A bound on the index of exponent-4 algebras in terms of the $u$-invariant

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Abstract. For a prime number $p$, an integer $e \geq 2$ and a field $F$ containing a primitive $p^e$-th root of unity, the index of central simple $F$-algebras of exponent $p^e$ is bounded in terms of the $p$-symbol length of $F$. For a nonreal field $F$ of characteristic different from 2, the index of central simple algebras of exponent 4 is bounded in terms of the $u$-invariant of $F$. Finally, a new construction for nonreal fields of $u$-invariant 6 is presented.

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1. Introduction

Let $F$ be a field and $n$ a positive integer. A central simple $F$-algebra of degree $n$ containing a subfield which is a cyclic extension of degree $n$ of $F$ is called cyclic or a cyclic $F$-algebra. Given a cyclic field extension $K/F$ of degree $n$, a generator $\sigma$ of its Galois group and an element $b \in F^\times$, the rules

$$j^n = b \quad \text{and} \quad xj = j\sigma(x) \quad \text{for all} \quad x \in K$$

determine a multiplication on the $K$-vector space $K \oplus jK \oplus \ldots \oplus j^{n-1}K$ turning it into a cyclic $F$-algebra of degree $n$, which is denoted by

$$[K/F; \sigma, b].$$

Any cyclic $F$-algebra is isomorphic to an algebra of this form; see [3, Theorem 5.9]. Furthermore, any central $F$-division algebra of degree 2 or 3 is cyclic; see [3, Theorem 11.5] for the degree-3 case.
Central simple $F$-algebras of degree 2 are called quaternion algebras. We refer to [10, p. 25] for a discussion of quaternion algebras, including their standard presentation by symbols depending on two parameters from the base field. If $\text{char } F \neq 2$, $a \in F^\times \setminus F^\times 2$ and $b \in F^\times$, then the $F$-quaternion algebra $(a, b)_F$ is equal to $[K/F, \sigma, b)$ for $K = F(\sqrt{a})$ and the nontrivial automorphism $\sigma$ of $K/F$.

We refer to [3] and [6] for the theory of central simple algebras, and to [4, Section 3] for a survey on the role of cyclic algebras in this context.

Before we approach the problem in the focus of our interest, we fix some notation. We set $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$. We denote by $\text{Br}(F)$ the Brauer group of $F$, and for $n \in \mathbb{N}^+$, we denote by $\text{Br}_n(F)$ the $n$-torsion part of $\text{Br}(F)$. Let $p$ always denote a prime number.

The following question was asked by Albert in [1, p. 126] and is still open in general.

**Question 1.1.** For $n \in \mathbb{N}^+$, is $\text{Br}_n(F)$ generated by classes of cyclic $F$-algebras of degree dividing $n$?

In view of the Primary Decomposition Theorem for central simple algebras (see e.g. [6, Corollary 9.11]), any such question can be reduced to the case where $n$ is a prime power. Each of the following two famous results gives a positive answer to Question 1.1 under additional hypotheses on $F$ in relation to $n$.

**Theorem 1.2** (Albert). Let $p$ be a prime number and assume that $\text{char } F = p$. Let $e \in \mathbb{N}^+$. Then $\text{Br}_{pe}(F)$ is generated by classes of cyclic $F$-algebras of degree dividing $p^e$.

**Proof.** See [3, Chapter VII, Section 9].

**Theorem 1.3** (Merkurjev-Suslin). Let $n \in \mathbb{N}^+$ and assume that $F$ contains a primitive $n$-th root of unity. Then $\text{Br}_n(F)$ is generated by the classes of cyclic $F$-algebras of degree dividing $n$.

**Proof.** See [13].

If $F$ contains a primitive $n$-th root of unity then $\text{char } F$ does not divide $n$. Hence, the hypotheses of Theorem 1.2 and Theorem 1.3 are mutually exclusive.

For $n = 2$, Theorem 1.3 was obtained by Merkurjev in [12]. Note that the hypothesis of Theorem 1.3 for $n = 2$ just means that $\text{char } F \neq 2$. Together with Theorem 1.2 this gives an unconditional positive answer to Question 1.1 for $n = 2$.

It was observed in [13, Proposition 16.6] that from the positive answer to Question 1.1 in the (highly nontrivial) case $n = 2$ one obtains (rather easily) an unconditional positive answer for $n = 4$. In Corollary 3.10, we obtain a different argument for this step.

Whenever we have a positive answer to Question 1.1, it is motivated to look at quantitative aspects of the problem. In the first place, this concerns the number of cyclic algebras needed for a tensor product representing a class in $\text{Br}(F)$ of given exponent. This leads to the notion and the study of symbol lengths.
For a central simple $F$-algebra $A$, the $n$-symbol length of $A$, denoted by $\lambda_n(A)$, is the smallest $m \in \mathbb{N}^+$ such that $A$ is Brauer equivalent to a tensor product of $m$ cyclic algebras of degree dividing $n$, if such an integer $m$ exists, otherwise we set $\lambda_n(A) = \infty$. The $n$-symbol length of $F$ is defined as

$$\lambda_n(F) = \sup\{\lambda_n(A) \mid [A] \in \text{Br}_n(F)\} \in \mathbb{N}^+ \cup \{\infty\}.$$ 

Note that the index of any central simple $F$-algebra of exponent $n$ is at most $n \lambda_n(F)$.

Let $p$ be a prime number. It seems plausible to take the $p$-symbol length of $F$ for a measure for the complexity of the whole $p$-primary part of the theory of central simple algebras over $F$. So in particular one might expect that $\lambda_{pe}(F)$ can be bounded in terms of $\lambda_p(F)$ for all $e \in \mathbb{N}^+$. When $F$ contains a primitive $p^e$-th root of unity, it follows from [17, Proposition 2.5] that $\lambda_{pe}(F) \leq e \lambda_p(F)$, but in general, this problem is still open.

In this article, we consider the following question.

**Question 1.4.** Let $e \in \mathbb{N}^+$. Can one bound the index of a central simple $F$-algebra of exponent $pe$ in terms of $e$ and $\lambda_p(F)$?

This is obviously true when $e = 1$. In the case where $F$ contains a primitive $p^e$-th root of unity, one can distill from the proof of [17, Proposition 2.5] an argument showing that the index of any central simple $F$-algebra of exponent $p^e$ is bounded by $p^{\frac{e(e+1)}{2}} \lambda_p(F)$. We retrieve this bound in Theorem 2.6 by means of a lifting argument formulated in Proposition 2.4.

In Section 3, we consider the case where $p^e = 4$ and make no assumption on roots of unity. For a nonreal field $F$, we obtain in Corollary 3.12 an upper bound on the index of exponent-4 algebras in terms of the $u$-invariant of $F$.

Section 4 is devoted to the construction of examples of nonreal fields with given $u$-invariant admitting a central simple algebra of given 2-primary exponent and of comparatively large index; see Proposition 4.3. If $F$ is nonreal and $u(F) = 4$, then by Corollary 3.12 the index of a central simple $F$-algebra of exponent 4 is at most 8, and we see in Example 4.4 that this is optimal. This example provides at the same time quadratic field extensions $K/F$ with $u(F) = 4$ and $u(K) = 6$; see Example 4.5. Hence, Section 4 provides also an alternative construction of fields of $u$-invariant 6.

### 2. Multiplication by a power of $p$ in the Brauer group

For a finite field extension $K/F$, let $N_{K/F} : K \to F$ denote the norm map.

**Theorem 2.1.** Let $\zeta \in F$ be a primitive $p$-th root of unity. Let $K/F$ be a cyclic field extension of degree $p^{e-1}$. Then $K/F$ embeds into a cyclic field extension of degree $p^e$ of $F$ if and only if $\zeta = N_{K/F}(x)$ for some $x \in K$.

**Proof.** See [2, Theorem 9.11].
Let $A$ and $B$ be central simple $F$-algebras. We write $A \sim B$ to indicate that $A$ and $B$ are Brauer equivalent. For $n \in \mathbb{N}^+$ we denote by $A^\otimes n$ the $n$-fold tensor product $A \otimes_F \ldots \otimes_F A$.

**Theorem 2.2** (Albert). Let $n, m \in \mathbb{N}$ with $m \leq n$ and $b \in F^\times$. Let $L/F$ be a cyclic field extension of degree $p^n$ and let $\sigma$ be a generator of its Galois group. Let $K$ be the fixed field of $\sigma p^{n-m}$ in $L$. Then

$$[L/F, \sigma, b] \otimes p^m \sim [K/F, |K, b]$$.

**Proof.** See [3, Theorem 7.14].

**Corollary 2.3.** Let $\zeta \in F$ be a primitive $p$-th root of unity. Let $e \in \mathbb{N}^+$. For $\alpha \in \text{Br}(F)$, the following are equivalent:

(i) $\alpha$ is the class of a cyclic $F$-algebra of degree $p^{e-1}$ containing a cyclic field extension $K/F$ of degree $p^{e-1}$ such that $\zeta = N_{K/F}(x)$ for some $x \in K$.

(ii) $\alpha = p\beta$ for the class $\beta \in \text{Br}(F)$ of a cyclic $F$-algebra of degree $p^e$.

**Proof.** (i $\Rightarrow$ ii) Assume that $K/F$ is a cyclic field extension of degree $p^{e-1}$, $\sigma$ a generator of its Galois group and $b \in F^\times$ is such that $\alpha$ is represented by $[K/F, \sigma, b]$. Assume further that $\zeta = N_{K/F}(x)$ for some $x \in K$. By Theorem 2.1, there exists a field extension $L/K$ of degree $p$ such that $L/F$ is cyclic. Then $\sigma$ extends to an $F$-automorphism $\sigma'$ of $L$, and it follows that $\sigma'$ generates the Galois group of $L/F$. Let $\beta$ be the class of the cyclic $F$-algebra $[L/F, \sigma', b]$. Since $[L : K] = p$ and $\sigma'|_K = \sigma$, we conclude by Theorem 2.2 that $p\beta = \alpha$.

(ii $\Rightarrow$ i) Assume that $\alpha = p\beta$ where $\beta \in \text{Br}(F)$ is the class of a cyclic $F$-algebra of degree $p^e$. Then $\beta$ is given by $[L/F, \sigma, b]$ for some cyclic field extension $L/F$ of degree $p^e$, a generator $\sigma$ of its Galois group and $b \in F^\times$. Let $K$ denote the fixed field of $\sigma p^{e-1}$ in $L$. Then $K/F$ is cyclic of degree $p^{e-1}$, and we obtain by Theorem 2.1 that $\zeta = N_{K/F}(x)$ for some $x \in K$. By Theorem 2.2, we have $[L/F, \sigma, b] \otimes p \sim [K/F, |K, b]$. Hence, $\alpha$ is given by $[K/F, |K, b]$. □

Given a central simple $F$-algebra $A$, we denote by $\deg A$, $\text{ind} A$ and $\exp A$, the degree, index and exponent of $A$, respectively. For $\alpha \in \text{Br}(F)$, we write $\text{ind} \alpha$ and $\exp \alpha$ for the index and the exponent of any central simple $F$-algebra representing $\alpha$.

Given a field extension $F'/F$ and $\alpha \in \text{Br}(F)$ we denote by $\alpha_{F'}$, the image of $\alpha$ under the natural map $\text{Br}(F) \to \text{Br}(F')$ induced by scalar extension.

Let $m \in \mathbb{N}^+$. We call $\alpha \in \text{Br}(F)$ an $m$-cycle if $\exp \alpha = m = [K : F]$ for some cyclic field extension $K/F$ for which $\alpha_K = 0$. Hence, given a central $F$-division algebra $D$, the class of $D$ in $\text{Br}(F)$ is an $m$-cycle if and only if $D$ is cyclic and $\exp D = \deg D = m$.

**Proposition 2.4.** Let $e, i \in \mathbb{N}^+$ with $i \leq e$ and such that every cyclic field extension of degree $p^i$ of $F$ embeds into a cyclic field extension of degree $p^e$ of $F$. Then every $p^i$-cycle in $\text{Br}(F)$ is of the form $p^{e-i} \beta$ for a $p^e$-cycle $\beta \in \text{Br}(F)$.

**Proof.** Let $\alpha \in \text{Br}(F)$ be a $p^i$-cycle. Hence, $\alpha$ is given by $D = [K/F, \sigma, b]$ for a cyclic field extension $K/F$ of degree $p^i$, a generator $\sigma$ of its Galois group and...
some \( b \in F^x \). In particular \( \deg D = p^i = \exp \alpha = \exp D \), whereby \( D \) is a division algebra. By the hypothesis, \( K/F \) embeds into a cyclic field extension \( L/F \) of degree \( p^i \). Then \( \sigma \) extends to an \( F \)-automorphism \( \sigma' \) of \( L \). It follows that \( \sigma' \) is a generator of the Galois group of \( L/F \). We set \( \Delta = [L/F, \sigma', b] \) and denote by \( \beta \) the class of \( \Delta \) in \( \text{Br}(F) \). We obtain by Theorem 2.2 that \( \Delta \otimes p^{e-1} \sim D \), whereby \( p^{e-1} \beta = \alpha \). Since \( \exp \alpha = p^i \), it follows that \( \exp \beta = p^e = \deg \Delta \). Since \( \beta_L = 0 \), we conclude that \( \beta \) is a \( p^e \)-cycle. \( \square \)

An element \( \alpha \in \text{Br}(F) \) is called a cycle if it is an \( m \)-cycle for some \( m \in \mathbb{N}^+ \) (given by \( m = \exp \alpha \)).

**Corollary 2.5.** Let \( e \in \mathbb{N}^+ \) be such that \( F \) contains a primitive \( p^e \)-th root of unity. Then every cycle in \( \text{Br}_{p^e}(F) \) is a multiple of a \( p^e \)-cycle.

**Proof.** Let \( \omega \in F \) be a primitive \( p^e \)-th root of unity and set \( \zeta = \omega^{p^{e-1}} \). Then \( \zeta \) is a primitive \( p \)-th root of unity. For any field extension \( K/F \) of degree \( p^i \) with \( 1 \leq i \leq e-1 \), we have that \( \zeta = (\omega^{p^{e-1}})^{p^i} = N_{K/F}(\omega^{p^{i-1}}) \). Hence, it follows by induction on \( i \) from Theorem 2.1 that every cyclic field extension of degree \( p^i \) of \( F \) embeds into a cyclic field extension of degree \( p^e \). Now the conclusion follows by Proposition 2.4. \( \square \)

The following bound can be easily derived from the proof of [17, Proposition 2.5]. To illustrate the general strategy, we include an argument.

**Theorem 2.6.** Let \( e \in \mathbb{N}^+ \) be such that \( F \) contains a primitive \( p^e \)-th root of unity. Then \( \text{Br}_{p^e}(F) \) is generated by the \( p^e \)-cycles. Furthermore, for every \( \alpha \in \text{Br}_{p^e}(F) \), we have \( \text{ind} \alpha = p^n \) for some \( n \in \mathbb{N}^+ \) with

\[
n \leq \frac{e(e+1)}{2} \lambda_p(F).
\]

**Proof.** Consider \( \alpha \in \text{Br}_{p^e}(F) \). By induction on \( e \) we will show at the same time that \( \alpha \) is a sum of \( p^e \)-cycles and that \( \text{ind} \alpha \) is of the claimed form.

We have \( p^e \alpha \in \text{Br}_{p^e}(F) \). It follows by Theorem 1.3 for \( n = p \) and by the definition of \( \lambda_p(F) \) that \( p^e \alpha = \sum_{i=1}^m y_i \) for some natural number \( m \leq \lambda_p(F) \) and classes \( y_1, \ldots, y_m \in \text{Br}(F) \) of cyclic \( F \)-division algebras of degree \( p \). Then \( y_1, \ldots, y_m \) are \( p \)-cycles. By Corollary 2.5, for \( 1 \leq i \leq m \), we have \( y_i = p^{e-1} \beta_i \) for a \( p^e \)-cycle \( \beta_i \in \text{Br}(F) \).

We set \( \alpha' = \alpha - \sum_{i=1}^m \beta_i \). Then \( \alpha' \in \text{Br}_{p^{e-1}}(F) \). If \( e = 1 \), then \( \alpha' = 0 \) and \( \alpha = \sum_{i=1}^m \beta_i \), and we obtain that \( \text{ind} \alpha = p^n \) for some positive integer \( n \leq m \leq \lambda_p(F) \), confirming the claims about \( \alpha \). Assume now that \( e > 1 \). By the induction hypothesis, \( \alpha' \) is equal to a sum of \( p^{e-1} \)-cycles and \( \text{ind} \alpha' = p^n \) for a natural number \( n' \leq \frac{(e-1)e}{2} \lambda_p(F) \). By Corollary 2.5, every cycle in \( \text{Br}_{p^e}(F) \) is a multiple of a \( p^e \)-cycle, hence in particular, a sum of \( p^e \)-cycles. We conclude that \( \alpha' \) is a sum of \( p^e \)-cycles, whereby \( \alpha \) is a sum of \( p^e \)-cycles. Furthermore \( \text{ind} \alpha \) divides \( \text{ind} \alpha' \cdot \text{ind} \beta_1 \cdots \text{ind} \beta_m = p^{ne+em} \). Hence, \( \text{ind} \alpha = p^n \) for some positive
integer
\[ n \leq n' + em \leq \frac{(e-1)e}{2} \lambda_p(F) + e\lambda_p(F) = \frac{e(e+1)}{2} \lambda_p(F). \]
This proves the claims about \( \alpha \).
\[\square\]

To obtain that \( \text{Br}_{p^e}(F) \) is generated by cycles, one can also conclude inductively on the basis of a weaker hypothesis on roots of unity than in Theorem 2.6.

**Proposition 2.7.** Let \( e \in \mathbb{N}^+ \) be such that \( p \text{Br}(F) \cap \text{Br}_{p^e-1}(F) \) is generated by elements \( p\beta \) with cycles \( \beta \in \text{Br}_{p^e}(F) \). Then \( \text{Br}_{p^e}(F) \) is generated by cycles.

**Proof.** Consider \( \alpha \in \text{Br}_{p^e}(F) \). Then \( p\alpha \in \text{Br}(F)\cap\text{Br}_{p^e-1}(F) \), so the hypothesis implies that \( p\alpha = \sum_{i=1}^n p\beta_i \) for some \( n \in \mathbb{N} \) and cycles \( \beta_1, \ldots, \beta_n \in \text{Br}_{p^e}(F) \).

Hence, \( \alpha - \sum_{i=1}^n \beta_i \in \text{Br}_{p^e}(F) \). By Theorem 1.3, \( \alpha - \sum_{i=1}^n \beta_i = \sum_{i=1}^m \gamma_i \) for some \( m \in \mathbb{N} \) and \( p \)-cycles \( \gamma_1, \ldots, \gamma_m \in \text{Br}(F) \). Hence, \( \alpha \) is a sum of cycles in \( \text{Br}_{p^e}(F) \).
\[\square\]

3. **Multiplying by 2 in the Brauer group**

From now on we assume that \( \text{char} F \neq 2 \). We show that the hypotheses of Proposition 2.7 for \( p = e = 2 \) are satisfied to retrieve the positive answer to Question 1.1 in the case where \( p^e = 4 \). The argument also yields bounds on the index of exponent-4 algebras in terms of the 2-symbol length, and hence an affirmative answer to Question 1.4 for these algebras.

We denote by \( S_2(F) \) the set of nonzero sums of two squares in \( F \). Note that \( S_2(F) \) is a subgroup of \( F \).

The following statement is essentially contained in [11, Corollary 5.14]. We include the argument for convenience.

**Proposition 3.1.** Let \( Q \) be an \( F \)-quaternion division algebra. The following are equivalent:

(i) \(-1\) is a norm in a quadratic field extension of \( F \) contained in \( Q \).
(ii) \(-1\) is a reduced norm of \( Q \).
(iii) \( Q \sim C^{\otimes 2} \) for some cyclic \( F \)-algebra \( C \) of degree 4.
(iv) \( Q \cong (s, t)F \) for certain \( s \in S_2(F) \) and \( t \in F^\times \).

**Proof.** Let \( \text{Nrd}_Q : Q \to F \) denote the reduced norm map. For any quadratic field extension \( K/F \) contained in \( Q \) and any \( x \in K \) we have \( \text{Nrd}_Q(x) = \text{N}_{K/F}(x) \).

Therefore, the implication \((i \Rightarrow ii)\) is obvious, and for \((ii \Rightarrow i)\), it suffices to observe that, since \( Q \) is a division algebra, every maximal commutative subring of \( Q \) is a quadratic field extension of \( F \).

The equivalence \((i \Leftrightarrow iii)\) corresponds to the equivalence formulated in Corollary 2.3 in the case where \( p = e = 2 \), taking for \( \alpha \in \text{Br}(F) \) the class of \( Q \).

To finish the proof, it suffices to show the equivalence \((i \Leftrightarrow iv)\). As char \( F \neq 2 \), any quadratic field extension of \( F \) is of the form \( F(\sqrt{s}) \) for some \( s \in F^\times \setminus F^{\times 2} \), and for such \( s \), we have that \(-1\) is a norm in \( F(\sqrt{s})/F \) if and only if the quadratic form \( X^2 + Y^2 - sz^2 \) over \( F \) is isotropic, if and only if \( s \in S_2(F) \). Finally, given a
quadratic field extension $K/F$ contained in $Q$ and $s \in F^x$ such that $K \simeq F(\sqrt{s})$, by [3, Theorem 5.9], we can find an element $t \in F^x$ such that $Q \simeq (s,t)_{F}$. □

We denote by $WF$ the Witt ring of $F$ and by $IF$ its fundamental ideal. For $n \in \mathbb{N}^+$, we set $I^n F = (IF)^n$, and we call a regular quadratic form over $F$ whose Witt equivalence class belongs to $I^n F$ simply a form in $I^n F$. Given a regular quadratic form $q$ over $F$, we denote by $\dim q$ its dimension (rank). By a torsion form we shall mean a regular quadratic form over $F$ whose class in $WF$ has finite additive order. A quadratic form $q$ such that $2 \times q$ is hyperbolic is called a 2-torsion form. The following statement describes 2-torsion forms in $I^2 F$.

**Lemma 3.2.** Let $q$ be a form in $I^2 F$. Let $m \in \mathbb{N}^+$ be such that $\dim q = 2m + 2$. Then $2 \times q$ is hyperbolic if and only if $q$ is Witt equivalent to $\bigoplus_{i=1}^{m} a_i \langle s_i, t_i \rangle$ for some $s_1, \ldots, s_m \in S_2(F)$ and $a_1, t_1, \ldots, a_m, t_m \in F^x$.

**Proof.** For $s \in S_2(F)$ and $t \in F^x$, the form $2 \times \langle s, t \rangle$ over $F$ is hyperbolic. This proves the right-to-left implication.

We prove the opposite implication by induction on $m$. If $m = 0$, then $q$ is a 2-dimensional quadratic form in $I^2 F$ and must therefore be hyperbolic. In particular, the statement holds in this case. Suppose now that $m \geq 1$. In view of the induction hypothesis, we may assume without loss of generality that $q$ is anisotropic. As the quadratic form $2 \times q$ is hyperbolic and hence in particular isotropic, it follows by [7, Lemma 6.24] that $q \simeq q_1 \perp q_2$ for certain regular quadratic forms $q_1$ and $q_2$ over $F$ such that $\dim q_1 = 2$ and $2 \times q_1$ is hyperbolic. We fix an element $a_1 \in F^x$ represented by $q_1$. Then $q_1 \simeq \langle a_1, -a_1 s_1 \rangle$ for some $s_1 \in F^x$. As $2 \times q_1$ is hyperbolic, so is $2 \times \langle 1, -s_1 \rangle$, whereby $s_1 \in S_2(F)$. We write $q_2 \simeq \langle a \rangle \perp q'$ with $a \in F^x$ and a $(2m - 1)$-dimensional regular quadratic form $q'$ over $F$. We set $q'' = q' \perp \langle s_1 a \rangle$ and $t_1 = -a_1 a$. We obtain that $q \perp -q''$ is Witt equivalent to $a_1 \langle s_1, t_1 \rangle$. Since $s_1 \in S_2(F)$, we have that $2 \times \langle s_1, t_1 \rangle$ is hyperbolic. Therefore, $2 \times q''$ is Witt equivalent to $2 \times q$, and hence equally hyperbolic. Furthermore, $q''$ is a form in $I^2 F$. Since $\dim q'' = 2m$ and $2 \times q''$ is hyperbolic, the induction hypothesis yields that there exist $s_2, \ldots, s_m \in S_2(F)$ and $a_2, t_2, \ldots, a_m, t_m \in F^x$ such that $q''$ is Witt equivalent to $\bigoplus_{i=2}^{m} a_i \langle s_i, t_i \rangle$. Then $q$ is Witt equivalent to $\bigoplus_{i=1}^{m} a_i \langle s_i, t_i \rangle$. This concludes the proof. □

By [7, Theorem 14.3], associating to a quadratic form its Clifford algebra induces a homomorphism

$$e_2 : \mathbb{I}^2F \to Br_2(F).$$

By Merkurjev’s Theorem [7, Theorem 44.1] together with [14, Theorem 4.1], the kernel of this homomorphism is precisely $I^2 F$.

For a quadratic field extension $K/F$, we denote by $\text{cor}_{K/F}$ the corestriction homomorphism $\text{Br}(K) \to \text{Br}(F)$ defined in [10, Section 3.B] (where it is denoted by $\text{N}_{K/F}$).

**Proposition 3.3.** Let $\beta \in Br_2(F)$. The following are equivalent:

(i) $\beta \in 2 \text{Br}(F)$. 

(ii) $\beta = e_2(q)$ for some 2-torsion form $q$ in $I^2 F$.

(iii) $\beta$ is given by $\bigotimes_{i=1}^{m} (s_i, t_i) F$ for some $m \in \mathbb{N}, s_1, \ldots, s_m \in S_2(F)$ and $t_1, \ldots, t_m \in F^\times$.

Moreover, if these conditions are satisfied and $\text{ind } \beta \leq 4$, then one can choose $m$ in (iii) such that $\text{ind } \beta = 2^m$.

**Proof.** The implication $(iii \Rightarrow i)$ follows by Proposition 3.1.

For $m \in \mathbb{N}, s_1, \ldots, s_m \in S_2(F)$ and $a_1, t_1, \ldots, a_m, t_m \in F^\times$, one has that $e_2(\bigoplus_{i=1}^{m} a_i <s_i, t_i>) \sim \bigotimes_{i=1}^{m} (s_i, t_i) F$. Hence, the equivalence $(ii \Leftrightarrow iii)$ follows by Lemma 3.2.

We show now the implication $(i \Rightarrow iii)$. If $-1 \in F^{x2}$, then $F^\times = S_2(F)$, so $(iii)$ holds by Theorem 1.3. Assume now that $-1 \not\in F^\times \setminus F^{x2}$ and that $(i)$ holds. We set $K = F(\sqrt{-1})$. As $\beta \in \text{Br}_2(F)$, it follows by Theorem 1.3 together with [11, Corollary A4] that $\beta \cup (-1) = 0$ in $H^3(F, \mu_2)$. By [7, Theorem 99.13], we obtain that $\beta = \text{cor}_{K/F} \beta'$ for some $\beta' \in \text{Br}_2(K)$. By Theorem 1.3 and [7, Proposition 100.2], $\text{Br}_2(K)$ is generated by the classes of $K$-quaternion algebras $(x, t)_K$ with $x \in K^\times$ and $t \in F^\times$, and the corestriction with respect to $K/F$ of such a class is given by $(N_{K/F}(x), t)_F$. Since $N_{K/F}(K^\times) \subseteq S_2(F)$ and $\beta = \text{cor}_{K/F} \beta'$, we obtain that $\beta$ is given by $\bigotimes_{i=1}^{m} (s_i, t_i)_F$ for some $m \in \mathbb{N}, s_1, \ldots, s_m \in S_2(F)$ and $t_1, \ldots, t_m \in F^\times$.

Hence, the equivalence of $(i) – (iii)$ is established and it remains to prove the supplementary statement under the assumption that $\text{ind } \beta \leq 4$. In this case $\beta$ is the class of an $F$-biquaternion algebra. It follows by [10, Section 16.A] that $\beta = e_2(q')$ for a 6-dimensional form $q'$ in $I^2 F$. By $(ii)$, there also exists a 2-torsion form $q$ in $I^2 F$ with $\beta = e_2(q)$. Then $q' \perp -q$ is a form in $I^2 F$ with $e_2(q' \perp -q) = 0$. As mentioned above, this implies that $q' \perp -q$ is a form in $I^2 F$. Since $2 \times q$ is hyperbolic, the Witt class of $2 \times q'$ lies in $I^4 F$. Note that dim $2 \times q' < 16$. Thus, $2 \times q'$ is hyperbolic, by [7, Theorem 23.7], and hence Lemma 3.2 yields the result. □

By Proposition 3.3, for $p = e = 2$, the hypotheses of Proposition 2.7 on $2 \text{Br}(F) \cap \text{Br}_2(F)$ are satisfied unconditionally. Hence, one gets a positive answer to Question 1.1 for $p^e = 4$. We will formulate this result together with a bound on the index of exponent-4 algebras in terms of the 2-symbol length.

For $\alpha \in \text{Br}_4(F)$, we denote by $\mu(\alpha)$ the smallest $m \in \mathbb{N}$ for which there exist $s_1, \ldots, s_m \in S_2(F)$ and $t_1, \ldots, t_m \in F^\times$ with $2\alpha = \sum_{i=1}^{m} [(s_i, t_i)_F]$, noticing that such a representation does exist in view of Proposition 3.3. We set further

$$\mu(F) = \sup \{ \mu(\alpha) \mid \alpha \in \text{Br}_4(F) \} \in \mathbb{N} \cup \{\infty\}.$$

**Remark 3.4.** If $S_2(F) = F^\times$, then $\mu(F) = \lambda_2(F)$.

The invariants $\lambda_2(F)$ and $\mu(F)$ are related to the existence of anisotropic torsion (respectively 2-torsion) forms over $F$ in certain dimensions. Recall that the $u$-invariant of $F$ is defined as

$$u(F) = \sup \{ \dim q \mid q \text{ anisotropic torsion form over } F \} \in \mathbb{N} \cup \{\infty\}.$$
We refer to [15, Chapter 8] for a general discussion of this invariant.

**Proposition 3.5.** If $F$ is nonreal, then $\lambda_2(F) \leq \max\left\{0, \frac{1}{2} u(F) - 1\right\}$.

**Proof.** See [9, Théorème 2].

In [15, Section 8.2], the following relative of the $u$-invariant is studied.

\[ u'(F) = \sup\{\dim q \mid q \text{ anisotropic } 2\text{-torsion form over } F\} \in \mathbb{N} \cup \{\infty\}. \]

Note that clearly $u'(F) \leq u(F)$.

**Proposition 3.6.** We have $\mu(F) \leq \max\left\{0, \frac{1}{2} u'(F) - 1\right\}$.

**Proof.** We need to show that $\mu(\alpha) \leq m$ holds for any $\alpha \in \text{Br}_4(F)$ and any $m \in \mathbb{N}^+$ with $u'(F) \leq 2m + 2$. Let $m \in \mathbb{N}^+$ be such that $u'(F) \leq 2m + 2$. Let $\alpha \in \text{Br}_4(F)$. By Proposition 3.3, we have $2\alpha = e_2(q)$ for some 2-torsion form $q$ in $F^\perp$. Then $\dim q \leq u'(F) \leq 2m + 2$. Hence, $q$ is even-dimensional and we obtain that $q$ is Witt equivalent to a quadratic form of dimension $2m + 2$. It follows by Lemma 3.2 that $q$ is Witt equivalent to $\bigwedge_{i=1}^m a_i \langle s_i, t_i \rangle$ for some $s_1, \ldots, s_m \in S_2(F)$ and $a_1, t_1, \ldots, a_m, t_m \in F^\times$. Then

\[ 2\alpha = e_2(q) = e_2 \left( \bigwedge_{i=1}^m a_i \langle s_i, t_i \rangle \right) = \sum_{i=1}^m \langle s_i, t_i \rangle, \]

whereby $\mu(\alpha) \leq m$. \hfill $\square$

The last statements motivate the following question.

**Question 3.7.** Is $\mu(F) \leq \lambda_2(F)$?

If $\lambda_2(F) \leq 2$, then a positive answer to Question 3.7 is obtained by Proposition 3.3. In the following example, the inequality in Proposition 3.6 is strict.

**Example 3.8.** Consider the iterated power series field $F = \mathbb{C}((x))((y))((z))$. The 8-dimensional quadratic form $\varphi = \langle 1, x, y, z, xy, xz, yz, xyz \rangle$ over $F$ is anisotropic. Since $-1$ is square in $F$ and $F^\times / F^\times 2$ is generated by the square-classes of $x, y$ and $z$, it is easy to see that every anisotropic quadratic form over $F$ is a subform of $\varphi$. This implies on the one hand that $u(F) = 8$, on the other hand that $\lambda_2(F) = 1$, because there is no anisotropic 6-dimensional form in $F^2$. Furthermore $-1 \in F^\times 2$, so $u'(F) = u(F) = 8$ and $\mu(F) = \lambda_2(F) = 1$.

**Proposition 3.9.** Let $\alpha \in \text{Br}_4(F)$. There exist a natural number $m \leq \mu(F)$ and 4-cycles $\alpha_1, \ldots, \alpha_m \in \text{Br}(F)$ such that $\alpha \equiv \sum_{i=1}^m \alpha_i \mod \text{Br}_2(F)$.

**Proof.** By Proposition 3.3 and the definition of $\mu(F)$, there exist a natural number $m \leq \mu(F)$ and $s_1, \ldots, s_m \in S_2(F)$ and $t_1, \ldots, t_m \in F^\times$ such that $2\alpha = \sum_{i=1}^m \langle s_i, t_i \rangle_F$. By Proposition 3.1, for $1 \leq i \leq m$, we can find a 4-cycle $\alpha_i \in \text{Br}_4(F)$ such that $2\alpha_i = \langle (s_i, t_i)_F \rangle$. We obtain that $2\alpha - \sum_{i=1}^m 2\alpha_i = 0$, whereby $\alpha - \sum_{i=1}^m \alpha_i \in \text{Br}_2(F)$. Therefore, $\alpha \equiv \sum_{i=1}^m \alpha_i \mod \text{Br}_2(F)$. \hfill $\square$
We retrieve [13, Proposition 6.16]:

**Corollary 3.10.** \( \text{Br}_4(F) \) is generated by cycles.

**Proof.** By Theorem 1.3, \( \text{Br}_2(F) \) is generated by classes of \( F \)-quaternion division algebras and thus by 2-cycles. The statement now follows by combining this fact with Proposition 3.9. \( \square \)

**Theorem 3.11.** We have \( \lambda_4(F) \leq \lambda_2(F) + \mu(F) \). Furthermore, for \( \alpha \in \text{Br}_4(F) \), there exist \( \beta \in \text{Br}_4(F) \) with \( \lambda_4(\beta) \leq \mu(\alpha) \) and \( \gamma \in \text{Br}_2(F) \) such that \( \alpha = \beta + \gamma \), and in particular \( \text{ind} \alpha = 2^n \) for some natural number \( n \leq \lambda_2(F) + 2\mu(F) \).

**Proof.** Let \( \alpha \in \text{Br}_4(F) \) and set \( m = \mu(\alpha) \). By Proposition 3.9, we obtain that \( \alpha = \sum_{i=1}^{m} \alpha_i + \gamma \) for some 4-cycles \( \alpha_1, \ldots, \alpha_m \in \text{Br}_4(F) \) and some \( \gamma \in \text{Br}_2(F) \). Set \( \beta = \sum_{i=1}^{m} \alpha_i \). Then \( \beta \in \text{Br}_4(F) \) and

\[
\lambda_4(\alpha) \leq \lambda_4(\gamma) + \lambda_4(\beta) \leq \lambda_2(\gamma) + m \leq \lambda_2(F) + \mu(F).
\]

Note that \( \text{ind} \beta \) divides \( \prod_{i=1}^{m} \text{ind} \alpha_i = 2^{2m} \). Since \( \text{ind} \gamma \) divides \( 2^{\lambda_2(\gamma)} \) and \( \text{ind} \alpha \) divides \( \text{ind} \beta \cdot \text{ind} \gamma \), we obtain that \( \text{ind} \alpha = 2^n \) for some \( n \in \mathbb{N} \) with \( n \leq \lambda_2(F) + 2\mu(F) \). \( \square \)

Note that when \( F \) contains a primitive 4-th root of unity, the bounds in Theorem 3.11 coincide with those in Theorem 2.6.

**Corollary 3.12.** Assume that \( F \) is nonreal. Let \( \alpha \in \text{Br}_4(F) \). Then \( \text{ind} \alpha = 2^n \) for some natural number \( n \leq \max \left\{ 0, 3 \left( \frac{1}{2} u(F) - 1 \right) \right\} \).

**Proof.** Since \( u'(F) \leq u(F) \), this follows by Theorem 3.11 together with Proposition 3.5 and Proposition 3.6. \( \square \)

**Proposition 3.13.** Let \( l = \lambda_3(F) \) and \( m = \mu(F) \) and assume that \( l + m < \infty \). Let \( D \) be a central \( F \)-division algebra of degree \( 2^{l+2m} \) for which \( D^{\otimes 4} \) is split. There exist \( F \)-quaternion algebras \( Q_1, \ldots, Q_l \) and cyclic \( F \)-algebras \( C_1, \ldots, C_m \) of degree 4 such that

\[
D \simeq \left( \bigotimes_{i=1}^{l} Q_i \right) \otimes \left( \bigotimes_{i=1}^{m} C_i \right).
\]

**Proof.** By Theorem 3.11, the class of \( D \) in \( \text{Br}(F) \) is represented by such a tensor product, and since the degrees coincide, the statement follows. \( \square \)

**Corollary 3.14.** Assume that \( F \) is nonreal and let \( m \in \mathbb{N} \) be such that \( u(F) = 2m + 2 \). Let \( D \) be a central \( F \)-division algebra such that \( D^{\otimes 4} \) is split and \( \deg D = 2^{3m} \). Then \( D \) is decomposable into a tensor product of \( m \) \( F \)-quaternion algebras and \( m \) cyclic \( F \)-algebras of degree 4.

**Proof.** Since \( u(F) = 2m + 2 \), we have \( \lambda_3(F) \leq m \), by Proposition 3.5, and further \( \mu(F) \leq m \), by Proposition 3.6. The statement follows by Proposition 3.13. \( \square \)
**Theorem 3.15.** Assume that $F$ is nonreal with $u(F) = 4$. Let $D$ be a central $F$-division algebra of degree 8 such that $D^\otimes 4$ is split. Then $D$ decomposes into a tensor product of a cyclic $F$-algebra of degree 4 and an $F$-quaternion algebra. Furthermore, $\text{ind} D^\otimes 2 = 2$, and $u(K) = 6$ holds for every quadratic field extension $K/F$ such that $(D^\otimes 2)_K$ is split.

**Proof.** The first part follows by Corollary 3.14 applied with $m = 1$.

Since $u(F) = 4$, we have $\lambda_2(F) \leq 1$, by Proposition 3.5. Hence, $\text{ind} C \leq 2$ for every central simple $F$-algebra $C$ such that $C^\otimes 2$ is split. Since $\text{ind} D > 2$ and $D^\otimes 4$ is split, we conclude that $\text{ind} D^\otimes 2 = 2$.

Consider now a quadratic field extension $K/F$ such that $(D^\otimes 2)_K$ is split. Note that $(D^\otimes 2)_K \cong (D_K)^\otimes 2$ and $\text{ind} D_K \geq \frac{1}{2} \text{ind} D = 4$. Hence, $D_K$ represents an element of $\text{Br}_2(K)$ which is not given by any $K$-quaternion algebra. This shows that $\lambda_2(K) \geq 1$. It follows by Proposition 3.5 that $u(K) \geq 6$. On the other hand, since $u(F) = 4$ and $[K : F] = 2$, it follows by [8, Theorem 4.3] that $u(K) \leq \frac{3}{2} u(F) \leq 6$. Therefore, $u(K) = 6$. \hfill \Box

**4. Examples of fields with $u$-invariant 6**

In this section, we provide a construction leading to an example which shows that the bound in Corollary 3.12 is optimal for fields of $u$-invariant 4. In particular this construction provides examples of nonreal fields of $u$-invariant 6.

Let $q$ be a regular quadratic form over $F$ of dimension $n \geq 2$. If $n = 2$, then assume that $q$ is not hyperbolic. Then as a polynomial in $F[X_1, \ldots, X_n]$, the quadratic form $q(X_1, \ldots, X_n)$ is irreducible. Thus, the ideal generated by $q(X_1, \ldots, X_n)$ in the polynomial ring $F[X_1, \ldots, X_n]$ is a prime ideal, and hence the quotient ring $F[X_1, \ldots, X_n]/(q(X_1, \ldots, X_n))$ is a domain. Its fraction field is denoted by $F(q)$ and called the function field of $q$ over $F$.

**Lemma 4.1.** Let $m, n \in \mathbb{N}^+$. Let $\alpha \in \text{Br}(F)$ be such that $\text{ind} \alpha = 2^n$. Let $q$ be a regular $(2m + 1)$-dimensional quadratic form over $F$ such that $\text{ind} \alpha_{F(q)} < \text{ind} \alpha$. Then $n \geq m$. Moreover, if $n > m$, then $\text{ind} 2\alpha \leq 2^{n-m-1}$.

**Proof.** Let $D$ be the central $F$-division algebra representing $\alpha$ in $\text{Br}(F)$. Then $\deg D = \text{ind} \alpha = 2^n$. Let $C_0(q)$ denote the even Clifford algebra of $q$. By [7, Proposition 11.6], the $F$-algebra $C_0(q)$ is central simple. As $\dim_F C_0(q) = 2^m$, we have $\deg C_0(q) = 2^m$. By [7, Example 11.3 and Proposition 11.4 (b)], $C_0(q)$ carries an $F$-linear involution. Therefore, $(C_0(q))^\otimes 2$ is split.

Since $\text{ind} D_{F(q)} = \text{ind} \alpha_{F(q)} < \text{ind} \alpha = \deg D$, it follows by [7, Proposition 30.5], that there exists an $F$-algebra homomorphism $C_0(q) \rightarrow D$. As $C_0(q)$ and $D$ are central simple $F$-algebras, it follows that $D \cong C_0(q) \otimes_F B$ for a central $F$-division algebra $B$. Hence, $2^n = \deg D = 2^m \cdot \deg B$, so in particular $n \geq m$.

Assume now that $n > m$. Then $\text{ind} B = \deg B = 2^{n-m} > 2$. Since $(C_0(q))^\otimes 2$ is split, the class $2\alpha \in \text{Br}_2(F)$ is given by $B^\otimes 2$. Hence, $\text{ind} 2\alpha = \text{ind} B^\otimes 2$. By [3, Lemma 5.7], we have $\text{ind} B^\otimes 2 \leq \frac{1}{2} \text{ind} B$. Therefore, $\text{ind} 2\alpha \leq 2^{n-m-1}$. \hfill \Box
Theorem 4.2. Let \( \mathcal{C} \) be a class of field extensions of \( F \) with the following properties:

(i) \( \mathcal{C} \) is closed under direct limits,
(ii) if \( L/F \in \mathcal{C} \) and \( K/F \) is a subextension of \( L/F \) then \( K/F \in \mathcal{C} \),
(iii) \( F/F \in \mathcal{C} \).

Then there exists a field extension \( K/F \in \mathcal{C} \) such that \( K(\varphi)/F \not\in \mathcal{C} \) for any anisotropic quadratic form \( \varphi \) over \( K \) of dimension at least 2.

Proof. See [5, Theorem 6.1]. \( \square \)

The following statement and its hypotheses are motivated by an application which we obtain in Example 4.4.

Proposition 4.3. Let \( m, e \in \mathbb{N}^+ \) with \( m \geq 2 \). Let \( \alpha \in \operatorname{Br}(F) \) be such that \( \exp \alpha = 2^e \), \( \operatorname{ind} \alpha = 2^{me-1} \) and \( \exp 2^i \alpha = 2^{me-i-1} \) for \( 0 \leq i \leq e-1 \). There exists a field extension \( K/F \) such that \( u(K) \leq 2m \), \( \exp \alpha_K = 2^e \) and \( \operatorname{ind} \alpha_K = 2^{me-1} \).

Proof. Let \( \mathcal{C} \) be the class of field extensions \( K/F \) such that \( \operatorname{ind} 2^i \alpha_K \geq 2^{me-mi-1} \) for \( 0 \leq i \leq e-1 \). Then \( \mathcal{C} \) satisfies the conditions of Theorem 4.2. Hence, there exists a field extension \( K/F \in \mathcal{C} \) such that \( K(\varphi)/F \not\in \mathcal{C} \) for any anisotropic quadratic form \( \varphi \) over \( K \) of dimension at least 2. As \( \exp 2^i \alpha_K \geq 2^{m-e}, m \geq 2 \) and \( \exp \alpha = 2^e \), we get that \( \exp \alpha_K = 2^e \). Since \( \operatorname{ind} \alpha_K \geq 2^{me-1} \) and \( \operatorname{ind} \alpha = 2^{me-1} \), we have that \( \operatorname{ind} \alpha_K = 2^{me-1} \).

Let \( \varphi \) be an arbitrary \((2m+1)\)-dimensional quadratic form over \( K \). We claim that \( \varphi \) is isotropic. Let \( \alpha_i = 2^i \alpha_K \) for \( 0 \leq i \leq e-1 \). We will check for \( 0 \leq i \leq e-1 \) that the inequality \( \operatorname{ind} \alpha_i \geq 2^{me-mi-1} \) is preserved under scalar extension from \( K \) to \( K(\varphi) \). Consider first the case where \( i = e-1 \). If \( \operatorname{ind} \alpha_{e-1} = 2^{m-1} \), then \( \operatorname{ind}(\alpha_{e-1})_{K(\varphi)} = 2^{m-1} \), by Lemma 4.1. Otherwise, \( \operatorname{ind} \alpha_{e-1} \geq 2^{m} \), and therefore \( \operatorname{ind}(\alpha_{e-1})_{K(\varphi)} \geq 2^{m-1} \). Consider now the case where \( 0 \leq i \leq e-2 \). Note that \( me-mi-1 \geq m+1 \), because \( m \geq 2 \). If \( \operatorname{ind} \alpha_i = 2^{me-mi-1} \), then since \( \operatorname{ind} 2\alpha_i = \operatorname{ind} \alpha_{i+1} \geq 2^{me-mi-1-m} \), we conclude by Lemma 4.1 that \( \operatorname{ind}(\alpha_i)_{K(\varphi)} = \operatorname{ind} \alpha_i \). Otherwise, \( \operatorname{ind} \alpha_i \geq 2^{me-mi} \), and hence \( \operatorname{ind}(\alpha_i)_{K(\varphi)} \geq 2^{me-mi-1} \). Therefore, we have \( \operatorname{ind}(\alpha_i)_{K(\varphi)} \geq 2^{me-mi-1} \) for \( 0 \leq i \leq e-1 \). This shows that \( K(\varphi)/F \in \mathcal{C} \). In view of the choice of \( K \), this implies that \( \varphi \) is isotropic. This argument shows that \( u(K) \leq 2m \). \( \square \)

We can now show that the bound in Corollary 3.12 is optimal when \( u(F) \leq 4 \).

Example 4.4. Let \( m, e \in \mathbb{N}^+ \) with \( m \geq 2 \). By [16, Construction 2.8], there exist a nonreal field \( F \) of characteristic different from 2 and a central \( F \)-division algebra \( D \) such that \( \exp D = 2^e \), \( \deg D = 2^{me-1} \) and \( \operatorname{ind} D^{2^i} = 2^{me-i-1} \) for \( 1 \leq i \leq e-1 \). Then Proposition 4.3 (applied to the the Brauer equivalence class of \( D \)) yields a field extension \( F'/F \) such that \( u(F') \leq 2m \), \( \exp D_{F'} = 2^e \) and \( \operatorname{ind} D_{F'} = 2^{me-1} \). In the case where \( m = 2 \), it follows that \( u(F') = 4 \).

Example 4.5. By Example 4.4, there exist a nonreal field \( F \) with \( \text{char} F \neq 2 \) together with an \( F \)-division algebra \( D \) of degree 8 such that \( u(F) = 4 \) and \( D^{\otimes 4} \)
is split. By Theorem 3.15, it follows that \( \text{ind } D^{\otimes 2} = 2 \) and that \( u(K) = 6 \) for every quadratic field extension \( K/F \) such that \( (D^{\otimes 2})_K \) is split.

**Acknowledgments**

We like to thank the referee for their comments. This work was supported by the *Fonds Wetenschappelijk Onderzoek – Vlaanderen (FWO)* in the FWO Odysseus Programme (project GOE6114N ‘Explicit Methods in Quadratic Form Theory’), the *Fondazione Cariverona* in the programme Ricerca Scientifica di Eccellenza 2018 (project ‘Reducing complexity in algebra, logic, combinatorics - REDCOM’), and by the *FWO-Tournesol programme* (project VS05018N).

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