On admissible square roots of non-negative $C^{2,2\alpha}$ functions

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Abstract. We establish a necessary and sufficient condition for $C^{1,\alpha}$ regularity of the admissible square root of a non-negative $C^{2,2\alpha}(\mathbb{R})$ function.

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1. Introduction

The paper concerns the following problem: the regularity of square root of $C^{2,2\alpha}$ non-negative functions. Nirenberg-Trèves’ gradient estimate for non-negative $C^{1,1}(\mathbb{R}^n)$ functions [14] implies square roots of these functions are Lipschitz. This estimate plays important roles in analysis of linear and nonlinear PDEs (e.g., [9], [1]). The sum of squares theorem of Fefferman and Phong [4, 5] stated that any non-negative $C^{3,1}$ function in $\mathbb{R}^n$ can be written as a sum of squares of $C^{1,1}$ functions. A detailed proof was given in [7] which was communicated by Fefferman (see also [3], [16]). This decomposition is crucial to obtain $C^2$ a priori estimates for degenerate real Monge-Ampère equations in [7] and complex Monge-Ampère equation in [15].

For functions of one variable, Glaeser [6] proved that if $0 \leq f \in C^2(\mathbb{R})$ is 2-flat on its zeroes (i.e., $f(x) = 0$ implies $f''(x) = 0$), then $f^{1/2} \in C^1(\mathbb{R})$. Mandai [13] proved that for any $0 \leq f \in C^2(\mathbb{R})$, $f$ always has an admissible square root $g \in C^1(\mathbb{R})$. In [3], Bony, Broglia, Colombini and Pernazza obtain a necessary and sufficient condition for a non-negative function $f \in C^4(\mathbb{R})$ to have an admissible square root in $C^2(\mathbb{R})$, which is only related to the non-zero

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local minimum points of $f$. Korobenko-Sawyer [12] consider higher regularity of square root functions under appropriate sufficient conditions.

The main result of this paper is the necessary and sufficient condition for optimal $C^{1,\alpha}$ regularity of square roots of $C^{2,2\alpha}(\mathbb{R})$ non-negative functions. In the rest of this paper, $C^{2,2\alpha}(\mathbb{R})$ indicates $C^{3,2\alpha-1}(\mathbb{R})$ if $1/2 < \alpha \leq 1$. Below is the statement of the main theorem.

**Theorem 1.1.** Let $0 \leq f \in C^{2,2\alpha}(\mathbb{R})$ with $\|f\|_{C^{2,2\alpha}(\mathbb{R})} \leq 1$. 0 < $\alpha$ ≤ 1. Define the set

$$\mathcal{A} = \{x_0 \in \mathbb{R} : f(x_0) > 0, f'(x_0) = 0, f''(x_0) > 0\}. \quad (1)$$

Then $f = g^2$ for some $g \in C^{1,\alpha}(\mathbb{R})$ if and only if there is a constant $M > 0$ such that

$$f''(x_0) \leq M \cdot (f(x_0))^{\frac{\alpha}{1+\alpha}}, \quad \forall x_0 \in \mathcal{A}. \quad (2)$$

Moreover, if (2) is satisfied, then $\|g\|_{C^{1,\alpha}(\mathbb{R})} \leq C$ for some universal $C > 0$, depending only on $\alpha$ and $M$.

**Remark 1.2.** The condition obtained by Bony, Broglia, Colombini and Pernazza in [3] is there is a continuous function $\gamma$ vanishing at every flat points of $f$ such that

$$f''(x_0) \leq \gamma(x_0) \cdot (f(x_0))^\frac{1}{2}, \quad \forall x_0 \in \mathcal{A}. \quad (3)$$

Condition (2) is a $C^{2,2\alpha}$ version of (3).

The main theorem is motivated by regularity problem associated to the isometric embedding problem. Guan and Li [8] showed that if $g$ is a $C^4$ Riemannian metric on $\mathbb{S}^2$ with Gauss curvature $K_g \geq 0$, then there exists a $C^{1,1}$ isometric embedding $X : (\mathbb{S}^2, g) \to (\mathbb{R}^3, g_{\text{Euc}})$. A natural question is, can the embedding $X$ be improved to $C^{2,1}$? A positive answer was given in Jiang [11] in the graph setting, under the assumption $X$ takes the form $X(x, y) = (x, y, u(x, y))$ in local coordinates. It relies on a square root regularity for square of monotone functions. It is a special case of Theorem 1.1 where $\alpha = 1$ and $\mathcal{A} = \emptyset$, which can be stated as follows.

**Corollary 1.3.** Let $I = [-1/2, 1/2]$. Assume $0 \leq f \in C^{3,1}(I)$ satisfies $\|f\|_{C^{3,1}(I)} \leq 1$. The zero set of $f$ in $I$ is a closed interval $N = [x'_0, x_0]$ (possibly $x'_0 = x_0$). $f$ is non-increasing in $[-1/2, x'_0)$ and $f$ is non-decreasing in $(x_0, 1/2]$. Then $\exists g \in C^{1,1}(I)$ such that $f = g^2$ in $I$, $g$ is non-decreasing in $I$ and $\|g\|_{C^{1,1}(I)} \leq C$ for some universal constant $C > 0$.

2. Fefferman-Phong’s Lemma for $C^{2,2\alpha}$ non-negative functions

The following lemma is well known (e.g. [16]). We provide a proof here for completeness.

**Lemma 2.1** (Even dominate odd, $C^{2,\alpha}$). Let $0 < \alpha \leq 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be a $C^2$ non-negative function such that $[f]_{C^{2,\alpha}(\mathbb{R})} \leq 1$. Then

$$|f'(x)| \leq \frac{3}{2} |f(x)|^{\frac{1+\alpha}{2\alpha}} + \frac{1}{2} |f''(x)| \cdot f(x)^{\frac{1}{2\alpha}} + f(x)^{\frac{\alpha}{2\alpha}} \cdot |f''(x)|^{\frac{1}{\alpha}} \quad \forall x \in \mathbb{R}. \quad (4)$$
Proof. We may assume \( f(x) \neq 0 \). By Taylor expansion, \( \forall x, h \in \mathbb{R}, \exists \xi \) between \( x, x + h \) such that
\[
0 \leq f(x + h) = f(x) + f'(x)h + \frac{1}{2} f''(x)h^2 + \frac{1}{2} \frac{f'''(\xi) - f''(x)}{|\xi - x|} |\xi - x|^2 h^2
\]
\[
\leq f(x) + f'(x)h + \frac{1}{2} f''(x)h^2 + \frac{1}{2} |h|^{2+\alpha}.
\]
Replacing \( h \) with \( \pm h \),
\[
|f'(x)h| \leq f(x) + \frac{1}{2} |f''(x)| h^2 + \frac{1}{2} |h|^{2+\alpha}.
\] (5)

Setting \( h = \frac{f(x)^\frac{2}{1+\alpha}}{f(x)^\frac{2}{1+\alpha} + |f''(x)|^\frac{1}{1+\alpha}} \) in (5) and using \( h \leq f(x)^\frac{1}{2+\alpha} \), we obtain (4). \( \square \)

Lemma 2.2 (Even dominate odd, \( C^{3,\alpha} \)). Let \( 0 < \alpha \leq 1 \). Let \( f : \mathbb{R} \to \mathbb{R} \) be a \( C^3 \) non-negative function such that \( [f]^\infty_1(\mathbb{R}) \leq 1 \). Then
\[
|f'(x)| \leq \frac{13}{6} f(x)^\frac{2\alpha}{3+\alpha} + \frac{3}{2} f(x)^\frac{1}{3+\alpha} |f''(x)| \frac{1}{1+\alpha} + f(x)^\frac{1}{3+\alpha} [f''(x)], \quad \forall x \in \mathbb{R}. \tag{6}
\]

\[
|f''(x)| \leq 6 f(x)^\frac{2}{3+\alpha} + 6 |f''(x)| \frac{2}{1+\alpha}, \quad \forall x \in \mathbb{R}. \tag{7}
\]

Proof. By Taylor expansion, \( \forall x \in \mathbb{R}, \)
\[
0 \leq f(x + h) \leq f(x) + f'(x)h + \frac{1}{2} f''(x)h^2 + \frac{1}{6} f'''(x)h^3 + \frac{1}{6} |h|^{3+\alpha}.
\] (8)

Replacing \( h \) with \( \pm h \),
\[
|f'(x)h + \frac{1}{6} f'''(x)h^3| \leq f(x) + \frac{1}{2} |f''(x)| h^2 + \frac{1}{6} |h|^{3+\alpha} =: A. \tag{9}
\]

Replacing \( h \) by \( 2h \) in (9), we have
\[
|2 \cdot f'(x)h + \frac{1}{6} f'''(x)h^3| \leq f(x) + \frac{1}{2} |f''(x)|(2h)^2 + \frac{1}{6} |2h|^{3+\alpha} =: B. \tag{10}
\]

Combining (9) and (10),
\[
|f'(x)h| \leq \frac{8A + B}{6}, \quad |\frac{1}{6} f'''(x)h^3| \leq \frac{2A + B}{6}. \tag{11}
\]

If \( f(x) = 0 \), then \( f'(x) = 0 \) since \( f \geq 0 \). Otherwise, setting \( h = \frac{f(x)^\frac{2}{1+\alpha}}{f(x)^\frac{2}{1+\alpha} + |f''(x)|^\frac{1}{1+\alpha}} \) in (11) and using \( h \leq f(x)^\frac{1}{2+\alpha} \), we have
\[
|f'(x)| \leq \frac{1}{6} \left( 9 \cdot f(x)^\frac{1}{h} + 6 \cdot |f''(x)| h + 4 \cdot |h|^{2+\alpha} \right)
\]
\[
\leq \frac{1}{6} \left( 9 \cdot f(x)^\frac{1+\alpha}{3+\alpha} (f(x)^\frac{1}{3+\alpha} + |f''(x)|^\frac{1}{3+\alpha}) + 6 \cdot |f''(x)| \cdot f(x)^\frac{1}{3+\alpha} + 4 \cdot f(x)^\frac{2\alpha}{3+\alpha} \right).
\]

Thus, (6) holds.
If \( f(x) = f''(x) = 0 \), then \( f'''(x) = 0 \) by (8). Otherwise, letting \( h = \max\{f(x)^{\frac{1}{3+\alpha}}, |f''(x)|^{\frac{1}{1+\alpha}}\} \) and using \( \max\{a, b\} \leq a + b \) in (11), and as \( (a + b)^{\alpha} \leq a^{\alpha} + b^{\alpha} \) for \( a, b \geq 0 \) and \( 0 < \alpha \leq 1 \), we have,

\[
|f'''(x)| \leq \frac{3f(x)}{h^3} + \frac{3|f''(x)|}{h} + \left( \frac{1}{3} + \frac{1}{6} \cdot 2^{3+\alpha} \right) \cdot |h|^\alpha \\
\leq 3f(x)^{\frac{\alpha}{3+\alpha}} + 3|f''(x)|^{\frac{\alpha}{1+\alpha}} + 3 \cdot \left( f(x)^{\frac{\alpha}{3+\alpha}} + |f''(x)|^{\frac{\alpha}{1+\alpha}} \right).
\]

Thus, (7) holds. \( \square \)

We define some constants which will be used in the rest of the paper.

\[
c_0 = 1/10, \quad C = 1000; \quad N(\alpha) = 2, \text{ if } 0 < \alpha \leq 1/2, \quad N(\alpha) = 3, \text{ if } 1/2 < \alpha \leq 1; \\
\epsilon_0 = \left( \frac{1}{10^5} \right)^{1/(2\alpha)}, \text{ if } 0 < \alpha \leq 1/2, \quad \epsilon_0 = \left( \frac{1}{10^5} \right)^{1/(2\alpha-1)}, \text{ if } 1/2 < \alpha \leq 1; \\
\tilde{c} = \frac{1}{10^5} \cdot \left( \frac{1}{10^5} \right)^{3/(2\alpha)}, \text{ if } 0 < \alpha \leq 1/2, \quad \tilde{c} = \frac{1}{10^5} \cdot \left( \frac{1}{10^5} \right)^{4/(2\alpha-1)}, \text{ if } 1/2 < \alpha \leq 1.
\] (12)

Denote the set of flat points of \( f \) by

\[
\mathcal{F} = \{ x \in \mathbb{R} : f(x) = f'(x) = f''(x) = 0 \}.
\] (13)

We note that if \( f \in C^3 \) and \( f \geq 0, x \in \mathcal{F} \) implies \( f'''(x) = 0 \).

The next lemma is a \( C^{2,2\alpha} \)-version of Fefferman-Phong’s lemma (see [4] and Lemma 18.6.9 of [10]).

**Lemma 2.3** (Fefferman-Phong’s Lemma). Let \( I = [-1/2, 1/2] \). If \( 0 \leq \phi \in C^{2,2\alpha}(I) \) such that

\[
|\phi^{(k)}(t)| \leq C \quad \forall t \in I \text{ for } k = 0, 1, \ldots, N(\alpha), \quad [\phi]_{C^{2,2\alpha}(I)} \leq 1 \quad (14)
\]

and

\[
\max[\phi(0), |\phi''(0)|] \geq \tilde{c},
\] (15)

where \( N(\alpha), C, \tilde{c} \) are defined in (12). Then there exist universal constants \( r_0 > 0, \hat{A} > 0, c_2 > 0 \) such that, for \( t \in (-r_0, r_0) \), either

\[
c_2 \leq \phi(t) \leq C, \quad \|\sqrt{\phi(t)}\|_{C^{1,\alpha}([-r_0, r_0])} \leq \hat{A};
\] (16)

or

\[
c_2 \leq \phi''(t) \leq C,
\] (17)

\[
\phi(t) = \phi(T) + (t - T)^2 \int_0^1 \phi''(t + s(T - t))s \, ds,
\] (18)

where \( t = T \) is the unique strict local minimum point of the function \( \phi \) in \((-r_0, r_0)\).
Moreover, the function
\[ g(t) = (t - T)(\int_0^1 \phi''(t + s(T - t))s \, ds)^{1/2} \]  
(19)
is in \( C^{1,\alpha}((-r_0, r_0)). \)

**Proof.** Set \( \mu = \min\{\frac{\varepsilon}{3c}, \left(\frac{\varepsilon}{3}\right)^\alpha\} \), where \( \varepsilon \) and \( C \) are defined in (12).

(i) If \( \phi(0) \geq \frac{1}{3}\varepsilon \), then
\[ \phi(t) \geq \frac{1}{3}\varepsilon, \quad \text{and} \quad |(\sqrt{\phi})'(t)| = \left| \frac{\phi'(t)}{2\sqrt{\phi(t)}} \right| \leq \frac{C}{2\sqrt{\frac{1}{3}\varepsilon}} =: b. \]  
(20)

By (14), (20), and the mean value theorem, for \( |t_1| < \mu \) and \( |t_2| < \mu, t_1 \neq t_2 \),
\[ 2|(\sqrt{\phi})'(t_1) - (\sqrt{\phi})'(t_2)|/|t_1 - t_2|^\alpha = \frac{\phi'(t_1)}{\sqrt{\phi(t_1)}} - \frac{\phi'(t_2)}{\sqrt{\phi(t_2)}}/|t_1 - t_2|^\alpha \]
\[ \leq \frac{\phi'(t_1)}{\sqrt{\phi(t_1)}}/|t_1 - t_2|^\alpha + \frac{\phi'(t_2)}{\sqrt{\phi(t_2)}}/|t_1 - t_2|^\alpha \]
\[ \leq \frac{1}{\sqrt{\frac{1}{3}\varepsilon}} \cdot \frac{|\phi''(\xi_1)||t_1 - t_2|^\alpha + C \cdot |\phi'(\xi_2)||t_1 - t_2|}{2\left(\frac{1}{3}\varepsilon\right)^{3/2}|t_1 - t_2|^2} \leq C_1 \]  
(21)
where \( b, C_1 > 0 \) are universal constants, and \( \xi_1, \xi_2 \) are some points between \( t_1, t_2 \).

(ii) Assume \( |\phi''(0)| \geq \varepsilon \).

(a) If \( \phi''(0) \leq -\varepsilon \), then for \( |t| < \mu \), \( \phi''(t) \leq -\frac{1}{3}\varepsilon \). For any \( |t_0| < \frac{1}{2}\mu \), expanding \( \phi \) near \( t_0 \), we have
\[ 0 \leq \phi(t_0 + h) + \phi(t_0 - h) \leq 2 \cdot \left( \phi(t_0) + \frac{1}{2} \cdot (-\frac{1}{3}\varepsilon) \cdot h^2 + \frac{1}{2} |h|^2 + 2\alpha \right). \]

Letting \( h = \frac{1}{2}\mu, \forall |t_0| < \frac{1}{2}\mu, \phi(t_0) \geq \frac{1}{6}\varepsilon h^2 - \frac{1}{2} |h|^2 + 2\alpha \geq \frac{1}{24}\mu^2\varepsilon(1 - 2^{-2\alpha}). \)

Similar to case (i), we have \( \sqrt{\phi} \in C^{1,\alpha}((\mu/2, \mu/2)). \)

(b) If \( \phi''(0) \geq \varepsilon \) and \( \phi(0) < c_1 \), where \( c_1 > 0 \) is a small and universal constant to be determined, then \( |\phi'(0)| \) is also small since \( \phi \geq 0 \). By expansion of \( \phi' \in C^{1,2\alpha}(I) \) near 0,
\[ \phi'(t) = \phi'(0) + \phi''(0)t + R(t), \quad \text{where} \quad |R(t)| \leq C|t|^{1+\alpha}. \]  
(22)
In particular, (22) shows that \( \phi'(r) > 0 \) and \( \phi'(-r) < 0 \) if
\[ \phi''(0)r > |\phi'(0)| + 2\alpha r^{1+\alpha}. \]  
(23)
Fix \( r = \min\{\frac{\varepsilon}{3c}, \left(\frac{\varepsilon}{3}\right)^\alpha\} \). As \( \phi'' \in C^{2\alpha}(I) \),
\[ \phi''(t) \geq \frac{1}{3}\varepsilon, \quad |t| \leq r. \]  
(24)
This implies $\phi'(t)$ is strictly increasing in $[-r, r]$, thus $\phi'(t) = 0$ has a unique solution $t = T$ in $B_r := (-r, r)$. By Taylor expansion of $\phi$ near $t = T$, we obtain in $B_r$,

$$
\phi(t) = \phi(T) + (t - T)^2 \int_0^1 \phi''(t + s(T - t)) s \, ds.
$$

(25)

We note $t = T$ is the unique strict local minimum point of the function $\phi$ in $B_r$. We will estimate Hölder seminorm of $g'$ where $g$ is defined in (19). Assume without loss of generality that $\phi(T) = 0$. Then in $B_r$, $g(t) = \sqrt{\phi(t)}$ if $t \geq T$ and $g(t) = -\sqrt{\phi(t)}$ if $t < T$. By Taylor expansion,

$$
\lim_{t \to T^+} \frac{g(t) - g(T)}{t - T} = \lim_{t \to T^+} \sqrt{\frac{1}{2} \phi''(T)(t - T)^2 + O(|t - T|^{2 + \min\{1, 2\alpha\}})} - \sqrt{0}
$$

$$
= \sqrt{\frac{1}{2} \phi''(T)}.
$$

We obtain the same value for the left limit and hence $g'(T) = \sqrt{\frac{1}{2} \phi''(T)}$.

If $t \neq T$, then by Taylor expansion,

$$
\phi(t) = \frac{1}{2} \phi''(T)(t - T)^2 + A_1,
$$

(26)

$$
\phi'(t) = \phi''(T)(t - T) + A_2,
$$

(27)

$$
\phi''(t) = \phi''(T) + A_3.
$$

(28)

By (24), (14) and $|t - T| \leq 2r$,

$$
|A_1| \leq C_1 \cdot |t - T|^{2 + \min\{1, 2\alpha\}} \leq \frac{1}{3} \phi''(T)(t - T)^2,
$$

$$
|A_2| \leq C_2 \cdot |t - T|^{1 + \min\{1, 2\alpha\}} \leq |\phi''(T)(t - T)|,
$$

$$
|A_3| \leq C_3 \cdot |t - T|^{\min\{1, 2\alpha\}}.
$$

(29)

Suppose $t > T$. By (26), (27), (29) and $\phi''(t) \sim 1$ in $B_r$, $\forall t \in B_r$,

$$
|g'(T) - g'(t)| = \left| \sqrt{\frac{1}{2} \phi''(T)} - \frac{\phi'(t)}{2\sqrt{\phi(t)}} \right|
$$

$$
\leq \left| \frac{\phi'(t)}{2\sqrt{\phi''(T)(t - T)^2}} \right| + \left| \frac{\phi'(t)}{2\sqrt{\phi''(T)(t - T)^2}} - \frac{\phi'(t)}{2\sqrt{\phi(t)}} \right|
$$

$$
= \frac{A_2}{\sqrt{2\phi''(T)(t - T)^2}}
$$

$$
+ \frac{1}{2} |\phi'(t)| \left| \frac{\phi'(t)}{\sqrt{\frac{1}{2} \phi''(T)(t - T)^2} \cdot \sqrt{\phi(t)} \cdot (\sqrt{\phi(t)} + \sqrt{\frac{1}{2} \phi''(T)(t - T)^2})} \right|
$$

$$
\leq b \cdot |T - t|^\alpha,
$$

(30)
where \( b > 0 \) is a universal constant. Proof is the same for \( t < T \).

By (26), (27), (28), and \( \phi''(t) \sim 1 \) in \( B_r \), there exists a universal \( c > 0 \) such
that, \( \forall t \in B_r \) with \( t \neq T \),
\[
|\phi''(t) \cdot \phi(t) - \frac{1}{2} \phi'(t)^2| = O(|t - T|^{2\min\{1, 2\alpha\}}),
\]
\[
l|g''(t)| = \frac{1}{2} \left| \frac{\phi''(t) \cdot \phi(t) - \frac{1}{2} \phi'(t)^2}{\phi(t)^{3/2}} \right| \leq c \cdot |t - T|^{\min\{0, 2\alpha - 1\}}. \tag{31}
\]

\( \forall t_1, t_2 \in B_r \), we want to estimate \( |g'(t_1) - g'(t_2)| \). By (30), we only need to deal with
\[
T < t_1 < t_2 \quad \text{or} \quad t_2 < t_1 < T, \quad \text{with} \quad |t_1 - t_2| \leq |t_1 - T|. \tag{32}
\]

We only consider the case \( T < t_1 < t_2 \) (\( t_1 < t_2 < T \) is similar). By assumption (32),
\[
|\xi - T| \geq |t_1 - T| \geq |t_1 - t_2|, \quad \forall \xi \in (t_1, t_2).
\]
By the mean value theorem, \( \exists \xi \in (t_1, t_2) \) such that
\[
|g'(t_1) - g'(t_2)| = |g''(\xi)||t_1 - t_2| \leq c \cdot |\xi - T|^{\min\{0, 2\alpha - 1\}} \cdot |t_1 - t_2| \leq c \cdot |t_1 - t_2|^\alpha.
\]
(c) If \( c_1 \leq \phi(0) < \epsilon \), then similar to case (i), we have \( \sqrt{\phi} \in C^{1, \alpha}((-\frac{c_1}{3}, \frac{c_1}{3})) \).

To summarize, case (i), (ii)(a) and (ii)(c) lead to (16). Case (ii)(b) leads to (17). \( \square \)

3. A Calderón-Zygmund decomposition

We use the Calderón–Zygmund decomposition, which was originally suggested by Fefferman in [7].

**Lemma 3.1.** Let \( 0 \leq f \in C^{2, 2\alpha}(\mathbb{R}) \) with \( \|f\|_{C^{2, 2\alpha}(\mathbb{R})} \leq 1 \). There is a countable collection of intervals \( \{Q_\nu\}_{\nu \geq 1} \) taking the form of \((a, b)\), whose interiors are disjoint, such that

\begin{enumerate}
  \item \( \mathbb{R} = \mathcal{F} \cup (\bigcup\limits_\nu Q_\nu) \) and \( \mathcal{F} \cap (\bigcup\limits_\nu Q_\nu) = \emptyset \), where \( \mathcal{F} \) is defined in (13).
  \item Let \( \delta_\nu = \text{diam}(Q_\nu) \). Then for any \( \nu, \delta_\nu \leq 1 \). With \( N(\alpha) \) defined in (12),
\[
\inf_{x \in Q_\nu} \left( \sum_{k=0}^{N(\alpha)} \delta_\nu^{k-(2+2\alpha)} |\nabla^k f(x)| \right) > N(\alpha) + 1. \tag{33}
\]
\end{enumerate}

**Proof.** We decompose \( \mathbb{R} \) into a (countable) collection of disjoint intervals \( (a_n, b_n] \) with the same length, and their common diameter is so large that
\[
\inf_{x \in Q_\nu} \left( \sum_{k=0}^{N(\alpha)} (\text{diam}(Q'))^{k-(2+2\alpha)} |\nabla^k f(x)| \right) \leq N(\alpha) + 1
\]
for every interval \( Q' \) in this collection. As \( \|f\|_{C^{2, 2\alpha}(\mathbb{R})} \leq 1 \), the common diameter can be chosen to be 1.

Let \( Q' \) be a fixed interval in this collection. By bisecting, we divide \( Q' \) into 2 congruent intervals. Let \( Q'' \) be one of these new intervals.
(1) If
\[
\inf_{x \in Q''} \left( \sum_{k=0}^{N(\alpha)} (\text{diam}(Q''))^k \cdot (2 + 2\alpha) |\nabla^k f(x)| \right) > N(\alpha) + 1,
\]
then we don't sub-divide $Q''$ any further, and $Q''$ is selected as one of
the intervals $Q_\alpha$.

(2) If
\[
\inf_{x \in Q''} \left( \sum_{k=0}^{N(\alpha)} (\text{diam}(Q''))^k \cdot (2 + 2\alpha) |\nabla^k f(x)| \right) \leq N(\alpha) + 1,
\]
then we proceed with the sub-division of $Q''$, and repeat this process
until we are forced to the case (i).

\[\square\]

**Lemma 3.2.** Let $3Q_\alpha$ be the interval of diameter $3\delta_\alpha$, with the same center as $Q_\alpha$, then
\[
\sum_{k=0}^{N(\alpha)} \delta_\alpha^{k-(2+2\alpha)} |\nabla^k f(x)| \leq C \quad \forall x \in 3Q_\alpha, \quad (34)
\]
where $C$ is defined in (12).

**Proof.** We prove the case where $1/2 < \alpha \leq 1$. $0 < \alpha \leq 1/2$ is similar.

Let $\tilde{Q}_\alpha$ be the step before we get $Q_\alpha$. Then $Q_\alpha \subset \tilde{Q}_\alpha$ and diameter of $\tilde{Q}_\alpha$ is $2\delta_\alpha$.
Since we didn’t stop at $\tilde{Q}_\alpha$, there is $x_0 \in \tilde{Q}_\alpha \subset 3Q_\alpha$ such that
\[
\sum_{k=0}^{3} (2\delta_\alpha)^{k-(2+2\alpha)} |\nabla^k f(x_0)| \leq 4.
\]
That is
\[
|\nabla^k f(x_0)| \leq 4(2\delta_\alpha)^{(2+2\alpha)-k}, \quad k = 0, 1, 2, 3. \quad (35)
\]
Using $\|f\|_{C^{2,2\alpha}(\mathbb{R})} \leq 1$ and $\text{dist}(x, x_0) \leq 3\delta_\alpha$, we get $\forall x \in 3Q_\alpha$,
\[
|\nabla^3 f(x)| \leq |\nabla^3 f(x_0)| + 1 \cdot |x - x_0|^{2\alpha-1}
\leq 4(2\delta_\alpha)^{2+2\alpha-3} + (3\delta_\alpha)^{2\alpha-1}
\leq 11\delta_\alpha^{2\alpha-1}. \quad (36)
\]
Using (35) and (36), we get $\forall x \in 3Q_\alpha$,
\[
|\nabla^2 f(x)| \leq \sup_{3Q_\alpha} |\nabla^3 f| \cdot |x - x_0| + |\nabla^2 f(x_0)|
\leq 11\delta_\alpha^{2\alpha-1} \cdot 3\delta_\alpha + 4(2\delta_\alpha)^{(2+2\alpha)-2}
\leq 49\delta_\alpha^{2\alpha}.
\]

Using (35) and (36), we get $\forall x \in 3Q_\alpha$,
\[
|\nabla^2 f(x)| \leq 179\delta_\alpha^{1+2\alpha} \quad \text{and} \quad |f(x)| \leq 601\delta_\alpha^{2+2\alpha}, \forall x \in 3Q_\alpha.
\]

\[\square\]
Lemma 3.3. Let $Q_\nu^+$ be the interval of diameter $(1 + \epsilon_0)\delta_\nu$, with the same center as $Q_\nu$, then
\begin{equation}
\inf_{x \in Q_\nu^+} \left( \sum_{k=0}^{N(\alpha)} \delta_\nu^{k-(2+2\alpha)} |\nabla^k f(x)| \right) \geq c_0, \tag{37}
\end{equation}
where $c_0, \epsilon_0$ are defined in (12).

Proof. Let $B = \{ x \in \mathbb{R} : \text{dist}(x, x_0) \leq \epsilon_0 \delta_\nu \}$.
We prove the case where $1/2 < \alpha \leq 1$, $0 < \alpha \leq 1/2$ is similar.
Assume not, then $\exists x_0 \in Q_\nu^+$ such that $\sum_{k=0}^{3} \delta_\nu^{3-k-(2+2\alpha)} |\nabla^k f(x_0)| < c_0$.
Using $\|f\|_{C^{2,2\alpha}(\mathbb{R})} \leq 1$ and the mean value theorem, we get
\begin{align*}
|\nabla^3 f(x)| &\leq |\nabla^3 f(x_0)| + 1 \cdot |x - x_0|^{2\alpha-1} \leq (c_0 + 1)\delta_\nu^{2\alpha-1} \quad \forall x \in B. \\
|\nabla^2 f(x)| &\leq \sup_B |\nabla^3 f| \cdot |x - x_0| + |\nabla^2 f(x_0)| \leq (2c_0 + 1)\delta_\nu^{2\alpha} \quad \forall x \in B.
\end{align*}
Going backwards, we get $|\nabla f(x)| \leq (3c_0 + 1)\delta_\nu^{1+2\alpha}$ and $|f(x)| \leq [(3c_0 + 1)\epsilon_0 + c_0]\delta_\nu^{2+2\alpha}$. Note $\epsilon_0 < \frac{1}{10^6}$, so for any $x \in B$, $\sum_{k=0}^{3} \delta_\nu^{k-(2+2\alpha)} |\nabla^k f(x)| < 4$, contradicting with (33).
\[ \square \]

Lemma 3.4. Let $\lambda = \epsilon_0/2$. Let $Q_\nu^+$ be the interval of diameter $(1 + \lambda)\delta_\nu$, with the same center as $Q_\nu$. Then for $z \in Q_\nu^+$, either
\begin{equation}
f(z) \geq \tilde{c}\delta_\nu^{2+2\alpha}, \tag{38}
\end{equation}
or
\begin{equation}
f(z) < \tilde{c}\delta_\nu^{2+2\alpha} \quad \text{and} \quad |\nabla^2 f(z)| \geq \tilde{c}\delta_\nu^{2\alpha}, \tag{39}
\end{equation}
where $\tilde{c}$ is defined in (12).

Proof. By translation we assume $z = 0$, with
\begin{equation}
f(0) < \tilde{c}\delta_\nu^{2+2\alpha} \quad \text{and} \quad |\nabla^2 f(0)| < \tilde{c}\delta_\nu^{2\alpha}. \tag{40}
\end{equation}
First, we assume $1/2 < \alpha \leq 1$. Let $c > 0$ be small such that $2c\delta_\nu < (\text{diam}(Q_\nu^+) - \text{diam}(Q_\nu^+))/2$. By Taylor expansion, (40) and $\|f\|_{C^{2,2\alpha}(\mathbb{R})} \leq 1$, for any $|x| < 2c\delta_\nu$,
\begin{equation}
0 \leq f(x) < \tilde{c}\delta_\nu^{2+2\alpha} + f'(0)x + \frac{1}{2!} \tilde{c}\delta_\nu^{2\alpha}x^2 + \frac{1}{6} f'''(0)x^3 + \frac{1}{6} |x|^{2+2\alpha}. \tag{41}
\end{equation}
Taking $x$ and $-x$ in (41), for any $|x| < 2c\delta_\nu$,
\begin{equation}
|f'(0)x + \frac{1}{6} f'''(0)x^3| \leq \tilde{c}\delta_\nu^{2+2\alpha} + \frac{1}{2} \tilde{c}\delta_\nu^{2\alpha}x^2 + \frac{1}{6} |x|^{2+2\alpha}. \tag{42}
\end{equation}
In particular, for any $|x| < c\delta_\nu$,
\begin{equation}
|f'(0)x + \frac{1}{6} f'''(0)x^3| \leq \tilde{c}\delta_\nu^{2+2\alpha} + \frac{1}{2} \tilde{c}\delta_\nu^{2\alpha}(c\delta_\nu)^2 + \frac{1}{6} |c\delta_\nu|^{2+2\alpha} := A\delta_\nu^{2+2\alpha}. \tag{43}
\end{equation}
On the other hand, by substituting \( x \) with \( 2x \) in (42), for any \( |x| < c\delta_v \),

\[
|f'(0)(2x) + \frac{1}{6}f''(0)(2x^3)| \leq \tilde{c}\delta_v^{2+2\alpha} + \frac{1}{2}c\tilde{c}\delta_v^{2\alpha}(2x)^2 + \frac{1}{6}|2x|^{2+2\alpha}
\]

\[
\leq \tilde{c}\delta_v^{2+2\alpha} + \frac{1}{2}c\tilde{c}\delta_v^{2\alpha}(2c\delta_v)^2 + \frac{1}{6}|2c\delta_v|^{2+2\alpha} =: B\delta_v^{2+2\alpha}.
\]

Combining (43) and (44), we obtain for any \( |x| < c\delta_v \),

\[
|f''(0)x| \leq \frac{1}{6}(8A + B)\delta_v^{2+2\alpha}, \quad |f'''(0)x^3| \leq \frac{1}{6}(2A + B)\delta_v^{2+2\alpha}.
\]

Thus, \( |f'(0)| \leq \frac{8A + B}{6c} \) and \( |f'''(0)| \leq \frac{2A + B}{c^3} \). If \( c = \epsilon_0/10, \tilde{c} = c^4 \), then

\[
\sum_{k=0}^{3} \delta_v^{k-(2+2\alpha)}|\nabla^k f(0)| \leq \tilde{c} + \frac{8A + B}{6c} + \tilde{c} + \frac{2A + B}{c^3} < 0.01^4 + 0.01 + 0.01^4 + 0.07 < c_0,
\]

contradicting with (37).

Next, we deal with the case \( 0 < \alpha < 1/2 \).

Let \( c > 0 \) be small such that \( 2c\delta_v < (diam(Q_v^+) - diam(Q_v^-))/2 \). By Taylor expansion, (40) and \( \|f\|_{C^{2,\alpha}([\epsilon])} \leq 1 \), for any \( |x| < 2c\delta_v \),

\[
0 \leq f(x) \leq \tilde{c}\delta_v^{2+2\alpha} + f'(0)x + \frac{1}{2}c\tilde{c}\delta_v^{2\alpha}x^2 + \frac{1}{2}|x|^{2+2\alpha}.
\]

(45)

If \( c = \epsilon_0/10, \tilde{c} = c^3 \), setting \( x = \pm c\delta_v \) in (45) yields

\[
|f'(0)| \leq (c^2 + \frac{1}{2}c^4 + \frac{1}{2}c^{1+2\alpha})\delta_v^{1+2\alpha} < 0.01\delta_v^{1+2\alpha}.
\]

Hence

\[
\sum_{k=0}^{2} \delta_v^{k-(2+2\alpha)}|\nabla^k f(0)| \leq \tilde{c} + 0.01 + \tilde{c} < 0.01^3 + 0.01 + 0.01^3 < c_0,
\]

contradicting with (37).

For any \( z \in Q_v^+ \), we apply Fefferman-Phong Lemma 2.3 to the function \( \phi(t) = \delta_v^{-(2+2\alpha)} \cdot f(z + t\delta_v) \).

**Corollary 3.5.** Let \( C = 1000 \). For \( z \in Q_v^+ \), there exist universal constants \( r_0 > 0, \tilde{A} > 0, c_2 > 0 \) such that, for \( x \in (z - r_0\delta_v, z + r_0\delta_v) \),

either

\[
c_2\delta_v^{2+2\alpha} \leq f(x) \leq C\delta_v^{2+2\alpha}, \quad (46)
\]

\[
\|\sqrt{f(x)}\|_{C^1((z-r_0\delta_v, z+r_0\delta_v))} \leq \tilde{A}\delta_v^{\alpha},
\]

\[
\|\sqrt{f(x)}\|_{C^2((z-r_0\delta_v, z+r_0\delta_v))} \leq \tilde{A};
\]

or

\[
c_2\delta_v^{2\alpha} \leq f''(x) \leq C\delta_v^{2\alpha},
\]

\[
f(x) = f(X) + (x - X)^2 \int_0^1 f''(x + t(X - x))t \, dt,
\]

(47)
where $x = X$ is the unique strict local minimum point of the function $f$ in $(z - r_0 \delta_x, z + r_0 \delta_x)$.

Moreover, $g(x) = (x - X) \left( \int_0^1 f''(x + t(X - x)) t \, dt \right)^{1/2}$ is in $C^{1, \alpha}((z - r_0 \delta_x, z + r_0 \delta_x))$ with $C^{1, \alpha}$ norm under control.

4. Proof of Theorem 1.1

Let $0 \leq f \in C^{2, \alpha}(\mathbb{R})$ with $\|f\|_{C^{2, \alpha}(\mathbb{R})} \leq 1$.

4.1. Proof of sufficiency.

4.1.1. Construction of $g$. We write $\mathbb{R} \setminus \mathcal{F}$ (where $\mathcal{F}$ is defined in (13)) as a countable union of disjoint open intervals, so that $\mathbb{R} \setminus \mathcal{F} = \bigcup_{k=1}^\infty I_k$. Note if $\exists x_0 \in I_k$ with $f(x_0) = 0$, then $f''(x_0) \neq 0$. (If $0 < \alpha \leq 1/2$, by Lemma 2.1, $|f(x_0)|$ and $|f''(x_0)|$ dominate $|f'(x_0)|$. If $1/2 < \alpha \leq 1$, by Lemma 2.2, $|f(x_0)|$ and $|f''(x_0)|$ dominate $|f'(x_0)|$ and $|f'''(x_0)|$.) For each $m, k \in \mathbb{N}$, define

$$I_{k,m} = \{x \in I_k : \text{dist}(x, \mathcal{F}) > \frac{1}{m}\}, \quad B = \{x \in \mathbb{R} : f(x) = 0, f''(x) \neq 0\}.$$

Lemma 4.1. $I_k \cap B$ is at most countable for each $k$, and

$$I_k \cap B = \{\ldots x_{-2} < x_{-1} < x_0 < x_1 < x_2 \ldots\}.$$

Proof. $\forall N > 0$, we claim that $I_{k,m} \cap B \cap [-N, N]$ is finite for each $m, k \in \mathbb{N}$. Assume $I_{k,m} \cap B \cap [-N, N]$ is infinite, then $\exists x_0 \in \mathbb{R}$ such that $x_0$ is an accumulation point of $I_{k,m} \cap B$. So, there is a sequence $\{x_n\}$ in $B$ such that $\lim_{n \to \infty} x_n = x_0$, and $f(x_0) = \lim_{n \to \infty} f(x_n) = 0$. Note $f \geq 0$, so $f'(x_0) = 0$.

If $f'''(x_0) \neq 0$, then $x = x_0$ is a strict local minimum point of $f$. However, by construction, near $x_0$ there is a point $x_1 \in B$, so that $f(x_1) = 0$, contradicting with strict local minimality.

If $f''(x_0) = 0$, then $x_0 \in \mathcal{F}$. However, $(x_0 - \frac{1}{2m}, x_0 + \frac{1}{2m}) \cap I_{k,m} = \emptyset$, contradiction.

Now since $I_k$ is an interval and $I_{k,m} \subset I_{k,m+1}$, any point in $I_{k,m+1} \setminus I_{k,m}$ is either on the left or right of $I_{k,m}$. The points in $I_k \cap B \cap [-N, N]$ can be ordered. The lemma follows by letting $N \to \infty$. □

We define the function $g$ as follows. If $x \in \mathcal{F}$, set $g(x) = 0$. For each $k$, if $I_k \cap B = \emptyset$ in $I_k$, then define $g(x) = \sqrt{f(x)}$ for $x \in I_k$. Otherwise,

$$I_k \cap B = \{\ldots x_{-2} < x_{-1} < x_0 < x_1 < x_2 \ldots\}.$$

Define $g(x) = (-1)^i \sqrt{f(x)}$ for $x \in [x_{i-1}, x_i]$. Note that $g$ changes sign when crossing each $x_i$ in $I_k$. 
4.1.2. $C^1$ regularity of $g$. $g$ is continuous in each $I_k = (a_k, b_k)$. It suffices to discuss the continuity at $x_0 \in \mathcal{I}$. By Taylor expansion of $f$ near $x_0$, $f(x) = O(|x - x_0|^{2+2\alpha})$, so that $|\pm \sqrt{f(x)}| = O(|x - x_0|^{1+\alpha}) \to 0$ as $x \to x_0$ and $\lim_{x \to x_0} g(x) = 0$.

**Lemma 4.2.** $g \in C^1(I_k)$ for each $k$.

**Proof.** If $I_k \cap B = \emptyset$, then $g' = \frac{f'}{2\sqrt{f}} \in C^0(I_k)$. If $I_k \cap B \neq \emptyset$, then for each $x_i \in I_k \cap B$, $x_i \in Q_\nu$ for some $\nu = \nu(x_i)$. By Corollary 3.5, only (47) holds and near $x_i$, $f$ can locally be written as

$$f(x) = (x - x_i)^2 \int_0^1 f''(x + t(x_i - x)) t \, dt,$$

with $\int_0^1 f''(x + t(x_i - x)) t \, dt \sim \delta_{\nu}^{2\alpha}$. By definition of $g$, near $x_i$, $g(x) = \pm (x - x_i)(\int_0^1 f''(x + t(x_i - x)) t \, dt)^{1/2}$ (the sign depends only on the choice of sign of $g$ near $x_0$), so that $g$ changes sign when crossing $x_i$. By Corollary 3.5, $g'$ is continuous at $x_i$. \hfill $\square$

The next is a key lemma to obtain uniform estimate for $g'$ under (2).

**Lemma 4.3.** Assume condition (2) is satisfied. There exists a universal constant $C_2 > 0$ such that, for any $x_0 \in I_k$ with $x_0 \in Q_\nu$ for some $\nu = \nu(x_0)$, then

$$|g'(x_0)| \leq C_2 \delta_{\nu}^{2\alpha}.$$  \hfill (49)

**Proof.** By Corollary 3.5, either (46) holds which implies (49); or

$$f(x) = f(X) + (x - X)^2 \int_0^1 f''(x + t(X - x)) t \, dt,$$  \hfill (50)

where $x = X$ is the unique strict local minimum point of the function $f$ in $(x_0 - r_0 \delta_\nu, x_0 + r_0 \delta_\nu)$.

If $f(X) = 0$, then $g(x) = \pm (x - X)(\int_0^1 f''(x + t(X - x)) t \, dt)^{1/2}$. By (47), local Hölder continuity of $g'$, and $g'(X) = \sqrt{\frac{1}{2} f''(X)}$, there is universal $b > 0$ such that,

$$|g'(x)| \leq |g'(X)| + b|x-X|^2 \leq \sqrt{\frac{1}{2} C\delta_\nu^{2\alpha} + b_0 \delta_\nu^2} \leq C_2 \delta_\nu^\alpha, \forall x \in (x_0 - r_0 \delta_\nu, x_0 + r_0 \delta_\nu).$$

If $f(X) \neq 0$, then by (2) and (47),

$$M \cdot (f(X))^{\alpha \nu} \geq f''(X) \geq c_2 \delta_\nu^{2\alpha},$$

so that (50) reads

$$f(x) \geq f(X) \geq \left( \frac{c_2}{M} \right)^{\frac{1}{2+2\alpha}} \cdot \delta_\nu^{2+2\alpha}.$$  

By (34), $f(x) \sim \delta_\nu^{2+2\alpha}$ and the computation is reduced to case (16). \hfill $\square$
**Corollary 4.4.** Assume \( I_k = (a_k, b_k) \), where \( b_k < +\infty \). Then
\[
\lim_{x \to b_k^-} g'(x) = 0.
\]
Similarly, if \( a_k > -\infty \), then \( \lim_{x \to a_k^+} g'(x) = 0 \).

**Proof.** By Corollary 3.5, for each \( x \in I_k \), \((x - r_0 \delta_{\nu}(x), x + r_0 \delta_{\nu}(x)) \subset I_k \). Hence \( \lim_{x \to b_k^-} \delta_{\nu}(x) = 0 \). By (49), \(|g'(x)| \leq C_2 \delta_{\nu}^{2\alpha} \to 0 \) as \( x \to b_k^- \). \( \square \)

**Corollary 4.5.** For any \( x_0 \in \mathcal{F} \), \( g'(x) \) is continuous at \( x_0 \), with
\[
\lim_{x \to x_0} g'(x) = g'(x_0) = 0.
\]

**Proof.** By Taylor expansion of \( f \) near \( x_0 \), \( f(x) = O(|x - x_0|^{2+2\alpha}) \), so that
\[
\frac{|g(x) - g(x_0)|}{|x - x_0|} = \frac{\pm \sqrt{f(x)}}{|x - x_0|} = O(|x - x_0|^2) \to 0 \quad \text{as} \quad x \to x_0.
\]
If \( x_0 \) has a neighbourhood which is contained in \( \mathcal{F} \), then the result is trivial. Otherwise, \( x_0 \) is the boundary point of some interval \( I_k = (a_k, b_k) \). Without loss of generality we assume \( x_0 = b_k < +\infty \).

If \( x_0 \) is discrete, then \( x_0 \) is the boundary point of two consecutive intervals \( I_k \) and \( I_{k+1} \), with \( a_k < b_k = x_0 = a_{k+1} < b_{k+1} \). By Corollary 4.4,
\[
\lim_{x \to b_k^-} g'(x) = \lim_{x \to a_{k+1}^-} g'(x) = 0.
\]
Otherwise, \( x_0 \in [x_0, a_{k+1}] \subset \mathcal{F} \) for some \( a_{k+1} \). By Corollary 4.4 again,
\[
\lim_{x \to a_{k+1}^-} g'(x) = \lim_{x \to x_0} g'(x) = 0.
\]
\( \square \)

To summarize, \( g \in C^1(\mathbb{R}) \), with \(|g'(x)| \leq C_2 \), \( \forall x \in \mathbb{R} \), since \( \delta_{\nu} \leq 1 \).

**4.1.3. Global Hölder estimate.** Let \( x, y \in \mathbb{R} \) with \( x \neq y \).

1. If \( \exists z \in \mathbb{R} \setminus \mathcal{F} \) such that \( x \) and \( y \) are both contained in \((z - r_0 \delta_{\nu}(z), z + r_0 \delta_{\nu}(z))\), then by Corollary 3.5, the Hölder estimate is trivial if (46) holds or (47) holds with \( f(X) = 0 \). If case (47) holds with \( f(X) \neq 0 \), then by (2),
\[
M \cdot (f(X))^{\frac{\alpha}{1+\alpha}} \geq f''(X) \geq c_2 \delta_{\nu}^{2\alpha},
\]
so that (48) reads
\[
f(x) \geq f(X) \geq \left( \frac{c_2}{M} \right)^{\frac{1+\alpha}{\alpha}} \cdot \delta_{\nu}^{-2+2\alpha} \text{ and } f(y) \geq \left( \frac{c_2}{M} \right)^{\frac{1+\alpha}{\alpha}} \cdot \delta_{\nu}^{2+2\alpha}.
\]
The computation is reduced to case (16), and \(|g'(x) - g'(y)| / |x - y|^{\alpha} \) is bounded by a constant depending only on \( M \) and \( \alpha \).

2. Assume \( \nexists z \in \mathbb{R} \setminus \mathcal{F} \) such that \( x \) and \( y \) are both contained in \((z - r_0 \delta_{\nu}(z), z + r_0 \delta_{\nu}(z))\).
Remark 4.6. By (34) and (54), for large $n$, \[ |g'(x) - g'(y)| = |g'(x)| \leq C_2 \delta_{\nu}^\alpha \leq \frac{C_2}{r_0} \cdot |x - y|^{\alpha}. \]

(c) If $x \not\in F$ and $y \in F$, then $x \in Q_{\nu}$ for some $\nu = \nu(x)$ and $|x - y| \geq r_0 \delta_{\nu(x)}$. By (49),

\[ |g'(x) - g'(y)| \leq |g'(x)| + |g'(y)| \leq C_2 \delta_{\nu(x)}^\alpha + C_2 \delta_{\nu(y)}^\alpha \leq \frac{2C_2}{r_0} \cdot |x - y|^{\alpha}. \]

Remark 4.6. $C^{1,\alpha}$ estimate of $g$ doesn't depend on the choice of sign of $g$ in each interval $I_k$.

4.2. Proof of necessity. Assume (2) doesn’t hold and $f = g^2$ for some $g \in C^{1,\alpha}(\mathbb{R})$, then there is a sequence $x_n$ in $\mathcal{A}$ such that

\[ f''(x_n) \geq nf^n \left( \frac{a}{1+\alpha} \right) (x_n), \quad \forall n \in \mathbb{N}. \]  

(52)

$f(x_n) > 0$, so $x_n \in Q_{\nu}$ for some $\nu = \nu(n)$.

In case (i) of Lemma 3.4, $f(x_n) \geq c \delta_{\nu}^{2+2\alpha}$ and $f''(x_n) < C \delta_{\nu}^{2\alpha}$. By (52),

\[ C \delta_{\nu}^{2\alpha} \geq n(c \delta_{\nu}^{2+2\alpha}) \frac{a}{1+\alpha}, \tag{53} \]

so that $\delta_{\nu}$ get cancelled. Letting $n \to \infty$ in (53), contradiction.

In case (ii) of Lemma 3.4, $f(x_n) < c \delta_{\nu}^{2+2\alpha}$ and $f''(x_n) \geq c \delta_{\nu}^{2\alpha}$. Define $s_n = \sqrt{\frac{f(x_n)}{f''(x_n)}}$. By (52) and $\delta_{\nu} \leq 1$,

\[ s_n = \sqrt{\frac{f(x_n)}{f''(x_n)}} \leq \sqrt{n \frac{f^n \left( \frac{a}{1+\alpha} \right) (x_n)}{f''(x_n)}} = \frac{\frac{1}{n^{\frac{a}{1+\alpha}}} (x_n)}{\sqrt{\frac{f''(x_n)}{n}}} \leq \frac{1}{\sqrt{2^{2\alpha}}} \cdot \frac{\delta_{\nu(n)}}{\sqrt{n}} \to 0 \text{ as } n \to \infty. \]  

(54)

If $1/2 < \alpha \leq 1$, by Taylor expansion and $\|f\|_{C^{2,2\alpha}(\mathbb{R})} \leq 1$,

\[ f(x_n + s_n) \geq f(x_n) + \frac{1}{2} f''(x_n) s_n^2 - \frac{1}{6} |f'''(x_n)| s_n^3 - \frac{1}{6} s_n^{2+2\alpha}. \]

\[ f(x_n + s_n) \leq f(x_n) + \frac{1}{2} f''(x_n) s_n^2 + \frac{1}{6} |f'''(x_n)| s_n^3 + \frac{1}{6} s_n^{2+2\alpha}. \]

By (34) and (54), for large $n$,

\[ |f'''(x_n)| s_n^3 = |f'''(x_n)| s_n \cdot \frac{f(x_n)}{f''(x_n)} \leq C \delta_{\nu}^{3+2\alpha-3} \cdot \frac{\delta_{\nu}}{\sqrt{n}} \cdot \frac{f(x_n)}{c \delta_{\nu}^{2\alpha}} \leq \frac{1}{2} f(x_n), \]

\[ s_n^{2+2\alpha} = s_n^{2\alpha} \cdot \frac{f(x_n)}{f''(x_n)} \leq (\frac{1}{2^{2\alpha}} \cdot \frac{\delta_{\nu}}{\sqrt{n}})^{2\alpha} \cdot \frac{f(x_n)}{c \delta_{\nu}^{2\alpha}} \leq \frac{1}{2} f(x_n). \]
So, \(4f(x_n) \geq f(x_n + s_n) \geq f(x_n) > 0\) for large \(n\). By the mean value theorem,

\[
2|g'(x_n + s_n) - g'(x_n)| = |\pm \frac{f'(x_n + s_n)}{\sqrt{f(x_n + s_n)}} - (\pm \frac{f'(x_n)}{\sqrt{f(x_n)}})|
\]

\[
= \frac{|f''(x_n + s_n) - f'(x_n)|}{\sqrt{f(x_n + s_n)}} = \frac{|f''(\xi_n)| \cdot s_n}{\sqrt{f(x_n + s_n)}},
\]

where \(\xi_n \in (x_n, x_n + s_n)\). By Taylor expansion of \(f''\), for large \(n\),

\[
f''(\xi_n) \geq f''(x_n) - |f'''(\xi_n)|s_n - s_n^{2\alpha} \geq \frac{1}{2} f''(x_n).
\]

If \(0 < \alpha \leq 1/2\), by expansion to the second order, we also have for large \(n\),

\[
4f(x_n) \geq f(x_n + s_n) \geq f(x_n) > 0 \text{ and } f''(\xi_n) \geq \frac{1}{2} f''(x_n).
\]

Therefore, for any \(0 < \alpha \leq 1\), by (52),

\[
2|g'(x_n + s_n) - g'(x_n)| = \frac{|f''''(\xi_n)| \cdot s_n}{\sqrt{f(x_n + s_n)}} \geq \frac{1}{2} \frac{f''(x_n) \cdot s_n}{\sqrt{f(x_n)}} = \frac{1}{4} \sqrt{f''(x_n)}
\]

\[
= \frac{1}{4} s_n^{\alpha} \cdot \frac{f''(x_n)^{1/2}}{s_n^{\alpha/2}} = \frac{1}{4} s_n^{\alpha} \cdot \left( \frac{f''(x_n)}{f(x_n)^{\alpha}} \cdot f''(x_n)^{\alpha} \right)^{1/2}
\]

\[
= \frac{1}{4} s_n^{\alpha} \cdot \left( \frac{f''(x_n)^{1+\alpha}}{f(x_n)^{\alpha}} \right)^{1/2} \geq \frac{1}{4} s_n^{\alpha} \cdot \sqrt{n}.
\]

Hence,

\[
|g'(x_n + s_n) - g'(x_n)|/s_n^\alpha \geq \frac{1}{8} \sqrt{n} \to \infty \text{ as } n \to \infty.
\]

Contradiction.

**References**


ON ADMISSIBLE SQUARE ROOTS OF NON-NEGATIVE $C^{2,2n}$ FUNCTIONS


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