)$ globally in a weak sense and locally in a strong sense when $\beta_i \to 0$ for $i = 1, \ldots, t < r$ and $\beta_j \to d_j \in (0, 1)$ for $j = t + 1, \ldots, r$.

**Proof.** By Theorem 2.6, the family of $\omega_{\phi_\beta}$ has uniformly bounded mass. Thus, the family of $\omega_{\phi_\beta}$ is relatively compact in the weak topology. The same arguments in the proof of Lemma 2.9 and elliptic estimates give respectively the $C^0$-estimate and all higher-order estimates for the family of $\omega_{\phi_\beta}$. Therefore, any weak limit $\omega_0$ is smooth on $X \setminus \text{supp}(D_\beta)$ and this $C^\infty_{loc}$-convergence indicates that such $\omega_0$ is Kähler–Einstein outside $D_\beta$. Lemma 2.10 shows any such $\omega_0$ also admits mixed cusp and edge singularities along $D_\beta$. Thus, by the uniqueness argument in this setting [7, Proposition 2.5], all such $\omega_0$ coincides with
the unique Kähler–Einstein metric on $X \setminus \text{supp}(D_{\beta})$ with mixed cusp and cone singularities along $D_{\beta}$. Hence we have shown the locally strong and globally weak convergence of $\omega_{\phi_{\beta}}$ to a Kähler–Einstein mixed cusp and edge metric as $\beta_i \to 0$ for $i = 1, \ldots, t$ and $\beta_j \to d_j$ for $j = t + 1, \ldots, r$. \qed

3. Asymptotic behavior near the divisors in the small angle limit

Theorem 2.11 only gives us the smooth convergence of $\omega_{\phi_{\beta}}$ to a Kähler–Einstein mixed cusp and edge metric away from the divisor when cone angles approach 0. In this section, we study the asymptotic behavior of $\omega_{\phi_{\beta}}$ near $D$ when some of the cone angles tend to 0. More precisely, consider a fixed point $p \in D_{\beta}$ with a holomorphic coordinate chart $(U, \{z_i\}_{i=1}^n)$ centered at $p$ such that $D_{\beta} \cap U = \{z_1 \cdots z_m = 0\}$, for $m \leq n$ and $D_j \cap U = \{z_j = 0\}$ for $j = 1, \ldots, m$. Let $\beta_i$ denote the cone angle along $D_i$ for each $i$. From now on, assume $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_m$. We allow other cone angles to tend to 0, but we always assume that $\beta_1$ goes to 0 in the fastest speed, i.e., $\beta_1/\beta_i \to +\infty$, for $i = 2, \ldots, m$.

3.1. A small neighborhood of $D_{\beta}$. By choosing an appropriate coordinate system [2, Lemma 4.1], whenever $D_{\beta}$ is locally given by $\{z_1 \cdots z_m = 0\}$, the reference metric $\Omega_{\beta}$ is equivalent to the following metric:

$$
\omega_{\beta, \text{mod}} := \frac{m}{2} \sum_{i=1}^m \frac{\beta_i^2 \sqrt{-1} dz_i \wedge d\bar{z}_i}{|z_i|^{2(1-\beta_i)}(1 - |z_i|^{2\beta_i})^2} + \sum_{j=m+1}^n \sqrt{-1} dz_j \wedge d\bar{z}_j. \tag{35}
$$

Thus, Theorem 2.6 tells us on $X \setminus \text{supp}(D_{\beta})$, there exists a uniform constant $C > 0$ such that

$$
C^{-1}\omega_{\beta, \text{mod}} \leq \omega_{\phi_{\beta}} \leq C\omega_{\beta, \text{mod}}. \tag{36}
$$

Thanks to (36), it is enough to consider $((\mathbb{C}^*)^m \times \mathbb{C}^{n-m}, \omega_{\beta, \text{mod}})$ when dealing with a small neighborhood of $D_{\beta}$ under the metric $\omega_{\phi_{\beta}}$. Let us fix a point $p \in D_{\beta}$. Let $(U, z_1, \ldots, z_n)$ be a holomorphic coordinate chart centered at $p$, such that $U \cap D_{\beta} = \{z_1 \cdots z_m = 0\}$ and $U \cap D_i = \{z_i = 0\}$ for $i = 1, \ldots, m$. Let $\mathbb{D} := \{|z_i| \leq 1, i = 1, \ldots, n\}$ be the unit polydisk. Then we claim that the distance function $d_{\beta}$ induced by the completion of $\omega_{\phi_{\beta}, \text{mod}}$ on $\mathbb{D}$ satisfies

$$
d_{\beta}(0, z) \simeq \frac{1}{2} \sum_{i=1}^m \frac{1}{\beta_i} \log \left( \frac{1 + |z_i|^{\beta_i}}{1 - |z_i|^{\beta_i}} \right) + \sum_{j=m+1}^n |z_j|, \quad z \in \mathbb{D}, \tag{37}
$$

where “$\simeq$” means “is equivalent up to a constant independent of $z$ to”. Indeed, $\frac{1}{2} \log \left( \frac{1 + x^{\beta_i}}{1 - x^{\beta_i}} \right)$ is the primitive of $\frac{\beta_i}{x^{1-\beta_i}(1 - x^{2\beta_i})}$, and (37) follows from this fact and (35). Summarizing the discussions above, it is enough to study the
polydisk in $\mathbb{C}^n$
\[
\left\{|z_i|^{\beta_i} < \frac{1 - e^{-2a}}{1 + e^{-2a}} \right\}, \quad i = 1, \ldots, m, z_j < a, j = m + 1, \ldots, n, \quad a > 0,
\]
when we study a neighborhood of $D_\beta$ given by the geodesic ball $B_{\omega_{\phi_\beta}}(p, a)$ centered at $p$ of radius $a$ with respect to the metric $\omega_{\phi_\beta}$.

### 3.2. The mixed cylinder and edge metric

In this section, we focus on a small neighborhood of $D_\beta$ and show that in a neighborhood of $D_\beta$, a renormalization of $\omega_{\phi_\beta, \text{mod}}$ locally converges to a mixed cylinder and edge metric (see Definition 3.1 below) in the $C^\infty$-sense.

**Definition 3.1.** A Kähler metric $\tilde{\omega}$ on $(\mathbb{C}^n)^m \times \mathbb{C}^{n-m}$ is called a mixed cylinder and edge metric if $\tilde{\omega}$ is quasi-isometric to the following metric:
\[
\omega_{\text{mix}} := \sum_{i=1}^{t} \frac{-1}{|z_i|^2} dz_i \wedge d\bar{z}_i + \sum_{j=t+1}^{m} \frac{\beta_j^2}{|z_j|^{2(1-\beta_j)}} dz_j \wedge d\bar{z}_j + \sum_{\ell=m+1}^{n} \frac{-1}{z_\ell} dz_\ell \wedge d\bar{z}_\ell,
\]
where $\beta_j \in (0, 1)$ for $j = t + 1, \ldots, m$.

Denote by $\mathbb{D}(a_1, \ldots, a_m, b)$ the set
\[
\{z \in (\mathbb{C}^n)^m \times \mathbb{C}^{n-m} : |z_i| < a_i, i = 1, \ldots, m, |z_j| < b, j = m + 1, \ldots, n\}.
\]
Let
\[
U_\beta := \mathbb{D}\left(e^{-\frac{1}{\beta_1}}, \left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{3\beta_2}}, \ldots, \left(\frac{\beta_1}{\beta_m}\right)^{\frac{1}{3\beta_m}}, 1\right).
\]
From section 3.1, one realizes $U_\beta$ as a neighborhood of $D_\beta$. We endow $U_\beta$ with $\frac{1}{\beta_1} \omega_{\phi_\beta, \text{mod}}$. Define a map
\[
\Psi_\beta : \mathbb{D}\left(e^{-\frac{1}{\beta_1}}, \left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{3\beta_2}}, \ldots, \left(\frac{\beta_1}{\beta_m}\right)^{\frac{1}{3\beta_m}}, 1\right) \to U_\beta = \mathbb{D}\left(e^{-\frac{1}{\beta_1}}, \left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{3\beta_2}}, \ldots, \left(\frac{\beta_1}{\beta_m}\right)^{\frac{1}{3\beta_m}}, 1\right)
\]
\[(w_1, \ldots, w_m, w_{m+1}, \ldots, w_n) \mapsto \left(e^{-\frac{1}{\beta_1}} w_1, \left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{3\beta_2}} w_2, \ldots, \left(\frac{\beta_1}{\beta_m}\right)^{\frac{1}{3\beta_m}} w_m, \beta_1 w_{m+1}, \ldots, \beta_1 w_n\right).
\]
On $\Psi^{-1}_\beta(U_\beta)$, the pull-back metric reads
\[
\psi_\beta^*(\frac{1}{\beta_1} \omega_{\phi_\beta, \text{mod}}) = \frac{e^{-2|w_1|^{2\beta_1}}}{(1 - e^{-2|w_1|^{2\beta_1}})^2} \sqrt{-1} dw_1 \wedge d\bar{w}_1 + \sum_{i=2}^{m} \frac{\sqrt{-1} dw_i \wedge d\bar{w}_i}{|w_1|^{2(1-\beta_i)}(1 - \frac{\beta_i^2}{\beta_1^2}|w_i|^{2\beta_i})^2} + \sum_{j=m+1}^{n} \sqrt{-1} dw_j \wedge d\bar{w}_j.
\]
Note that for $\omega_1, \omega_m, \omega_m+1, \omega_n) \in (C^*)^m \times C^{n-m}$, $\omega_i^{2\beta_i} \to 1$ as $\beta_i \to 0$ and $\beta_i |\omega_i|^{2\beta_i} \to 0$ as $\beta_i \to 0$. Moreover, for any compact subset $K \subset (C^*)^m \times C^{n-m}$, when $\beta_1$ is small enough, $K \subset \Psi_{\beta}^{-1}(U_{\beta})$. Hence we have indeed shown the following result.

**Lemma 3.2.** The pull-back of $\frac{1}{\beta_i \omega_{\beta,mod}}$ by $\Psi_{\beta}$ on any compact subset $K \subset (C^*)^m \times C^{n-m}$ converges to a mixed cylindrical and conical metric in $C^\infty(K)$ when $\beta_1 \to 0$ and $\beta_i$ does not converge to 0 for each $i = 2, \ldots, m$.

**Proof.** Summarizing the discussions above, $\Psi_{\beta}^*(\frac{1}{\beta_i \omega_{\beta,mod}})$ converges to

$$\frac{e^{-2}}{(1 - e^{-2})^2} \cdot \frac{\sqrt{-1}d\omega_1 \wedge d\bar{\omega}_1}{|\omega_1|^2} + \sum_{i=2}^m \frac{\sqrt{-1}d\omega_i \wedge d\bar{\omega}_i}{|\omega_i|^{2(1 - \beta_i)}} + \sum_{j=m+1}^n \frac{\sqrt{-1}d\omega_j \wedge d\bar{\omega}_j}{|\omega_j|^{2(1 - \beta_j)}} =: \omega,$$

which is a mixed cylinder and edge metric by Definition 3.1, in $C^\infty(K)$ as $\beta_1 \to 0$ and $\beta_i \to 0$, $\forall i = 2, \ldots, m$. \qed

### 3.3. The convergence of renormalized $\omega_{\phi, \beta}$ near $D_\beta$

For a Kähler metric $\xi$ on $\mathbb{C}^n$, let us denote $\xi := \Psi_{\beta}^*(\frac{1}{\beta_1^2} \xi)$. 

**Theorem 3.3.** Let $\{\beta_{1,k}\}_{k \in \mathbb{N}}$ be a sequence of positive numbers converging to 0. Assume further that $\lim_{k \to \infty} \beta_{1,k} > 0$ for each $i = 2, \ldots, r$. Assume all $\beta_{i,k} \in (0, \frac{1}{2}]$. Let $\omega_{\phi_{\beta,k}}$ be the (negatively curved) Kähler–Einstein crossing edge metric on $(X, D_k = \sum_{i=1}^r (1 - \beta_{i,k})D_i)$. Then there exists a subsequence of the metric spaces $(U_{\beta_{i,k}}, \frac{1}{\beta_{1,k}^2} \omega_{\phi_{\beta,k}})$ which converges in pointed Gromov–Hausdorff topology to $((C^*)^m \times C^{n-m}, \tilde{\omega}_\omega)$, where $\tilde{\omega}_\omega$ is a mixed cylindrical and conical metric. Indeed, a subsequence of $\tilde{\omega}_{\phi_{\beta,k}}$ converges in $C^\infty_{\text{loc}}((C^*)^m \times C^{n-m})$-topology to $\tilde{\omega}_\omega$.

**Proof.** First note that $\omega_{\phi, \beta}$ admits a potential function on $U_{\beta}$ since $\omega_{\beta,mod}$ admits one. Thus, $\omega_{\phi, \beta}$ admits a potential function on $\Psi_{\beta}^{-1}(U_{\beta})$, denoted by $\Psi_{\beta}$. The proof consists of three steps. We first deduce the $C^0$-estimate of $\Phi_{\beta}$ using Theorem 2.6. Then we derive the $C^{2, \alpha}$-estimates for $\Phi_{\beta}$ by the standard regularization arguments for Monge–Ampère equations. This combining with Arzela–Ascoli Theorem gives us a cluster value of the metrics. Finally, we use that smooth convergence to conclude the pointed Gromov–Hausdorff convergence as wanted.

**Step 1: $C^0$-estimates.**

Discussions in section 3.1 indicate that there exists a uniform constant $C > 0$ (independent of $\beta \in (0, \frac{1}{2}]$) such that

$$C^{-1} \omega_{\beta, \text{mod}} \leq \omega_{\phi_{\beta}} \leq C \omega_{\beta, \text{mod}}$$
on $U_\beta$. Thus

$$C^{-1}\Psi^r_\beta\left(\frac{1}{\beta_1^2}\omega_{\beta,\text{mod}}\right) \leq \hat{\omega}_{\phi, \beta} \leq C\Psi^r_\beta\left(\frac{1}{\beta_1^2}\omega_{\beta,\text{mod}}\right).$$

By Lemma 3.2, $\Psi^r_\beta\left(\frac{1}{\beta_1^2}\omega_{\beta,\text{mod}}\right)$ converges in $C^\infty(K)$ to $\hat{\omega}$ for any compact $K \subset (\mathbb{C}^*)^m \times \mathbb{C}^{n-m}$. Hence, there exists a constant $C_K > 0$ (independent of $\beta$) such that

$$C^{-1}_K \hat{\omega} \leq \hat{\omega}_{\phi, \beta} \leq C_K \hat{\omega}. \quad (40)$$

By $\Psi^r_\beta\left(\frac{1}{\beta_1^2}\omega_{\beta,\text{mod}}\right)$ converges in $C^\infty(K)$ to $\hat{\omega}$ for any compact $K \subset (\mathbb{C}^*)^m \times \mathbb{C}^{n-m}$. Hence, there exists a constant $C_K > 0$ (independent of $\beta$) such that

$$C^{-1}_K \hat{\omega} \leq \hat{\omega}_{\phi, \beta} \leq C_K \hat{\omega}. \quad (40)$$

By (40), $\hat{\omega}_{\phi, \beta}$ is uniformly bounded on $K$. Then by the weak compactness of positive currents and the equivalence between this and the $L^1$-convergence of potential functions, we can find a normalized potential function $\hat{\phi}_\beta$ such that it has a uniform $L^1_{\text{loc}}$ bound, hence uniform $L^p_{\text{loc}}$ bounds, for any $p > 1$. By (40), $\Delta \hat{\phi}_\beta$ is uniformly bounded. Thus by standard elliptic regularity results, $[5$, Theorem 8.17$]$, there exists a constant $C$ independent of $\beta$ such that

$$||\hat{\phi}_\beta||_{C^0(K)} \leq C, \quad \text{for } C = C(K). \quad (41)$$

**Step 2: Higher-order estimates and the smooth local convergence.**

Since $\omega_{\phi, \beta}$ satisfies the Kähler–Einstein equation outside $D_\beta$, $\hat{\omega}_{\phi, \beta}$ satisfies

$$\text{Ric} \hat{\omega}_{\phi, \beta} = -\beta_1^2 \hat{\omega}_{\phi, \beta} \text{ on } K. \quad (42)$$

Let $dV_{\text{eucl}}$ denote the Euclidean volume form on $(\mathbb{C}^*)^m \times \mathbb{C}^{n-m}$. Define

$$H_\beta := \log \frac{\omega_{\phi, \beta}^n e^{-\beta_1^2 \hat{\phi}_\beta}}{dV_{\text{eucl}}}. \quad (43)$$

$H_\beta$ is pluriharmonic by (42). By the definition of $H_\beta$,

$$(\sqrt{-1} \partial \bar{\partial} \hat{\phi}_\beta)^n = e^{\beta_1^2 \hat{\phi}_\beta + H_\beta} dV_{\text{eucl}}. \quad (43)$$

By (40),

$$||\beta_1^2 \hat{\phi}_\beta + H_\beta||_{C^0(K)} < +\infty. \quad (44)$$

Combining (41) and (44), we see $||H_\beta||_{C^0(K)} < +\infty$. Then by gradient estimates for pluriharmonic functions,

$$||H_\beta||_{C^k(K)} < C(k, K), \quad \text{where } C(k, K) \text{ only depends on } k \text{ and } K \text{ not on } \beta. \quad (45)$$

Define

$$\Phi : \phi \mapsto \log \frac{\sqrt{-1} \partial \bar{\partial} \phi^n e^{-\beta_1^2 \phi}}{dV_{\text{eucl}}}. \quad (46)$$

$\Phi$ is a uniform elliptic concave operator as a function of $\partial \bar{\partial} \phi$. Hence by the Evans–Krylov theory, $||\Phi||_{C^{2, \alpha}(K')} \text{ is controlled by } ||\hat{\phi}_\beta||_{C^0(K')}, ||\Delta \hat{\phi}_\beta||_{C^0(K')} \text{, and } ||\Phi(\hat{\phi}_\beta)||_{C^{0, \alpha}(K')} \text{ for some } K' \ni K$. Since $\Phi(\hat{\phi}_\beta) = H_\beta$, by (45), (41) and the fact
By standard bootstrapping arguments, every derivative of $\phi_{\beta}$ is uniformly bounded on $K$. Then Arzelà-Ascoli theorem indicates $(\phi_{\beta_k})_{k \in \mathbb{N}}$ has a convergent subsequence in $C^\infty(K)$-topology. Recall (42), letting $\beta_1 \to 0$ then due to the $C^\infty$-convergence above we get a cluster value $\omega_\infty$ such that

$$\text{Ric} \omega_\infty = 0.$$ 

By (40), $\omega_\infty$ is quasi-isometric to $\omega$, and therefore is a mixed cylinder and edge metric.

**Step 3: Pointed Gromov–Hausdorff convergence**

It remains to show a subsequence of $(U_{\beta_k}, \frac{1}{\beta_{i,k}} \omega_{\phi_{\beta_k}})$ converges in pointed Gromov-Hausdorff topology to $((C^*)^m \times C^{n-m}, \omega_\infty)$. Fix $q \in (C^*)^m \times C^{n-m}$ and fix a radius $a > 0$. First note that by construction, $B_{\omega_{\phi_{\beta_k}}}(q,a)$ is isometric to $B_{\frac{1}{\beta_{i,k}} \omega_{\phi_{\beta_k}}}(\Psi_{\beta_k}(q),a)$. Secondly, by letting the index $k \in \mathbb{N}$ be large enough, we have $B_{\omega_\infty}(q,2a) \subset \Psi^{-1}(U_{\beta_k})$. Finally, due to the local $C^\infty$-convergence, $B_{\omega_{\phi_{\beta_k}}}(q,a) \subset B_{\omega_\infty}(q,2a)$ and $B_{\omega_{\phi_{\beta_k}}}(q,a)$ converges to $B_{\omega_\infty}(q,a)$ in the Gromov-Hausdorff topology. Therefore $(U_{\beta_k}, \frac{1}{\beta_{i,k}} \omega_{\phi_{\beta_k}})$ converges (up to a subsequence) in pointed Gromov-Hausdorff topology to $((C^*)^m \times C^{n-m}, \omega_\infty)$ by [1, Definition 8.1.1].

If we further allow more than one cone angles converge to 0, then we have the following results by modifying the result of Lemma 3.2.

**Theorem 3.4.** Let $\{\beta_{1,k}\}_{k \in \mathbb{N}}$ be a sequence of positive numbers converging to 0. Assume further that for any $i = 2, \ldots, r$ such that $\{\beta_{i,k}\}_{k \in \mathbb{N}}$ also converges to 0, there holds $\lim_{k \to \infty} \beta_{i,k} \in [0,1]$. Assume $\beta_{i,k} \in (0, \frac{1}{2}]$ for $i = 1, 2, \ldots, r$ and all $k \in \mathbb{N}$. Let $\omega_{\phi_{\beta_k}}$ be the (negatively curved) Kähler–Einstein crossing edge metric on $(X, D_k = \sum_{i=1}^{r} (1 - \beta_{i,k})D_i)$. Then there exists a subsequence of the metric spaces $(U_{\beta_k}, \frac{1}{\beta_{i,k}} \omega_{\phi_{\beta_k}})$ converging in pointed Gromov-Hausdorff topology to $((C^*)^m \times C^{n-m}, \omega_\infty)$, where $\omega_\infty$ is a mixed cylinder and edge metric with cylindrical part along components whose cone angles converge to 0 and conical part along other components.

**Proof.** Without loss of generality, assume

$$\lim_{k \to \infty} \beta_{i,k} = 0, \quad \text{for } i = 1, \ldots, t, t < m, \quad (46)$$

$$\lim_{k \to \infty} \beta_{j,k} := \beta_{j,\infty} > 0, \quad \text{for } j = t + 1, \ldots, m. \quad (47)$$
Moreover, assume
\[
\lim_{k \to \infty} \frac{\beta_{1,k}}{\beta_{\ell,k}} = c_{\ell} \in (0, 1), \quad \text{for } \ell = 1, \ldots, s, s < t, \quad (48)
\]
\[
\lim_{k \to \infty} \frac{\beta_{1,k}}{\beta_{\ell,k}} = 0, \quad \text{for } \ell = s + 1, \ldots, t. \quad (49)
\]
Recall in Lemma 3.2, we denote by $\mathbb{D}(a_1, \ldots, a_m, b)$ the set
\[
\{z \in (C^{*})^m \times \mathbb{C}^{n-m} : |z_i| < a_i, i = 1, \ldots, m, |z_j| < b, j = m + 1, \ldots, n\},
\]
and
\[
\Psi_{\beta} : \mathbb{D} \left( e^{\frac{i}{n+1}}, (\frac{\beta_1}{\beta_1})^{\frac{1}{n+1}}, \ldots, (\frac{\beta_m}{\beta_1})^{\frac{1}{n+1}}, \frac{1}{\beta_1} \right) \to U_{\beta} := \mathbb{D} \left( e^{\frac{i}{n+1}}, (\frac{\beta_1}{\beta_2})^{\frac{1}{n+1}}, \ldots, (\frac{\beta_1}{\beta_m})^{\frac{1}{n+1}}, 1 \right)
\]
\[
(w_1, \ldots, w_m, w_{m+1}, \ldots, w_n) \mapsto \left( e^{\frac{1}{n+1}} w_1, (\frac{\beta_1}{\beta_2})^{\frac{1}{n+1}} w_2, \ldots, (\frac{\beta_1}{\beta_m})^{\frac{1}{n+1}} w_m, \beta_1 w_{m+1}, \ldots, \beta_1 w_n \right).
\]
Now let us modify $\Psi_{\beta}$ by defining
\[
V_{\beta} := \mathbb{D} \left( e^{\frac{i}{n+1}}, e^{\frac{i}{n+2}}, \ldots, e^{\frac{i}{n+1}}, (\frac{\beta_1}{\beta_1})^{\frac{1}{n+1}}, \ldots, (\frac{\beta_1}{\beta_m})^{\frac{1}{n+1}}, 1 \right),
\]
and
\[
\Phi_{\beta} : \mathbb{D} \left( e^{\frac{i}{n+1}}, e^{\frac{i}{n+2}}, \ldots, e^{\frac{i}{n+1}}, (\frac{\beta_1}{\beta_1})^{\frac{1}{n+1}}, \ldots, (\frac{\beta_1}{\beta_m})^{\frac{1}{n+1}}, \frac{1}{\beta_1} \right) \to V_{\beta},
\]
\[
\Phi_{\beta}(w_1, \ldots, w_s, w_{s+1}, \ldots, w_m, w_{m+1}, \ldots, w_n) = \left( e^{-\frac{1}{n+1}} w_1, e^{-\frac{1}{n+2}} w_2, \ldots, e^{-\frac{1}{n+1}} w_s, (\frac{\beta_1}{\beta_{s+1}})^{\frac{1}{n+1}} w_{s+1}, \ldots, (\frac{\beta_1}{\beta_m})^{\frac{1}{n+1}} w_m, \beta_1 w_{m+1}, \ldots, \beta_1 w_n \right).
\]
Then
\[
\Phi_{\beta} \left( \frac{1}{\beta_1^2} \omega_{\beta, \text{mod}} \right) = \sum_{i=1}^{s} \sum_{j=1}^{m} \sum_{k=1}^{n} \frac{|w_i|^{2\beta_i}}{\beta_i^2} \frac{\sqrt{-1} dw_i \wedge d\bar{w}_i}{|w_i|^2} 
+ \sum_{j=m+1}^{n} \frac{\sqrt{-1} dw_j \wedge d\bar{w}_j}{|w_j|^2} 
+ \sum_{\ell=s+1}^{m} \frac{\sqrt{-1} dw_\ell \wedge d\bar{w}_\ell}{|w_\ell|^2}.
\]
Denote
\[
\beta_k := (\beta_{1,k}, \ldots, \beta_{r,k}), \quad \text{for } k \in \mathbb{N}.
\]
For a compact $K \subset (\mathbb{C}^*)^m \times \mathbb{C}^{n-m}$, there exists a large enough $k$ such that $K \subset \Phi_{\beta_k}^{-1}(V_{\beta_k})$. By assumptions (46) and (48), $\Phi_{\beta_k}^* \left( \frac{1}{\beta_k^2} \omega_{\beta_k,\text{mod}} \right)$ converges in $C^\infty(K)$ to

$$\sum_{i=1}^s \frac{e^{-2}}{c_i^2} \frac{1}{|w_i|^2} \sqrt{-1} dw_i \wedge d\bar{w}_i + \sum_{j=s+1}^t \frac{1}{|w_j|^2} \sqrt{-1} dw_j \wedge d\bar{w}_j + \sum_{\ell=t+1}^m \frac{1}{|w_\ell|^{2(1-\beta_{\ell,\infty})}} \sqrt{-1} dw_\ell \wedge d\bar{w}_\ell + \sum_{q=m+1}^n \frac{1}{|w_q|^{2(1-\beta_{q,\infty})}} \sqrt{-1} dw_q \wedge d\bar{w}_q =: \tilde{\omega},$$

which is a mixed cylindrical and conical metric by Definition 3.1. The cylindrical parts are along $D_i$, $i = 1, \ldots, t$, whose cone angle goes to 0. The remainder of the proof is similar to that of Theorem 3.3.

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