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Large totally symmetric sets

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ABSTRACT. A totally symmetric set is a subset of a group such that every permutation of the subset can be realized by conjugation in the group. The (non-)existence of large totally symmetric sets obstruct homomorphisms, so bounds on the sizes of totally symmetric sets are of particular use. In this paper, we prove that if a group has a totally symmetric set of size k, it must have order at least $(k+1)!$. We also show that with three exceptions, $\{(1\ i) \mid i = 2, \dots, n\} \subset S_n$ is the only totally symmetric set making this bound sharp, a fact which gives a new perspective on the automorphism group of S_n.

CONTENTS

1. Introduction

Kordek—Margalit [3] introduced the notion of a totally symmetric set in a group as a means to study homomorphisms. Briefly, a subset $X \subset G$ of a group is totally symmetric if any permutation of X can be realized by conjugation in G—for instance, the set of transpositions

$$X_n = \{(1\ i) \mid i = 2, \dots, n\} \subset S_n$$

is totally symmetric. Understanding the totally symmetric sets of groups G, H immediately yields constraints on homomorphisms $G \to H$, and in some cases give complete classifications. Kordek—Margalit [3] classified homomorphisms $\rho : B_n' \to B_n$ with essentially this strategy: they first classify totally symmetric sets in B_n, then use this classification to deduce the image of a well-chosen totally symmetric set in B_n'. This general strategy has been used by Chen–Mukherjea [5] to classify maps from braid groups to mapping class groups, and

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by Scherich–Verberne [6], Caplinger–Kordek [4] and Chudnovsky–Kordek–Li–Partin [1] to understand finite quotients of braid groups. Classifications of totally symmetric sets and upper bounds on their sizes are of particular interest in this scheme, as they obstruct homomorphisms. Our two main results are directly in this vein.

Theorem 1.1. *Let G be a group, and $X \subset G$ a totally symmetric set of cardinality $k > 3$. Then $|G| \geq (k+1)!$. If $|G| = (k+1)!$, then $G \cong S_{k+1}$.*

This result should be compared to [1, Proposition 2.2] which gives the bound $|G| \geq k! \cdot 2^{k-1}$ under the additional hypothesis that elements of X pairwise commute. The totally symmetric set $X_n = \{(1\ i) \mid i = 2, \dots, n\}$ shows that the bound in Theorem 1.1 is sharp. Our next theorem shows that X_n is the only such example (with three exceptions for small n).

Theorem 1.2. *Let $Y = \{y_1, \dots, y_k\}$ be a totally symmetric set in S_n of cardinality k.*

(1) *If $n \notin \{3, 4, 6\}$ and $k = n - 1$, then Y is conjugate to X_n.*
(2) *If $n = 6$ and $k = 5$, then Y is conjugate to either X_6 or $\rho(X_6)$ where $\rho \in \mathrm{Out}(S_6)$ is non-trivial.*
(3) *If $n = 4$ and $k = 3$, then either Y is conjugate to X_4 or $\{(1\ 2), (1\ 3), (2\ 3)\}$, or Y is equal to $\{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$.*
(4) *If $n = 3$, then Y may be any subset of any conjugacy class of S_3. In particular $k \leq 3$, and equality is realized by $\{(1\ 2), (1\ 3), (2\ 3)\}$.*

Both the braid group B_n and the general linear group $\mathrm{GL}_n(\mathbb{C})$ have similarly rigid maximal totally symmetric sets (see [3, Lemma 2.6] and [7, Theorem B] respectively). This is not a completely general phenomenon—$\mathbb{Z}_2 \times S_n$ contains two non-conjugate maximal totally symmetric sets—but it raises the question of what general properties lead to such rigidity. This question can also be asked about other "totally symmetric" objects (see Definition 2.2). The totally symmetric multicurves of [3] and the totally symmetric arrangements of [7] both exhibit similar rigidity properties and both give rise to their corresponding rigidity theorems.

As a sample application of Theorem 1.2, we use this result to give a short, conceptually simple proof of the well-known classification of homomorphisms $S_n \to S_m$ for $n \geq m$ due to Hölder [8]. The basic idea is that Theorem 1.2 determines all possible images of $f(X_n)$.

Theorem 1.3 (Hölder)**.** *Let $n \geq m > 2$ and $f : S_n \to S_m$ be a homomorphism. Then*

(1) *If $n > m$ and $(n, m) \neq (4, 3)$, then $\mathrm{Im}(f)$ is cyclic.*
(2) *If $n = m \notin \{4, 6\}$ and $\mathrm{Im}(f)$ is non-cyclic, then f is an inner automorphism.*
(3) *If $n = m = 6$ and $\mathrm{Im}(f)$ is non-cyclic, then f is an automorphism. Furthermore, $\mathrm{Out}(S_6) \cong \mathbb{Z}/2\mathbb{Z}$.*

(4) *If* $(n, m) = (4, 3)$ *and* $\mathrm{Im}(f)$ *is non-cyclic, then* f *is conjugate to the exceptional map* $g : S_4 \to S_3$ *defined by* $g(1\,4) = (1\,2)$, $g(2\,4) = (1\,3)$ *and* $g(3\,4) = (2\,3)$.

(5) *If* $n = m = 4$ *and* $\mathrm{Im}(f)$ *is non-cyclic, then* f *is either an inner automorphism or conjugate to the exceptional map above composed with the inclusion* $S_3 \to S_4$.

We are careful to note that our proof of Theorem 1.2 uses certain facts (see item 2, page 936) which comprise the major step in a common proof of Theorem 1.3 (which can be found in [9]). Our proof of Theorem 1.3 should therefore not be regarded as entirely independent from the standard proof. We include it here both because it is conceptually satisfying and also to illustrate the more general approach of applying totally symmetric sets to the study of homomorphisms.

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2. Totally symmetric sets

Informally, a totally symmetric set is a subset $Y \subset G$ of a group such that every permutation of Y can be realized by conjugation in G—that is, the conjugation action contains every symmetry of Y.

Definition 2.1 (Totally symmetric set of a group)**.** *Let* G *be a group. A subset* $Y = \{y_1, \dots, y_k\} \subset G$ *is said to be* ***totally symmetric*** *if for every* $\sigma \in S_k$*, there is some* $g_\sigma \in G$ *such that* $g_\sigma y_i g_\sigma^{-1} = y_{\sigma(i)}$.

The definition of total symmetry given by Kordek-Margalit [3] made the additional constraint that the elements pairwise commute. In [7], Salter and the author generalized this definition to arbitrary G-sets and in particular required the use of non-commuting totally symmetric sets. We do not require the full apparatus of the definition from [7], but we will make use of this broader notion of total symmetry.

Definition 2.2 (General totally symmetric set)**.** *Let* Z *be a* G*-set. A subset* $Y = \{y_1, \dots, y_k\} \subset Z$ *is said to be* ***totally symmetric*** *if for every* $\sigma \in S_k$*, there is a* $g_\sigma \in G$ *such that* $g_\sigma \cdot y_i = y_{\sigma(i)}$.

This group action perspective will be quite useful in our analysis. The stabilizer of a totally symmetric set (under the induced action on subsets) will play a central role in the proofs of Theorems 1.1 and 1.2. In fact, Definition 2.2 can be reformulated in terms of the stabilizer of the action on subsets of Z: a subset $Y \subset Z$ is totally symmetric if and only if the natural map $\mathrm{Stab}(Y) \to S_Y$ is surjective.

The utility of totally symmetric sets stems from the "fundamental lemma" of [3], which we now state in the language of Definition 2.2.

Lemma 2.3 (Collision implies collapse)**.** *Let G be a group, Z_1, Z_2 be G-sets, and $Y = \{y_1, \dots, y_k\} \subset Z_1$ be a totally symmetric set. Let $f : Z_1 \to Z_2$ be a G-equivariant map. Then either $|f(Y)| = |Y|$ or $|f(Y)| = 1$. Furthermore, $f(Y)$ is totally symmetric.*

Proof. For $|Y| \leq 2$, the result is clear. Say $|Y| > 2$ and $f(y_i) = f(y_j)$ for distinct $y_i, y_j \in Y$. For every $y_m \in Y$ distinct from y_i, y_j, there is some $g_{(j\,m)} \in G$ realizing the transposition $(j\,m)$ on Y. Then

$$f(y_i) = f(g_{(j\,m)} \cdot y_i) = g_{(j\,m)} \cdot f(y_i) = g_{(j\,m)} \cdot f(y_j) = f(g_{(j\,m)} \cdot y_j) = f(y_m).$$

Any singleton is vacuously totally symmetric. If $|f(Y)| = |Y|$, then $g_\sigma \cdot f(y_i) = f(y_{\sigma(i)})$, so $f(Y)$ is also totally symmetric. ☐

When Z_1, Z_2 are groups under the action of conjugation and f is a homomorphism, we recover [3, Lemma 2.1], which states that if $f : G \to H$ is a homomorphism, and $X \subset G$ is totally symmetric, then $f(X)$ is also totally symmetric and has cardinality 1 or $|f(X)|$. This is the primary way totally symmetric sets are used to study homomorphisms. As a sample demonstration, we will prove Theorem 1.3 parts 1 and 2 assuming Theorem 1.2. Conceptually, this proof is quite simple—in order to find all maps $f : S_n \to S_m$, we need only find the possible images of X_n, which are listed in Theorem 1.2. In part 1, there are no suitable images (so the map is cyclic), and in part 2 there is only one (up to conjugation, so the map is inner).

Proof of Theorem 1.3 parts 1 and 2 assuming Theorem 1.2. Let $n \geq m > 2$ be integers and $f : S_n \to S_m$ be a homomorphism. We prove only parts 1 (if $n > m$, then $\mathrm{Im}(f)$ is cyclic) and 2 (if $n = m \notin \{4, 6\}$ and $\mathrm{Im}(f)$ is non-cyclic, then f is an inner automorphism). Parts 3, 4 and 5 are similar.

Consider the totally symmetric set $f(X_n)$, which has cardinality 1 or $n - 1$ by Lemma 2.3. If $n > m$ with $(n, m) \neq (4, 3)$, Theorem 1.2 says that S_m has no totally symmetric sets of size $n - 1$. Then $|f(X_n)| = 1$, which implies that $\mathrm{Im}(f)$ is cyclic. We will proceed assuming $\mathrm{Im}(f)$ is non-cyclic. Then $f(X_n)$ has cardinality $n - 1$ and therefore must be one of the totally symmetric sets listed in Theorem 1.2.

If $n \notin \{3, 4, 6\}$ and $n = m$, then Theorem 1.2 gives some $\sigma \in S_n$ so that $f(X_n) = \tilde{\sigma}(X_n)$, where $\tilde{\sigma}$ is the inner automorphism corresponding to σ. Then $(\tilde{\sigma}^{-1} \circ f)(X_n) = X_n$, that is $\tilde{\sigma}^{-1} \circ f$ permutes X_n. Total symmetry now gives an element $\tau \in S_n$ realizing this permutation so that $\tilde{\tau}^{-1} \circ \tilde{\sigma}^{-1} \circ f$ is the identity map on X_n, which generates S_n. Then $f = \widetilde{\sigma\tau}$ is an inner automorphism as desired. ☐

3. Proof of Theorem 1.1

Let X be a totally symmetric set of cardinality k in a group G. The proof of Theorem 1.1 is essentially an orbit-stabilizer argument applied to the action of conjugation on totally symmetric sets.

Proof of Theorem 1.1. By total symmetry, the natural map $\phi : \mathrm{Stab}(X) \to S_k$ is surjective, and therefore $|\mathrm{Stab}(X)| \geq k!$. It remains to show $|\mathrm{Orb}(X)| \geq k+1$. This is accomplished in two steps: first, we argue that some $Y \in \mathrm{Orb}(X)$ intersects X non-trivially, then we use this intersection to produce k additional totally symmetric sets in $\mathrm{Orb}(X)$.

Assume every $Y \in \mathrm{Orb}(X)$ not equal to X is disjoint from X. In particular, any $a \in X$ satisfies $aXa^{-1} = X$, so $X \subset \mathrm{Stab}(X)$. This means X is a totally symmetric set in $\mathrm{Stab}(X)$. Since $\mathrm{Stab}(X)$ acts transitively on X by conjugation, and all elements fix X setwise, X is an entire conjugacy class of $\mathrm{Stab}(X)$. If $\phi(X) = \{e\}$, then the elements of X pairwise commute, and we may apply [1, Proposition 2.2] which gives the bound $|G| \geq k! \cdot 2^{k-1} > (k+1)!$ for any totally symmetric set with pairwise commuting elements. Then we may additionally assume $\phi(X) \neq \{e\}$. Because $\phi : \mathrm{Stab}(X) \to S_k$ is surjective and X is an entire conjugacy class, $\phi(X)$ is also an entire (non-trivial) conjugacy class of S_k. Moreover, the elements of $\phi(X)$ must fix at least one point, since conjugation by $a \in X$ fixes a. For $k > 3$, such conjugacy classes in S_k have cardinality larger than k. This contradicts $|X| = k$.

Let $Y \in \mathrm{Orb}(X)$ intersect X non-trivially. We can use total symmetry to produce $\binom{k}{|X\cap Y|} \geq k$ other elements of $\mathrm{Orb}(X)$ as follows. For each $|X \cap Y|$-element subset $A \subset X$, let $g_A \in \mathrm{Stab}(X)$ be such that $g_A(X \cap Y)g_A^{-1} = A$. Then the $g_A Y g_A^{-1}$ are all distinct, as they have different intersections with X. This proves $|G| \geq (k+1)!$.

We will now prove the second part of Theorem 1.1, which states that S_{k+1} is the only group G of order $(k+1)!$ with a totally symmetric set X of cardinality $k > 3$. The basic strategy is to produce an action of G on a $(k+1)$-element set which is isomorphic to the action of S_{k+1}.

From the proof of part 1, the equality $|G| = (k+1)!$ is achieved exactly when $\phi : \mathrm{Stab}(X) \to S_k$ is an isomorphism and $|\mathrm{Orb}(X)| = k+1$. In this case, every $Y \in \mathrm{Orb}(X)$ with $Y \neq X$ satisfies $|X \cap Y| = 1$ or $|X \cap Y| = k+1$. We deal with the case $|X \cap Y| = 1$; the other case is nearly identical.

Write $X = \{x_1, \ldots, x_k\}$, let $f : G \to S_{\mathrm{Orb}(X)} \cong S_{k+1}$ be the action of G on $\mathrm{Orb}(X)$ and let $Y_i \in \mathrm{Orb}(X)$ denote the unique totally symmetric set satisfying $X \cap Y_i = \{x_i\}$. Then $\mathrm{Orb}(X) = \{X, Y_1, \ldots Y_k\}$. Furthermore, $S_k \cong \mathrm{Stab}(X) \subset G$ acts on $\{Y_1, \ldots, Y_k\}$ by permuting indices. Any $g \notin \mathrm{Stab}(X)$ does not fix X, so $f(\mathrm{Stab}(X))$ and $f(g)$ generate S_{k+1}. □

4. Proof of Theorem 1.2

Let $Y = \{y_1, \ldots, y_k\} \subset S_n$ be a totally symmetric set. In this section, we will prove that if k takes the largest value allowed by Theorem 1.1, then it must be one of the totally symmetric sets listed in Theorem 1.2. The case $n = 3$ is dealt with by noting that any subset of any conjugacy class of S_3 is totally symmetric. The cases $n = 4$ and $n \geq 5$ will be treated separately. The proof requires the following two facts about permutation groups, both of which can be found in [9, page 2,].

(1) Any proper subgroup $H \subset S_n$ not equal to A_n satisfies $|S_n : H| \geq n$.
(2) For $n \neq 6$, every index n subgroup of S_n is a point stabilizer (that is, a subgroup $S_{n-1} \subset S_n$ fixing a point in $[n] = \{1, \ldots, n\}$). If $n = 6$, there is one additional conjugacy class of point stabilizers found by applying an outer automorphism to a point stabilizer.

4.1. The generic case: $n \geq 5$. We first claim that $\mathrm{Stab}(Y)$ is a point stabilizer or else $n = 6$ and $\mathrm{Stab}(Y)$ is the image of a point stabilizer under an outer automorphism of S_6. Since Y is totally symmetric, the natural map $\mathrm{Stab}(Y) \to S_k \cong S_{n-1}$ is surjective. Counting orders shows that $|S_n : \mathrm{Stab}(Y)| \leq n$. Then $\mathrm{Stab}(Y)$ is a proper subgroup of S_n not equal to A_n, meaning $|S_n : \mathrm{Stab}(Y)| \geq n$ and therefore $|S_n : \mathrm{Stab}(Y)| = n$. Item 2 above now proves the claim.

We now show that if $\mathrm{Stab}(Y)$ is a point stabilizer, then Y is conjugate to $X_n = \{(1\ i) \mid i = 2, \ldots, n\}$. For future notational simplicity, we will actually show Y is conjugate to $\{(i\ \ n) \mid i = 1, \ldots, n-1\}$. This suffices even when $n = 6$ — if $\mathrm{Stab}(Y)$ is not a point stabilizer, then let $[\rho] \in \mathrm{Out}(S_6)$ and consider the totally symmetric set $\rho(Y)$ and its stabilizer $\mathrm{Stab}(\rho(Y)) = \rho(\mathrm{Stab}(Y))$.

Without loss of generality, assume $\mathrm{Stab}(Y)$ fixes the point $n \in [n]$. Set $a_i = y_i(n)$, and for $\sigma \in S_Y$, let $g_\sigma \in \mathrm{Stab}(Y)$ be such that $g_\sigma y_i g_\sigma^{-1} = y_{\sigma(i)}$. Then

$$g_\sigma(a_i) = g_\sigma y_i(n) = g_\sigma y_i g_\sigma^{-1}(n) = y_{\sigma(i)}(n) = a_{\sigma(i)}. \tag{1}$$

In other words, the set $\{a_i \mid i \in [k]\} \subset [n]$ is totally symmetric in the sense of Definition 2.2. The association $y_i \to a_i$ is moreover $\mathrm{Stab}(Y)$-equivariant.

We next claim that $\{a_i \mid i \in [k]\} = \{1, \ldots, n-1\}$. This will be done in two steps: first show that no $a_i = n$, second, show that no $a_i = a_j$. If any a_i is equal to n, then every a_i is equal to n by Equation 1 (recall every $g_\sigma \in \mathrm{Stab}(Y)$ fixes n). If $a_i = a_j$ for distinct $i, j \in [k]$, then Lemma 2.3 tells us $\{a_m \mid m \in [k]\}$ is the singleton $\{a_i\}$. Equation 1 now says, $\mathrm{Stab}(Y)$ fixes $a_i \neq n$ in addition to n. Then $\mathrm{Stab}(Y)$ has order at most $(n-2)!$ and therefore cannot surject onto $S_k \cong S_{n-1}$. Thus, $\{a_i \mid i \in [k]\}$ is an $(n-1)$-element subset of $[n]$ not containing n, so it is $\{1, \ldots, n-1\}$ as claimed.

By conjugating, we may assume without loss of generality that $a_i = i$. Equation 1 now says $g_\sigma = \sigma$. At this stage, we know $y_i(n) = i$ and want to show that $y_i = (i\ n)$. This will be accomplished in two steps:

(1) Show that $y_i(i) = n$
(2) Show that if $j \notin \{i, n\}$, then $y_i(j) = j$

Step 1. Suppose $y_i(i) = j \neq n$. Let $k \notin \{n, i, j\}$, and consider the element $g_{(j\ k)} = (j\ k)$. By total symmetry,

$$(j\ k) y_i (j\ k) = g_{(j\ k)} y_i g_{(j\ k)}^{-1} = y_i.$$

But the left hand side of this equation sends $i \to k$, while the right hand side sends $i \to j$. Hence, $y_i(i) = n$.

Step 2. Assume that y_i does not fix some $j \notin \{i, n\}$, that is $y_i(j) = k$ for $j, k \notin \{i, n\}$. We will use the same trick: Because $n \geq 5$, there is an $m \notin \{i, n, j, k\}$. Just as before,

$$(m\ j)y_i(m\ j) = g_{(m\ j)}y_i g_{(m\ j)}^{-1} = y_i.$$

But the left hand side takes $m \to k$, while the right hand side takes $j \to k$. Then $y_i = (i\ n)$ as required. Note that this step fails for the totally symmetric set $\{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\} \subset S_4$.

4.2. The exceptional case: $n = 4$. We will check 3 element subsets of conjugacy classes directly. Elements of a totally symmetric set are conjugate, so there are four cycle types to consider:

(1) $(*\,*)(*\,*)$
(2) $(*\,*)$
(3) $(*\,*\,*)$
(4) $(*\,*\,*\,*)$

Case 1: $(**)(**)$. We claim the conjugacy class of cycle type $(**)(**)$ given by $Y = \{y_1 = (12)(34), y_2 = (13)(24), y_3 = (14)(23)\}$ is totally symmetric. Indeed, $g_{(1\ 2)} = (2\,3)$, $g_{(1\ 3)} = (1\,3)$ and $g_{(2\ 3)} = (2\ 3)$ realize all three transpositions of Y.

We are then left to consider totally symmetric sets $Y = \{c_1, c_2, c_3\}$ of cycles. By total symmetry, the intersection pattern of subsets of Y must be the same: $|c_1 \cap c_2| = |c_2 \cap c_3| = |c_1 \cap c_3|$ (here we think of cycles as subsets of $[4] = \{1, 2, 3, 4\}$). This is a consequence of the more general fact[1] that $g_\sigma c_i = c_{\sigma(i)}$ and therefore $g_\sigma(c_i \cap c_j) = g_\sigma c_i \cap g_\sigma c_j = c_{\sigma(i)} \cap c_{\sigma(j)}$, where again we think of the cycles as subsets of $[4]$.

Case 2: $(*\,*)$. There are no three element sets of transpositions in S_4 which do not intersect. Then c_1 and c_2 share a single element. By the previous discussion, c_3 must intersect both c_1 and c_2. Then either c_3 contains the point $c_1 \cap c_2$ or it contains the points $c_1 \setminus (c_1 \cap c_2)$ and $c_2 \setminus (c_1 \cap c_2)$. In the first case, Y is conjugate to X_4. In the second, Y is conjugate to $\{(1\ 2), (1\ 3), (2\ 3)\}$.

Case 3: $(*\,*\,*)$. If $c_1 = c_2$ as subsets of $[4]$, then c_3 is a 3-cycle on the same three elements. But there are only two distinct 3-cycles in S_3. Then $c_1 \neq c_3$ as subsets of $[4]$, and $|c_1 \cap c_2| = 2$. Then c_3 intersects $c_1 \cap c_2$ in exactly one point—if it contained $c_1 \cap c_2$, then the third point (which lies in either c_1 or c_2) would break the symmetry of intersection patterns. Let p be the unique point in $c_1 \cap c_2 \cap c_3$, and let $g_{(1\ 2)}$ realize the permutation $(1\,2)$ on the totally symmetric set $Y = \{c_1, c_2, c_3\}$. The centralizer of a 3-cycle in S_4 is the group generated by that 3-cycle. Then $g_{(1\ 2)}$ is a power of c_3, but also fixes p. Then $g_{(1\ 2)} = e$, which does not realize $(1\ 2)$.

[1]In the language of [7], the set of intersections $c_i \cap c_j$ forms a totally symmetric set under the action of S_4 on two elements subsets of $[4]$. This is a more general notion of total symmetry than used in this paper.

Case 4: $(****)$. Any group element c and its inverse cannot appear in a three element totally symmetric set—one cannot move c by conjugation while fixing c^{-1}. There are only three inverse pairs of four-cycles in S_4, so $c_1 = (1\ 2\ 3\ 4)^{\pm 1}$, $c_2 = (1\ 2\ 4\ 3)^{\pm 1}$, and $c_3 = (1\ 3\ 2\ 4)^{\pm 1}$. Without loss of generality, assume $c_1 = (1\,2\,3\,4)$. We will show $c_2 = (1\,2\,4\,3)$ is impossible—the case $c_2 = (1\,2\,4\,3)^{-1}$ is nearly identical.

As before, the element $g_{(2\,3)}$ realizing the permutation $(2\,3)$ must be a power of c_1. Denote this power $p \in \{0, 1, 2, 3\}$. If $p = 3$, then $(c_1^3)^3 = c_1$ also realizes the permutation $(2\ 3)$, so we need only check $p = 1$ and $p = 2$. If $p = 1$, we compute

$$c_3 = c_1 c_2 c_1^{-1} = (2\,3\,1\,4) \qquad \text{and} \qquad c_2 = c_1 c_3 c_1^{-1} = (3\,4\,2\,1) \neq c_2.$$

Then $p = 2$, and

$$c_3 = c_1^2 c_2 c_1^{-2} = (3\,4\,2\,1) = c_2^{-1}.$$

But Y cannot contain both c_2 and c_2^{-1}.

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