

Completion preserves homotopy fibre squares of connected nilpotent spaces

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ABSTRACT. We prove that completion preserves homotopy fibre squares of connected nilpotent spaces. As an application, we deduce the Hasse fracture square associated to a connected nilpotent space. Along the way, we record some closure properties of the category of T -complete nilpotent groups, where T is a non-empty set of primes.

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1. Introduction

Let T be a non-empty set of primes. By an \mathbb{F}_T -equivalence, we mean a map of spaces which induces an isomorphism on homology with coefficients in \mathbb{F}_p , for every $p \in T$. By a T -completion, we mean an \mathbb{F}_T -equivalence into a T -complete space, as defined in, say, [1, Subsection 10.2]. Let $N(f, g)$ denote the double mapping path space of maps f and g with the same codomain. The main result of this paper now states:

Theorem 1.1. *Let $f : X \rightarrow A$ and $g : Y \rightarrow A$ be maps between connected nilpotent spaces such that $N(f, g)$ is connected. If we have a commutative diagram:*

Received October 19, 2023.

2010 *Mathematics Subject Classification.* 55P60, 55R05, 20F18.

Key words and phrases. nilpotent, spaces, groups, localisation, completion, homotopy, homotopy fibre square, homotopy pullback, T -complete, fracture theorem.

The author gratefully acknowledges funding from the University of Warwick and the UK Engineering and Physical Sciences Research Council (Grant number: EP/R513374/1).

$$\begin{array}{ccccc}
 X & \xrightarrow{f} & A & \xleftarrow{g} & Y \\
 \downarrow & & \downarrow & & \downarrow \\
 \widehat{X}_T & \xrightarrow{\widehat{f}_T} & \widehat{A}_T & \xleftarrow{\widehat{g}_T} & \widehat{Y}_T
 \end{array}$$

such that the vertical maps are T -completions, then the induced map $N(f, g) \rightarrow N(\widehat{f}_T, \widehat{g}_T)$ is a T -completion.

It follows from Theorem 1.1 that any functorial T -completion preserves homotopy fibre squares of connected nilpotent spaces, where by a homotopy fibre square we mean a strictly commutative square such that the canonical map to the double mapping path space is a weak equivalence. For example, this applies to the functorial completions obtained via Bousfield localisation at the homology theory $H_*(-; \bigoplus_{p \in T} \mathbb{F}_p)$, or via use of the Bousfield-Kan completion functor.

Similar results to Theorem 1.1 can be found in the literature. In particular, in [2, Theorem 1.1], Farjoun proves an analogue of Theorem 1.1 in the case of disconnected spaces. However, one can only conclude directly from Theorem 1.1 of [2] that the comparison map $\widehat{N(f, g)}_T \rightarrow N(\widehat{f}_T, \widehat{g}_T)$ has homotopically discrete fibre. Our result can also be viewed as a generalisation of the connected fibre lemma of Bousfield and Kan, [8, Ch. II Lemma 4.8], which is the special case $Y = Y' = *$ of Theorem 1.1.

Our main reason for being interested in Theorem 1.1 is that it provides a natural context in which to deduce the following well-known fracture theorem, sometimes known as the Hasse square:

Theorem 1.2. *Let X be a T -local connected nilpotent space. Then any commutative square:*

$$\begin{array}{ccc}
 X & \xrightarrow{\widehat{\phi}_T} & \widehat{X}_T \\
 \psi \downarrow & & \downarrow \phi \\
 X_0 & \xrightarrow{(\widehat{\phi}_T)_0} & (\widehat{X}_T)_0
 \end{array}$$

with $\widehat{\phi}_T$ a T -completion and ψ, ϕ rationalisations, is a homotopy fibre square.

For the standard proof, see [4, Theorem 4.4]. We explain how the Hasse square can be deduced from Theorem 1.1 at the end of this paper.

Finally, at the beginning of the paper, we derive some basic properties of the category of T -complete nilpotent groups, thereby allowing us to clarify some misleading statements in the literature. For example, we will show that the kernel and, if the image is normal, the cokernel of a homomorphism between

T -complete nilpotent groups are T -complete, whereas it was claimed that this is not necessarily the case on page 218 of [1]. It goes without saying that such a flexible result is useful when dealing with exact sequences of T -complete nilpotent groups.

1.1. Counterexamples. We now say a few words about the hypotheses of Theorem 1.1. Consideration of the path-space fibration associated to $K(G, 1)$ for any G with $\pi_2(\widehat{K(G, 1)}_T) \neq 0$, justifies the connectivity assumptions on $N(f, g)$. For example, $\pi_2(K(\mathbb{Z}[p^{-1}]/\mathbb{Z}, 1)_p) \cong \widehat{\mathbb{Z}}_p$, [1, Example 10.1.14].

We also have the following counterexample, due to Sullivan, when the spaces involved are not nilpotent. The counterexample is based on a non-nilpotent space Z satisfying the following properties - see [9, pg. 104] and [8, Ch. VII. 3.6] for details of the construction:

- (i) $\pi_1(Z) = \mathbb{Z}/n\mathbb{Z}$, where n can be any integer dividing $p - 1$,
- (ii) $\pi_2(Z) = \widehat{\mathbb{Z}}_p$,
- (iii) $\pi_i(Z) = 0$ for $i \geq 3$,
- (iv) $\Omega\widehat{\mathbb{Z}}_p \simeq \widehat{S}_p^{2n-1}$.

For large values of n and p , it is clear from these properties that p -completion cannot preserve the fibre sequence:

$$K(\widehat{\mathbb{Z}}_p, 2) \rightarrow Z \rightarrow K(\mathbb{Z}/n\mathbb{Z}, 1)$$

1.2. Notation. We use throughout the notations and conventions of [1]. Indeed, anybody who has read [1] has more than enough background to understand this paper. For example, we have the definitions $\mathbb{E}_T G := \pi_1(\widehat{K(G, 1)}_T)$ and $\mathbb{H}_T G := \pi_2(\widehat{K(G, 1)}_T)$. If G is abelian, then $\mathbb{H}_T G$ and $\mathbb{E}_T G$ are the first and zeroth derived functors of T -adic completion, respectively. In the abelian case, $\mathbb{H}_T G = \text{Hom}(\mathbb{Z}[T^{-1}]/\mathbb{Z}, G)$ and $\mathbb{E}_T G = \text{Ext}(\mathbb{Z}[T^{-1}]/\mathbb{Z}, G)$, which justifies the notation. We also assume that all rings are commutative and unital.

Acknowledgements. I would like to thank the referee for a close reading of this paper, and a number of helpful suggestions.

2. Completion of nilpotent groups

2.1. Properties of T -complete nilpotent groups. We begin the paper with some basic results about the category \mathcal{B}_T of T -complete abelian groups, and their nilpotent analogues. Let G and H be nilpotent groups equipped with central series of the same length, $\{G_i\}$ and $\{H_i\}$. Let $f : G \rightarrow H$ be a group homomorphism such that, for all i , $f(G_i) \subset H_i$, and let K denote the kernel of f and, if the image is normal, let $\pi : H \rightarrow Q$ denote the cokernel of f . Then, there are induced central series $\{G_i \cap K\}$ and $\{\pi(H_i)\}$ expressing K and, if the image is normal, Q as nilpotent groups. Recall from the proof of [8, Ch.

III, Lemma 5.8], that there is a singly graded spectral sequence which inductively computes the induced central series for K and Q , starting from the maps $f : G_i/G_{i-1} \rightarrow H_i/H_{i-1}$. Its E^0 -page is given by:

$$E_i^0 = G_i/G_{i-1} \oplus H_i/H_{i-1}$$

and has differential defined by $d^0(g, h) = (0, f(g))$. It converges after finitely many pages to its E^∞ -page, which is defined by:

$$E_i^\infty = (G_i \cap K)/(G_{i-1} \cap K) \oplus H_i/(f(G) \cap H_i)H_{i-1}$$

Of course, when the image of f is normal, the second term of the sum can be identified with $\pi(H_i)/\pi(H_{i-1})$. If \mathcal{C} is a class of abelian groups, then we call a nilpotent group, G , \mathcal{C} -nilpotent if there exists a central series expressing G as a nilpotent group, such that each quotient G_i/G_{i-1} is in \mathcal{C} . We have:

Lemma 2.1. *Let \mathcal{C} be a class of abelian groups which is closed under taking kernels and cokernels of abelian group homomorphisms between members of \mathcal{C} . Let $f : G \rightarrow H$ be a group homomorphism between \mathcal{C} -nilpotent groups. Then, the kernel and, if the image is normal, the cokernel of f are \mathcal{C} -nilpotent. Moreover, if $\{G_i\}$ expresses G as a \mathcal{C} -nilpotent group, then $\{G_i \cap K\}$ expresses K as a \mathcal{C} -nilpotent group. Similarly, if the image is normal, and $\{H_i\}$ expresses H as a \mathcal{C} -nilpotent group, then $\{\pi(H_i)\}$ expresses Q as a \mathcal{C} -nilpotent group.*

Proof. Let $\{G_i\}$ and $\{H_i\}$ express G and H as \mathcal{C} -nilpotent groups. We can reindex these central series so that they have the same length, $f(G_i) \subset H_i$, and, for any i , either G_i/G_{i-1} or H_i/H_{i-1} is 0. The result now follows directly from the spectral sequence discussed above. \square

A word of caution is required regarding the hypotheses of Lemma 2.1. In particular, if R is a ring, we cannot, in general, take \mathcal{C} to be a category of R -modules, even if we require that each $f : G_i/G_{i-1} \rightarrow H_i/H_{i-1}$ is an R -module homomorphism. This is because, even though the differentials d^0 are R -module homomorphisms, there is no guarantee that the differentials d^1 are R -module homomorphisms, as the following example shows:

Example 2.2. Suppose that $\{G_i\}_{i=0}^2$ and $\{H_i\}_{i=0}^2$ are central series of length 2 representing G and H as nilpotent groups, and that we have a commutative diagram of group homomorphisms:

$$\begin{array}{ccccccc} 1 & \longrightarrow & G_1 & \longrightarrow & G_2 & \longrightarrow & G_2/G_1 & \longrightarrow & 1 \\ & & \downarrow f_1 & & \downarrow f & & \downarrow f_2 & & \\ 1 & \longrightarrow & H_1 & \longrightarrow & H_2 & \longrightarrow & H_2/H_1 & \longrightarrow & 1 \end{array}$$

Then a summand of the differential d^1 can be identified with the connecting homomorphism $\partial : \ker(f_2) \rightarrow \operatorname{coker}(f_1)$ induced by the Snake Lemma, assuming that all images are normal. Taking $\mathcal{C} = \mathbb{C}$ -modules, let $\psi : \mathbb{C} \rightarrow \mathbb{C}$ denote complex conjugation, and consider the diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{C} & \xrightarrow{\psi} & \mathbb{C} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow 1 & & \downarrow & & \\
1 & \longrightarrow & \mathbb{C} & \xrightarrow{1} & \mathbb{C} & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

Then the connecting homomorphism $\partial : \mathbb{C} \rightarrow \mathbb{C}$ can be identified with complex conjugation which is not a map of \mathbb{C} -modules, even though $f_1 = f_2 = 0$ are. Note that we could swap the position of ψ with the vertical identity map to produce a counterexample.

If $\mathcal{C} = \widehat{\mathbb{Z}}_p$ -modules, we can replace complex conjugation by the identity map $\widehat{\mathbb{Z}}_p \otimes \widehat{\mathbb{Z}}_p \rightarrow \widehat{\mathbb{Z}}_p \otimes \widehat{\mathbb{Z}}_p$, viewing the domain as a $\widehat{\mathbb{Z}}_p$ -module via multiplication in the left factor, and the codomain as a $\widehat{\mathbb{Z}}_p$ -module via multiplication in the right factor. Then $\psi = 1$ is not a map of $\widehat{\mathbb{Z}}_p$ -modules, since $\widehat{\mathbb{Z}}_p$ is not a solid ring, see [3, Definition 2.1] and Remark 2.4. To see this more directly, note that including $\widehat{\mathbb{Z}}_p \otimes \widehat{\mathbb{Z}}_p$ into $\widehat{\mathbb{Q}}_p \otimes \widehat{\mathbb{Q}}_p$ shows that $r \in \widehat{\mathbb{Z}}_p$ is in the core $c(\widehat{\mathbb{Z}}_p) := \{r \in \widehat{\mathbb{Z}}_p \mid r \otimes 1 = 1 \otimes r \in \widehat{\mathbb{Z}}_p \otimes \widehat{\mathbb{Z}}_p\}$ iff $mr \in \mathbb{Z}$ for some non-zero integer m . This shows that $c(\widehat{\mathbb{Z}}_p) = \mathbb{Z}_p$.

Therefore, [1, Lemma 4.3.4] and preceding discussion are incorrect as stated, where it was claimed that the kernel and, if the image is normal, the cokernel of a ' $\widehat{\mathbb{Z}}_p$ -map' between ' $\widehat{\mathbb{Z}}_p$ -nilpotent groups' are $\widehat{\mathbb{Z}}_p$ -nilpotent. For a counterexample, consider the map $\widehat{\mathbb{Z}}_p \xrightarrow{1 \otimes \widehat{\phi}_p - \widehat{\phi}_p \otimes 1} \widehat{\mathbb{Z}}_p \otimes \widehat{\mathbb{Z}}_p$, which can be viewed as a $\widehat{\mathbb{Z}}_p$ -map with $f_1 = f_2 = 0$ as in Example 2.2. Its kernel is $c(\widehat{\mathbb{Z}}_p) = \mathbb{Z}_p$, which is not $\widehat{\mathbb{Z}}_p$ -nilpotent, since no non-trivial subgroup admits a $\widehat{\mathbb{Z}}_p$ -module structure. This is because no non-trivial maps $\widehat{\mathbb{Z}}_p \rightarrow \widehat{\mathbb{Z}}_p$ factor through the core \mathbb{Z}_p .

We note, however, that there are alternative definitions of a $\widehat{\mathbb{Z}}_p$ -nilpotent group in the literature, such as [7, Definition 10.4], which is a generalisation of the notion of a $\widehat{\mathbb{Z}}_p$ -module. For our purposes, we have the following result:

Lemma 2.3. *The following classes of abelian groups are closed under kernels and cokernels of abelian group homomorphisms between members of the class:*

- (i) *the class of R -modules, where R is a solid ring (such as $\mathbb{Z}[T^{-1}]$ or \mathbb{F}_p , see [3, Definition 2.1]),*
- (ii) *the class, \mathcal{B}_T , of T -complete abelian groups,*
- (iii) *the class of $f\widehat{\mathbb{Z}}_T$ -modules (i.e., the class of finitely generated $\widehat{\mathbb{Z}}_T$ -modules),*
- (iv) *the class of T -complete abelian groups A , such that, for every $p \in T$, \widehat{A}_p is an $f\widehat{\mathbb{Z}}_p$ -module.*

Proof. i) This is the case originally dealt with in [8, Ch. III, Lemma 5.8]. The key points are that, if R is solid, then an abelian group has at most one R -module structure, and any homomorphism of abelian groups between R -modules is an R -module homomorphism, [3, 2.4].

ii) An abelian group is T -complete iff $\text{Hom}(\mathbb{Z}[T^{-1}], A) = \text{Ext}(\mathbb{Z}[T^{-1}], A) = 0$, [1, Prop. 10.1.18], and it follows from this that the image and, therefore, the kernel and cokernel of an abelian group homomorphism between T -complete abelian groups are T -complete.

iii) The universal property of T -completion implies that any T -complete abelian group has a $\widehat{\mathbb{Z}}_T$ -module structure, and that this $\widehat{\mathbb{Z}}_T$ -module structure is unique. The universal property also implies that an abelian group homomorphism between T -complete abelian groups is a $\widehat{\mathbb{Z}}_T$ -module homomorphism - in fact, that it is isomorphic to a product of $\widehat{\mathbb{Z}}_p$ -module homomorphisms between the individual p -completions. Now, any $f\widehat{\mathbb{Z}}_T$ -module is isomorphic to a product of $f\widehat{\mathbb{Z}}_p$ -modules which can each be generated by less than q elements, where q is an integer which doesn't depend on p . Moreover, any such product, with a common bound on the size of a minimal generating set, admits the structure of an $f\widehat{\mathbb{Z}}_T$ -module. Therefore, since each $\widehat{\mathbb{Z}}_p$ is a PID, the kernel and cokernel of a homomorphism between $f\widehat{\mathbb{Z}}_T$ -modules are $f\widehat{\mathbb{Z}}_T$ -modules.

iv) Similarly to iii), this follows from the fact that each $\widehat{\mathbb{Z}}_p$ is a PID, and the fact that a homomorphism of abelian groups between T -complete abelian groups is a product of $\widehat{\mathbb{Z}}_p$ -module maps between the individual p -completions. \square

Remark 2.4. If R is a ring with the property that any abelian group homomorphism between R -modules is an R -module homomorphism, then R is solid. To see this, consider the two R -module structures on $R \otimes_{\mathbb{Z}} R$ given by multiplication on the left and right factors, and the identity map as the homomorphism of abelian groups. The fact that the identity is an R -module homomorphism implies that $r \otimes_{\mathbb{Z}} 1 = 1 \otimes_{\mathbb{Z}} r$ for all $r \in R$, which is the definition of solidity. Conversely, if R is solid, then it is straightforward to show that any abelian group homomorphism between R -modules is an R -module homomorphism, as we used in the proof of Lemma 2.3 i).

Recall that a nilpotent group is T -complete iff it is \mathcal{B}_T -nilpotent, [1, Lemma 10.4.1]. We briefly sketch how the argument goes. If G is \mathcal{B}_T -nilpotent, then it is T -complete due to the construction of a Postnikov tower for $K(G, 1)$ from a \mathcal{B}_T -central series for G , as well as the fact that such towers are T -complete via co-HELP, [1, Theorem 3.3.7]. Conversely, if G is T -complete, then we can inductively T -complete any Postnikov tower for $K(G, 1)$. The fact that the abelian homotopy groups of the building blocks $\widehat{K(B, 2)}_T$ are T -complete is the starting point for an inductive proof that G is \mathcal{B}_T -nilpotent, using Lemma 2.3 ii) and the closure of \mathcal{B}_T -nilpotent groups under central extensions, [1, Lemma 3.1.3]. Therefore, we have:

Corollary 2.5. *The kernel and, if the image is normal, the cokernel of a homomorphism between T -complete nilpotent groups are T -complete.*

Before moving on to the proof of our main result, we record some consequences of Lemmas 2.1 and 2.3 to the theory of T -complete nilpotent groups:

Lemma 2.6. *Let G be a T -complete nilpotent group, and H a T -complete subgroup. Then:*

- (i) *there is a subnormal series $H = H_0 \leq H_1 \leq \dots \leq H_k = G$, where each H_i is T -complete,*
- (ii) *if G is $f\widehat{\mathbb{Z}}_T$ -nilpotent, then so is H ,*
- (iii) *if T is a finite set of primes and G is $f\widehat{\mathbb{Z}}_T$ -nilpotent, then G satisfies the ascending chain condition (ACC) for T -complete subgroups,*
- (iv) *if G is a T -torsion $f\widehat{\mathbb{Z}}_T$ -nilpotent group, then G is finite.*

Proof. i) We will induct on the nilpotency class of G , noting that the result is trivial if G is abelian. Let $e = G_0 \leq \dots \leq G_q = G$ represent G as a \mathcal{B}_T -nilpotent group. Let:

$$H/(H \cap G_1) = K_0 \leq K_1 \leq \dots \leq K_k = G/G_1$$

be a subnormal series as is guaranteed to exist by the inductive hypothesis. Note, for example, that $H \cap G_1$ is T -complete since it is the kernel of $H \rightarrow G/G_1$. Let $\pi : G \rightarrow G/G_1$ denote the quotient, and define $H_{i+1} = \pi^{-1}(K_i)$. Then, each H_i is T -complete and H_i is a normal subgroup of H_{i+1} . Moreover, H is a normal subgroup of H_1 , since G_1 is central in G .

ii) If $\{G_i\}$ represents G as an $f\widehat{\mathbb{Z}}_T$ -nilpotent group, then $\{H \cap G_i\}$ represents H as an $f\widehat{\mathbb{Z}}_T$ -nilpotent group, since a T -complete submodule of an $f\widehat{\mathbb{Z}}_T$ -module is an $f\widehat{\mathbb{Z}}_T$ -module, by the discussion in the proof of Lemma 2.3 iii). Here, the fact that each $H \cap G_i$ is T -complete follows from i).

iii) When G is abelian, this follows from the fact that $\widehat{\mathbb{Z}}_T$ is Noetherian when T is a finite set of primes. In general, we can induct on the length q of a central series, $\{G_i\}_{i=0}^q$, expressing G as an $f\widehat{\mathbb{Z}}_T$ -nilpotent group. If $\{H_i\}_{i=0}^\infty$ is an ascending chain of T -complete subgroups of G , then $\{H_i \cap G_1\}$ and $\{\pi(H_i)\}$ are ascending chains of T -complete subgroups of G_1 and G/G_1 , respectively, where $\pi : G \rightarrow G/G_1$ is the quotient map. These chains both terminate by the inductive hypothesis, and this implies that $\{H_i\}$ also terminates, as desired.

iv) Since each \widehat{G}_p is $\{p\}$ -local, and $G = \prod_{p \in T} \widehat{G}_p$, we must have $\widehat{G}_p = 1$ for all but finitely many primes, in order for G to be T -torsion. Therefore, we can reduce to the case where T is a finite set of primes. When G is abelian, the ACC implies that there is a product of primes in T , r , such that $rg = 0$ for all $g \in G$. Tensoring $\mathbb{Z}/r\mathbb{Z}$ with a suitable $\widehat{\mathbb{Z}}_T$ -free resolution of G , we conclude that G is finite. The general case then follows by induction. \square

3. Completion and fibre squares

3.1. Proof of the main theorem. The cocellular construction of the completion of a nilpotent space X , [1, Theorem 10.3.4], can be modified in the following way. First, when replacing X by a Postnikov tower, we can modify the construction to ensure that the coattaching maps are cofibrations, using the methods of [10, Ch. II, Theorem 1.2]. Then, to construct the completion, we inductively use commutative squares of the form:

$$\begin{array}{ccc}
 X_i & \longrightarrow & \widehat{X}_i \\
 \downarrow & & \downarrow \\
 K(B, m) & \longrightarrow & \widehat{K(B, m)}_T
 \end{array}$$

where the left hand vertical map is a cofibration, and the map $X_{i+1} \rightarrow \widehat{X}_{i+1}$ is then defined as the canonical map between homotopy fibres. Note that the inductive construction of completion implies that each \widehat{X}_i is also nilpotent. Our strategy of proof will use the refinement of the Zeeman comparison theorem due to Hilton and Roitberg, [5], and to use this we will need the following lemma, which states that a nilpotent fibration is quasi-nilpotent in the language of Hilton and Roitberg:

Lemma 3.1. *If $f : X \rightarrow Y$ is a map of connected spaces such that Ff is connected and $\pi_1(X)$ acts nilpotently on $\pi_*(Ff)$, then $\pi_1(Y)$ acts nilpotently on $H_*(Ff)$.*

Proof. See [6, Corollary 2.2]. □

In particular, the lemma holds if $f : X \rightarrow Y$ is a map of connected nilpotent spaces inducing a surjection on fundamental groups. We can now prove our main theorem:

Theorem 3.2. *Let $f : X \rightarrow A$ and $g : Y \rightarrow A$ be maps between connected nilpotent spaces such that $N(f, g)$ is connected. If we have a commutative diagram:*

$$\begin{array}{ccccc}
 X & \xrightarrow{f} & A & \xleftarrow{g} & Y \\
 \downarrow & & \downarrow & & \downarrow \\
 \widehat{X}_T & \xrightarrow{\widehat{f}_T} & \widehat{A}_T & \xleftarrow{\widehat{g}_T} & \widehat{Y}_T
 \end{array}$$

such that the vertical maps are T -completions, then the induced map $N(f, g) \rightarrow N(\widehat{f}_T, \widehat{g}_T)$ is a T -completion.

Proof. Via a straightforward diagram chase, we can reduce to the case where A is the limit of a Postnikov tower with coattaching maps, $A_i \rightarrow K(B, m)$ ($m \geq 2$), which are cofibrations, and the T -completion $A \rightarrow \widehat{A}_T$ is as described above. We define a filtration of $P := N(f, g)$ by pullbacks P_i :

$$\begin{array}{ccc}
 P_i & \longrightarrow & (A \times A) \times_{A_i \times A_i} A_i^{I_+} \\
 \downarrow & & \downarrow \\
 X \times Y & \longrightarrow & A \times A
 \end{array}$$

and let Q_i denote the pullbacks in the corresponding filtration of $Q := N(\widehat{f}_T, \widehat{g}_T)$. Observe that we have maps $P_{i+1} \rightarrow P_i$ which fit into a larger diagram of pullbacks as shown below:

$$\begin{array}{ccccccc}
 P_{i+1} & \longrightarrow & A^2 \times_{A_{i+1}^2} A_{i+1}^{I_+} & \longrightarrow & A_{i+1}^{I_+} & \longrightarrow & PK(B, m)^{I_+} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 P_i & \longrightarrow & A^2 \times_{A_i^2} A_i^{I_+} & \longrightarrow & A_{i+1}^2 \times_{A_i^2} A_i^{I_+} & \longrightarrow & (PK(B, m))^2 \times_{K(B, m)^2} K(B, m)^{I_+} \\
 \downarrow & & \downarrow & & & & \\
 X \times Y & \longrightarrow & A^2 & & & &
 \end{array}$$

Here, the inclusion $I \wedge \{0, 1\}_+ \cup \{0\}_+ \wedge I_+ \rightarrow I \wedge I_+$, where I has basepoint 1, is homeomorphic to the inclusion $S^1 \rightarrow D^2 \cong I \wedge S^1$, and so the right hand vertical map is homeomorphic to $P\Omega K(B, m) \rightarrow \Omega K(B, m)$.

We have maps $P_i \rightarrow Q_i$ and we will inductively show they are T -completions. Since $P_0 = X \times Y$, the base case holds by assumption. Assume that $P_i \rightarrow Q_i$ is T -completion. Comparing the diagram above with the corresponding diagram for Q , we have a commutative square:

$$\begin{array}{ccc}
 P_i & \longrightarrow & Q_i \\
 \downarrow & & \downarrow \\
 \Omega K(B, m) & \longrightarrow & \widehat{\Omega K(B, m)}_T
 \end{array}$$

and the induced map between homotopy fibres is the map $P_{i+1} \rightarrow Q_{i+1}$.

The Zeeman comparison theorem, or more specifically a refinement due to Hilton and Roitberg, [5], can now be used either because $m > 2$ and so the base space is simply connected, or because $m = 2$ and Lemma 3.1 implies that B and $\mathbb{E}_T B$ act nilpotently on the homology of the respective fibres. For the left hand fibration, this follows since P being connected implies that P_i is connected and $\pi_1(P_i) \rightarrow B$ is surjective. This is because if P_i were disconnected, then consideration of the fibre sequence $P_{i+1} \rightarrow P_i \rightarrow \Omega K(B, m)$ shows that P_{i+1} would also be disconnected, and similarly for P_{i+2}, \dots, P . For the right hand

fibration, we are assuming that $P_i \rightarrow Q_i$ is T -completion. Therefore, $\pi_1(Q_i) \rightarrow \mathbb{E}_T B$ is surjective, since \mathbb{E}_T is right exact.

Therefore, we have that $P_{i+1} \rightarrow Q_{i+1}$ is an \mathbb{F}_T -equivalence. We can also use this filtration of Q to show that the homotopy groups of Q_{i+1} are T -complete, using Lemma 2.3, as well as the closure of T -complete nilpotent groups under extensions, [1, Corollary 10.4.5], and so Q_{i+1} is T -complete, by [1, Theorem 11.1.1], and $P_{i+1} \rightarrow Q_{i+1}$ is T -completion as desired. \square

3.2. Fracture theorem as a consequence. In order to deduce the fracture theorem as a corollary of our results, we first need to show that the homotopy pullback in question is connected. This is the content of the next lemma. To prove it we will use the fact that if G is a nilpotent group, then $\mathbb{E}_p G = 1$ iff G is p -divisible, [1, Proposition 10.4.7 iii)]. Recall also that a nilpotent group G is said to be T -local if it is uniquely p -divisible for all $p \notin T$.

Lemma 3.3. *If G is a nilpotent group, then the function $\varphi : \mathbb{E}_T G \times G_0 \rightarrow (\mathbb{E}_T G)_0$, defined via composition with $(g, h) \rightarrow gh^{-1}$, is surjective.*

Proof. We first assume that G is abelian. Let J denote the image of φ , and P denote the kernel of φ , which is also the pullback of $\mathbb{E}_T G \rightarrow (\mathbb{E}_T G)_0 \leftarrow G_0$. We will first show that J is rational. Since J is a subgroup of $(\mathbb{E}_T G)_0$, we have $\mathbb{H}_T J = 0$. Therefore, we have a short exact sequence:

$$0 \rightarrow \mathbb{E}_T P \rightarrow \mathbb{E}_T G \rightarrow \mathbb{E}_T J \rightarrow 0$$

The first map is split surjective, with right inverse given by applying \mathbb{E}_T to the induced map $G \rightarrow P$. It follows that $\mathbb{E}_T J = 0$, and since J is also torsion-free and T -local, p -divisibility for $p \in T$ implies that J is rational. Now φ_0 is surjective and factors through J , so $J = (\mathbb{E}_T G)_0$ and φ is surjective.

The result for general nilpotent groups G can now be proven inductively on the nilpotency class of G , using [1, Lemma 7.6.1]. A key point is that the image of $\mathbb{E}_T Z(G)$ in $\mathbb{E}_T G$ is a central subgroup - this can be seen from the Postnikov tower construction of completion applied to the upper central series of G . \square

Finally, we give the proof of the fracture theorem that we have been building toward:

Theorem 3.4. *Let X be a T -local connected nilpotent space. Then any commutative square:*

$$\begin{array}{ccc} X & \xrightarrow{\hat{\phi}_T} & \hat{X}_T \\ \psi \downarrow & & \downarrow \phi \\ X_0 & \xrightarrow{(\hat{\phi}_T)_0} & (\hat{X}_T)_0 \end{array}$$

with $\hat{\phi}_T$ a T -completion and ψ, ϕ rationalisations, is a homotopy fibre square.

Proof. We can assume that ϕ is a fibration. Then, by Lemma 3.3, the pullback P is connected and we have a comparison map $f : X \rightarrow P$. By Theorem 3.2, applying functorial rationalisation and completion shows that f_0 and \hat{f}_T are weak equivalences, respectively. It follows that $\tilde{H}_*(Cf; \mathbb{Q}) = \tilde{H}_*(Cf; \bigoplus_{p \in T} \mathbb{F}_p) = 0$. Therefore, $\tilde{H}_*(Cf)$ is local away from T and has trivial rationalisation. Moreover, $\tilde{H}_*(Cf)$ is T -local, since X and P are. Therefore, $\tilde{H}_*(Cf) = 0$ and f is a homology isomorphism between connected nilpotent spaces, so must be a weak equivalence. \square

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This paper is available via <http://nyjm.albany.edu/j/2024/30-20.html>.