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The sigma invariants for the golden mean Thompson group

Lewis Molyneux, Brita Nucinkis and Yuri Santos Rego

ABSTRACT. We use a method of Bieri, Geoghegan and Kochloukova to calculate the BNSR-invariants for the irrational slope Thompson’s group \( F_\tau \). To do so we establish conditions under which the Sigma invariants coincide with those of a subgroup of finite index, addressing a problem posed by Strebel.

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1. Introduction

The study of what is now known as the first Sigma invariant or the Bieri–Neumann–Strebel invariant \( \Sigma^1(G) \) for a finitely generated group \( G \) goes back to [5, 3] and was later extended by Bieri and Renz to a sequence of homotopical invariants

\[
\cdots \subseteq \Sigma^n(G) \subseteq \Sigma^{n-1}(G) \subseteq \cdots \subseteq \Sigma^1(G) \subseteq S(G)
\]

and homological invariants

\[
\cdots \subseteq \Sigma^n(G, R) \subseteq \Sigma^{n-1}(G, R) \subseteq \cdots \subseteq \Sigma^1(G, R) \subseteq S(G),
\]

where \( R \) is a commutative ring; cf. [22, 4].

In this note, we compute the Sigma invariants for the Golden Mean Thompson group \( F_\tau \) defined by Cleary in [10], see also [9]. We prove:

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Theorem 1.1. Let \( \lambda, \rho : F_\tau \to \mathbb{R} \) be the characters given by:

\[
\lambda(f) = \log_\tau(f'(0)) \quad \text{and} \quad \rho(f) = \log_\tau(f'(1)).
\]

Then the Sigma invariants of \( F_\tau \) are as follows:

1. \( \Sigma^1(F_\tau) = \Sigma^1(F_\tau, \mathbb{Z}) = S(F_\tau) \setminus \{[-\lambda], [-\rho]\} \), and
2. \( \Sigma^\infty(F_\tau) = \Sigma^\infty(F_\tau, \mathbb{Z}) = \Sigma^2(F_\tau) = \Sigma^1(F_\tau) \setminus \{[-a\lambda - b\rho] \mid a, b > 0\} \).

Note that \( \Sigma^1(F_\tau) \) was already known, see Citation 1.3 below. The computation of \( \Sigma^1 \) and higher Sigma invariants is of interest for various topological reasons; see, for instance, [3, 4, 21, 2, 24, 15, 14, 19]. Particularly, we obtain the following information about coabelian subgroups of \( F_\tau \).

Corollary 1.2. Let \( N \trianglelefteq F_\tau \) be a normal subgroup of homological type \( \text{FP}_2 \) for which the quotient \( F_\tau / N \) is abelian. Then \( N \) is of homotopical type \( \text{FP}_\infty \).

Proof. Immediate from Theorem 1.1, [22, Satz C] and [4, Theorem B]. \qed

Theorem 1.1 confirms that, similarly to the case of R. Thompson’s original group \( F \) [2], the Sigma invariants of \( F_\tau \) are determined by an integral polytope (in the sense of [14]). The same behaviour is seen in other Thompson groups that ‘resemble’ \( F \) (e.g., [2, 25, 26, 19]), though not all of them; see [23].

While no unexpected phenomenon for the Sigma invariants of \( F_\tau \) is observed, their computation slightly diverges from those in the above mentioned works. More precisely, as a first step we consider the behaviour of the Sigma invariants \( \Sigma^n(G) \) under passage to subgroups of finite index — which, to our knowledge, was not needed so far for other Thompson groups. Using this, the computations for the Sigma invariants for \( F_\tau \) then follow from methods similar to those of Bieri–Geoghegan–Kochloukova in [2].

Throughout the paper, we denote by \( G \) a finitely generated group and by \( G_{ab} \cong H_1(G; \mathbb{Z}) \) its abelianisation. We consider nontrivial characters \( \chi \in \text{Hom}(G, \mathbb{R}) \cong H^1(G; \mathbb{R}) \). Define an equivalence relation by \( \chi \sim \chi' \) if and only if there exists an \( a \in \mathbb{R}_{>0} \) such that \( \chi = a\chi' \). The set of equivalence classes is a sphere in \( \mathbb{R}^n \), called the character sphere \( S(G) \). Its dimension is determined by the torsion-free rank \( r_0(G_{ab}) \) of \( G_{ab} \) (equivalently, the first Betti number \( b_1(G) \) of the group \( G \)) and given by \( r_0(G_{ab}) - 1 \); see [6, Lemma 1.1]. Now consider the following subset of the Cayley graph \( \Gamma(G) \) with respect to some finite generating set: \( \Gamma_\chi(G) \) is the subgraph of \( \Gamma(G) \) consisting of those vertices with \( \chi(g) > 0 \), and edges that have both initial and terminal vertices in \( \Gamma_\chi(G) \). The first homotopical Sigma invariant is now defined as

\[
\Sigma^1(G) = \{[\chi] \in S(G) \mid \Gamma_\chi(G) \text{ is connected}\}.
\]

Note that this is independent of the choice of finite generating set for the Cayley graph [3]. For certain groups of homeomorphisms of the real line, including Thompson’s group \( F \) and the Golden Mean Thompson’s group \( F_\tau \), we have a complete description of \( \Sigma^1(G) \):
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Citation 1.3 ([6, Chapter IV, Corollary 3.4]). Let $G$ be an irreducible subgroup of the group of piecewise linear homeomorphisms of the interval $[0, 1]$. Take the characters $\chi_1(g) = \ln(g'(0))$ as the natural log of the right derivative of an element $g \in G$ at 0 and $\chi_2(g) = \ln(g'(1))$ as the natural log of the left derivative of that element at 1. If $G = \ker \chi_1 \cdot \ker \chi_2$, then $\Sigma^1(G)^c = \{[\chi_1], [\chi_2]\}$.

In the 1990s, Bieri and Strebel gave a formula to compute the complement $\Sigma^1(G)^c$ using $\Sigma^1(H)^c$ and a subsphere of $S(H)$ in case $H$ is a subgroup of finite index in $G$; see [6, Chapter III, Proposition 2.9] and [24, Proposition B1.11]. In higher dimensions, a related formula was recently considered by Koban–Wong in [15]. In his notes [24, Section B1.2c], Strebel goes on to wonder about the applicability of this formula, and poses the following.

Citation 1.4 ([24, Problem B1.13]). Find situations where one is interested in $\Sigma^1(G)$ with $G$ admitting a subgroup of finite index which is easier to deal with and for which $\Sigma^1$ can be computed.

We give a positive contribution towards Strebel's problem and find a sufficient condition for 'equality' of Sigma invariants with those of subgroups of finite index.

Theorem 1.5. Let $G$ be a group of type $F_n$ with $H \leq G$ a subgroup of finite index and write $\iota : H \hookrightarrow G$ for the inclusion. If $r_0(G_{ab}) = r_0(H_{ab})$, then $\iota^* : S(G) \to S(H)$ is a well-defined homeomorphism and for all $n$ it holds

$$\iota^*(\Sigma^n(G)) = \Sigma^n(H).$$

Theorem 1.6. Let $A$ be a $\mathbb{Z}G$-module of type $F^n$. Suppose $H \leq G$ is a subgroup of finite index and write $\iota : H \hookrightarrow G$ for the inclusion. If $r_0(G_{ab}) = r_0(H_{ab})$, then $\iota^* : S(G) \to S(H)$ is a well-defined homeomorphism and for all $n$ it holds

$$\iota^*(\Sigma^n(G, A)) = \Sigma^n(H, A).$$

Recalling the definition of $\iota^*$, any character $\chi \in \text{Hom}(G, \mathbb{R})$ can be restricted to a character of $H$, and we set $\iota^*(\chi) = \chi|_H \in \text{Hom}(H, \mathbb{R})$. In general, this map does not induce a function between character spheres. Thus, the above statements also mean that the assignment $\iota^*(|\chi|) = |\chi|_H$ can be made on the level of character spheres, and we abuse notation also denoting this map by $\iota^* : S(G) \to S(H)$. We refer the reader to Section 3 for the proofs of Theorems 1.5 and 1.6 — the main issue, as should be known to experts, is whether characters of the subgroup $H$ can be extended to characters of the whole group $G$. Examples 1.7, 1.8, and 3.2 illustrate how the equalities $\iota^*(\Sigma^n(G)) = \Sigma^n(H)$ and $\iota^*(\Sigma^n(G, A)) = \Sigma^n(H, A)$ can fail.

We note that the problems of extending characters and of computing Sigma invariants from those of given subgroups appear in various guises in the literature; see, for example, [6, 21, 15, 17, 13, 18]. However, we were unable to find explicit references of statements along the lines of Theorems 1.5 and 1.6. We will make use of the homological result in [21] (included here as Citation 3.5 in Section 3) during our proof of Theorem 1.6.
Expanding on some related work, the authors in [15] study Sigma invariants for finite-index normal subgroups \( N \triangleleft G \), obtaining the image of \( \Sigma^n(G) \) as an intersection of \( \Sigma^n(N) \) with a certain subset of \( \text{Hom}(N, \mathbb{R}) \) invariant under a \( G/N \)-action. In [17], extensions of characters from coabelian normal subgroups play a central role. More recently in [13, 18], the authors give conditions under which one can extend a character from certain normal subgroups of infinite index. Our formulation of Theorems 1.5 and 1.6 gives a simple, easy-to-check condition on the Sigma invariants for (not necessarily normal) finite-index subgroups. We also remark that, over \( \mathbb{Z} \) or a field, Theorem 1.6 can be alternatively proved using techniques from Novikov homology and a recent generalisation of Sikorav’s theorem due to Fisher [12]; see Remark 3.6. Our proof, in turn, uses only elementary methods.

We stress that neither the equality \( b_1(G) = r_0(G) = r_0(H) = b_1(H) \) nor finite index alone suffice as hypotheses, as the following examples show.

**Example 1.7.** Note that \( r_0(G) = r_0(H) \) is insufficient to show an embedding of character spheres via \( \iota^* \). As a counterexample, consider Thompson’s original group \( F = \langle x_0, x_1, \ldots | x_i^{-1}x_jx_i = x_{j+1} \text{ for } 0 \leq i < j \rangle \) and the subgroup \( F[1] = \langle x_1, x_2, \ldots | x_i^{-1}x_jx_i = x_{j+1} \text{ for } 1 \leq i < j \rangle \). Clearly \( F \cong F[1] \), and so \( r_0(F) = r_0(F[1]) = 2 \). But any character \( \chi \in \text{Hom}(F, \mathbb{R}) \) with \( \chi(x_1) = 0 \) restricts to the trivial character on \( F[1] \), and all other character classes in \( S(F) \) restrict to \( [\pm \chi_1] \), where \( \chi_1(x_0) = 0, \chi_1(x_1) = 1 \). Hence \( \iota^* \) is only defined on a proper subset of \( S(F) \), and the character classes in \( \Sigma^n(F) \) on which \( \iota^* \) is defined are mapped to a proper subset of \( \Sigma^n(F[1]) \).

**Example 1.8.** Similarly, \( |G : H| < \infty \) alone does not guarantee the existence of a bijection between Sigma invariants of \( G \) and \( H \). For instance, the infinite dihedral group \( D_{\infty} \cong \mathbb{Z} \rtimes C_2 \) contains \( \mathbb{Z} \) as a subgroup of index two. While \( S(\mathbb{Z}) = \Sigma^1(\mathbb{Z}) \) is the 0-sphere (and thus consists of two points), one has that \( S(D_{\infty}) \) — thus also \( \Sigma^n(D_{\infty}) \) — is empty as the abelianisation of \( D_{\infty} \) is finite.

Also note that the implications in Theorems 1.5 and 1.6 cannot be reversed:

**Example 1.9.** Let \( F_n \) denote the free group on \( n \) letters. It is known, see [6, Proposition III.4.5], that \( \Sigma^1(F_n) = \emptyset \) for all \( n \geq 2 \). Furthermore, \( F_n \) embeds with finite index in \( F_2 \) [20, Proposition I.3.9]. However, the torsion-free ranks of these groups are not equal as long as \( n > 2 \).

We begin by establishing facts about both the Sigma invariants and \( F_\tau \). In Section 3 we prove Theorems 1.5 and 1.6. And finally, in Section 4 we compute the Sigma invariants for \( F_\tau \).

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2. Background

2.1. Higher homotopical sigma invariants. We will begin with recalling some general definitions and facts that can be found, for example, in [11]. An Eilenberg–MacLane space, denoted $K(G,1)$, is an aspherical CW-complex $Y$ with $\pi_1(Y) = G$. Its universal cover $X$ is contractible and has $G$ acting freely by deck transformations. Such a space is also called a model for $EG$ and is unique up to $G$-homotopy. A group $G$ is said to be of type $\text{F}_n$ if there is a model for $EG$ with finite $n$-skeleton modulo the $G$-action, in which case we also say that this model has $G$-finite $n$-skeleton. Finally, $G$ is said to be of type $\text{F}_\infty$ if it is of type $\text{F}_n$ for all $n \in \mathbb{N}$.

From now on, let $G$ be of type $\text{F}_n$ and let $X$ be a model for $EG$ with $G$-finite $n$-skeleton. The following construction is due to Renz [22, Kapitel II, Abschnitt 2], see also [6, Appendix B, Section B1.1]: For a given character $\chi \in \text{Hom}(G, \mathbb{R})$, one defines an action of $G$ on $\mathbb{R}$ by $g \cdot r = r + \chi(g)$ for all $g \in G$ and $r \in \mathbb{R}$, which can be extended to a corresponding continuous $G$-equivariant map $h_\chi : X \rightarrow \mathbb{R}$, also called a height function. Any such height function gives rise to an $\mathbb{R}$-filtration of $X$ given by the closed subspaces $h_\chi^{-1}([r, \infty))$. We shall consider $X_{h_\chi}^{[r,+\infty)}$, defined as the largest subcomplex of $X$ such that

$$x \in X_{h_\chi}^{[r,+\infty)} \implies h_\chi(x) \in [r, +\infty).$$

When considering $X_{h_\chi}^{(0,+\infty)}$, we shall use the notation $X_{h_\chi}$.

**Definition 2.1** ([22, Kapitel II, Definition 3.4] or [6, Appendix B, Definition in p. 194]). Let $G$ be of type $\text{F}_n$. Then the $n$-th Sigma invariant $\Sigma^n(G) \subseteq S(G)$ is defined as follows: $[\chi] \in \Sigma^n(G)$ if there exists a model $X$ for $EG$ with $G$-finite $n$-skeleton and a corresponding height function $h_\chi$ on $X$ such that $X_{h_\chi}$ is $(n - 1)$-connected.

There are a priori different ways of extending the character $\chi$ to a $G$-equivariant height function $h_\chi$, though Renz shows that this distinction is immaterial and $\Sigma^n(G)$ is well-defined; cf. [22, Kapitel II, Bemerkungen 3.5]. This allows us to write $h$ instead of $h_\chi$ for an admissible height function extending a character $\chi$, if no confusion arises.

While the connectivity condition in Definition 2.1 might not hold for every model of $EG$ with $G$-finite $n$-skeleton, Renz [22] also showed that the model may be arbitrary if one considers essential connectivity instead.

**Definition 2.2** ([22, Kapitel II, Definition 3.6] or [6, Appendix B, Section B1.2]). For $X_{h}^{[r,+\infty)}$ as defined above, we say that $X_{h}^{[r,+\infty)}$ is essentially $k$-connected for $k \in \mathbb{Z}_{\geq -1}$ if there is a real number $d \geq 0$ such that the map $i_j : \pi_j(X_{h}^{[r,+\infty)}) \rightarrow \mathbb{Z}$ is...
\( \pi_j(X_h^{[r-d, +\infty]}) \) induced by the inclusion \( \iota : X_h^{[r, +\infty]} \hookrightarrow X_h^{[r-d, +\infty]} \) is the zero map for all \( j \leq k \).

**Citation 2.3** ([22, Kapitel IV, Satz 3.4] or [6, Appendix B, Theorem B1.1]). Let \( G \) be a group of type \( F_n \) and let \( X \) be an arbitrary model for \( EG \) with \( G \)-finite \( n \)-skeleton. Let \( \chi : G \rightarrow \mathbb{R} \) be a nontrivial character and \( h : X \rightarrow \mathbb{R} \) a corresponding height function as above. Then

\[
[\chi] \in \Sigma^n(G) \iff X_h \text{ is essentially } (n - 1)\text{-connected}.
\]

### 2.2. The homological invariant \( \Sigma^n(G, A) \)

We will now give a brief overview of the definition and essential properties of the homological invariants \( \Sigma^n(G, A) \), where \( A \) is a \( \mathbb{Z}G \)-module, see [4]. We follow the convention of Bieri, Renz, and Strebel [1, 22, 4, 6] of working with left modules. In particular, a group or monoid acting on a module acts on the left.

**Definition 2.4** ([7, Chapter VIII.4]). Given a unital ring \( R \), a (left) \( R \)-module \( A \) is said to be of type \( FP_n \) over \( R \) if it admits a resolution of the form

\[
P : \ldots \rightarrow P_i \rightarrow P_{i-1} \rightarrow \ldots \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0
\]

where the \( P_i \) are free (left) \( R \)-modules which are finitely generated for \( i \leq n \). In case \( G \) is a group or monoid, we say that \( G \) is of type \( FP_n \) if the trivial \( \mathbb{Z}G \)-module \( A = \mathbb{Z} \) is of type \( FP_n \) over \( R = \mathbb{Z}G \).

One can analogously define '(right) type \( FP_n \)', i.e., using right actions and right modules, and a group being of type \( FP_n \) does not depend on whether one works from the left or right; cf. [1]. However, the same is not true in the case of monoids (see, for instance, [16]), whence the importance of fixing a convention for the actions when working with both groups and monoids.

**Definition 2.5** ([4, Section 1.3]). Let \( G \) be a group and \( A \) a \( \mathbb{Z}G \)-module of type \( FP_n \). The \( n \)-th homological Sigma invariant \( \Sigma^n(G, A) \subseteq S(G) \) is defined as follows:

\[
[\chi] \in \Sigma^n(G, A) \iff A \text{ is of type } FP_n \text{ over the subring } \mathbb{Z}G_{\chi} \subseteq \mathbb{Z}G,
\]

where \( G_{\chi} \) is the submonoid \( G_{\chi} = \{ g \in G \mid \chi(g) \geq 0 \} \).

Now let \( G \) be a group of type \( F_n \). Then (cf. [22, Satz B] and also [4]) it holds:

\[
\Sigma^1(G) = \Sigma^1(G; \mathbb{Z})
\]

\[
\Sigma^n(G) = \Sigma^2(G) \cap \Sigma^n(G; \mathbb{Z}) \text{ for } n \geq 2
\]

(1)

Similarly to the homotopical case, it was shown that the definition of \( \Sigma^n(G; A) \) does not depend on the partial finitely generated free resolution of the \( \mathbb{Z}G \)-module \( A \), see [4, Theorem 3.2].
2.3. Background on the golden mean Thompson group $F_\tau$. Let $\tau$ denote the small Golden Ratio, that is, the positive solution $\tau = \frac{\sqrt{5} - 1}{2}$ to the equation $x^2 + x = 1$.

Definition 2.6 ([10]). The group $F_\tau$ is defined as the subgroup of piecewise linear, orientation-preserving homeomorphisms of the interval $[0, 1]$ with slopes in the group $(\tau)$ and breakpoints in $\mathbb{Z}[\tau]$.

Citation 2.7 ([9, Theorem 4.4]). $F_\tau$ has the (infinite) presentation

$$F_\tau \cong \langle x_i, y_i \mid a_j b_i = b_i a_{j+1}, y_i^2 = x_i x_{i+1} ; a, b \in \{x, y\}, 0 \leq i < j \rangle. \tag{2}$$

In the above, $i, j \in \mathbb{Z}_{\geq 0}$. We can write the generators of $F_\tau$ as functions on the interval $[0, 1]$ in the following forms:

$$x_i(n) = \begin{cases} n & \text{for } 0 \leq n \leq 1 - \tau^i, \\ \tau n + \tau^2 & \text{for } 1 - \tau^i \leq n \leq 1 - \tau^i + \tau^{i+4}, \\ n + \tau^{i+3} & \text{for } 1 - \tau^i + \tau^{i+4} \leq n \leq 1 - \tau^{i+1}, \\ \tau n + \tau^2 & \text{for } 1 - \tau^{i+1} \leq n \leq 1, \end{cases} \tag{3}$$

$$y_i(n) = \begin{cases} n & \text{for } 0 \leq n \leq 1 - \tau^i, \\ \tau^{-1} n - \tau^{-1}(1 - \tau^i) & \text{for } 1 - \tau^i \leq n \leq 1 - \tau^{i+1}, \\ \tau n + \tau^2 & \text{for } 1 - \tau^{i+1} \leq n \leq 1. \end{cases}$$

These elements can also be understood as equivalence classes of ordered tree-pairs, as described in [9, Section 4]. As for the original Thompson group $F$, the elements of $F_\tau$ have a unique normal form [9, Theorem 7.3]. We shall use the following normal form:

Citation 2.8 ([9, Section 7]). Any element $f \in F_\tau$ can be uniquely expressed in the form

$$f = x_{i_0} y_{j_0} x_{i_1} y_{j_1} \cdots x_{i_n} y_{j_n} x_{i_{n-1}} \cdots x_{i_0},$$

where $i_k, j_k \in \mathbb{Z}_{\geq 0}$, $\varepsilon_k \in \{0, 1\}$, $0 \leq k \leq n$, and moreover the following hold for all $k$:

1. If $i_k \neq 0 \neq j_k$, then at least one of $i_{k+1}$, $j_{k+1}$, $\varepsilon_{k+1}$ is nonzero;
2. In case $f$ contains a subword of the form $x_k y_k x_{k+2} u x_{k+1} x_k^{-1}$, then the middle subword $u$ contains a generator indexed either by $k + 1$ or $k + 2$.

Like $F$, the group $F_\tau$ also enjoys the strong homotopical and homological finiteness properties.

Citation 2.9 ([10]). The Golden Mean Thompson group $F_\tau$ is of type $\mathcal{F}_\infty$.

3. Sigma invariants and finite index

In this section, we prove Theorems 1.5 and 1.6. We begin by discussing, for $H \leq G$, maps between $H^1(H; \mathbb{R}) \cong \text{Hom}(H, \mathbb{R})$ and $H^1(G; \mathbb{R}) \cong \text{Hom}(G, \mathbb{R})$. 
Lemma 3.1. Suppose $G$ is a finitely generated group, let $H \leq G$, and write $\pi : G \to G_{ab}$ for the canonical projection and $\iota : H \hookrightarrow G$ for the inclusion. Then the following hold.

1. If $|G : H| < \infty$, then the map $\iota^* : \text{Hom}(G, \mathbb{R}) \to \text{Hom}(H, \mathbb{R})$ induced by the inclusion is injective.

2. If the image $\pi(H)$ is infinite, then there exists a nontrivial morphism $e : \text{Hom}(H, \mathbb{R}) \to \text{Hom}(G, \mathbb{R})$. That is, any character $\psi$ of $H \leq G$ gives rise to a character $e(\psi)$ of $G$ and the image $e(\text{Hom}(H, \mathbb{R})) \subseteq \text{Hom}(G, \mathbb{R})$ is a nonzero subspace.

Lemma 3.1(2) was observed by Kochloukova–Vidussi; cf. [18, Proof of Theorem 1.1]. Kochloukova and Vidussi work with characters in $G$ that are already assumed to be extensions of characters of a subgroup $H \leq G$. However, in the form we state Lemma 3.1, the character $e(\psi) \in \text{Hom}(G, \mathbb{R})$ need not be a valid extension of the original character $\psi \in \text{Hom}(H, \mathbb{R})$. That is, it might be the case that $\iota^* e(\psi) \neq \psi$; see Example 3.2 below.

From now on, when working in the abelianisation of a group, we will write the group operation additively.

Proof. Part (1): Take a nonzero character $\chi \in \text{Hom}(G, \mathbb{R})$ and suppose that $\iota^*(\chi) = \chi|_H = 0$. As $\chi(G) \neq 0$, there exists $g \in G$ such that $\chi(g) \neq 0$, but as $\chi(H) = 0$ one has $g \not\in H$. Furthermore, we can say $g^n \not\in H$ for all $n \in \mathbb{N}$, as

$$g^n \in H \implies \chi(g^n) = 0$$

$$\iff n \chi(g) = 0$$

$$\iff \chi(g) = 0,$$

contradicting $\chi(g) \neq 0$. Thus $g^nH$ are all distinct cosets of $H$, which means that $H$ is not finite index, contradicting our assumption. Hence, $\chi(H) \neq 0$.

Part (2): Consider the (finite dimensional) $\mathbb{Q}$-vector space $V = G_{ab} \otimes_{\mathbb{Z}} \mathbb{Q}$. Since the image $\pi(H) \subseteq G_{ab}$ is infinite, the set $\pi(H)$ contains some torsion-free element and thus $\pi(H) \otimes_{\mathbb{Z}} \mathbb{Q}$ contains a partial basis for $V$, say $\mathcal{B}' = \{\overline{h}_1, ... , \overline{h}_m\}$, where each $\overline{h}_i$ is the image in $G_{ab}$ of some $h_i \in H$. Extend this to a basis $\mathcal{B} = \{\overline{h}_1, ... , \overline{h}_m, \overline{g}_{m+1}, ... , \overline{g}_r\}$ of $V$, again with $\overline{g}_j$ being the image of some $g_j \in G$. Since the image of characters of a group factors through their abelianisation, we may define

$$e(\psi)(g) := \sum_{i=1}^{m} a_i \psi(h_i),$$

where the $a_x$ with $x \in \mathcal{B}$ are the coordinates of the image of $g$ in $G_{ab} \otimes_{\mathbb{Z}} \mathbb{Q}$ in the basis $\mathcal{B}$. It is straightforward to check that $e$ is a homomorphism from $\text{Hom}(H, \mathbb{R})$ to $\text{Hom}(G, \mathbb{R})$. Again because $\pi(H) \subseteq G_{ab}$ is infinite and $G$ is finitely generated, the induced map $H \to \pi(H) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^m$ gives a nontrivial character, call it $\psi \in \text{Hom}(H, \mathbb{R})$, by projecting onto the line spanned by a
nonzero vector of \( \pi(H) \otimes \mathbb{Z} \mathbb{R} \). By construction, the character \( e(\psi) \in \text{Hom}(G, \mathbb{R}) \) is also nontrivial.

**Example 3.2.** As mentioned above, the proof of Lemma 3.1(2) might yield an 'extension' of a character \( \psi \) of \( H \) such that \( r^0 e(\psi) \neq \psi \). For example, let \( H = \mathbb{Z} \times \mathbb{Z} \leq G = D_\infty \times \mathbb{Z} \) and take \( \psi \) to be a character of \( H \) which is nonzero on the first coordinate.

However, provided \( H \) is of finite index in \( G \) and their first Betti numbers agree, one can always construct a lift from \( \text{Hom}(H, \mathbb{R}) \) to \( \text{Hom}(G, \mathbb{R}) \) that circumvents these problems. We summarise these properties in the following.

**Proposition 3.3.** Let \( G \) be a finitely generated group and \( H \leq G \) a subgroup of finite index. Then the following are equivalent:

1. \( r_0(G) = r_0(H) \).
2. \( r^* : \text{Hom}(G, \mathbb{R}) \to \text{Hom}(H, \mathbb{R}) \) is an isomorphism of \( \mathbb{R} \)-vector spaces.
3. The assignment \( r^*([\chi]) := [\chi_H] \) is defined on all character classes \( [\chi] \in S(G) \), and the corresponding map \( r^* : S(G) \to S(H) \) is a homeomorphism.
4. Every character \( \chi \in \text{Hom}(H, \mathbb{R}) \) admits a lift \( \chi' \in \text{Hom}(G, \mathbb{R}) \) such that \( \chi'|_H = \chi \) and \( \chi \neq 0 \iff \chi' \neq 0 \).

**Proof.** The equivalences of (1), (2), and (3) are immediate from Lemma 3.1(1) as \( \text{dim}_{\mathbb{R}}(\text{Hom}(\Gamma, \mathbb{R})) = r_0(\Gamma) \) for any group \( \Gamma \). Item (4) is equivalent to (2) as the function \( e : \text{Hom}(H, \mathbb{R}) \to \text{Hom}(G, \mathbb{R}) \) given by \( e(\chi) = \chi' \) is a right inverse to \( r^* \).

**Example 3.4.** It is not hard to explicitly construct the 'extension map' \( e : \text{Hom}(H, \mathbb{R}) \to \text{Hom}(G, \mathbb{R}) \) of Proposition 3.3. Let \( \{x_1, \ldots, x_n\} \) be a generating set for \( G \) and write \( r_0(G) = r_0(H) = k \leq n \). Without loss of generality one can assume that \( \{x_1, \ldots, x_k\} \) generates \( G_{ab} \). Since \( |G : H| < \infty \), for each \( i = 1, \ldots, n \), there exists an \( \alpha_i \in \mathbb{N} \) such that \( x_i^{\alpha_i} \in H \). Hence, using functoriality of abelianisations, and the fact that \( x_i \) has infinite order in \( G_{ab} \), we have that \( 0 \neq x_i, x_i^\infty \in (H_{ab})_0 \) for all \( i = 1, \ldots, k \). Let \( \alpha = \text{lcm}(\alpha_1, \ldots, \alpha_k) \). Given a character \( \chi : H \to \mathbb{R} \), we define its lift \( e(\chi) = e' : G \to \mathbb{R} \) by

\[
\chi'(x_i) = \frac{1}{\alpha} \chi(x_i^{\alpha_i}), \text{ for all } i = 1, \ldots, n.
\]

To finish off Theorem 1.6, we make use of the following:

**Citation 3.5 ([21, Proposition 9.3]).** Suppose that \( H \leq G \) is a subgroup of finite index and \( A \) a \( \mathbb{Z}G \)-module of type \( \text{FP}_n \), and suppose that \( \chi : G \to \mathbb{R} \) restricts to a nonzero homomorphism of \( H \). Then

\[
[\chi]|_H \in \Sigma^n(H, A) \iff [\chi] \in \Sigma^n(G, A).
\]

**Proof of Theorem 1.6.** Immediate from Proposition 3.3 and Citation 3.5.
Remark 3.6. In case $A = \mathbb{Z}$ or a field, Theorem 1.6 can also be proved as follows: by change of rings [1] and noting that the Novikov ring $\hat{A}[G]$ is isomorphic to the tensor product $\hat{A}[H] \otimes A[H] A[G]$, an application of Proposition 3.3 combined with the equivalence $(1) \iff (5)$ from a result of Fisher [12, Theorem 5.3] proves the claim.

For completeness, we now give an elementary proof of the homotopical part, which needs the following.

Proposition 3.7. Let $G$ be a group of type $\mathbb{F}_n$ and $H$ a subgroup of finite index such that $r_0(G) = r_0(H)$. With the notation of Proposition 3.3 we have

$\left[ \chi \right] \in \Sigma^n(G) \implies \left[ \chi | H \right] \in \Sigma^n(H),$

and

$\left[ \chi \right] \in \Sigma^n(H) \implies \left[ \chi' \right] \in \Sigma^n(G).$

Proof. To prove the first claim, consider a model $X$ for $EG$ with $G$-finite $n$-skeleton. Now suppose $\left[ \chi \right] \in \Sigma^n(G)$, hence $X^{[0,+\infty]}_h$ is $(n-1)$-connected for the height function $h_\chi$ corresponding to $\chi$. Since $H$ is finite index in $G$, the space $X$ is also a model for $EH$ with $H$-finite $n$-skeleton, and $h_\chi = h_\chi | H$. Hence, $\left[ \chi | H \right] \in \Sigma^n(H)$.

Let us now assume $\left[ \chi \right] \in \Sigma^n(H)$. Again using $|G:H| < \infty$, choose a model for $X$ for $EH$ as above: $X$ is a simplicial complex with $G$-finite $n$-skeleton and one $G$-orbit of zero-cells labeled by $G$.

We now fix a set $T = \{t_0, \ldots, t_{m-1}\}$ of coset representatives of $H$ in $G$, put $t_0 = e$, and construct an $H$-equivariant height function $h_\chi : X \to \mathbb{R}$ on the vertices of $X$ as follows: For $y \in H$ we put $h_\chi(y) = \chi(y)$ and set $h_\chi(t_i) = 0$. Hence, since every $g \in G$ has a unique expression as $g = t_i y$, we get

$\chi'(g) = h_\chi(t_i) + h_\chi(y) = \chi(y).$

Finally, we extend this function piecewise linearly to the entire $n$-skeleton on $X$. Hence $X^{[0,+\infty]}_h$ is essentially $(n-1)$-connected, see Citation 2.3.

It remains to show that this connectivity property remains true using a height function $h_\chi'$ corresponding to a lift $\chi'$ of $\chi$. Note that $\chi'(t_i)$ is not necessarily equal to 0. Define $d = \min \{ \chi(t_i) \}$.

We claim that, for every $g \in G$, $h_\chi(g) \geq 0$ if and only if $\chi'(g) \geq d$. To see this, write $g = t_i y$ as above. Since $h_\chi(g) = \chi(y)$ and $\chi'(g) = \chi'(t_i y) = \chi'(t_i) + \chi(y)$, we get

$h_\chi(g) \geq 0 \iff \chi'(y) \geq 0 \iff \chi'(t_i) + \chi(y) \geq d + 0 \iff \chi'(g) \geq d,$

as required.

This now implies that the 0-skeleton of $X^{[0,+\infty]}_h$ is precisely the same as the 0-skeleton of $X^{[d, +\infty]}_h$. As the space $X^{[r, +\infty]}_h$ is defined as the maximal subcomplex of $X$ contained in $h_\chi^{-1}([r, +\infty))$, where an $m$-cell is included if all
of its boundary cells are included [6, Appendix B, p. 194], we have shown that
\[ X_{h_x}^{[0, +\infty)} = X_{h_x}^{[d, +\infty)} \]. Hence \([\chi'] \in \Sigma^u(G)\) again by Citation 2.3. □

**Proof of Theorem 1.5.** This follows from Propositions 3.3 and 3.7. □

4. The sigma invariants for \(F_\tau\)

We begin by collecting some properties of \(F_\tau\) and its characters as well as
exhibiting a finite index subgroup which satisfies the assumptions of Theorems 1.5 and 1.6.

It was shown in [9, Chapter 5] that
\[(F_\tau)_{ab} \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}.\]

Hence,
\[ S(F_\tau) = S^1. \]

Similarly to the original Thompson’s group case, we have the two linearly in-
dependent characters \(\lambda\) and \(\rho\) given by some logarithm of the slopes at 0 and
1 respectively, such that \([\lambda]\) and \([\rho]\) span \(S(F_\tau)\). In particular, these, for every
\(f \in F_\tau\), are given by
\[ \lambda(f) = \log_{\tau}(f'(0)) \quad \text{and} \quad \rho(f) = \log_{\tau}(f'(1)). \]

By taking appropriate subdivisions of \([0, 1]\), one can construct elements \(f \in F_\tau\) with support in \([0, b] \cap \mathbb{Z}[\tau]\) for some \(b < 1\) and such that \(f'(0) = \tau\). Analog-
ously, one can find \(g \in F_\tau\) with support in \([a, 1]\) for some \(a > 0\) and with
\(g'(1) = \tau\). Hence \(\lambda(f) = 1 = \rho(g), \lambda(g) = 0 = \rho(f)\) and thus \(\lambda\) and \(\rho\) are
linearly independent.

As an example, we can use the following elements:

**Example 4.1.**

\[
\begin{align*}
  f(x) &= \begin{cases} 
    \tau x & \text{for } 0 \leq x \leq \tau^2 \\
    \tau^{-1}x - \tau^2 & \text{for } \tau^2 \leq x \leq \tau \\
    x & \text{for } \tau \leq x \leq 1
  \end{cases} \\
  g(x) &= \begin{cases} 
    x & \text{for } 0 \leq x \leq \tau^2 \\
    \tau^{-1}x - \tau^3 & \text{for } \tau^2 \leq x \leq \tau \\
    \tau x + \tau^2 & \text{for } \tau \leq x \leq 1
  \end{cases}
\end{align*}
\]

**Proposition 4.2.** Let \(K\) denote the subgroup of \(F_\tau\), generated by
\(\{x_0, x_1, y_1, x_2, y_2, \ldots\}\). Then \(|F_\tau : K| = 2\) and \(K_{ab} \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}\).

**Proof.** We claim \(F_\tau = K \cup y_0 K\). To do so, consider \(g \in F_\tau\) in normal form, see Citation 2.8:

\[
g = x_0^{i_0} y_0^{\epsilon_0} x_1^{i_1} y_1^{\epsilon_1} \cdots x_n^{i_n} y_n^{\epsilon_n} x_{m-1}^{i_{m-1}} y_{m-1} \cdots x_0^{i_0}
\]

where \(i_0, \ldots, i_n, j_0, \ldots, j_m \in \mathbb{Z}_{\geq 0}\) and \(\epsilon_0, \ldots, \epsilon_m \in \{0, 1\}\). Hence,
\[
gK = x_0^{i_0} y_0^{\epsilon_0} K.
\]
When \( \epsilon_0 = 0 \) we have \( g \in K \), and when \( \epsilon_0 = 1 \) a repeated application of the following computation gives \( g \in y_0K \):
\[
x_0^i y_0 = x_0^{i-1} x_0 y_0 = x_0^{i-1} x_0 x_1 x^{-1}_0 y_0 = x_0^{i-1} y_0 x_1 x^{-1}_0 y_0 = x_0^{i-1} y_0 y_0 x_0 x_1 x^{-1}_0 = x_0^{i-1} y_0 x_0 x_1 y_0 x_1^{-1}.
\]

Consider any word in the generators \( x_i \) and \( y_j (i \geq 0, j \geq 1) \) in \( F_r \). The relations of \( F_r \), see Eq. (2), imply that in any other expression on this element, the occurrence of \( y_0 \) will have \( k \) an even integer. Hence, in the normal form of Citation 2.8 such an element will have no occurrence of \( y_0 \). This implies that \( K \) is a proper subgroup of \( F_r \), and moreover \( |F_r : K| = 2 \).

To determine the abelianisation, we do a similar calculation to that in [9, Section 5]: We denote the obvious presentation of \( F_r \) by \( \hat{\tau} \) and write the group operation additively. From the relations, it follows immediately that \( \hat{x}_i = \hat{x}_{i+1} \) and that \( 2\hat{y}_i = 2\hat{x}_i \) for all \( i \geq 1 \). Substituting \( \hat{z} = \hat{y}_1 - \hat{x}_1 \), we have the two infinite order generators \( \hat{x}_0 \) and \( \hat{x}_1 \) as well an order 2 generator \( \hat{z} \) as required.

Let \( H \) be a group and \( \sigma : H \to H \) a monomorphism. An ascending HNN extension (with base \( H \)) is a group given by the presentation
\[
H \ast_{t,\sigma} = (H, t \mid t h t^{-1} = \sigma(h); h \in H).
\]

We now consider the subgroup \( F_r[1] \leq F_r \) generated by \( \{x_1, y_1, x_2, y_2, \ldots\} \). In analogy to Thompson’s \( F \), there is a well-known monomorphism \( \sigma : F_r \to F_r \) given by \( \sigma(x_n) = x_{n+1} \) and \( \sigma(y_n) = y_{n+1} \), whose image is clearly \( F_r[1] \subseteq F_r \).

Restricting to \( F_r[1] \) gives a monomorphism \( \sigma : F_r[1] \to F_r[1] \) whose image is the proper subgroup \( F_r[2] \subseteq F_r[1] \) generated by \( \{x_2, y_2, x_3, y_3, \ldots\} \), and so on. Hence, any \( F_r[m] \) is isomorphic to \( F_r \) and thus of type \( F_{\infty} \). Much like \( F \) is an HNN extension over a copy of itself (cf. [8, Proposition 1.7]), the group \( K \) — which contains \( F_r[1] \) by definition — differs from its subgroup \( F_r[1] \cong F_r \) by the stable letter \( x_0 \).

**Lemma 4.3.** The subgroup \( K \leq F_r \) is isomorphic to the HNN extension
\[
K \cong F_r[1] \ast_{t,\sigma} = \langle F_r[1], t \mid tgt^{-1} = \sigma(g); g \in F_r[1] \rangle
\]
by mapping \( t \) to \( x_0^{-1} \) and \( F_r[1] \) to itself.

**Proof.** For this proof, we implicitly use standard facts about presentations and HNN extensions; cf. [20, Chapter IV, Section 2].

Let \( \langle X \mid R \rangle \) denote the obvious presentation of \( F_r[1] \), that is, the same as that of \( F_r \) from Eq. (2) but with decorated generating set \( X = \{\bar{x}_i, \bar{y}_i \mid i \geq 1\} \) and indices starting from 1. The HNN extension \( F_r[1] \ast_{t,\sigma} \) is thus given by the (abstract) group presentation
\[
F_r[1] \ast_{t,\sigma} \cong L := \langle X, t \mid R, t\bar{x}_i t^{-1} = \bar{x}_{i+1}, t\bar{y}_i t^{-1} = \bar{y}_{i+1} \text{ for all } i \geq 1 \rangle.
\]

The obvious map
\[
\phi : L \to K \text{ induced by } t \mapsto x_0^{-1}, \bar{x}_i \mapsto x_i, \bar{y}_i \mapsto y_i
\]
is a well-defined group homomorphism since all defining relations in \( L \) hold in \( K \). It is surjective by construction, and we want to check that it is also injective. Note that, since \( L \) is an HNN extension, the group \( F_\tau[1] \) effectively embeds in \( L \) as its obvious subgroup \( \langle X \rangle \). The restriction of \( \phi \) to \( \langle X \rangle \) is thus an isomorphism onto its image \( F_\tau[1] \subseteq K \). In particular, if \( g \in \langle X \rangle \), the isomorphisms \( \langle X \rangle \cong F_\tau[1] \cong F_\tau \) and Citation 2.8 yield a (unique) normal form for \( g \) matching the (unique) normal form of \( \phi(g) \in K \subseteq F_\tau \) (by dropping the tildes), and such a normal form of \( \phi(g) \) in \( K \) does not involve the generator \( x_0 \).

Now let \( w \in \ker(\phi) \subseteq L \). As \( L \) is an HNN extension, we may write \( w \) in normal form

\[
w = g_0t^{i_1}g_1t^{i_2} \cdots g_{n-1}t^{i_n}g_n
\]

with each \( i_j \in \{\pm 1\} \) and \( g_i \in \langle X \rangle \). If \( i_j = -1 \), repeated applications of the defining relations in \( L \) yield

\[
g_jt^{-1} = t^{-1}g_j' \text{ for some } g_j' \in \langle \{\tilde{x}, \tilde{y} \mid j \geq 2\} \rangle \leq \langle X \rangle.
\]

Similarly, if \( i_j = 1 \), then

\[
tg_j = g_j't \text{ for some } g_j' \in \langle \{\tilde{x}, \tilde{y} \mid j \geq 2\} \rangle \leq \langle X \rangle.
\]

Thus, writing \( a = \#\{i \mid i_j < 0\} \geq 0 \) and \( b = \#\{i \mid i_j > 0\} \geq 0 \), the word \( w \) can be rewritten as

\[
w = t^{-a}g'b \text{ where } g' \in \langle X \rangle.
\]

As \( g' \in \langle X \rangle \), we may replace it by its (unique) normal form in \( \langle X \rangle \cong F_\tau[1] \), if necessary. Mapping over to \( K \), we obtain

\[
\phi(w) = \phi(t)^{-a}\phi(g')\phi(t)^b = x_0^a\phi(g')x_0^{-b},
\]

where the subword \( \phi(g') \) lies in \( F_\tau[1] \) and is written in its (unique) normal form, not involving the letter \( x_0 \). In particular, the word \( x_0^a\phi(g')x_0^{-b} \in K \subseteq F_\tau \) can be written in a normal form as in Citation 2.8.

Suppose first that \( x_0^a\phi(g')x_0^{-b} \) is already in normal form, see Citation 2.8. Since \( 1 = \phi(w) = x_0^a\phi(g')x_0^{-b} \) by assumption, the above considerations imply that \( g' = 1 \) and \( a = b \), whence \( w \) is trivial in \( L \).

If \( x_0^a\phi(g')x_0^{-b} \) is not in normal form, then \( \phi(g') \) has no occurrences of the letters \( x_1 \) or \( y_1 \). We can assume that \( a \geq b \). Hence \( x_0^a\phi(g')x_0^{-b} = x_0^{a-b}\phi(g'[b]) \), where \( g'[b] \) denotes the word \( g' \) with the indices of the \( x_i \) and \( y_i \) increased by \( b \). This is now in normal form as in Citation 2.8, and as above it means that \( g'[b] = 1 \), hence \( g' = 1 \), and that \( a - b = 0 \). Again, \( w \) is trivial in \( L \). This finishes the proof. \( \square \)

We can finally adapt the calculations for Thompson’s group \( F \) as in [2] to compute the Sigma invariants for \( F_\tau \).

**Citation 4.4** ([2, Theorem 2.1]). Let \( G \) decompose as an ascending HNN extension \( H \ast_{t,c} \). Let \( \chi \) be a character such that \( \chi(H) = 0, \chi(t) = 1 \).

- Suppose \( H \) is of type \( F_n \), then \( [\chi] \in \Sigma^n(G) \).
- Suppose \( H \) is of type \( FP_n \), then \( [\chi] \in \Sigma^n(G; \mathbb{Z}) \).
If $H$ is finitely generated and $\sigma$ is not surjective, then $[-\chi] \not\in \Sigma^1(G)$.

**Lemma 4.5.** Let $\lambda$ and $\rho$ be the characters defined at the beginning of this section. Then

$$\left[\lambda,\rho\right] \in \Sigma^\infty(K) \cap \Sigma^\infty(F_\tau) \quad \text{and} \quad [-\lambda],[-\rho] \not\in \Sigma^1(K) \cup \Sigma^1(F_\tau),$$

$$\left[\lambda,\rho\right] \in \Sigma^\infty(K;\mathbb{Z}) \cap \Sigma^\infty(F_\tau;\mathbb{Z}) \quad \text{and} \quad [-\lambda],[-\rho] \not\in \Sigma^1(K;\mathbb{Z}) \cup \Sigma^1(F_\tau;\mathbb{Z}).$$

**Proof.** We begin by determining the result for $[\lambda]$ and $[-\lambda]$. The support of $F_\tau[1]$ lies in $[\tau^2,1]$ and hence $\lambda(F_\tau[1]) = 0$. The slope of $x_0$ at 0 is $\tau^{-2}$. Hence, taking the character $\chi := \frac{1}{2} \lambda \in [\lambda]$, we obtain $\chi(t) = 1$. We can thus apply Citation 4.4 to conclude that $[\lambda] \in \Sigma^\infty(K)$ and $[-\lambda] \not\in \Sigma^1(K)$. By Theorem 1.5, it follows that $[\lambda] \in \Sigma^\infty(F_\tau)$ and $[-\lambda] \not\in \Sigma^1(F_\tau)$.

As in [2, Section 1.4], we now consider a specific automorphism $\nu$ of $F_\tau$ to clear the case of $\rho$. Viewing the group $F_\tau$ as a group of PL homeomorphisms of the unit interval, $\nu$ is given by conjugation by $t \mapsto 1 - t$. This induces a homeomorphism of the character sphere that in particular swaps $[\lambda]$ with $[\rho]$, and also $[-\lambda]$ with $[-\rho]$, thus proving the lemma for $F_\tau$. A further application of Theorem 1.5 also yields the result for $K$.

The homological variant of the lemma follows similarly; see also Eq. (1). □

We shall now consider the arcs between $[-\lambda]$ and $[-\rho]$ on the character sphere $S(F_\tau)$. Since $[-\lambda]$ and $[-\rho]$ are not antipodal points, there is a unique (closed) geodesic segment in $S(F_\tau)$ connecting them, which we denote by $\text{conv}([-\lambda], [-\rho])$. In the other direction, there is a unique local geodesic from $[-\lambda]$ and $[-\rho]$, which we call the long arc, whose union with $\text{conv}([-\lambda], [-\rho])$ yields the great circle in $S(F_\tau)$ containing $[-\lambda]$ and $[-\rho]$, in particular in this one-dimensional case, this is just $S(F_\tau)$ itself. We will need the following:

**Citation 4.6 ([2, Theorem 2.3]).** Let $G$ decompose as an ascending HNN extension $G = H *_{t,\sigma}$. Let $\chi$ be a character of $G$ such that $\chi|_H \neq 0$. If $H$ is of type $F_\infty$ and $\chi|_H \in \Sigma^\infty(H)$, then $\chi \in \Sigma^\infty(G)$.

**Proposition 4.7.** All of $S(F_\tau)$, except possibly the closed geodesic

$$\text{conv}([-\lambda], [-\rho]),$$

lies in $\Sigma^\infty(F_\tau)$ and in $\Sigma^\infty(F_\tau;\mathbb{Z})$.

**Proof.** Again, we use our previous expression of the subgroup $K$ as an HNN extension of $H = F_\tau[1]$. By Lemma 4.5, we know that $[\rho] \in \Sigma^\infty(K) \cap \Sigma^\infty(F_\tau)$. Now let $\chi \in \text{Hom}(F_\tau, \mathbb{R})$ be arbitrary. We claim that

$$\chi(x_1) > 0 \iff \chi|_H \in [\rho|_H]. \quad (5)$$

In effect, $\chi = r\lambda + s\rho$ for some (unique) $r, s \in \mathbb{R}$ as $\lambda$ and $\rho$ are linearly independent and $\dim_{\mathbb{R}}(\text{Hom}(F_\tau, \mathbb{R})) = 2$. Since $\lambda(x_1) = \lambda(y_1) = 0$ and $a_j = a_0a_{j+1}a_0^{-1}$ for any $j \geq 1$ and $a \in \{x,y\}$, it follows that $\chi(w) = s\rho(w)$ for any $w \in H = F_\tau[1]$. This means that $\chi|_H \in \{[\rho|_H],[\rho|_H]\}$. Finally, $\rho(x_1) = 1$ implies that $\chi(x_1) = s$, whence $\chi(x_1) > 0$ if and only if $\chi|_H \in [\rho|_H]$. 
From here, we highlight that $H = F_\tau[1]$ is isomorphic to $F_\tau$, via the isomorphism $\gamma$ such that $\gamma(x_i) = x_{i-1}$ and $\gamma(y_i) = y_{i-1}$ for $i \geq 1$. The homeomorphism $S(F_\tau[1]) \cong S(F_\tau)$ induced by $\gamma$ maps $[\rho]_{H[1]}$ to $[\rho]$. As $[\rho] \in \Sigma^\infty(F_\tau)$, this means $[\rho]_{H[1]} \in \Sigma^\infty(F_\tau[1])$. In particular, if $\chi \in \text{Hom}(F_\tau, \mathbb{R})$ is positive on $x_1$, Claim (5) yields $[\chi]_{H[1]} = [\rho]_{H[1]} \in \Sigma^\infty(F_\tau[1])$. From here, we can apply Citation 4.6 to conclude that $[\chi|_K] \in \Sigma^\infty(K)$. Thus, $\chi(x_1) > 0$ $\implies [\chi|_K] \in \Sigma^\infty(K)$, whence $[\chi] \in \Sigma^\infty(F_\tau)$ by Theorem 1.5.

A straightforward computation shows that any character $\chi$ on the straight line from $\lambda$ to $\rho$ in $\text{Hom}(F_\tau, \mathbb{R})$ satisfies $\chi(x_1) > 0$. The same holds for any character on the straight line from $\rho$ to $-\lambda$. Hence, we have that the open arc in $S(F_\tau)$ from $[\lambda]$ to $[-\lambda]$ that contains $[\rho]$ actually lies in $\Sigma^\infty(F_\tau)$. Arguing again with the symmetry in $S(F_\tau)$ given by the automorphism $\nu$ induced by conjugation with $t \mapsto 1 - t$, we conclude that the open arc from $[\rho]$ to $[-\rho]$ containing $[\lambda]$ is also in $\Sigma^\infty(F_\tau)$. Altogether, the long (open) arc from $[-\lambda]$ to $[-\rho]$ is in $\Sigma^\infty(F_\tau)$, as claimed. The homological version follows directly from Eq. (1).

It now remains to consider the remaining short arc $\text{conv}([-\lambda], [-\rho])$. To do this we will follow the approach of [2, Section 2.3]. We need the following two results:

**Citation 4.8** ([2, Corollary 1.2]). The kernel of a nonzero discrete character $\chi$ has type $FP_n$ over the ring $R$ if and only if both $[\chi]$ and $[-\chi]$ lie in $\Sigma^\infty(G, R)$.

**Citation 4.9** ([2, Theorem 2.7]). Assume $G$ contains no nonabelian free subgroups and is of type $FP_2$ over a ring $R$. Let $\overline{\chi} : G \to \mathbb{R}$ be a nonzero discrete character. Then $G$ decomposes as an ascending HNN extension $H \ast_{i, \sigma} \rho$, where $H$ is a finitely generated subgroup of $\text{ker}(\overline{\chi})$ and $\overline{\chi}(t)$ generates the image of $\overline{\chi}$.

**Proposition 4.10.** Let $R$ be a ring. Then

$$\text{conv}([-\lambda], [-\rho]) \cap \Sigma^2(F_\tau, R) = \emptyset.$$

**Proof.** It suffices to show that no discrete character $\chi \in \text{conv}([-\lambda], [-\rho])$ lies in $\Sigma^2(F_\tau, R)$ because such characters are dense in $\text{conv}([-\lambda], [-\rho])$ and $\Sigma^2(F_\tau, R)$ is open; see, e.g., [2, Proposition 2.9]. Observe further that $[-\lambda], [-\rho] \not\in \Sigma^2(F_\tau, R)$ by Lemma 4.5.

So let $\chi$ be a discrete character of the form $\chi = a\lambda + b\rho$, with $a, b \in \mathbb{Q} \setminus \{0\}$. Using the elements $f, g \in F_\tau$ of Example 4.1, we can construct elements $t \in F_\tau$ with the following properties:

$$\lambda(t) = mb \quad \text{and} \quad \rho(t) = -ma \quad \text{for some } m \in \mathbb{Q} \setminus \{0\}. \quad (6)$$

In particular, $\chi(t) = 0$. Since $\lambda$ has discrete image in $\mathbb{R}$ and $a \neq 0$, there exists $t_0$ satisfying condition (6) such that $|\lambda(t_0)|$ is minimal among all elements $t$ fulfilling the properties listed in (6). Moreover, $\lambda(t_0) \neq 0$ for otherwise $t_0$ would not fulfill (6).

Let $G = \text{ker}(\chi)$. Then, since the abelianisation of $F_\tau$ is $\mathbb{Z}^2 \times \mathbb{Z}/2\mathbb{Z}$, we have that $G = \langle \sqrt{F_\tau}, t_0 \rangle = \sqrt{F_\tau} \rtimes \langle t_0 \rangle$, where $\sqrt{F_\tau} := \{ f \in F_\tau | f^n \in F_\tau \text{ for some } n \}$. 

Note that $\lambda|_G$ is a discrete nonzero character vanishing on the subgroup $\sqrt{F_\tau} \leq G$ and such that $\text{im}(\lambda|_G)$ is generated by $\lambda(t_0)$.

Now suppose $G$ has type $\text{FP}_2$ over a ring $R$. By Citation 4.9, we can decompose $G$ as the HNN extension $H \ast_{t,\sigma}$, where $H$ is a finitely generated subgroup of $\sqrt{F_\tau}$. As $H$ is generated by a finite set of elements of $F_\tau$, and each generator has support away from 0 and 1, there exists a value $\varepsilon'' > 0$ such that all elements of $H$ are supported in the interval $[\varepsilon'', 1 - \varepsilon'']$. Similarly, as $t_0$ has finitely many breakpoints, there is a value $\varepsilon' > 0$ such that $t_0$ is linear on the intervals $[0, \varepsilon']$ and $[1 - \varepsilon', 1]$. Let $\varepsilon = \min\{\varepsilon', \varepsilon''\}$, giving us a value with both of these properties.

Since $\sqrt{F_\tau} \rtimes \langle t_0 \rangle = G \cong H \ast_{t,\sigma}$, we can say that $\sqrt{F_\tau} = \bigcup_{n \geq 1} t^n H t^{-n}$. Hence for each $f \in \sqrt{F_\tau}$, there is a value $n$ such that $t^{-n} f t^n \in H$, hence $t^{-n} f t^n$ is supported in $[\varepsilon, 1 - \varepsilon]$. From here, we can see that any $f \in \sqrt{F_\tau}$ must be supported in $[t_0^n(\varepsilon), t_0^n(1 - \varepsilon)]$ for some $n$. As $\sqrt{F_\tau}$ has support $(0, 1)$, we can see that $(t_0^n(\varepsilon))_{n \in \mathbb{N}}$ must have a subsequence that converges to 0 and $(t_0^n(1 - \varepsilon))_{n \in \mathbb{N}}$ must have a subsequence that converges to 1. As $t_0$ is linear on the intervals $[0, \varepsilon]$ and $[1 - \varepsilon, 1]$, it holds $t_0(\varepsilon) < \varepsilon$ and $t_0(1 - \varepsilon) > 1 - \varepsilon$. Hence $t_0$ must have slope smaller than 1 near 0 and slope bigger than 1 near 1. Therefore, $ab < 0$. Given that we started with the assumption that $G = \ker(\chi)$ was of type $\text{FP}_2$, we obtain the implication

$$\chi = a\lambda + b\rho \quad \text{and} \quad \ker(\chi) \text{ of type } \text{FP}_2 \implies ab < 0$$

whenever $a, b \in \mathbb{Q} \setminus \{0\}$. The contrapositive of this is that $ab > 0$ implies $\ker(\chi)$ is not of type $\text{FP}_2$. Combining this with Citation 4.8, we see that we cannot have both $[\chi]$ and $[-\chi]$ in $\Sigma^2(F_\tau, R)$. In particular, if the antipodal point $[-\chi]$ lies in $\Sigma^2(F_\tau, R)$, then by Proposition 4.7 we have that $[\chi] \not\in \Sigma^2(F_\tau, R)$.

Transferring this result to the homotopical invariant with the use of Eq. (1), we conclude that if $[\chi] \not\in \Sigma^2(F_\tau, R)$, then $[\chi] \not\in \Sigma^2(F_\tau)$.

This finishes off the proof of Theorem 1.1.}

References


[548] LEWIS MOLYNEUX, BRITA NUCINKIS AND YURI SANTOS REGO


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(Lewis Molyneux) Royal Holloway, University of London, Department of Mathematics, McCrea Building, TW20 0EX Egham, UK
lewis.molyneux@rhul.ac.uk

(Brita Nucinkis) Royal Holloway, University of London, Department of Mathematics, McCrea Building, TW20 0EX Egham, UK
brita.nucinkis@rhul.ac.uk

(Yuri Santos Rego) University of Lincoln, Charlotte Scott Research Centre for Algebra, School of Mathematics and Physics, Brayford Pool, LN6 7TS Lincoln, UK
ysantosrego@lincoln.ac.uk

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