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Classification of real approximate interval C^*-algebras

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Abstract. A functorial classification, up to approximate unitary equivalence, is given of unital real *-homomorphisms from a real C^*-algebra arising as an inductive limit of real forms on finite direct sums of matrix algebras over the continuous complex valued functions on the unit interval to another such algebra. The invariant consists of a diagram $Cu(A) \to Cu(A \otimes_{\mathbb{R}} \mathbb{C}) \to Cu(A \otimes_{\mathbb{R}} \mathbb{H})$ of Cuntz semigroups with distinguished elements. As a corollary, a classification, up to *-isomorphism, of such real approximate interval algebras is obtained. Also, unital real *-homomorphisms from real approximately finite dimensional C^*-algebras to a certain general class of real C^*-algebras are classified, up to approximate unitary equivalence, by a diagram $K_0(A) \to K_0(A \otimes_{\mathbb{R}} \mathbb{C}) \to K_0(A \otimes_{\mathbb{R}} \mathbb{H})$ of ordered K_0 groups with distinguished elements. As a corollary, a new proof of the already known classification of real approximately finite dimensional algebras in terms of this invariant is obtained.

Contents

1. Introduction

Following the classification of AF C^*-algebras with the scaled, ordered K_0 group by Elliott, this invariant was generalised in many different ways to study more complicated objects (cf [17]). Adjoining the K_1 group and tracial data has proved very successful for classifying large classes of simple C^*-algebras. In the case of non-simple C^*-algebras, the Cuntz semigroup has been used to classify *-homomorphisms up to approximate unitary equivalence when the

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C^*-algebras in question are inductive limits of sufficiently tractable building blocks (cf [3], [4], [14]). In particular, the Cuntz semigroup can be used to classify not necessarily simple approximate interval algebras.

Shortly after the classification of AF C^*-algebras by Elliott, the problem of classifying real AF algebras was solved by Giordano [6] and Stacey [18]. The invariant used in [18], and studied extensively in [7], consists of the diagram $K_0(A) \to K_0(A \otimes_{\mathbb{R}} \mathbb{C}) \to K_0(A \otimes_{\mathbb{R}} \mathbb{H})$ of ordered K_0 groups with distinguished elements. In light of the success of the Cuntz semigroup as an invariant for classifying non-simple C^*-algebras, it seems natural to consider the diagram $Cu(A) \to Cu(A \otimes_{\mathbb{R}} \mathbb{C}) \to Cu(A \otimes_{\mathbb{R}} \mathbb{H})$ of Cuntz semi-groups with distinguished elements as an invariant for real, not necessarily simple, AI algebras. The results of this paper bear this out.

The organisation of this paper is as follows. In the next section, real structures on C^*-algebras are introduced and the building blocks of real AI algebras described. Two invariants are introduced, one involving dimension ranges and the other Cuntz semi-groups. These are computed for the basic building blocks of real AI algebras. Since one of the basic building blocks of real AI algebras does not have stable rank one, care must be taken when working with different pictures of the Cuntz semi-group. Some general results on real AI algebras and their invariants are also gathered in this section. In the third section, we give the existence lemmas, first for homomorphisms from finite dimensional algebras, using the invariant with dimension ranges, and then for the case of real interval algebras mapping into real AI algebras. The fourth section follows the same pattern, but with the uniqueness lemmas, and the resulting classification theorems are given in the fifth section.

2. Preliminaries and definitions

We begin this preliminary section by fixing some notation and terminology for real C^*-algebras.

We define a *real C^*-algebra* to be a real Banach $*$-algebra that is isometrically $*$-isomorphic to a norm closed, self-adjoint sub-algebra of the bounded linear operators on a real Hilbert space. Given such an algebra, A, it can be shown that the norm on A extends to the complexification $A \otimes_{\mathbb{R}} \mathbb{C}$ in a unique way so as to make $A \otimes_{\mathbb{R}} \mathbb{C}$ into a complex C^*-algebra. One then has a map $\tau : A \otimes_{\mathbb{R}} \mathbb{C} \to A \otimes_{\mathbb{R}} \mathbb{C}$ given by $\tau(x+iy) = x^* + iy^*$ for $x, y \in A$. This map τ is a complex linear, $*$-preserving, involutive anti-automorphism of $A \otimes_{\mathbb{R}} \mathbb{C}$, called the *real structure* on $A \otimes_{\mathbb{R}} \mathbb{C}$ associated to A. One then has $A = \{z \in A \otimes_{\mathbb{R}} \mathbb{C} \mid \tau(z) = z^*\}$. One could also use the map $x + iy \mapsto x - iy = (\tau(x + iy))^* = \tau((x + iy)^*)$. If we denote this map $z \mapsto \bar{z}$, we can describe the real structure by $\tau(z) = (\bar{z})^*$, and one has $A = \{z \in A \otimes_{\mathbb{R}} \mathbb{C} \mid z = \bar{z}\}$. We shall use both pictures as convenient.

Alternatively, given a complex C^*-algebra A, and a complex liner involutive *-anti-automorphism τ of A, one may define the *real form on A associated to τ* by $A_\tau = \{z \in A \mid \tau(z) = z^*\}$. Then A_τ is a real C^*-algebra in the sense defined above, and $A \cong A_\tau \otimes_{\mathbb{R}} \mathbb{C}$. (see [12])

These two pictures, real C^*-algebras vs real structures on complex C^*-algebras, are essentially equivalent, and which one uses is a matter of taste. We shall use whichever terminology seems more convenient in the given situation.

Notation We shall denote the three canonical pairwise anti-commuting generators of $\mathbb{H}$ α, β and γ, so elements of $\mathbb{H}$ are of the form $r + a\alpha + b\beta + c\gamma$ where $r, a, b, c \in \mathbb{R}$ and $\alpha^2 = \beta^2 = \gamma^2 = -1$. We shall consider the canonical embedding of $\mathbb{C}$ into $\mathbb{H}$ to be the one that sends $i \in \mathbb{C}$ to α. It will often be convenient for us to view the complexification $A \otimes_{\mathbb{R}} \mathbb{C}$ of a real C^*-algebra A as sitting inside $A \otimes_{\mathbb{R}} \mathbb{H}$, so we will write elements of $A \otimes_{\mathbb{R}} \mathbb{C}$ as $a + b\alpha$ with $a, b \in A$. Note that we then have $(\bar{x}) = Ad\,\beta(x)$ for any $x \in A \otimes_{\mathbb{R}} \mathbb{C}$.

Notice that having an anti-self-adjoint unitary in a real C^*-algebra A is exactly equivalent to having a unital *-homomorphism from $\mathbb{C}$ into A. The following lemma will be useful in constructing such *-homomorphisms.

Lemma 2.1. [Embeddings of $\mathbb{C}$] *If A is a unital real C^*-algebra and r is a projection in $A \otimes_{\mathbb{R}} \mathbb{C}$ such that $r + \bar{r} = 1$, then we can write $r = \frac{1}{2}(1 + u\alpha)$ where u is a unitary in A such that $u^* = -u$. Conversely, if u is a unitary in A such that $u^* = -u$, then $r = \frac{1}{2}(1 + u\alpha)$ is a projection in $A \otimes_{\mathbb{R}} \mathbb{C}$ such that $r + \bar{r} = 1$.*

Proof. Suppose r is a projection in $A \otimes_{\mathbb{R}} \mathbb{C}$ such that $r + \bar{r} = 1$. Write $r = s + t\alpha$ with $s, t \in A$. Then $1 = r + \bar{r} = (s + t\alpha) + (s - t\alpha) = 2s$. Using $r = r^*$ gives $t^* = -t$. Using $r = r^2$ gives $\frac{1}{4} = tt^* = -t^2 = t^*t$, and the result follows. Conversely, if u is a unitary in A such that $u^* = -u$, and we set $r = \frac{1}{2}(1 + u\alpha)$ then it is easy to check that $r = r^* = r^2$ and $r + \bar{r} = 1$. □

Definition 2.2. *We use the following notations for real forms on interval algebras:*

$$A(n, \mathbb{R}) = \{f \in C([0,1], M_n(\mathbb{C})) \mid f(1) \in M_n(\mathbb{R})\}$$

$$A(n, \mathbb{H}) = \{f \in C([0,1], M_{2n}(\mathbb{C})) \mid f(1) \in M_n(\mathbb{H})\}$$

$$M_n(C_{\mathbb{F}}[0,1]) = M_n(\{f : [0,1] \to \mathbb{F} \mid f \text{ is continuous}\})$$

For $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$.

It is shown in [19] that any real form on a finite direct sum of matrix algebras over the continuous complex valued functions on the unit interval is a finite direct sum of the above five types. We shall therefore refer to these five types as the basic building blocks for real approximate interval algebras, and to finite direct sums of these as real interval algebras. Note that in the case of $A = M_n(C_{\mathbb{C}}[0,1])$, we have $A \otimes_{\mathbb{R}} \mathbb{C} \cong M_n(C_{\mathbb{C}}[0,1]) \oplus M_n(C_{\mathbb{C}}[0,1])$, and in the other four cases we have $A \otimes_{\mathbb{R}} \mathbb{C} \cong M_m(C_{\mathbb{C}}[0,1])$, where $m = n$ or $2n$.

Definition 2.3. *Let A be a C^*-algebra, either real or complex, and let a, b be positive elements of A. We say that a is Cuntz sub-equivalent to b, and write $a \precsim b$*

if there exists a sequence $d_n \in A$ such that $d_n b d_n^* \to a$. We write $a \sim b$ if $a \precsim b$ and $b \precsim a$. Then $\sim$ is an equivalence relation on the set of positive elements of A, called Cuntz equivalence.

Definition 2.4. *Let A be a separable C^*-algebra, either real or complex. Let $Cu(A)$ denote the set of Cuntz equivalence classes of positive elements of $A \otimes_{\mathbb{R}} \mathcal{K}_{\mathbb{R}}$, where $\mathcal{K}_{\mathbb{R}}$ is the real C^*-algebra of compact operators on a separable real Hilbert space. Fix an isomorphism of $\mathcal{K}_{\mathbb{R}}$ with $M_2(\mathcal{K}_{\mathbb{R}})$, and define addition on $Cu(A)$ by $[a]+[b] = [\begin{pmatrix} a & 0 \\ o & b \end{pmatrix}]$ (this does not depend on the choice of isomorphism). Define a partial order on $Cu(A)$ by $[a] \leq [b]$ if, and only if, $a \precsim b$ (this does not depend on choice of representatives). With these definitions, $Cu(A)$ becomes a partially ordered abelian semi-group with neutral element.*

Remark 2.5. *There are a couple of different definitions of Cuntz semigroup in the literature. We have chosen the one which is most appropriate for classification of approximate interval algebras, and most convenient for generalising to real C^*-algebras.*

Lemma 2.6. *Given a real C^*-algebra A with canonical real structure τ on $A \otimes_{\mathbb{R}} \mathbb{C}$, one may extend τ to a real structure (also to be denoted τ) on $(A \otimes_{\mathbb{R}} \mathbb{C})^{**}$. If $x \in A$, then when forming the polar decomposition $x = v(x^*x)^{1/2} \in (A \otimes_{\mathbb{R}} \mathbb{C})^{**}$, one may take the partial isometry $v \in (A \otimes_{\mathbb{R}} \mathbb{C})^{**}_{\tau}$.*

Proof. To see this, embed A into $B(L^2(\mathbb{R}))$ and $A \otimes_{\mathbb{R}} \mathbb{C}$ into $B(L^2(\mathbb{C}))$ with the transpose as real structure. One then has $x = v(x^*x)^{1/2} = \bar{v}(x^*x)^{1/2}$, so $v = \bar{v}$. Thus we have $v^{\tau} = v^*$. □

Lemma 2.7. *Let A be a real C^*-algebra, and let a, b be positive elements in A. Let $\varepsilon > \| a - b \|$ be given. Then there exists a contraction $d \in A$ such that $dbd^* = (a - \varepsilon)_+$.*

Proof. This is analogous to lemma 2.2 in [9]. With the observation above on polar decompositions, the proof in [9] carries over verbatim. □

Remark 2.8. *With the lemma above in place of lemma 2.2 in* [9]*, the proofs in* [5] *of theorems 1 and 2 of that paper carry over to the real case essentially unchanged.*

Definition 2.9. *Given a partially ordered abelian semigroup $\mathcal{S}$ in which increasing sequences have suprema, we define a relation $x << y$ on $\mathcal{S}$ called compact containment, as follows. We say that $x << y$ if whenever we have an increasing sequence $y_1 \leq y_2 \leq y_3 \leq ...$ with $y \leq \sup_n(y_n)$, then we have $x \leq y_m$ for some m.*

Remark 2.10. *Given a unital real C^*-algebra A such that $A \otimes_{\mathbb{R}} \mathbb{C}$ has stable rank one, it need not be the case that the invertible elements of A are dense in A. To see this, one need only consider the example of $C_{\mathbb{R}}[0,1]$. While $C_{\mathbb{C}}[0,1]$ has stable rank one, the element $f(t) = t - 1/2$ is not norm close to any invertible element in $C_{\mathbb{R}}[0,1]$.*

Notation For a C^*-algebra A, either real or complex, we shall write $D(A)$ for the dimension range of A, i.e. the abelian partial semigroup of Murray-v. Neumann equivalence classes of projections in A.

Definition 2.11. [The Class C] *Let C denote the class of separable unital real C^*-algebras A such that A has cancelation of projections, $D(A)$ is unperforated and has interpolation, and $A \otimes_{\mathbb{R}} \mathbb{C}$ and $A \otimes_{\mathbb{R}} \mathbb{H}$ have the same properties.*

Definition 2.12. [Inv_1 and Inv_2] *Given a unital real C^*-algebra A, Invariant # 1, denoted $Inv_1(A)$, consists of the diagram*

$$D(A,[1]) \to D(A \otimes_{\mathbb{R}} \mathbb{C},[1]) \to D(A \otimes_{\mathbb{R}} \mathbb{H},[1])$$

of dimension ranges with distinguished elements, where the connecting maps are induced by the inclusions. A morphism of invariants $\eta : Inv_1(A) \to Inv_1(B)$ consists of a triple (η_r, η_c, η_h) of unital homomorphisms of abelian partial semigroups such that the following diagram commutes:

$$\begin{array}{ccccc}
(D(A),[1]) & \longrightarrow & (D(A \otimes_{\mathbb{R}} \mathbb{C}),[1]) & \longrightarrow & (D(A \otimes_{\mathbb{R}} \mathbb{H}),[1]) \\
\downarrow \eta_r & & \downarrow \eta_c & & \downarrow \eta_h \\
(D(B),[1]) & \longrightarrow & (D(B \otimes_{\mathbb{R}} \mathbb{C}),[1]) & \longrightarrow & ((D(B \otimes_{\mathbb{R}} \mathbb{H}),[1])).
\end{array}$$

Given a unital real C^*-algebra A, Invariant # 2, denoted $Inv_2(A)$, consists of the diagram $Cu(A,[1]) \to Cu(A \otimes_{\mathbb{R}} \mathbb{C},[1]) \to Cu(A \otimes_{\mathbb{R}} \mathbb{H},[1])$ of Cuntz semigroups with distinguished elements, where the connecting maps are induced by the inclusions. A morphism of invariants $\eta : Inv_2(A) \to Inv_2(B)$ consists of a triple (η_r, η_c, η_h) of unital homomorphisms of ordered abelian partial semigroups preserving suprema of increasing sequences, zero elements, and compact containment such that the following diagram commutes:

$$\begin{array}{ccccc}
(Cu(A),[1]) & \longrightarrow & (Cu(A \otimes_{\mathbb{R}} \mathbb{C}),[1]) & \longrightarrow & (Cu(A \otimes_{\mathbb{R}} \mathbb{H}),[1]) \\
\downarrow \eta_r & & \downarrow \eta_c & & \downarrow \eta_h \\
(Cu(B),[1]) & \longrightarrow & (Cu(B \otimes_{\mathbb{R}} \mathbb{C}),[1]) & \longrightarrow & ((Cu(B \otimes_{\mathbb{R}} \mathbb{H}),[1])).
\end{array}$$

Lemma 2.13. [Density of elements with distinct eigenvalues] *The set of self-adjoint elements of $M_n(C_{\mathbb{R}}[0,1])$ with distinct eigenvalues in every fibre is dense in the set of all self-adjoint elements.*

Proof. This is part of lemma 2.4 (a) in [19]. □

Lemma 2.14. [Density of elements with distinct eigenvalues] *The set of self-adjoint elements of $M_n(C_{\mathbb{H}}[0,1])$ with distinct eigenvalues in every fibre is dense in the set of all self-adjoint elements.*

Proof. This follows from lemma 2.4 (c) in [19]. □

Lemma 2.15. [Equivalence of positive elements within the real part] *Let A be a unital real approximate interval algebra, and let a and b be two positive elements of A such that a is approximately unitarily equivalent to b in $A \otimes_{\mathbb{R}} \mathbb{C}$. Then a is approximately unitarily equivalent to b in A, i.e. we may choose the unitaries to lie in A.*

Proof. Let A, a, and b be as in the statement of the lemma. Let u be a unitary $u \in A \otimes_{\mathbb{R}} \mathbb{C}$ such that $\| b - uau^* \|$ is small in norm. Write $A = \lim\{A_n, \varphi_{nm}\}$ where A_n are finite direct sums of real interval algebras and φ_{nm} are unital real *-homomorphisms. We may choose n and $a_n, b_n \in A_n$ and $u_n \in A_n \otimes_{\mathbb{R}} \mathbb{C}$ such that a_n and b_n are positive, u_n is unitary, and $\| b_n - u_n a_n u_n^* \|$, $\| \varphi_{n\infty}(a_n) - a \|$, $\| \varphi_{n\infty}(b_n) - b \|$, and $\| \varphi_{n\infty}(u_n) - u \|$ are all as small in norm as we like. The problem reduces to showing that we can replace u_n with a unitary in A_n. Since A_n is a finite direct sum real interval algebras, we may consider each type of basic building block separately.

Case 1: ($A_n \cong M_m(C_{\mathbb{C}}[0,1])$) In this case, we have $A_n \otimes_{\mathbb{R}} \mathbb{C} \cong M_m(C_{\mathbb{C}}[0,1])^2$ with real structure $(x,y)^\tau = (y^{tr}, x^{tr})$. Let p denote the central projection that is a unit for the first summand. Then $a_n = pa_n + (pa_n)^\tau$, and $b_n = pb_n + (pb_n)^\tau$. Replacing $u_n = pu_n + (1-p)u_n$ with $v_n = pu_n + (pu_n)^\tau$, we have $\| v_n a_n v_n^* - b_n \| \leq \| u_n a_n u_n^* - b_n \|$ and $v_n \in A_n$.

Case 2: ($A_n \cong A(m, \mathbb{R})$) In this case, we have $A_n \otimes_{\mathbb{R}} \mathbb{C} \cong M_m(\overline{C_{\mathbb{C}}}[-1,1])$ with real structure $f^\tau(t) = f(-t)^{tr}$ and $A_n = \{f \in A_n \otimes \mathbb{C} \mid f(-t) = \overline{f(t)}\}$. If u_n satisfies $u_n(0) \in M_n(\mathbb{R})$, then we may replace u_n by v_n defined by $v_n(t) = u_n(t)$ for $t \leq 0$, and $v_n(t) = \overline{u_n(-t)}$ for $t \geq 0$. We would then have $\| v_n a_n v_n^* - b_n \| \leq \| u_n a_n u_n^* - b_n \|$ and $v_n \in A_n$.

We may assume, by making an arbitrarily small adjustment if necessary, that a_n and b_n are in standard form, i.e. they are unitary conjugates of diagonal elements. Choose a system of matrix units with respect to which b_n is diagonal and $b_n(0) \in M_n(\mathbb{R})$. Then $u_n a_n u_n^*$ is approximately diagonal with respect to these matrix units, so multiplying u_n by a unitary close to 1 we may assume that $u_n a_n u_n^*$ is exactly diagonal at 0. It follows that the columns of $u_n(0)$ are eigenvectors for the self-adjoint matrix $a_n(0)$. Since $a_n(0)$ is self-adjoint, the real and imaginary parts of these column vectors are also eigenvectors for $a_n(0)$, and for each column, one or the other is non-zero. Thus there exists a unitary matrix $w(0)$ such that $w(0)$ commutes with $a_n(0)$ and $u_n(0)w(0) \in M_n(\mathbb{R})$. Since a_n is in standard form, we may connect $w(0)$ to 1 with a path $w(|t|)$ such that $w(t)$ commutes with $a_n(t)$, and thus adjust $u_n(t)$ all along its length to meet our requirements.

Case 3: ($A_n \cong A(m, \mathbb{H})$) In this case, we have $A_n \otimes \mathbb{C} \cong M_{2m}(C_{\mathbb{C}}[-1,1])$ with real structure $f^\tau(t) = f(-t)^\#$. By [19] lemma 2.4 (b), we may find approximants to a_n and b_n that have distinct eigenvalues in every fibre except the endpoint, where they have multiplicity two, and the eigenprojections are continuous. Lemma 2.5 (a) in [19] now provides unitaries in $A_n \cong A(m, \mathbb{H})$ that conjugate these approximants to real diagonal elements. We may assume the eigenvalues are arranged in increasing order. Now a_n being approximately unitarily close to b_n implies that their spectra are approximately contained in each other in each fibre, so the above diagonal elements are norm close to each other.

Case 4: ($A_n \cong M_m(C_{\mathbb{R}}[0,1])$) Using the lemma on distinct eigenvalues, we can approximate a_n and b_n by elements $a'_n = \lambda_1(t)P_1(t) + \cdots + \lambda_m(t)P_m(t)$ and $b'_n = \gamma_1(t)Q_1(t) + \cdots + \gamma_m(t)Q_m(t)$ where λ_i and γ_i are continuous real functions and $\{P_i\}$ and $\{Q_i\}$ are families of pairwise orthogonal minimal projections in $M_m(C_{\mathbb{R}}[0,1])$ and the λ_i and γ_i are distinct and arranged in increasing order in each fibre. We may choose partial isometries $w_i \in M_m(C_{\mathbb{R}}[0,1])$ such that $w_i^* w_i = P_i$ and $w_i w_i^* = Q_i$ for each i. Then $w = w_1 + \ldots w_n$ is a unitary in $M_m(C_{\mathbb{R}}[0,1])$. By construction, we have that a'_n is approximately equal to a unitary conjugate of b'_n in $M_m(C_{\mathbb{C}}[0,1])$. It follows that their corresponding eigenvalues, when listed in increasing order are close to each other. Thus a'_n is approximately equal to $wb'_n w^*$, and $wb_n w^*$ approximates a_n.

Case 5: ($A_n \cong M_m(C_{\mathbb{H}}[0,1])$) The proof in this case is the same as for $A_n \cong M_m(C_{\mathbb{R}}[0,1])$ with the lemma on density of elements with distinct eigenvalues for $\mathbb{H}$ in place of that for $\mathbb{R}$. □

Lemma 2.16. *Let A be either $\mathcal{K}_{\mathbb{R}}$ or $\mathbb{H} \otimes \mathcal{K}_R$ and let B be a hereditary subalgebra of A. Let $x, y \in A$ be such that*

$$x^*x = y^*y \text{ and } xx^*, yy^* \in B.$$

Then for every $\epsilon > 0$ there is a unitary $u \in B^{\sim}$ such

$$\|x - uy\| < \epsilon.$$

Proof. Let B, x, and y be as above, and let $\mathbb{F}$ denote $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Then there exists a projection $p \in B(H_{\mathbb{F}})$ such that $B = p\mathcal{K}_{\mathbb{F}}p$. Let $x = v|x|$, $y = w|y| = w|x|$ be the polar decompositions of x and y in $B(H_{\mathbb{F}})$. Let $q = vv^*$, $r = ww^*$, and let f denote the support projection of $|x|$ in $B(H_{\mathbb{F}})$. Let $e = vw^*$. Then $e \in B(H_{\mathbb{F}})$, $e^*e = ww^* = r$, $ee^* = vv^* = q$, and $x = ey$. We have $r, q \leq p$. Now choose a finite rank spectral projection $t \leq f$ of $|x|$ such that $t|x| \simeq |x|$. Let $t' = wtw^*$. Then $wt = t'w$ and $x \simeq xt = eyt = ew|y|t = ewt|y| = (et')w|y| = (et')y$. Let $s = (et')$. Then s is a finite rank partial isometry in B with $s^*s = t$. Using that $K_0(\mathbb{F}) = \mathbb{Z}$, by adding another partial isometry in B to s, we may construct a partial unitary u_1 in B with $u_1 y \simeq sy$, and $u = u_1 + (1 - u_1^* u_1)$ is then a unitary that meets our requirements. □

Lemma 2.17. *Let A be a real C^*-algebra of stable rank 1 and let $a, b \in (A \otimes \mathcal{K}_{\mathbb{R}})_+$. Then $a \precsim b$ if and only if there is $x \in A \otimes \mathcal{K}_{\mathbb{R}}$ such that*

$$a = x^*x \text{ and } xx^* \in \mathrm{Her}(b).$$

Proof. One can check that the proofs of proposition 2.2, lemma 2.3, and proposition 2.4 of [16], proposition 1.4.5 of [10], the relevant portions of [11], and of proposition 1.1 of [1], all work for real C^*-algebras, giving the statement above. □

Theorem 2.18. [Cuntz semigroups of the basic building blocks] *The maps*

$$\begin{aligned}
&\mathrm{Cu}(C_{\mathbb{R}}[0,1]) \longrightarrow \mathrm{Lsc}([0,1],\mathbb{N}\cup\{\infty\}),\\
&\mathrm{Cu}(C_{\mathbb{C}}[0,1]) \longrightarrow \mathrm{Lsc}([0,1],\mathbb{N}\cup\{\infty\}),\\
&\mathrm{Cu}(C_{\mathbb{H}}[0,1]) \longrightarrow \mathrm{Lsc}([0,1],2\mathbb{N}\cup\{\infty\}),\\
&\mathrm{Cu}(A(1,\mathbb{R})) \longrightarrow \mathrm{Lsc}([0,1],\mathbb{N}\cup\{\infty\}),\\
&\mathrm{Cu}(A(1,\mathbb{H})) \longrightarrow \{f \in \mathrm{Lsc}([0,1],\mathbb{N}\cup\{\infty\}) \mid f(1) \in 2\mathbb{N}\}
\end{aligned}$$

given by the rank function

$$[a] \mapsto \mathrm{Rank}(a)$$

are isomorphisms in the category Cu.

Proof. That the first three maps in the theorem are isomorphisms in the category Cu follow using the same proof of Theorem 1.1 of [13]. The proof that the last two maps in the theorem are isomorphisms in the category Cu follow the same steps. Hence we will provide the proof for the last map.

The us show that the map

$$\begin{aligned}
(A(1,\mathbb{H})) &\longrightarrow \{f \in \mathrm{Lsc}([0,1],\mathbb{N}\cup\{\infty\}) \mid f(1) \in 2\mathbb{N}\}\\
[a] &\mapsto \mathrm{Rank}(a),
\end{aligned}$$

is an isomorphism in the category Cu. Let $a, b \in A(1,\mathbb{H})) \otimes \mathcal{K}_{\mathbb{R}}$ be positive contractions such that

$$\mathrm{Rank}(a) \leq \mathrm{Rank}(b).$$

Then using that $A(1,\mathbb{H}))$ is a subalgebra of $C([0,1],\mathrm{M}_2(\mathbb{C}))$ and that the second map in the statement of the theorem is an order-embedding it follows that $a \precsim b$ in $C([0,1],\mathrm{M}_2(\mathbb{C}))\otimes\mathcal{K}_{\mathbb{R}}$. Then using that $C([0,1],\mathrm{M}_2(\mathbb{C}))\otimes\mathcal{K}_{\mathbb{R}}$ has stable rank one and Lemma 2.17 we can find an element $x \in C([0,1],\mathrm{M}_2(\mathbb{C})) \otimes \mathcal{K}_{\mathbb{R}}$ such that

$$a = x^*x \text{ and } xx^* \in \overline{b(C([0,1],\mathrm{M}_2(\mathbb{C})) \otimes \mathcal{K}_{\mathbb{R}})b}.$$

We have that

$$\mathrm{Rank}(a(1)) \leq \mathrm{Rank}(b(1)).$$

Hence, since the map from $\mathrm{Cu}(\mathbb{H}) \longrightarrow \mathbb{N}\cup\{\infty\}$ given by the rank function is an isomorphism in the category Cu, it follows that $a(1) \precsim b(1)$ in $\mathrm{Cu}(\mathbb{H})$. Therefore by Lemma 2.17 there is $y \in \mathbb{H} \otimes \mathcal{K}_{\mathbb{R}}$ such that

$$a(1) = y^*y \text{ and } yy^* \in \overline{b(1)(\mathbb{H} \otimes \mathcal{K}_{\mathbb{R}})b(1)} \subset \overline{b(1)(\mathrm{M}_2(\mathbb{C}) \otimes \mathcal{K}_{\mathbb{R}})b(1)}.$$

Let $0 < \epsilon < 1$. Then by Lemma 2.16 applied to $x(1)$ and y and

$$B = \overline{b(1)(\mathrm{M}_2(\mathbb{C}) \otimes \mathcal{K}_{\mathbb{R}})b(1)}$$

there is a unitary u in the unitization of

$$\overline{b(1)(\mathrm{M}_2(\mathbb{C}) \otimes \mathcal{K}_{\mathbb{R}})b(1)}$$

such that $\|ux(1) - y\| < \epsilon$. Using now that the map

$$\mathrm{K}_1(\overline{b(C([0,1],\mathrm{M}_2(\mathbb{C})) \otimes \mathcal{K}_{\mathbb{R}})b}) \to \mathrm{K}_1(\overline{\mathrm{b}(1)(\mathrm{M}_2(\mathbb{C}) \otimes \mathcal{K}_{\mathbb{R}})\mathrm{b}(1)}) = 0$$

induced by the quotient map is surjective and that

$$\overline{b(C([0,1],\mathrm{M}_2(\mathbb{C}))\otimes\mathcal{K}_{\mathbb{R}})b}$$

has stable rank one, there is a unitary v in the unitization of

$$\overline{b(C([0,1],\mathrm{M}_2(\mathbb{C}))\otimes\mathcal{K}_{\mathbb{R}})b}$$

such that $v(1)=u$. Choose z in

$$\overline{b(C([0,1],\mathrm{M}_2(\mathbb{C}))\otimes\mathcal{K}_{\mathbb{R}})}$$

such that $z(1)=y$ and $\|vx-z\|<\epsilon$. Then $z\in A(1,\mathbb{H}))\otimes\mathcal{K}_{\mathbb{R}}$,

$$\|a-z^*z\|=\|(vx)^*(vx)-z^*z\|\le\|(vx)^*(vx-z)+(vx-z)^*z\|<\epsilon+2\epsilon=3\epsilon,$$

and

$$zz^*\in\overline{b(C([0,1],\mathrm{M}_2(\mathbb{C}))\otimes\mathcal{K}_{\mathbb{R}})b}$$

It follows that

$$(a-3\epsilon)_+\precsim z^*z\sim zz^*\precsim b$$

in $A(1,\mathbb{H}))\otimes\mathcal{K}_{\mathbb{R}}$ Since ϵ is an arbitrary number between 0 and 1 we get $[a]=\sup_\epsilon[(a-3\epsilon)_+]\le[b]$ in $\mathrm{Cu}(A(1,\mathbb{H})))$. This shows that the map given by the rank function is an order-embedding.

Let

$$f\in\{f\in\mathrm{Lsc}([0,1],\mathbb{N}\cup\{\infty\})\mid f(1)\in2\mathbb{N}\}\subset\mathrm{Lsc}([0,1],\mathbb{N}\cup\{\infty\}).$$

Using that second map in the statement of the theorem is surjective we can find a positive contraction $a\in C([0,1],\mathrm{M}_2(\mathbb{C}))$ such that $[a]=f$. Also, using that the map $\mathrm{Cu}(\mathbb{H})\to2\mathbb{N}\cup\{\infty\}$ given by the rank function is an isomorphism in the category Cu we can find a positive contraction $b\in\mathbb{H}$ such that $[b]=f(1)$. It follows that $a(1)\sim b$ in $\mathrm{M}_2(\mathbb{C})\otimes\mathcal{K}_{\mathbb{R}}$. □

Lemma 2.19. [Facts about $Cu(A)$**]** *If A is a unital real AI algebra, multiplication by 2 in the Cuntz semi-group $Cu(A)$ is injective, and the maps $Cu(A)\to Cu(A\otimes_{\mathbb{R}}\mathbb{C})$ and $Cu(A)\to Cu(A\otimes_{\mathbb{R}}\mathbb{H})$ induced by the canonical inclusions are injective.*

Proof. Each of the three statements is easily seen to be true for the basic building blocks by our computations of the Cuntz semi-groups of those above. □

Notation: Let $S=\{f:[0,1]\to\mathbb{N}\cup\{\infty\}\,|\,f$ is lower semi-continuous$\}$, $S_0=\{f:[0,1]\to\mathbb{N}\cup\{\infty\}\,|\,f$ is lower semi-continuous and $f(0)=0\}$, and $S_2=\{f:[0,1]\to\mathbb{N}\cup\{\infty\}|f$ is lower semi-continuous and $f(1)$ is even or $\infty\}$.

Lemma 2.20. [Unitisations] *Let A be a real AI algebra. Suppose $\psi,\phi:S\to Cu(A)$ are two Cuntz semi-group morphisms whose restrictions to S_0 are equal. Then $\psi=\phi$.*

Proof. From the computations of the Cuntz semigroups of the basic building blocks above, one sees that the map on Cuntz semi-groups induced by the inclusion of A into $A \otimes_{\mathbb{R}} \mathbb{C}$ is an order embedding in the case where A is a real interval algebra, i.e. $Cu(\iota)(x) \leq Cu(\iota)(y)$ if and only if $x \leq y$. This passes to inductive limits, so the same holds for real AI algebras. From the note added in proof to [3], the Cuntz semi-group of a complex AI algebra has the following cancelation property: $a + c << b + c \Rightarrow a \leq b$. It follows that the Cuntz semi-group of a real AI algebra has the same cancelation property. In the note in [3], it is shown that if $Cu(A)$ is a Cuntz semi-group with this property, two Cuntz morphisms from S to $Cu(A)$ that agree on S_0 are equal. □

Lemma 2.21. *Let A be a real AI algebra. Then the natural inclusion of $D(A)$ into $Cu(A)$ is injective and its image consists of exactly those elements between* 0 *and* [1] *that are compactly contained in themselves.*

Proof. This is well known for the building block $Cu(C_{\mathbb{C}}[0,1])$. For $Cu(C_{\mathbb{R}}[0,1])$, $Cu(C_{\mathbb{H}}[0,1])$, and $Cu(A(1,\mathbb{R}))$, it follows from the computation of their invariants above and the inclusion of $D(A)$ having the same image in each case. For $Cu(A(1,\mathbb{H}))$, it follows from the above computation and the image of $D(A)$ being the even valued constant functions. The property in the lemma is clearly preserved under taking direct sums of the algebras, so it remains to check that it is preserved under taking inductive limits.

In [5], it is shown that inductive limits of sequences always exist in the category Cu, and an explicit construction is given, as follows. Given an inductive system $\{G_n, \varphi_{mn}\}$ in Cu, Let T denote the set of sequences (s_n) such that $s_n \in G_n$ and $s_{n+1} \geq \varphi_{n(n+1)}(s_n)$ for all n. Define addition of sequences element wise: $(s_n) + (t_n) = (s_n + t_n)$. Then the relation defined by $(s_n) \leq (t_n)$ if for any n and $s \in G_n$ with $s << s_n$, eventually $\varphi_{nm}(s) \leq t_m$ in G_m is a preorder on T. Forming the quotient of T with respect to the associated order relation relation then gives a partially ordered semi-group, call it S, that is in the category Cu. This new semi-group G is the inductive limit of the system in the category Cu, when the maps $\varphi_{n\infty} : G_n \to G$ are given by $s \mapsto (0, 0, 0, \dots s, \varphi_{n(n+1)}(s), \dots)$. (Note, the equivalence relation includes tail equivalence.)

Since the maps $\varphi_{n\infty} : G_n \to G$ are morphisms in the category Cu, they preserve the relation of an element being compactly contained in itself. If $A = \lim A_n$ is an inductive limit decomposition of A as a real AI algebra, then $D(A) = \cup_n(i_{n*}D(A_n))$, so it follows that $D(A)$ is contained in the set of elements of $Cu(A)$ that are compactly contained in themselves. Suppose now that $s = [(s_n)] \in Cu(A)$ and that $s << s$. It is shown in [5] that we may assume $s = \sup \varphi_{n\infty}(s_n)$ and that $\varphi(s_{n(n+1)}) << s_{n+1}$ for all n. It follows that $s = \varphi_{m\infty}(s_m)$ for some m and that $s_{m+1} << s_{m+1}$, so s is the image of an element of $D(A)$, as required.

It remains to see that the inclusion is injective. We have this at each finite stage. representing our element of G with a sequence of elements with $s_n \in D(A_n)$, i.e. with elements compactly contained in themselves, we see that the equivalence relation for S for such elements reduces to tail equivalence, which is the same as that for $D(A)$. □

Remark 2.22. *If A and B are real AI algebras and $\eta : Inv_2(A) \to Inv_2(B)$ is a morphism, then η restricts to a morphism from $Inv_1(A)$ to $Inv_1(B)$.*

3. Existence lemmas

In this section, we introduce the existence lemmas, beginning with the finite dimensional cases.

3.1. Finite dimensional algebras.

Lemma 3.1. [Existence for $\mathbb{H}$] *Let A be a real C^*-algebra in the class C. Suppose there exist three homomorphisms η_r, η_c, and η_h of unital abelian partial semigroups such that the following diagram commutes:*

$$\begin{array}{ccccc} (D(\mathbb{H}),[1]) & \longrightarrow & (D(M_2(\mathbb{C}),[1]) & \longrightarrow & (D(M_4(\mathbb{R})),[1]) \\ \downarrow{\scriptstyle \eta_r} & & \downarrow{\scriptstyle \eta_c} & & \downarrow{\scriptstyle \eta_h} \\ (D(A),[1]) & \longrightarrow & (D(A \otimes_{\mathbb{R}} \mathbb{C}),[1]) & \longrightarrow & ((D(A \otimes_{\mathbb{R}} \mathbb{H}),[1])) \end{array}$$

*Then there exists a unital *-homomorphism $\varphi : \mathbb{H} \to A$ such that φ gives rise to η_r, η_c, and η_h.*

Proof. Let A, η_r, η_c, and η_h be as above. The map η_h gives the existence in $A \otimes_{\mathbb{R}} \mathbb{H}$ of four mutually orthogonal, mutually Murray-von Neumann equivalent projections adding up to 1, so there exists a unital *-homomorphism $\psi : M_4(\mathbb{R}) \to A \otimes_{\mathbb{R}} \mathbb{H}$. We may view ψ as a map from $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}$ to $A \otimes_{\mathbb{R}} \mathbb{H}$. Tensoring with $\mathbb{H}$ we get a map $\psi \otimes_{\mathbb{R}} id$ from $M_4(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{H}$ to $(A \otimes_{\mathbb{R}} \mathbb{H}) \otimes_{\mathbb{R}} \mathbb{H} \cong A \otimes_{\mathbb{R}} M_4(\mathbb{R})$. Since they are the same on the dimension ranges, the two maps from $M_4(\mathbb{R})$ to $A \otimes_{\mathbb{R}} M_4(\mathbb{R})$ given by $x \mapsto 1 \otimes_{\mathbb{R}} x$ and $x \mapsto \psi \otimes_{\mathbb{R}} id(x \otimes_{\mathbb{R}} 1)$ are equivalent, by a unitary W say. We then have that $Ad\, W \circ (\psi \otimes_{\mathbb{R}} id)$ must carry $1 \otimes_{\mathbb{R}} \mathbb{H}$ into $A \otimes_{\mathbb{R}} 1$, since it must take the commutant of the copy of $M_4(\mathbb{R})$ into the commutant of its image. This gives us the existence of the desired map φ. That it has the right image on the invariant is forced, since it is unital. □

Lemma 3.2. [Existence for $\mathbb{C}$] *Let A be a real C^*-algebra in the class C. Suppose there exist three homomorphisms η_r, η_c, and η_h of unital abelian partial semigroups such that the following diagram commutes:*

$$\begin{array}{ccccc} (D(\mathbb{C}),[1]) & \longrightarrow & (D(\mathbb{C} \oplus \mathbb{C}),[1]) & \longrightarrow & (D(M_2(\mathbb{C})),[1]) \\ \downarrow{\scriptstyle \eta_r} & & \downarrow{\scriptstyle \eta_c} & & \downarrow{\scriptstyle \eta_h} \\ (D(A),[1]) & \longrightarrow & (D(A \otimes_{\mathbb{R}} \mathbb{C}),[1]) & \longrightarrow & ((D(A \otimes_{\mathbb{R}} \mathbb{H}),[1])) \end{array}$$

*Then there exists a unital *-homomorphism $\varphi : \mathbb{C} \to A$ such that φ gives rise to η_r, η_c, and η_h.*

Proof. Let A, η_r, η_c, and η_h be as above. Let e_1 and e_2 be the units of the two minimal direct summands of $\mathbb{C}\oplus\mathbb{C}$, so $e_1+e_2 = 1$. Then we have $\eta_c([e_1]_{D(\mathbb{C}\oplus\mathbb{C})})+\eta_c([e_2]_{D(\mathbb{C}\oplus\mathbb{C})}) = [1]_{D(A\otimes_{\mathbb{R}}\mathbb{C})}$. By commutativity of the second square of the diagram, we have $\eta_c([e_1]_{D(\mathbb{C}\oplus\mathbb{C})}) \neq 0$ and $\eta_c([e_2]_{D(\mathbb{C}\oplus\mathbb{C})}) \neq 0$. We can find non-zero projections $p_1, p_2 \in A \otimes_{\mathbb{R}} \mathbb{C}$ such that $p_1 \neq 0$, $p_2 \neq 0$, $p_1 + p_2 = 1$ and $[p_1] = [p_2]$ in $D(A \otimes_{\mathbb{R}} \mathbb{H})$. We can thus find a partial isometry $f_{12} \in A \otimes_{\mathbb{R}} \mathbb{H}$ such that $f_{12}f_{12}^* = p_1$ and $f_{12}^*f_{12} = p_2$. Since $A \otimes_{\mathbb{R}} \mathbb{C}$ is a complex algebra, it contains a central copy of $\mathbb{C}$. Consider the partial unitary $u = ip_1 = p_1 i$. the element $v = u + f_{12}^* u f_{12} \in A \otimes_{\mathbb{R}} \mathbb{H}$ is an anti-selfadjoint unitary such that $f_{12}v = vf_{12}$. It follows that the real *-algebra generated by f_{12} and v is isomorphic to $M_2(\mathbb{C})$, so we have a unital *-homomorphism $\psi : M_2(\mathbb{C}) \to A \otimes_{\mathbb{R}} \mathbb{H}$. Writing $M_2(\mathbb{C}) \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$, we get a unital *-homomorphism $\psi \otimes_{\mathbb{R}} id$ mapping $(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}) \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{C} \otimes_{\mathbb{R}} M_4(\mathbb{R})$ to $(A \otimes_{\mathbb{R}} \mathbb{H}) \otimes_{\mathbb{R}} \mathbb{H} \cong A \otimes_{\mathbb{R}} M_4(\mathbb{R})$. The two unital inclusions of $M_4(\mathbb{R})$ into $A \otimes_{\mathbb{R}} M_4(\mathbb{R})$ are equivalent, via a unitary W say. It follows that $Ad\, W \circ (\psi \otimes_{\mathbb{R}} id)$ maps $\mathbb{C} \otimes_{\mathbb{R}} 1_{M_4(\mathbb{R})}$ into $A \otimes_{\mathbb{R}} 1_{M_4(\mathbb{R})}$, since it carries the image of the commutant of $1 \otimes_{\mathbb{R}} M_4(\mathbb{R})$ into the commutant of its image. We thus have the existence of a unital *-homomorphism $\varphi : \mathbb{C} \to A$.

It remains to see that it can be adjusted to give rise to η_r, η_c, and η_h. In the first case this is trivial, since the map is unital, and in the third case it follows from the map being unital and the dimension ranges being torsion-free. From the existence of a unital *-homomorphism from $\mathbb{C}$ to A, we get that there exists a projection $p \in A \otimes_{\mathbb{R}} \mathbb{C}$ such that $\bar{p} = 1 - p$. Since $\bar{p} = \beta p \beta^*$, we have a 2×2 system of matrix units in $A \otimes_{\mathbb{R}} \mathbb{H}$ such that $e_{11} = p$, $e_{22} = \bar{p}$, and $e_{12} = p\beta$. We get an injective *-homomorphism from $A \otimes_{\mathbb{R}} \mathbb{H}$ into $M_2(p(A \otimes_{\mathbb{R}} \mathbb{C})p)$ that sends $a \in p(A \otimes_{\mathbb{R}} \mathbb{C})p$ to $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, sends β to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and sends α to $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Now suppose $x = \eta_c(e_1)$ in the diagram above. Since $x \leq [p] + [\bar{p}] = [1]$, by Riesz interpolation there exist $x_1, x_2 \in D(A \otimes_{\mathbb{R}} \mathbb{C})$ such that $x = x_1 + x_2$, $x_1 \leq [p]$, and $x_2 \leq [\bar{p}]$. Let q_1 be a projection in $p(A \otimes_{\mathbb{R}} \mathbb{C})p$ and let q_2 be a projection in $\bar{p}(A \otimes_{\mathbb{R}} \mathbb{C})\bar{p}$ such that $[q_1] = x_1$ and $[q_2] = x_2$ in $D(A \otimes_{\mathbb{R}} \mathbb{C})$. We have $[1] = 2[p] = 2x$ in $D(A \otimes_{\mathbb{R}} \mathbb{H})$, so $x = p$ in $D(A \otimes_{\mathbb{R}} \mathbb{H})$. We have $x = [q_1] + [q_2]$ in $D(A \otimes_{\mathbb{R}} \mathbb{C})$ and $x = [q_1] + [p - q_1]$ in $D(A \otimes_{\mathbb{R}} \mathbb{H})$, so $[q_2] = [p - q_1]$ in $D(A \otimes_{\mathbb{R}} \mathbb{H})$. Thus $[q_2] = [\overline{p - q_1}]$ in $D(A \otimes_{\mathbb{R}} \mathbb{H})$. Using the embedding of $A \otimes_{\mathbb{R}} \mathbb{H}$ into $M_2(p(A \otimes_{\mathbb{R}} \mathbb{C})p)$ above, we have that $[q_2] = [\overline{p - q_1}]$ in $M_2(p(A \otimes_{\mathbb{R}} \mathbb{C})p)$ as well. Since both projections are in the e_{22} corner, they are equivalent in that corner as well, however, this corner is just $\bar{p}(A \otimes_{\mathbb{R}} \mathbb{C})\bar{p}$, so we have $[q_2] = [\overline{p - q_1}]$ in $D(A \otimes_{\mathbb{R}} \mathbb{C})$. Now let $r = q_1 + \overline{p - q_1}$. Then we have $r + \bar{r} = 1$ and $[r] = [q_1] + [\overline{p - q_1}] = [q_1] + [q_2] = x$ all in $D(A \otimes_{\mathbb{R}} \mathbb{C})$, so the unital *-homomorphism from $\mathbb{C}$ to A associated with the projection r (see Lemma 2.1 in the preliminaries above) has the right image on the invariant. □

Lemma 3.3. [Existence for Amplifications I] *Let A be a unital real C^*-algebra in the class C, and suppose that A has the property that if B is any unital real C^*-algebra in the class C and (η_r, η_c, η_h) is a morphism of invariants from $Inv_1(A)$ to $Inv_1(B)$, then there exists a unital *-homomorphism $\varphi : A \to B$ such that*

$(\eta_r, \eta_c, \eta_h) = Inv_1(\varphi)$. *Then for any natural number* n, $M_n(A)$ *has the same property.*

Proof. Let A and B be as above, and suppose that (η_r, η_c, η_h) is a morphism of invariants from $Inv_1(M_n(A))$ to $Inv_1(B)$. Consider $\eta_r([e_{11}]) \in D(B)$. Since B is in the class C, there exists a projection $p \in B$ such that $[p] = \eta_r([e_{11}])$. Then (η_r, η_c, η_h) defines a morphism of invariants from $Inv_1(A)$ to $Inv_1(pBp)$, where we have identified A with $e_{11}M_n(A)e_{11}$. Thus there exists a unital *-homomorphism $\psi : A \to pBp$ giving rise to (η_r, η_c, η_h). Since $n[p] = [1]$ in $D(B)$, we have that $\varphi = \psi \otimes_{\mathbb{R}} id : A \otimes_{\mathbb{R}} M_n(\mathbb{R}) \to (pBp) \otimes_{\mathbb{R}} M_n(\mathbb{R}) \cong B$ meets our requirements. □

Lemma 3.4. [Existence for Direct Sums I] *Let* A_1 *and* A_2 *be two unital real* C^* *algebras, and suppose they have the property that if* B *is any unital real* C^**-algebra in the class* C *and* (η_r, η_c, η_h) *is a morphism of invariants from* $Inv_1(A_i)$ *to* $Inv_1(B)$, *then there exists a unital *-homomorphism* $\varphi : A_i \to B$ *such that* $(\eta_r, \eta_c, \eta_h) = Inv_1(\varphi)$. *Then* $A_1 \oplus A_2$ *has the same property.*

Proof. Let A_1 and A_2 be as in the statement of the lemma, suppose B is in the class C, and $\eta : Inv_1(A_1 \oplus A_2) \to Inv_1(B)$ is a morphism of invariants. We have $\eta_r([1_{A_1}]) + \eta_r([1_{A_2}]) = [1_B]_{D(B)}$, so there exists a projection $p \in B$ with $\eta_r([1_{A_1}]) = [p]_{D(B)}$ and $\eta_r([1_{A_2}]) = [1-p]_{D(B)}$. Both pBp and $(1-p)B(1-p)$ are in the class C. The morphism induced by the inclusion of pBp into B is injective in all three components, so we may view the pieces of the invariant for pBp as subsets of those for B. Let i^1 and i^2 denote the inclusions of A_1 and A_2 respectively into $A_1 \oplus A_2$. We get morphisms of invariants $\eta \circ i^1_*$ and $\eta \circ i^2$ from $Inv_1(A_1)$ to $Inv_1(pBp)$ and from $Inv_1(A_2)$ to $Inv_1((1-p)B(1-p))$ respectively. From our hypothesis on A_1 and A_2, there exist unital $*$-homomorphisms $\varphi_1 : A_1 \to pBp$ and $\varphi_2 : A_2 \to (1-p)B(1-p)$ that induce these maps on the invariants. Then $\varphi = \varphi_1 \oplus \varphi_2$ meets our requirements. □

Theorem 3.5. [Existence for Finite Dimensional Algebras] *Let* A *be a finite dimensional real* C^**-algebra and let* B *be a real* C^**-algebra in the class* C. *Then if* $\eta = (\eta_r, \eta_c, \eta_h)$ *is a morphism of invariants from* $Inv_1(A)$ *to* $Inv_1(B)$, *there exists a unital *-homomorphism* $\varphi : A \to B$ *such that* $\eta = Inv_1(\varphi)$.

Proof. The case in which $A \cong \mathbb{R}$ is trivial, as we just send the unit to the unit. The case of $A \cong M_n(\mathbb{R})$ then follows from the lemma for amplifications. The cases of $A \cong \mathbb{C}$ or $A \cong \mathbb{H}$ were covered in the lemmas above, and the case for matrix algebras over these follows from the lemma for amplifications. Finally, since any finite dimensional real C^* algebra is a finite direct sum of these cases, the theorem follows from the lemma for direct sums. □

3.2. Interval algebras.

Lemma 3.6. *If* A *is a real unital AI algebra,* e *is a positive contraction in* A, *and* x *is an element of* A *such that* $ex = xe = x$, *then* $x^*x + e$ *is approximately unitarily equivalent to* $xx^* + e$ *in* A.

Proof. From the results of [4] and [14], it follows that x^*x+e is approximately unitarily equivalent to xx^*+e in $A \otimes_{\mathbb{R}} \mathbb{C}$. From our lemma on equivalence of positive elements within the real part, it follows that we take the unitaries to lie in A. $\square$

Lemma 3.7. [Existence for $C_0((0,1],\mathbb{R})$] *Let $a(t)=t$ denote the canonical generator of $C_0((0,1],\mathbb{R})$. Let B be a unital real AI algebra, and let (η_r,η_c,η_h) be a triple of Cu morphisms such that the following diagram commutes,*

$$\begin{array}{ccccc} Cu(C_0((0,1],\mathbb{R})) & \longrightarrow & Cu(C_0((0,1],\mathbb{C})) & \longrightarrow & Cu(C_0((0,1],\mathbb{H})) \\ \downarrow{\scriptstyle \eta_r} & & \downarrow{\scriptstyle \eta_c} & & \downarrow{\scriptstyle \eta_h} \\ Cu(B) & \longrightarrow & Cu(B\otimes_{\mathbb{R}}\mathbb{C}) & \longrightarrow & Cu(B\otimes_{\mathbb{R}}\mathbb{H}) \end{array}$$

and $[\eta_r(a)] \leq [1]$. Then there exists a real $*$-homomorphism $\varphi : C_0((0,1],\mathbb{R}) \to B$ such that $(\eta_r,\eta_c,\eta_h) = (Cu(\varphi), Cu(\varphi\otimes_{\mathbb{R}} id_{\mathbb{C}}), Cu(\varphi\otimes_{\mathbb{R}} id_{\mathbb{H}}))$.

Proof. If $\varphi : C_0((0,1],\mathbb{R}) \to B$ is a $*$-homomorphism such that $Cu(\varphi)=\eta_r$, then commutativity of the diagram and the fact the morphisms in the top row are isomorphisms implies that $Cu(\varphi\otimes_{\mathbb{R}} id_{\mathbb{C}}) = \eta_c$ and $Cu(\varphi\otimes_{\mathbb{R}} id_{\mathbb{H}}) = \eta_h$ as well, so we only need consider the first term in the invariant in this case. The required existence now follows from the proof of Theorem 2 in [14], which carries over verbatim to the real case here, and the lemma above. $\square$

Lemma 3.8. [Existence for $C_0((0,1],\mathbb{C})$] *Let $b(t)=it$ denote the canonical generator of $C_0((0,1],\mathbb{C})$. Let B be a unital real AI algebra, and let (η_r,η_c,η_h) be a triple of Cu morphisms such that the following diagram commutes,*

$$\begin{array}{ccccc} Cu(C_0((0,1],\mathbb{C})) & \longrightarrow & Cu(C_0((0,1],\mathbb{C})^2) & \longrightarrow & Cu(M_2(C_0((0,1],\mathbb{C}))) \\ \downarrow{\scriptstyle \eta_r} & & \downarrow{\scriptstyle \eta_c} & & \downarrow{\scriptstyle \eta_h} \\ Cu(B) & \longrightarrow & Cu(B\otimes_{\mathbb{R}}\mathbb{C}) & \longrightarrow & Cu(B\otimes_{\mathbb{R}}\mathbb{H}) \end{array}$$

and $[\eta_r(|b|)] \leq [1]$. Then there exists a real $*$-homomorphism $\varphi : C_0((0,1],\mathbb{C}) \to B$ such that $(\eta_r,\eta_c,\eta_h) = (Cu(\varphi), Cu(\varphi\otimes_{\mathbb{R}} id_{\mathbb{C}}), Cu(\varphi\otimes_{\mathbb{R}} id_{\mathbb{H}}))$.

Proof. Let B, b, and (η_r,η_c,η_h) be as above. Let η_c^1 (respectively, η_c^2) denote the restriction of η_c to the first (respectively, second) direct summand. Let x denote the canonical generator $t \mapsto t$ of $C_0((0,1],\mathbb{C})$. From [4] there exists a *-homomorphism $\psi : C_0((0,1],\mathbb{C}) \oplus C_0((0,1],\mathbb{C}) \to B\otimes\mathbb{C}$ such that $Cu(\psi)=\eta_c$. Let ψ^1 (respectively, ψ^2) denote the restriction of ψ to the first (respectively, second) direct summand. Let $a_+ = \psi^1(x)$ and $a_- = \psi^2(x)$, and let $a = a_+ - a_-$.

Next, we lift a approximately to a self adjoint element $a_n \in B_n \otimes_{\mathbb{R}} \mathbb{C}$ such that $\| a_{n+} \| \leq \| a_+ \|$ and $\| a_{n-} \| \leq \| a_- \|$. We get a *-homomorphism ψ_n : $C_0((0,1],\mathbb{C}) \oplus C_0((0,1],\mathbb{C}) \to B_n \otimes_{\mathbb{R}} \mathbb{C}$ such that $\psi_n^1(x) = \psi_n(x,0) = a_{n+}$ and $\psi_n^2(x) = \psi_n(0,x) = a_{n-}$.

By the commutativity of the second square in the diagram in the theorem, the two maps η_c^1 and η_c^2 become equal when pushed forward to $Cu(B\otimes_{\mathbb{R}}\mathbb{H})$. Pushing forward further to the complexification, we see they give rise to equal maps to $Cu(B \otimes_{\mathbb{R}} M_2(\mathbb{C}))$. From [4] it follows that ψ_1 and ψ_2 are approximately unitarily equivalent via unitaries in $M_2(B \otimes_{\mathbb{R}} \mathbb{C})$, and from our lemma on equivalence of positive elements within the real part, they are approximately unitarily equivalent via unitaries in $B \otimes_{\mathbb{R}} \mathbb{H}$. We may approximately lift these unitaries to unitaries in $B_n \otimes_{\mathbb{R}} \mathbb{H}$, and pushing forward in the inductive system if necessary, we may assume there exists a unitary $v_n \in B_n \otimes_{\mathbb{R}} \mathbb{H}$ such that $\| a_{n+} - v_n a_{n-} v_n^* \|$ is as small as we like.

Suppose now that B is a real interval algebra. Notice that by commutativity of the first square in the diagram, if we write $\eta_c = \eta_c^1 + \eta_c^2$, as above, the map η_c^2 is determined by η_c^1 on the finite valued functions. Since Cuntz semi-group maps preserve suprema, it follows that η_c^1 determines η_c^2. Now suppose that we have a lift ψ of η_c to a *-homomorphism. If we may choose ψ_1 such that $Ad\,\beta \circ \psi_1 \perp \psi_1$, then replacing ψ_2 with $Ad\,\beta \circ \psi_1 \circ Ad\,\beta$ we get a real *-homomorphism $\zeta = \psi_1 + Ad\,\beta \circ \psi_1 \circ Ad\,\beta$ that induces the same map on the Cuntz semi-group. One way we can do this is if we may choose ψ_1 such that for some projection $p \in B_n$ we have $\psi_1(x) \leq p$ and $p \perp \bar{p}$. Since the Cuntz semi-group respects direct summands, it will suffice to consider each of our basic building blocks in turn.

Recall from Section 2 that

$$S = \{f : [0,1] \to \mathbb{N} \cup \{\infty\} \mid f \text{ is lower semi-continuous}\},$$

$$S_0 = \{f : [0,1] \to \mathbb{N} \cup \{\infty\} \mid f \text{ is lower semi-continuous and } f(0) = 0\},$$

and

$$S_2 = \{f : [0,1] \to \mathbb{N} \cup \{\infty\} \mid f \text{ is lower semi-continuous and } f(1) \text{ is even or } \infty\}.$$

Case 1: $(B_n \cong M_m(C_{\mathbb{R}}[0,1])$. In this case, we have the following diagram of invariants:

$$\begin{array}{ccccc} S_0 & \xrightarrow{x \mapsto (x,x)} & S_0 \oplus S_0 & \xrightarrow{(x,y) \mapsto x+y} & S_0 \\ \downarrow{\scriptstyle \eta_r} & & \downarrow{\scriptstyle \eta_c} & & \downarrow{\scriptstyle \eta_h} \\ S & \xrightarrow{x \mapsto x} & S & \xrightarrow{x \mapsto x} & S \end{array}$$

From the commutativity of the first square, we get

$$\eta_r([|b|]) = \eta_c^1([|b|]) + \eta_c^2([|b|]),$$

and from commutativity of the second square we get $\eta_c^1([|b|]) = \eta_c^2([|b|])$, so $2\eta_c^1([|b|]) \leq [1]$. The rank one projection

$$p_2 = (1/2)(e_{11} - ie_{12} + ie_{21} + e_{22}) \in M_2(\mathbb{C})$$

has the property that $p_2 + \bar{p}_2 = 1$. Letting p be a suitable direct sum of copies of p_2, viewed as a constant matrix valued function, we have a projection in $M_m(C_{\mathbb{C}}[0,1])$ that meets our requirements.

Case 2: $(B_n \cong M_m(C_{\mathbb{H}}[0,1])$. In this case, we have the following diagram of invariants:

$$\begin{array}{ccccc} S_0 & \xrightarrow{x\mapsto(x,x)} & S_0\oplus S_0 & \xrightarrow{(x,y)\mapsto x+y} & S_0 \\ \downarrow{\scriptstyle \eta_r} & & \downarrow{\scriptstyle \eta_c} & & \downarrow{\scriptstyle \eta_h} \\ S & \xrightarrow{x\mapsto 2x} & S & \xrightarrow{x\mapsto 2x} & S \end{array}$$

The commutativity of the first square gives $2\eta_r(|b|) = \eta_c^1([|b|]) + \eta_c^2([|b|])$, so $\eta_c^1([|b|]) + \eta_c^2([|b|]) \leq [1]$ in $Cu(M_m(C_{\mathbb{C}}[0,1]))$, and the commutativity of the second square gives $\eta_c^1([|b|]) = \eta_c^2([|b|])$, so $2\eta_c^1([|b|]) \leq [1]$. Once again we have the rank one projection $p_2 = (1/2)(e_{11} - ie_{12} + ie_{21} + e_{22}) \in M_2(\mathbb{C})$, which again has the property that $\bar{p} = p^{\#*} = 1-p$. Again letting p be a suitable direct sum of copies of p_2, viewed as a constant matrix valued function, we have a projection in $M_m(C_{\mathbb{C}}[0,1])$ that meets our requirements.

Case 3: ($B_n \cong M_m(C_{\mathbb{C}}[0,1])$. We have the following diagram of invariants:

$$\begin{array}{ccccc} S_0 & \xrightarrow{x\mapsto(x,x)} & S_0\oplus S_0 & \xrightarrow{(x,y)\mapsto x+y} & S_0 \\ \downarrow{\scriptstyle \eta_r} & & \downarrow{\scriptstyle \eta_c} & & \downarrow{\scriptstyle \eta_h} \\ S & \xrightarrow{x\mapsto(x,x)} & S\oplus S & \xrightarrow{(x,y)\mapsto x+y} & S \end{array}$$

Write $\eta_c^1 = \eta_c^{11} + \eta_c^{12}$ and $\eta_c^2 = \eta_c^{21} + \eta_c^{22}$ for the decomposition of η_c into partial maps between the direct summands. From the commutativity of the first square, we have $\eta_c^{11} + \eta_c^{12} = \eta_r$ and $\eta_c^{12} + \eta_c^{22} = \eta_r$, when all these are viewed as maps from S_0 to S. From the commutativity of the second square, we get $\eta_h = \eta_c^{11} + \eta_c^{12} = \eta_c^{21} + \eta_c^{22}$. From the commutativity of the large square, we have $2\eta_r = 2\eta_h$, so $\eta_r = \eta_h$.

Combining these, we have that $\eta_c = \begin{pmatrix} \eta_c^{11} & \eta_c^{12} \\ \eta_c^{12} & \eta_c^{11} \end{pmatrix}$.

From [4] there exists a complex *-homomorphism ψ_1 with $Cu(\psi) = \eta_c^{11} + \eta_c^{12}$. Writing $\psi_1 = \psi_{11} + \psi_{12}$, we may assume that $\psi_{12} \perp \psi_{11}$ It follows that defining ψ_2 to be $Ad\,\beta \circ \psi_1 \circ Ad\,\beta$ the map $\psi = \psi_1 + \psi_2$ is a real $*$-homomorphism meeting our requirements.

Case 4: ($B_n \cong A(m, \mathbb{H})$). Given a function f on $[0,1]$, define a new function f^s on $[0,1]$ by $f^s(t) = f(2t)$ for $0 \leq t \leq 1/2$ and $f^s(t) = f(2-2t)$ for $1/2 \leq t \leq 1$. Define new functions f^b and f^h by $f^b(t) = f(1-t)$ and $f^h(t) = f(t/2)$. Then we have the following diagram of invariants:

$$\begin{array}{ccccc} S_0 & \xrightarrow{x\mapsto(x,x)} & S_0\oplus S_0 & \xrightarrow{(x,y)\mapsto x+y} & S_0 \\ \downarrow{\scriptstyle \eta_r} & & \downarrow{\scriptstyle \eta_c} & & \downarrow{\scriptstyle \eta_h} \\ S_2 & \xrightarrow{x\mapsto x^s} & S & \xrightarrow{x\mapsto(f+f^b)^h} & S \end{array}$$

From the commutativity fo the first square we get $\eta_c^1(x) + \eta_c^2(x) = (\eta_r(x))^s$, and from commutativity of the second square we get $(\eta_c^1 + (\eta_c^1)^b)^h(x) = \eta_h(x) =$

$(\eta_c^2 + (\eta_c^2)^b)^h(x)$, which implies $(\eta_c^1 + \eta_c^1)(x) = (\eta_c^2 + \eta_c^2)(x)$. Together these give $2\eta_c^1 = 2(\eta_c^2)^b$, and so $\eta_c^1 = (\eta_c^2)^b$.

We have in this case that $n = 2m$ and $B_n \otimes \mathbb{C} \cong M_n(C[0,1])$. We have that $f^\tau(t) = (f(t-1))^\#$ for $f \in B_n \otimes \mathbb{C}$. The constant rank one projection $p_2 = (1/2)(e_{11} - ie_{12} + ie_{21} + e_{22})$ defined above satisfies $\bar{p} = p^{\#*} = 1 - p$ in the case $m = 1$. For general m, we may choose a system of matrix units $\{e_{ij}\}$ in $B_n \otimes \mathbb{C}$ such that $\overline{e_{11}} = e_{nn}, \overline{e_{22}} = e_{(n-1)(n-1)}, \ldots \overline{e_{mm}} = e_{(m+1)(m+1)}$, so that the bar operation flips the matrix units down the diagonal.

From [4] there exists a *-homomorphism ψ_1 lifting η_c^1. We may assume that ψ_1 is diagonal with respect to our chosen system of matrix units, and that the eigenvalues of the canonical positive generator of $C_{\mathbb{C}}[0,1]$ are listed in decreasing order down the diagonal, with zeros at the bottom. Applying $Ad\,\beta$ to ψ_1 both flips the argument in the interval, and the order down the diagonal, so the eigenvalues are now in increasing order, with the zeros at the top. Since $Ad\,\beta \circ \psi_1 \circ Ad\,\beta$ is a lift of η_c^2, our condition $\eta_c^1(x) + \eta_c^2(x) = (\eta_r(x))^s \leq [1]$ now ensures that $Ad\,\beta \circ \psi_1 \circ Ad\,\beta \perp \psi_1$.

Case 5: $(B_n \cong A(m, \mathbb{R}))$

We have the following diagram of invariants:

$$\begin{array}{ccccc}
S_0 & \xrightarrow{x \mapsto (x,x)} & S_0 \oplus S_0 & \xrightarrow{(x,y) \mapsto x+y} & S_0 \\
\downarrow{\scriptstyle \eta_r} & & \downarrow{\scriptstyle \eta_c} & & \downarrow{\scriptstyle \eta_h} \\
S & \xrightarrow{x \mapsto x^s} & S & \xrightarrow{x \mapsto (f+f^b)^h} & S_2
\end{array}$$

This case is very similar to case 4 above. The only difference is that in the case where n is odd, there will be a fixed matrix unit in the diagonal flip implemented by $Ad\,\beta$. The conditions imply that this will be a zero eigen-projection for one of ψ_1 or $Ad\,\beta \circ \psi_1 \circ Ad\,\beta$, so again there is no overlap. □

Lemma 3.9. [Existence for Unitisations] *Let A be $C_0((0,1], \mathbb{C})$ or $C_0((0,1], \mathbb{R})$ and let B be a unital real AI algebra. Then the unitisation $\tilde{A}$ has the property that if η is a morphism from $Inv_2(\tilde{A})$ to $Inv_2(B)$, then there exists a unital $*$-homomorphism $\tilde{\varphi} : \tilde{A} \to B$ such that $\eta = Inv_2(\tilde{\varphi})$.*

Proof. The inclusion of A into $\tilde{A}$ induces injective maps on the three Cuntz semigroups, so by our hypothesis and the two lemmas above, there exists a $*$-homomorphism $\varphi : A \to B$ such that $Cu(\varphi) = \eta \circ Cu(\iota)$ in each of the three cases, where ι is the inclusion of A into $\tilde{A}$. We now get a well defined unital real $*$-homomorphism $\tilde{\varphi} : \tilde{A} \to B$ by $\tilde{\varphi}(a + \lambda 1) = \varphi(a) + \lambda 1_B$. That $Inv_2(\tilde{\varphi}) = \eta$ now follows from our lemma on S, S_0 (unitisations). □

Lemma 3.10. [Existence for $A(1, \mathbb{R})$] *Let A be a unital real AI algebra, and let $\eta : Inv_2(A(1, \mathbb{R})) \to Inv_2(A)$ be a morphism of invariants. Then there exists a unital real $*$-homomorphism $\varphi : A(1, \mathbb{R}) \to A$ such that $Inv_2(\varphi) = \eta$.*

Proof. This follows from the lemmas above, since $A(1,\mathbb{R})$ is the unitisation of $C_0((0,1],\mathbb{C})$. □ □

Lemma 3.11. [Existence for $C_{\mathbb{R}}[0,1]$] *Let A be a unital real AI algebra, and let $\eta : Inv_2(C_{\mathbb{R}}[0,1]) \to Inv_2(A)$ be a morphism of invariants. Then there exists a unital real $*$-homomorphism $\varphi : C_{\mathbb{C}}[0,1] \to A$ such that $Inv_2(\varphi) = \eta$.*

Proof. This follows from the lemmas above, since $C_{\mathbb{R}}[0,1]$ is the unitisation of $C_0((0,1],\mathbb{R})$. □ □

Lemma 3.12. [Existence for $C_{\mathbb{C}}[0,1]$] *Let A be a real AI algebra, and let $\eta : Inv_2(C_{\mathbb{C}}[0,1]) \to Inv_2(A)$ be a morphism of invariants. Then there exists a unital real $*$-homomorphism $\varphi : C_{\mathbb{C}}[0,1] \to A$ such that $Inv_2(\varphi) = \eta$.*

Proof. Since a morphism of Inv_2 restricts to one of Inv_1, and AI algebras are in the class C, we have that there exists a unital real $*$-homomorphism $\varphi_0 : \mathbb{C} \to A$ such that $Inv_1(\varphi_0) = \eta \circ Inv_1(\iota)$, where ι is the inclusion of $\mathbb{C}$ into $C_{\mathbb{C}}[0,1]$ as constant functions. From this we get the canonical projections $p = (1/2)(1 - \varphi_0(i)\alpha)$ and $\bar{p} = (1/2)(1 + \tilde{\varphi}_0(i)\alpha)$ in $A \otimes_{\mathbb{R}} \mathbb{C}$ with $1 = p + \bar{p}$.

The algebra $C_{\mathbb{C}}[0,1]$ is generated as a real C^*-algebra by the constant functions and the element $a(t) = t$, so to construct a $*$-homomorphism from it, after specifying where the constant functions go, one need only choose a positive contraction commuting with the image of the constants to send a to.

From [4], there exists a unital $*$-homomorphism $\varphi_1 : C_{\mathbb{C}}[0,1] \to p(A \otimes_{\mathbb{R}} \mathbb{C})p$ such that $Inv_2(\varphi_1) = \eta \circ Inv_2(j_1)$, where j_1 is the inclusion of $C_{\mathbb{C}}[0,1]$ as the first summand in $C_{\mathbb{C}}[0,1] \otimes_{\mathbb{R}} \mathbb{C}$. Define $\varphi : C_{\mathbb{C}}[0,1] \to A \otimes_{\mathbb{R}} \mathbb{C}$ to be the unique real $*$-homomorphism that agrees with φ_0 on the constant functions and sends a to $\varphi_1(a) + \overline{\varphi_1(a)}$. This is well defined, since φ_1 and $\overline{\varphi_1}$ have orthogonal images, so a is being sent to a positive contraction, and the given element commutes with p and $\bar{p}$, and so with $\varphi_0(\mathbb{C})$. Moreover, since this element is invariant under conjugation, it lies in A, so we get that φ is a homomorphism into A.

It remains to check that φ induces the right map on Inv_2. From [4] again, there exists a unital $*$-homomorphism $\varphi_2 : C_{\mathbb{C}}[0,1] \to p(A \otimes_{\mathbb{R}} \mathbb{C})p$ such that $Inv_2(\varphi_1) = \eta \circ Inv_2(j_2)$, where j_2 is the inclusion of $C_{\mathbb{C}}[0,1]$ as the second summand in $C_{\mathbb{C}}[0,1] \otimes \mathbb{C}$. As in the proof of the existence lemma for $\mathbb{C}$, we have an embedding of $A \otimes_{\mathbb{R}} \mathbb{H}$ into $M_2(p(A \otimes_{\mathbb{R}} \mathbb{C})p)$, and both $\overline{\varphi_1}$ and φ_2 map into the e_{22} corner. By the commutativity of the diagram, these two maps have the same invariants as maps into $A \otimes_{\mathbb{R}} \mathbb{H}$, and therefore also into $M_2(p(A \otimes_{\mathbb{R}} \mathbb{C})p)$. They are therefore approximately unitarily equivalent in $M_2(p(A \otimes_{\mathbb{R}} \mathbb{C})p)$. Since they live in a cutdown by a projection, they are approximately unitarily equivalent in the cutdown. They therefore have the same invariant as maps into $A \otimes_{\mathbb{R}} \mathbb{C}$, so our map φ has the right complex component on the invariant. That it does the right thing on the real part now follows from commutativity of the first square in the diagram and the injectivity of the horizontal maps in that square. That it does the right thing on the quaternion component follows from the commutativity of the second square, and the surjectivity of the top horizontal map in that square. □

Lemma 3.13. [Existence for $C_{\mathbb{H}}[0,1]$] *Let A be a unital real AI algebra, and let $\eta : Inv_2(C_{\mathbb{H}}[0,1]) \to Inv_2(A)$ be a morphism of invariants. Then there exists a unital real $*$-homomorphism $\varphi : C_{\mathbb{H}}[0,1] \to A$ such that $Inv_2(\varphi) = \eta$.*

Proof. As in the case of $C_{\mathbb{C}}[0,1]$ above, there exists a unital $*$-homomorphism $\varphi_0 : \mathbb{H} \to A$ such that $Inv_1(\varphi_0) = \eta \circ Inv_1(\iota)$, where ι is the inclusion of $\mathbb{H}$ into $C_{\mathbb{H}}[0,1]$ as constant functions.

The algebra $C_{\mathbb{H}}[0,1]$ is generated as a real C^*-algebra by the constant functions and the element $a(t) = t$. To construct a $*$-homomorphism from it, having one from $\mathbb{H}$ already, one need only choose a positive contraction commuting with the already chosen copy of $\mathbb{H}$.

In $C_{\mathbb{H}}[0,1] \otimes_{\mathbb{R}} \mathbb{H} \cong M_4(C_{\mathbb{R}}[0,1])$, our generator $a(t)$ is $diag(t,t,t,t)$. Apply the existence lemma for $C_0((0,1],\mathbb{R})$ above to get a real $*$-homomorphism $\varphi_1 : C^*(te_{11}) \to \varphi_0 \otimes_{\mathbb{R}} id_{\mathbb{H}}(e_{11})(A \otimes_{\mathbb{R}} \mathbb{H})\varphi_0 \otimes_{\mathbb{R}} id_{\mathbb{H}}(e_{11})$ such that $Inv_2(\varphi_1) = \eta \circ Inv_2(j)$, where j is the inclusion $C_0((0,1],\mathbb{R}) \to C^*(te_{11})$. Let $a_{11} = \varphi_1(te_{11})$, and let $\tilde{a} = a_{11} + \varphi_0 \otimes_{\mathbb{R}} id_{\mathbb{H}}(e_{21})a_{11}\varphi_0 \otimes_{\mathbb{R}} id_{\mathbb{H}}(e_{12}) + \varphi_0 \otimes_{\mathbb{R}} id_{\mathbb{H}}(e_{31})a_{11}\varphi_0 \otimes_{\mathbb{R}} id_{\mathbb{H}}(e_{13}) + \varphi_0 \otimes_{\mathbb{R}} id_{\mathbb{H}}(e_{41})a_{11}\varphi_0 \otimes_{\mathbb{R}} id_{\mathbb{H}}(e_{14})$. Then $\tilde{a}$ commutes with the image of $\varphi_0 \otimes id_{\mathbb{H}}$, so $\tilde{a} \in A$, and it also commutes with the image of φ_0 in A. We therefore get a real $*$-homomorphism $\varphi : C_{\mathbb{H}}[0,1] \to A$ such that $\varphi(a) = \tilde{a}$ and $\varphi|_{\mathbb{H}} = \varphi_0$.

It remains to check that φ induces the right map on Inv_2. By construction, the map agrees with η_h. Since the maps from $Cu(A) \to Cu(A \otimes_{\mathbb{R}} \mathbb{H})$ are injective, this implies that $Inv_2(\varphi)_r = \eta_r$. Commutativity of the diagram now gives that $Inv_2(\varphi)_c = \eta_c$ on all multiples of two in $Cu(M_2(C_{\mathbb{C}}[0,1]))$. Since multiplication by 2 is injective, this implies that they agree. □

Lemma 3.14. [Existence for $A(1,\mathbb{H})$] *Let A be a unital real AI algebra, and let $\eta : Inv_2(A(1,\mathbb{H})) \to Inv_2(A)$ be a morphism of invariants. Then there exists a unital real $*$-homomorphism $\varphi : A(1,\mathbb{H}) \to A$ such that $Inv_2(\varphi) = \eta$.*

Proof. As in the case of $C_{\mathbb{H}}[0,1]$ above, there exists a unital $*$-homomorphism $\varphi_0 : \mathbb{H} \to A$ such that $Inv_1(\varphi_0) = \eta \circ Inv_1(\iota)$, where ι is the inclusion of $\mathbb{H}$ into $C_{\mathbb{H}}[0,1]$ as constant functions.

The algebra $A(1,\mathbb{H})$ is generated as a real C^*-algebra by the constant functions and the element $b(t) = it$. To construct a $*$-homomorphism from it, having one from $\mathbb{H}$ already, we need to choose a real $*$-homomorphism from $C_0((0,1],\mathbb{C})$ to A whose image commutes with the already given copy of $\mathbb{H}$. We do this in a similar fashion to the case of $C_{\mathbb{H}}[0,1]$ above, but using the existence lemma for $C_0((0,1],\mathbb{C})$ in place of the one for $C_0((0,1],\mathbb{R})$ and $b(t)$ in place of $a(t)$.

It remains to check that φ induces the right map on Inv_2. As the maps agree on the $\mathbb{H}$ component, and the horizontal maps in the top row are injective, this follows exactly as in the case for $C_{\mathbb{H}}[0,1]$ above. □

Lemma 3.15. [Existence for Amplifications II] *Let A be a real interval algebra, and suppose that A has the property that if B is any unital real AI algebra* *and (η_r, η_c, η_h) is a morphism of invariants from $Inv_2(A)$ to $Inv_2(B)$, then there exists a unital *-homomorphism $\varphi : A \to B$ such that $(\eta_r, \eta_c, \eta_h) = Inv_2(\varphi)$. Then for any natural number n, $M_n(A)$ has the same property.*

Proof. This is essentially the same as the proof for Inv_1. One just has to notice that pBp is also a real interval algebra, and that the φ constructed in that lemma also works for Inv_2. □

Lemma 3.16. [Existence for Direct Sums II] *Let A_1 and A_2 be two unital real C^* algebras, and suppose they have the property that if B is any unital real interval algebra and η is a morphism of invariants from $Inv_1(A_i)$ to $Inv_1(B)$, then there exists a unital *-homomorphism $\varphi : A_i \to B$ such that $\eta = Inv_1(\varphi)$. Then $A_1 \oplus A_2$ has the same property.*

Proof. The proof is essentially the same as for Inv_1. One has only to notice that pBp and $(1-p)B(1-p)$ are real interval algebras, and the inclusion of pBp into B induces injective morphisms on all three Cuntz semi-groups. □

Theorem 3.17. [Existence for Interval Algebras] *Let A be a real interval algebra and let B be a unital real AI algebra. Then if η is a morphism of invariants from $Inv_2(A)$ to $Inv_2(B)$, there exists a unital *-homomorphism $\varphi : A \to B$ such that $\eta = Inv_1(\varphi)$.*

Proof. This follows as every real interval algebra is a finite direct sum of amplifications of the basic ones covered in the lemmas above. □

4. Uniqueness lemmas

In this section, we introduce our uniqueness lemmas, again beginning with the finite dimensional cases.

4.1. Finite imensional algebras.

Lemma 4.1. [Uniqueness for $\mathbb{H}$] *Let A be a real C^*-algebra in the class C and suppose φ and ψ are two unital *-homomorphisms from $\mathbb{H}$ to A with $Inv_1(\varphi) = Inv_1(\psi)$. Then there exists a unitary $u \in A$ such that $\psi = Adu \circ \varphi$.*

Proof. Let φ and ψ be two unital *-homomorphisms from $\mathbb{H}$ to A with $Inv_1(\varphi) = Inv_1(\psi)$. Then $\varphi \otimes_{\mathbb{R}} id, \psi \otimes_{\mathbb{R}} id : \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \to A \otimes_{\mathbb{R}} \mathbb{H}$ are the same on D. Since $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong M_4(\mathbb{R})$, there exists a unitary $w \in A \otimes_{\mathbb{R}} \mathbb{H}$ such that $Adw \circ (\varphi \otimes_{\mathbb{R}} id) = \psi \otimes_{\mathbb{R}} id$. We have that for any $h \in \mathbb{H}$, $Adw(1 \otimes_{\mathbb{R}} h) = Adw(\varphi(1) \otimes_{\mathbb{R}} h) = Adw \circ \varphi \otimes_{\mathbb{R}} id(1 \otimes_{\mathbb{R}} h) = \psi \otimes_{\mathbb{R}} id(1 \otimes_{\mathbb{R}} h) = (1 \otimes_{\mathbb{R}} h)$. Thus w is in the commutant of $1 \otimes_{\mathbb{R}} \mathbb{H} \subseteq A \otimes_{\mathbb{R}} \mathbb{H}$, which is $A \otimes_{\mathbb{R}} 1$. We can write $w = u \otimes_{\mathbb{R}} 1$ for some unitary $u \in A$, and we have $Adu \circ \varphi = \psi$. □

Lemma 4.2. *Suppose A is a real C^*-algebra and u and v are two anti-self adjoint unitaries in A that are unitarily conjugate in $A \otimes_{\mathbb{R}} \mathbb{C}$. Then u and v are unitarily conjugate in A.*

Proof. Let A, u and v be as above. Let w be a unitary in $A \otimes_{\mathbb{R}} \mathbb{C}$ such that $u = wvw^*$. Write $w = w_r + w_c\alpha$ with $w_r, w_c \in A$. We deduce relations involving w_r and w_c. From $uw = wv$ we get: $uw_r = w_rv$, $uw_c = w_cv$, $w_c^*u = vw_c^*$, and $w_r^* = vw_r^*$. From w being unitary we get: $w_cw_r^* - w_rw_c^* = 0$, $w_r^*w_c - w_c^*w_r = 0$, $w_rw_r^* + w_cw_c^* = 1$, and $w_r^*w_r + w_c^*w_c = 1$. From $wvw^* = u$ we get $w_cvw_r^* - w_rvw_c^* = 0$ and $w_cvw_c^* + w_rvw_r^* = u$. On easily checks that u commutes with $(w_cw_c^*)$ and $(w_rw_r^*)$, and that v commutes with $(w_r^*w_r)$ and $(w_c^*w_c)$. Now consider $z = (w_r + w_cv)$. This is an element of A. We have $uz = u(w_r + w_cv) = uw_r + uw_cv = uw_r + w_cv^2 = uw_r - w_c$, and $zv = w_rv + w_cv^2 = uw_r - w_c = uz$. We also have $(w_r + w_cv)^*(w_r + w_cv) = (w_r^*w_r - v(w_c^*w_c)v) + (w_r^*w_cv - vw_c^*w_r) = (w_r^*w_r + w_c^*w_c) + v(w_r^*w_c - w_c^*w_r) = 1$ and $(w_r + w_cv)(w_r + w_cv)^* = (w_rw_r^* - w_cv^2w_c^*) + (w_cvw_r^* - w_rvw_c^*) = (w_rw_r^* + w_cw_c^*) + u(w_cw_r^* - w_rw_c^*) = 1$, so z is a unitary meeting our requirements. □

Lemma 4.3. [Uniqueness for $\mathbb{C}$] *Let A be a real C^*-algebra in the class C, and suppose φ and ψ are two unital *-homomorphisms from $\mathbb{C}$ to A with $Inv_1(\varphi) = Inv_1(\psi)$. Then there exists a unitary $u \in A$ such that $\psi = Adu \circ \varphi$.*

Proof. Let $r_1 = (1/2)(1 + \varphi(i)\alpha)$ and $r_2 = (1/2)(1 + \psi(i)\alpha)$. Then, since φ and ψ have the same invariant, r_1 and r_2 are uinitarily conjugate in $A \otimes_{\mathbb{R}} \mathbb{C}$. It follows that $\varphi(i)$ and $\psi(i)$ are unitarily conjugate in $A \otimes_{\mathbb{R}} \mathbb{C}$. From the lemma above, they are conjugate in A. Thus φ and ψ are unitarily conjugate in A. □

Lemma 4.4. [Uniqueness for Amplifications I] *Let A be a real C^*-algebra, and suppose that A has the property that if B is any unital real C^*-algebra in the class C and ψ and φ are two unital *-homomorphisms from A to B with $Inv_1(\varphi) = Inv_1(\psi)$, then there exists a unitary u in B such that $\varphi = Ad\, u \circ \psi$. Then for any natural number n, $M_n(A)$ has the same property.*

Proof. Suppose that A has the above property, that B is in the class C, and that $\psi, \varphi : M_n(A) \to B$ are two unital *-homomorphisms with the same invariants. Consider $\varphi(e_{11})$ and $\psi(e_{11})$. These two projections have the same class in $K_0(B)$, so there exists a unitary in $w \in B$ such that $Ad\, w(\varphi(e_{11})) = \psi(e_{11})$. The unital *-homomorphisms $Ad\, w \circ \varphi|_A$ and $\psi|_A$ from $A \cong e_{11}Ae_{11}$ to $\psi(e_{11})B\psi(e_{11})$ now have the same invariants, and $\psi(e_{11})B\psi(e_{11})$ is in the class C, so there exists a unitary $v \in \psi(e_{11})B\psi(e_{11})$ such that $Ad\, v \circ (Ad\, w \circ \varphi|_A) = \psi|_A$. Now consider $u = \sum_{i=1}^n \psi(e_{i1})vw\varphi(e_{1i}) \in B$. Straightforward calculations show that $uu^* = u^*u = 1$ and that $u\varphi(x)u^* = \psi(x)$ for all $x \in M_n(A)$, as required. □

Lemma 4.5. [Uniqueness for Direct Sums I] *Let A_1 and A_2 be two unital real C^*-algebras and suppose that they both have the property that if B is a real C^*-algebra in the class C and $\psi, \varphi : A_i \to B$ are unital $*$-homomorphisms with $Inv_1(\psi) = Inv_1(\varphi)$, then there exists a unitary $u \in B$ with $Ad\, u \circ \varphi = \psi$. Then $A_1 \oplus A_2$ has the same property.*

Proof. Let A_1, A_2, and B be as in the statement of the lemma, and let $\varphi, \psi : A_1 \oplus A_2 \to B$ be two unital $*$-homomorphisms with $Inv_1(\varphi) = Inv_1(\psi)$. Let $p = \varphi(1_{A_1})$ and $q = \psi(1_{A_1})$. Then $[p] = [q]$ in $D(B)$, so there exists a unitary

$v \in B$ such that $q = vpv^*$. The two maps $Ad\,v \circ (\varphi|_{A_1}) : A_1 \to qBq$ and $\psi|_{A_1} : A_1 \to qBq$ are the same on Inv_1, and qBq is in C, so there exists a unitary $w_1 \in qBq$ such that $Ad\,w_1 \circ Ad\,v \circ (\varphi|_{A_1}) = \psi|_{A_1}$. Similarly, there exists a unitary $w_2 \in (1-q)B(1-q)$ such that $Ad\,w_2 \circ Ad\,v \circ (\varphi|_{A_2}) = \psi|_{A_2}$. Setting $u = (w_1 + w_2)v \in B$ gives a unitary in B such that $Ad\,u \circ \varphi = \psi$ as required. □

Theorem 4.6. [Uniqueness for Finite Dimensional Algebras] *Let A be a finite dimensional real C^*-algebra and let B be a real C^*-algebra in the class C. Then if $\varphi, \psi : A \to B$ are two unital $*$-homomorphisms with $Inv_1(\varphi) = Inv_1(\psi)$, there exists a unitary $u \in B$ with $\psi = Ad\,u \circ \varphi$.*

Proof. The case in which $A \cong \mathbb{R}$ is trivial, as both maps just send the unit to the unit. The case of $A \cong M_n(\mathbb{R})$ then follows from the lemma for amplifications. The cases of $A \cong \mathbb{C}$ or $A \cong \mathbb{H}$ were covered in the lemmas above, and the case for matrix algebras over these follows from the lemma for amplifications. Finally, since any finite dimensional real C^* algebra is a finite direct sum of these cases, the theorem follows from the lemma for direct sums. □

4.2. Interval algebras.

Lemma 4.7. [Uniqueness for $C_0((0,1],\mathbb{R})$] *Let A be a unital real AI algebra, and let $\varphi, \psi : C_0((0,1],\mathbb{R}) \to A$ be two $*$-homomorphisms such that $Cu(\varphi) = Cu(\psi)$, $Cu(\varphi \otimes_{\mathbb{R}} id_{\mathbb{C}}) = Cu(\psi \otimes_{\mathbb{R}} id_{\mathbb{C}})$, and $Cu(\varphi \otimes_{\mathbb{R}} id_{\mathbb{H}}) = Cu(\psi \otimes_{\mathbb{R}} id_{\mathbb{H}})$. Then φ and ψ are approximately unitarily equivalent (via unitaries in A).*

Proof. Let A, φ, and ψ be as in the statement of the lemma. Let $f(t) = t$ be the canonical positive self-adjoint generator of $C_0((0,1],\mathbb{R})$, and let $a = \varphi(f)$ and $b = \psi(f)$. By [4] we may choose a unitary $u \in A \otimes_{\mathbb{R}} \mathbb{C}$ so that $\| b - uau^* \|$ is small. Write $A = \lim\{A_n, \zeta_{nm}\}$ where A_n are finite direct sums of real interval algebras and ζ_{nm} are unital real *-homomorphisms. We may choose n and $a_n, b_n \in A_n$ and $u_n \in A_n \otimes_{\mathbb{R}} \mathbb{C}$ such that a_n and b_n are positive, u_n is unitary, and $\| b_n - u_n a_n u_n^* \|$, $\| \zeta_{n\infty}(a_n) - a \|$, $\| \zeta_{n\infty}(b_n) - b \|$, and $\| \eta_{n\infty}(u_n) - u \|$ are small. The problem reduces to showing that we can replace u_n with a unitary in A_n. Since A_n is a finite direct sum real interval algebras, we may consider each type of basic building block separately.

Case 1: ($A_n \cong M_m(C_{\mathbb{C}}[0,1])$) In this case, we have $A_n \otimes_{\mathbb{R}} \mathbb{C} \cong M_m(C_{\mathbb{C}}[0,1])^2$ with real structure $(x,y)^\tau = (y^{tr}, x^{tr})$. Let p denote the central projection that is a unit for the first summand. Then $a_n = pa_n + (pa_n)^\tau$, and $b_n = pb_n + (pb_n)^\tau$. Replacing $u_n = pu_n + (1-p)u_n$ with $v_n = pu_n + (pu_n)^\tau$, we have $\| v_n a_n v_n^* - b_n \| \leq \| u_n a_n u_n^* - b_n \|$ and $v_n \in A_n$.

Case 2: ($A_n \cong A(m,\mathbb{R})$) In this case, we have $A_n \otimes_{\mathbb{R}} \mathbb{C} \cong M_m(C_{\mathbb{C}}[-1,1])$ with real structure $f^\tau(t) = f(-t)^{tr}$ and $A_n = \{f \in A_n \otimes_{\mathbb{R}} \mathbb{C} \mid f(-t) = \overline{f(t)}\}$. If u_n satisfies $u_n(0) \in M_n(\mathbb{R})$, then we may replace u_n by v_n defined by $v_n(t) = u_n(t)$ for $t \leq 0$, and $v_n(t) = \overline{u_n(-t)}$ for $t \geq 0$. We would then have $\| v_n a_n v_n^* - b_n \| \leq \| u_n a_n u_n^* - b_n \|$ and $v_n \in A_n$.

We may assume, by making an arbitrarily small adjustment if necessary, that a_n and b_n are in standard form, i.e. they are unitary conjugates of diagonal elements. Choose a system of matrix units with respect to which b_n is diagonal and $b_n(0) \in M_n(\mathbb{R})$. Then $u_n a_n u_n^*$ is approximately diagonal with respect to these matrix units, so multiplying u_n by a unitary close to 1 we may assume that $u_n a_n u_n^*$ is exactly diagonal at 0. It follows that the columns of $u_n(0)$ are eigenvectors for the self-adjoint matrix $a_n(0)$. Since $a_n(0)$ is self-adjoint, the real and imaginary parts of these column vectors are also eigenvectors for $a_n(0)$, and for each column, one or the other is non-zero. Thus there exists a unitary matrix $w(0)$ such that $w(0)$ commutes with $a_n(0)$ and $u_n(0)w(0) \in M_n(\mathbb{R})$. Since a_n is in standard form, we may connect $w(0)$ to 1 with a path $w(|t|)$ such that $w(t)$ commutes with $a_n(t)$, and thus adjust $u_n(t)$ all along its length to meet our requirements.

Case 3: ($A_n \cong A(m, \mathbb{H})$) In this case, we have $A_n \otimes_{\mathbb{R}} \mathbb{C} \cong M_{2m}(C_{\mathbb{C}}[-1, 1])$ with real structure $f^\tau(t) = f(-t)^\#$. By [19] lemma 2.4 (b), we may find approximants to a_n and b_n that have distinct eigenvalues in every fibre except the endpoint, where they have multiplicity two, and the eigenprojections are continuous. Lemma 2.5 (a) in [19] now provides unitaries in $A_n \cong A(m, \mathbb{H})$ that conjugate these approximants to real diagonal elements. We may assume the eigenvalues are arranged in increasing order. Now a_n being approximately unitarily close to b_n implies that their spectra are approximately contained in each other in each fibre, so the above diagonal elements are norm close to each other.

Case 4: ($A_n \cong M_m(C_{\mathbb{R}}[0, 1])$) Using the lemma on distinct eigenvalues, we can approximate a_n and b_n by elements $a_n' = \lambda_1(t)P_1(t) + \cdots + \lambda_m(t)P_m(t)$ and $b_n' = \gamma_1(t)Q_1(t) + \cdots + \gamma_m(t)Q_m(t)$ where λ_i and γ_i are continuous real functions and $\{P_i\}$ and $\{Q_i\}$ are families of pairwise orthogonal minimal projections in $M_m(C_{\mathbb{R}}[0, 1])$ and the λ_i and γ_i are distinct and arranged in increasing order in each fibre. We may choose partial isometries $w_i \in M_m(C_{\mathbb{R}}[0, 1])$ such that $w_i^* w_i = P_i$ and $w_i w_i^* = Q_i$ for each i. Then $w = w_1 + \ldots w_n$ is a unitary in $M_m(C_{\mathbb{R}}[0, 1])$. By construction, we have that a_n' is approximately equal to a unitary conjugate of b_n' in $M_m(C_{\mathbb{C}}[0, 1])$. It follows that their corresponding eigenvalues, when listed in increasing order are close to each other. Thus a_n' is approximately equal to $wb_n'w^*$, and wb_nw^* approximates a_n.

Case 5: ($A_n \cong M_m(C_{\mathbb{H}}[0, 1])$) The proof in this case is the same as for $A_n \cong M_m(C_{\mathbb{R}}[0, 1])$ with the lemma on density of elements with distinct eigenvalues for $\mathbb{H}$ in place of that for $\mathbb{R}$. □

Lemma 4.8. **[Uniqueness for** $C_0((0, 1], \mathbb{C})$**]** *Let A be a unital real C^*-algebra in the class C and suppose $\varphi, \psi : C_0((0, 1], \mathbb{C}) \to A$ are two *-homomorphisms such that $Cu(\varphi) = Cu(\psi)$, $Cu(\varphi \otimes id_{\mathbb{C}}) = Cu(\psi \otimes_{\mathbb{R}} id_{\mathbb{C}})$, and $Cu(\varphi \otimes_{\mathbb{R}} id_{\mathbb{H}}) = Cu(\psi \otimes_{\mathbb{R}} id_{\mathbb{H}})$. Then φ and ψ are approximately unitarily equivalent (via unitaries in A).*

Proof. As in the case of $C_0((0,1],\mathbb{R})$ above, we reduce to the cases of the basic building blocks. Let A, φ, and ψ be as in the statement of the lemma. Let $g(t) = it$ be the canonical anti-self-adjoint generator of $C_0((0,1],\mathbb{C})$, and let $a = \varphi(g)$ and $b = \psi(g)$. By [4] there exists a unitary $u \in A_\tau \otimes \mathbb{C}$ such that $\| b - uau^* \|$ is small. Using the same notation as in the lemma above, we may choose n and $a_n, b_n \in A_n$ and $u_n \in A_n \otimes_{\mathbb{R}} \mathbb{C}$ such that a_n and b_n are anti-self-adjoint, u_n is unitary, and $\| b_n - u_n a_n u_n^* \|$, $\| \eta_{n\infty}(a_n) - a \|$, $\| \eta_{n\infty}(b_n) - b \|$, and $\| \eta_{n\infty}(u_n) - u \|$ are small. As before, the problem reduces to showing that we can replace u_n with a unitary in A_n, and since A_n is a finite direct sum real interval algebras, we may consider each type of basic building block separately.

Case 1: ($A_n \cong M_m(C_{\mathbb{C}}[0,1])$) This case is handled exactly as in the lemma for $C_0((0,1],\mathbb{R})$ above.

Case 2: ($A_n \cong M_m(C_{\mathbb{R}}[0,1])$) From lemma 2.4(a) in [19], we may approximate a_n and b_n with elements having distinct complex eigenvalues in every fibre. From lemma 2.5 (d) in [19], it follows that these approximations are conjugate, via unitaries in $A_n \cong M_m(C_{\mathbb{R}}[0,1])$, to elements of the form

$$d = diag(\begin{pmatrix} 0 & c_1 \\ -c_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & c_l \\ -c_l & 0 \end{pmatrix}, 0 \dots 0)$$

with $c_1 > c_2 \cdots > c_l > 0$. Now a_n being approximately unitarily close to b_n implies that their spectra are approximately contained in each other in each fibre, so the above diagonal forms are norm close to each other.

Case 3: ($A_n \cong A(m,\mathbb{R})$) This is similar to the case above, only now the unitaries provided by lemma 2.5 (d) in [19] lie in $A_n \cong A(m,\mathbb{R})$.

Case 4: ($A_n \cong M_m(C_{\mathbb{H}}[0,1])$) This is similar to the case above. From lemma 2.4(a) in [19], we may approximate a_n and b_n with elements in $A_n \cong M_m(C_{\mathbb{H}}[0,1])$ having distinct complex eigenvalues in every fibre. From lemma 2.5 (c), it follows that these approximations are conjugate, via unitaries in $A_n \cong M_m(C_{\mathbb{H}}[0,1])$, to pure imaginary diagonal elements, which we may assume are arranged in complex conjugate pairs, ordered by the size of the element in the upper half plane. Now a_n being approximately unitarily close to b_n implies that their spectra are approximately contained in each other in each fibre, so the above diagonal forms are norm close to each other.

Case 5: ($A_n \cong A(m,\mathbb{H})$) This is similar to case 4 above. By lemma 2.4(a) in [19], we may approximate a_n and b_n with elements in $A_n \cong M_m(C_{\mathbb{H}}[0,1])$ having distinct complex eigenvalues in every fibre. From lemma 2.5 (c), it follows that these approximations are conjugate, via unitaries in $A_n \cong A(m,\mathbb{H})$ to pure imaginary diagonal elements. In the fibre at 1 we may assume that these are arranged in complex conjugate pairs, ordered by the size of the element in the upper half plane. As the eigenvalues are distinct in every fibre, this determines their order completely. Again, a_n being approximately unitarily close to b_n implies that their spectra are approximately contained in each other in each fibre, so these diagonal forms are norm close to each other. □

Lemma 4.9. [Uniqueness for Unitisations] *Let A be a non-unital real C^*-algebra, and suppose A has the property that if B is a real AI algebra and $\varphi, \psi : A \to B$ are two real $*$-homomorphisms such that $Cu(\varphi) = Cu(\psi)$, $Cu(\varphi \otimes_{\mathbb{R}} id_{\mathbb{C}}) = Cu(\psi \otimes_{\mathbb{R}} id_{\mathbb{C}})$, and $Cu(\varphi \otimes_{\mathbb{R}} id_{\mathbb{H}}) = Cu(\psi \otimes_{\mathbb{R}} id_{\mathbb{H}})$, then φ is approximately unitarily equivalent to ψ. Then the unitisation $\tilde{A}$ has the property that if B is a real AI algebra and $\tilde{\varphi}, \tilde{\psi} : A \to B$ are two unital real $*$-homomorphisms such that $Inv_2(\tilde{\varphi}) = Inv_2(\tilde{\psi})$, then $\tilde{\varphi}$ is approximately unitarily equivalent to $\tilde{\psi}$.*

Proof. The unitaries that work for the restrictions of $\tilde{\varphi}$ and $\tilde{\psi}$ to A work for the unitisations as well. □

Lemma 4.10. [Uniqueness for $A(1, \mathbb{R})$] *Let A be a unital real AI algebra, and let $\varphi, \psi : A(1, \mathbb{R}) \to A$ be two unital $*$-homomorphisms such that $Inv_2(\varphi) = Inv_2(\psi)$. Then φ and ψ are approximately unitarily equivalent (via unitaries in A).*

Proof. This follows from the lemmas above, since $A(1, \mathbb{R})$ is the unitisation of $C_0((0, 1], \mathbb{C})$. □

Lemma 4.11. [Uniqueness for $C_{\mathbb{C}}[0, 1]$] *Let A be a unital real AI algebra, and let $\varphi, \psi : C_{\mathbb{C}}[0, 1] \to A$ be two unital $*$-homomorphisms such that $Inv_2(\varphi) = Inv_2(\psi)$. Then φ and ψ are approximately unitarily equivalent (via unitaries in A).*

Proof. Let A, φ and ψ be as above. Since a morphism of Inv_2 restricts to a morphism of Inv_1, if we let $\iota : \mathbb{C} \to C_{\mathbb{C}}[0, 1]$ be the inclusion of the constant functions, we have $Inv_1(\varphi \circ \iota) = Inv_1(\psi \circ \iota)$. Since real AI algebras are in the class C, the uniqueness for $\mathbb{C}$ lemma applies, and there exists a unitary $u \in A$ such that $\psi \circ \iota = Ad\, u \circ \varphi \circ \iota$. Thus we may assume that ψ and φ agree on the constant functions.

The algebra $C_{\mathbb{C}}[0, 1]$ is generated as a real algebra by the constant functions and the function $f(t) = t$. Let $a = \varphi(f)$ and $b = \psi(f)$. Then a and b are positive elements of A that commute with $u = \varphi(i) = \psi(i)$. Since $a, b \in A$, they commute with $\alpha, \beta \in A \otimes_{\mathbb{R}} \mathbb{H}$. Since they commute with u and α, they commute with the projections $p = (1/2)(1 - u\alpha)$ and $\bar{p} = (1/2)(1 + u\alpha)$ in $A \otimes_{\mathbb{R}} \mathbb{C}$.

As in the proof of the existence lemma for $\mathbb{C}$, we get a 2×2 system of matrix units in $A \otimes_{\mathbb{R}} \mathbb{H}$ with $p = e_{11}$, $\bar{p} = e_{22}$, and $\beta = e_{12} - e_{21}$ giving an embedding of $A \otimes_{\mathbb{R}} \mathbb{H}$ into $M_2(p(A \otimes_{\mathbb{R}} \mathbb{C})p)$ for which α has the form $\begin{pmatrix} \alpha_{11} & 0 \\ 0 & -\alpha_{11} \end{pmatrix}$ for a central anti-self adjoint unitary $\alpha_{11} \in p(A \otimes_{\mathbb{R}} \mathbb{C})p$. Since they commute with p, the elements a and b are diagonal in this picture, and since they commute with β they have the form $a = diag(a_{11}, a_{11}), b = diag(b_{11}, b_{11})$. in other words, $a = a_{11} + \overline{a_{11}}$ for an $a_{11} \in p(A \otimes_{\mathbb{R}} \mathbb{C})p$, and similarly for b. Since the Cuntz semi-groups for AI algebras are torsion free, it follows from theorem 1.1 in [4], that a_{11} is approximately unitarily equivalent to b_{11} in $p(A \otimes_{\mathbb{R}} \mathbb{C})p$. Given a unitary $v_{11} \in p(A \otimes_{\mathbb{R}} \mathbb{C})p$ such that $Ad\, v_{11}(a_{11})$ approximates b_{11}, we can let

$v = v_{11} + \overline{v_{11}}$ to get a unitary in $A \otimes_{\mathbb{R}} \mathbb{C}$ such that $Ad\, v(a)$ approximates b. Since v commutes with β, $v \in A$. Since v commutes with p and $\bar{p}$, v commutes with u. Thus $Ad\, v \circ \varphi$ approximates ψ. □

Lemma 4.12. [Uniqueness for $C_{\mathbb{H}}[0,1]$] *Let A be a unital real AI algebra, and let $\varphi, \psi : C_{\mathbb{H}}[0,1] \to A$ be two unital $*$-homomorphisms such that $Inv_2(\varphi) = Inv_2(\psi)$. Then φ and ψ are approximately unitarily equivalent (via unitaries in A).*

Proof. Let A, φ and ψ be as above. As in the case for $\mathbb{C}$, using the uniqueness lemma for $\mathbb{H}$, we may assume that φ and ψ agree on the constant functions. As with $\mathbb{C}$ above, the algebra $C_{\mathbb{H}}[0,1]$ is generated as a real algebra by the constant functions and the function $f(t) = t$. Let $a = \varphi(f)$ and $b = \psi(f)$. We have embeddings of $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong M_4(\mathbb{R})$ into $A \otimes_{\mathbb{R}} \mathbb{H}$ given by $\varphi \otimes_{\mathbb{R}} id_{\mathbb{H}}$ and $\psi \otimes_{\mathbb{R}} id_{\mathbb{H}}$. Since a and b commute with both copies of $\mathbb{H}$, if we use this embedded copy of $M_4(\mathbb{R})$ for a system of matrix units in $A \otimes_{\mathbb{R}} \mathbb{H}$, we have that $a = diag(a_{11}, a_{11}, a_{11}, a_{11})$ for some element $a_{11} \in e_{11}(A \otimes_{\mathbb{R}} \mathbb{H})e_{11}$, and similarly for b. Now we use the uniqueness for $C_{\mathbb{R}}((0,1])$ above to get a unitary $w_{11} \in e_{11}(A \otimes_{\mathbb{R}} \mathbb{H})e_{11}$ such that $Ad\, w_{11}(a)$ approximates b_{11}. Then we can use $w = diag(w_{11}, w_{11}, w_{11}, w_{11})$. This commutes with all of $M_4(\mathbb{R})$, so it commutes with the second copy of $\mathbb{H}$, and is therefore in A, and it commutes with the first copy of $\mathbb{H}$, so conjugating by it does not alter φ on the constant functions, so $Ad\, w \circ \varphi$ approximates ψ. □

Lemma 4.13. [Uniqueness for $A(1, \mathbb{H})$] *Let A be a unital real AI algebra, and let $\varphi, \psi : A(1, \mathbb{H}) \to A$ be two unital $*$-homomorphisms such that $Inv_2(\varphi) = Inv_2(\psi)$. Then φ and ψ are approximately unitarily equivalent (via unitaries in A).*

Proof. The proof in this case is very similar to that for $C_{\mathbb{H}}[0,1]$ above. Let A, φ and ψ be as in the statement of the lemma. As in the case above, using the uniqueness lemma for $\mathbb{H}$, we may assume that φ and ψ agree on the constant functions. The algebra $A(1, \mathbb{H})$ is generated as a real algebra by the constant functions and the function $g(t) = it$. Let $z = \varphi(g)$ and $w = \psi(g)$. Now we proceed exactly as in the case of $C_{\mathbb{H}}[0,1]$, except that we use the uniqueness lemma for $C_{\mathbb{C}}((0,1])$ in place of the one for $C_{\mathbb{R}}((0,1])$. □

Lemma 4.14. [Uniqueness for Amplifications II] *Suppose A is a unital real C^*-algebra and that A has the property that for any unital real AI algebra B, if $\varphi, \psi : A \to B$ are two unital *-homomorphisms with $Inv_2(\varphi) = Inv_2(\psi)$, then φ is approximately unitarily equivalent to ψ in B. Then $M_n(A)$ has the same property.*

Proof. The proof is essentially the same as for Inv_1. One has to notice that the cut-downs are real AI algebras, and instead of one unitary v, we have a sequence v_n that approximately conjugates one to the other. The sequence of unitaries $u_n = \sum_{i=1}^{n} \psi(e_{i1}) v w \varphi(e_{1i}) \in B$ meets our requirements. □

Lemma 4.15. [Uniqueness for Direct Sums II] *Suppose A_1 and A_2 are unital real C^*-algebras such that A_1 and A_2 have the property that, for any unital real*

*AI algebra B, if $\varphi, \psi : A_i \to B$ are two unital *-homomorphisms with $Inv_2(\varphi) = Inv_2(\psi)$, then φ is approximately unitarily equivalent to ψ via unitaries in B. Then $A_1 \oplus A_2$ has the same property.*

Proof. This is similar to the proof for the finite dimensional case. Let A_1, A_2, and B be as in the statement of the lemma, and let $\varphi, \psi : A_1 \oplus A_2 \to B$ be two unital $*$-homomorphisms with $Inv_2(\varphi) = Inv_2(\psi)$. Let $p = \varphi(1_{A_1})$ and $q = \psi(1_{A_1})$. Then, since a morphism of Inv_2 restricts to one of Inv_1, $[p] = [q]$ in $D(B)$, so there exists a unitary $v \in B$ such that $q = vpv^*$. The two maps $Ad\, v \circ (\varphi|_{A_1}) : A_1 \to qBq$ and $\psi|_{A_1} : A_1 \to qBq$ are the same on Inv_2, and qBq is an AI algebra, so there exists a sequence of unitary $w_n^1 \in qBq$ such that $Ad\, w_n^1 \circ Ad\, v \circ (\varphi|_{A_1}) \to \psi|_{A_1}$ as $n \to \infty$. Similarly, there exists a sequence of unitaries $w_n^2 \in (1-q)B(1-q)$ such that $Ad\, w_n^2 \circ Ad\, v \circ (\varphi|_{A_2}) \to \psi|_{A_2}$ as $n \to \infty$. Setting $u_n = (w_n^1 + w_n^2)v \in B$ gives a sequence of unitaries in B such that $Ad\, u_n \circ \varphi \to \psi$ as required. □

Theorem 4.16. [Uniqueness for Real Interval Algebras] *Let A be a real interval algebra and let B be a real AI algebra. If $\varphi, \psi : A \to B$ are two unital $*$-homomorphisms with $Inv_2(\varphi) = Inv_2(\psi)$, Then φ and ψ are approximately unitarily equivalent (via unitaries in the real C^*-algebra B).*

Proof. This follows as every real interval algebra is a finite direct sum of amplifications of the basic ones covered in the lemmas above. □

5. Classification theorems

In this final section, we prove our main theorems, and include a new proof of the already known classification of real AF algebras from [6] and [18].

Theorem 5.1. [AF Algebras] *Let A be a real AF algebra, and let B be a real C^*-algebra in the class C. Then if $(\eta_r, \eta_c, \eta_h) : Inv_1(A) \to Inv_1(B)$ is a morphism of invariants, there exists a unital $*$-homomorphism $\varphi : A \to B$ such that $K_0(\varphi) = \eta_r$, $K_0(\varphi \otimes_{\mathbb{R}} id_{\mathbb{C}}) = \eta_c$, and $K_0(\varphi \otimes_{\mathbb{R}} id_{\mathbb{H}}) = \eta_h$. Moreover, if $\varphi, \psi : A \to B$ are two unital $*$-homomorphisms with $Inv_1(\varphi) = Inv_1(\psi)$, then φ and ψ are approximately unitarily equivalent.*

Proof. Let A, B and $\eta = (\eta_r, \eta_c, \eta_h)$ be as in the statement of the theorem. Write $A = \overline{\cup_{n=1}^{\infty} A_n}$ where each A_n is a finite dimensional real C^*-algebra and $1 \in A_n \subseteq A_{n+1}$ for each n. We get a commuting diagram of invariants:

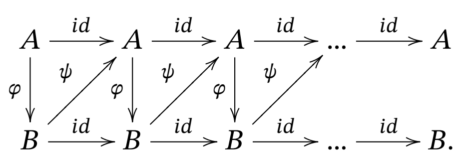
$$\begin{array}{ccccccccc}
Inv_1(A_1) & \xrightarrow{i_*} & Inv_1(A_2) & \xrightarrow{i_*} & Inv_1(A_3) & \xrightarrow{i_*} & \dots & \xrightarrow{i_*} & Inv_1(A) \\
\downarrow{\scriptstyle \eta_1} & & \downarrow{\scriptstyle \eta_2} & & \downarrow{\scriptstyle \eta_3} & & & & \downarrow{\scriptstyle \eta} \\
Inv_1(B) & \xrightarrow{id} & Inv_1(B) & \xrightarrow{id} & Inv_1(B) & \xrightarrow{id} & \dots & \longrightarrow & Inv_1(B).
\end{array}$$

Now using the existence theorem for finite dimensional algebras, we may lift each of the maps η_n to a unital $*$-homomorphism φ_n from A_n to B. Moving from left to right through the diagram, we may use the uniqueness theorem for finite dimensional algebras to adjust each φ_n by an inner automorphism of B to achieve a commuting diagram that still induces the same diagram on the invariants. The resulting limit homomorphism from A to B. meets the requirements of the existence part of the theorem.

Now suppose A and B are as in the statement of the theorem and $\varphi, \psi : A \to B$ are two unital $*$-homomorphisms with $Inv_1(\varphi) = Inv_1(\psi)$. Again, write $A = \overline{\cup_{n=1}^{\infty} A_n}$ where each A_n is a finite dimensional real C^*-algebra and $1 \in A_n \subseteq A_{n+1}$ for each n. Using the uniqueness theorem for finite dimensional algebras, for each n, we get a unitary $u_n \in B$ such that $\psi|_{A_n} = Ad\, u \circ (\varphi|_{A_n})$. We then have $\psi(x) = \lim (Ad\, u_n \circ \varphi)(x)$ for every $x \in A$, so the two $*$-homomorphisms are approximately unitarily equivalent. □

Corollary 5.2. [AF Algebras] *If A and B are two unital real AF algebras and $(\eta_r, \eta_c, \eta_h) : Inv_1(A) \to Inv_1(B)$ is an isomorphism of invariants, there exists a $*$-isomorphism $\varphi : A \to B$ such that $Inv_1(\varphi) = (\eta_r, \eta_c, \eta_h)$.*

Proof. Let A, B, and (η_r, η_c, η_h) be as above. From the AF homomorphism theorem above, there exist unital *-homomorphisms $\varphi : A \to B$ and $\psi : B \to A$ such that $Inv_1(\varphi) = (\eta_r, \eta_c, \eta_h)$ and $Inv_1(\psi) = (\eta_r, \eta_c, \eta_h)^{-1}$. From the homomorphism theorem above, we have that $\varphi \circ \psi$ and $\psi \circ \varphi$ are approximately unitarily equivalent to the identity maps on B and A respectively. We may form a diagram:



Moving left to right through the diagram, we may adjust each vertical map by an inner automorphism to achieve an approximately commuting diagram that induces the same diagram on the invariants. The resulting limit maps $\varphi : A \to B$ and $\psi : B \to A$ are then a pair of inverse isomorphisms that meet our requirements. □

Theorem 5.3. [AI Algebras] *Let A and B be unital real AI algebras. Then if $(\eta_r, \eta_c, \eta_h) : Inv_2(A) \to Inv_2(B)$ is a morphism of invariants, there exists a unital $*$-homomorphism $\varphi : A \to B$ such that $Cu(\varphi) = \eta_r$, $Cu(\varphi \otimes_{\mathbb{R}} id_{\mathbb{C}}) = \eta_c$, and $Cu(\varphi \otimes_{\mathbb{R}} id_{\mathbb{H}}) = \eta_h$. Moreover, if $\varphi, \psi : A \to B$ are two unital $*$-homomorphisms with $Inv_2(\varphi) = Inv_2(\psi)$, then φ and ψ are approximately unitarily equivalent.*

Proof. This is proved in the same fashion as the AF case above, but with the existence and uniqueness theorems for interval algebras in place of those for finite dimensional ones. □

Corollary 5.4. [AI Algebras] *If A and B are two unital real AI algebras and $(\eta_r, \eta_c, \eta_h) : Inv_2(A) \to Inv_2(B)$ is an isomorphism of invariants, there exists a $*$-isomorphism $\varphi : A \to B$ such that $Inv_2(\varphi) = (\eta_r, \eta_c, \eta_h)$.*

Proof. The proof of this is similar to the proof of the analogous corollary for AF algebras given above, but using the AI homomorphism theorem. □

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