Hilbert modules, rigged modules, and stable isomorphism

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Abstract. Rigged modules over an operator algebra are a generalization of Hilbert modules over a $C^*$-algebra. We characterize the rigged modules over an operator algebra $\mathcal{A}$ which are orthogonally complemented in $C_\infty(\mathcal{A})$, the space of infinite columns with entries in $\mathcal{A}$. We show that every such rigged module ‘restricts’ to a bimodule of Morita equivalence between appropriate stably isomorphic operator algebras.

Contents

1. Introduction 29
2. Orthogonally complemented modules and $\sigma\Delta$-rigged modules 32
3. Doubly $\sigma\Delta$-rigged modules 37
4. Morita equivalence of rigged modules 44
References 49

1. Introduction

Let $X$, $Y$ be operator spaces. We call them stably isomorphic if the spatial tensor products $X \otimes \mathcal{K}$, $Y \otimes \mathcal{K}$ are completely isometrically isomorphic, where $\mathcal{K}$ is the algebra of compact operators acting on an infinite dimensional Hilbert space. We also denote by $C_\infty(X)$ the operator space of infinite columns with entries in $X$. In the case where $X$ is a right rigged module over an operator algebra, $\mathcal{A}$, so is $C_\infty(X)$.

The notion of a Hilbert $C^*$-module was introduced and developed in the early 1970s by Paschke and Rieffel, see [14, 18]. A Hilbert module over a $C^*$-algebra $\mathcal{A}$ is a right $\mathcal{A}$-module $Y$ together with a map $(\cdot, \cdot) : Y \times Y \to \mathcal{A}$ which is linear in the second variable, and which also satisfies the following conditions:

1. $(y, y) \geq 0$ for all $y \in Y$,
2. $(y, y) = 0 \Leftrightarrow y = 0$,
3. $(y, za) = (y, z) a$, for all $y, z \in Y$, $a \in \mathcal{A}$,

Received September 13, 2021.
2010 Mathematics Subject Classification. 47L30 (primary), 46L05, 46L08, 47L05, 47L25, 16D90 (secondary).

Keywords and phrases. Operator algebras, $C^*$-algebras, Hilbert modules, TRO, stable isomorphism, Morita equivalence.
(4) \((y,z)^* = \langle z,y \rangle\) for all \(y, z \in Y\),
(5) \(Y\) is complete in the norm \(\|y\| = \|\langle y, y \rangle\|^{1/2}\).

Observe that the space \(I_A(Y)\), which is the closure of the linear span of the set \(\{(y,z) \in A \mid y, z \in Y\}\), is an ideal of \(A\).

Consider the \(C^*\)-algebra \(K_A(Y)\) of the ‘compact’ adjointable operators from \(Y\) to \(Y\). It is known that \(Y\) is a bimodule of Morita equivalence between \(I_A(Y)\) and \(K_A(Y)\). But these \(C^*\)-algebras are not always stably isomorphic.

Let \(Y\) be a right Hilbert \(A\)-module. In case there exists a sequence \((y_k)_{k \in \mathbb{N}} \subseteq Y\) such that

\[ y = \sum_{k=1}^{\infty} y_k \langle y_k, y \rangle, \quad \forall y \in Y\]

where the series converges in the norm of \(Y\), we say that \((y_k)_{k \in \mathbb{N}}\) is a right quasibasis for \(Y\). It follows by the Brown–Kasparov stabilization theorem, see [3, Corollary 8.20], that the spaces \(I_A(Y)\), \(K_A(Y)\), \(Y\) are all stably isomorphic.

Let \(Y\) be a right Hilbert \(A\)-module. We call it countably generated if there exists a sequence \((y_k)_{k \in \mathbb{N}} \subseteq Y\) such that

\[ Y = \overline{\text{span}}(\{y_k a \mid k \in \mathbb{N}, a \in A\}).\]

If \(Y\) has a right quasibasis, then \(Y\) is countably generated and conversely. Every countably generated Hilbert \(A\)-module is isomorphic as a Hilbert \(A\)-module with an orthogonally complemented bimodule of \(C_\infty(A)\).

Blecher in [1] generalized the notion of Hilbert modules to the setting of non-selfadjoint operator algebras. He called these modules rigged modules, see the definition below. Hilbert modules over a \(C^*\)-algebra are rigged modules in terms of this definition. Using the notion of a ternary ring of operators, we introduce a new category of \(A\)-rigged modules, where \(A\) is an operator algebra, the \(\sigma\Delta\)-\(A\)-rigged modules. We prove that an \(A\)-rigged module is a \(\sigma\Delta\)-\(A\)-rigged module if and only if it is isomorphic with an orthogonally complemented module in \(C_\infty(A)\). We also introduce a subcategory of the \(\sigma\Delta\)-\(A\)-rigged modules, the doubly \(\sigma\Delta\)-\(A\)-rigged modules. In the case of \(C^*\)-algebras, these two categories coincide. Every doubly \(\sigma\Delta\)-\(A\)-rigged module implements a stable isomorphism between the corresponding operator algebras. Conversely, if \(A\) and \(B\) are stably isomorphic operator algebras, there exists a doubly \(\sigma\Delta\)-\(A\)-rigged module \(Y\) which is a bimodule of strong Morita equivalence (BMP-Morita equivalence) for \(A\) and \(B\) in the sense of Blecher, Muhly and Paulsen, [4]. Every \(\sigma\Delta\)-\(A\)-rigged module has a ‘restriction’ which is a doubly \(\sigma\Delta\)-\(A\)-rigged module. Thus, every orthogonally complemented rigged module in \(C_\infty(A)\), has a ‘restriction’ making it into a bimodule of BMP-Morita equivalence over some operator algebras \(\mathcal{E}\), \(\mathcal{D}\). Furthermore, \(\mathcal{E}\) and \(\mathcal{D}\) are stably isomorphic.

In Section 4, we will develop a theory of Morita equivalence for rigged modules. If \(A\), \(B\) are operator algebras, \(E\) is a right \(B\)-rigged module, and \(F\) is a right \(A\)-rigged module, we call \(E\) and \(F\) \(\sigma\)-Morita equivalent if there exists a doubly \(\sigma\Delta\)-\(A\)-rigged module \(Y\) such that

(i) \(A \cong Y \otimes^h_B Y\),
\(B \cong Y \otimes_{\mathcal{B}} \tilde{Y},\)

(iii) \(F \cong E \otimes_{\mathcal{B}} Y,\)

where \(\tilde{Y}\) is the counterpart bimodule of \(Y.\) In this case we write \(E \sim_{\mathcal{A}} F.\) We will prove that if \(E \sim_{\mathcal{A}} F,\) then \(E\) and \(F\) are stably isomorphic.

This paper has been written with an emphasis on the theory of non-selfadjoint operator algebras, but the conclusions for \(C^*\)-algebras follow easily.

At this point, we recall some definitions, notation and lemmas which will be useful for what follows.

**Definition 1.1.** [1]

Let \(\mathcal{A}\) be an approximately unital operator algebra, i.e. an operator algebra with a contractive approximate identity, and let \(Y\) be a right \(\mathcal{A}\)-operator module. Suppose there is a net \((n(b))_{b \in B}\) of positive integers and right \(\mathcal{A}\)-module maps

\[
\Phi_b : Y \to C_{n(b)}(\mathcal{A}), \; \Psi_b : C_{n(b)}(\mathcal{A}) \to Y, \; b \in B
\]

such that

(i) the maps \(\Phi_b,\) \(\Psi_b\) are completely contractive,

(ii) \(\Psi_b \circ \Phi_b \to I_{\mathcal{A}}\) strongly on \(Y,\)

(iii) the maps \(\Psi_b,\) \(b \in B\) are right \(\mathcal{A}\)-essential maps (that is, \(\Psi_b e_i \to \Psi_b\) for a bounded approximate identity \((e_i)_{i \in I}\) of \(\mathcal{A}\)),

(iv) \(\Phi_c \circ \Psi_b \circ \Phi_b \to \Phi_c, \forall c \in B\) (uniformly in norm)

Then we say that \(Y\) is a right \(\mathcal{A}\)-rigged module.

We denote by \(\mathcal{B}(H, K)\) the space of all linear and bounded operators from the Hilbert space \(H\) to the Hilbert space \(K.\) If \(H = K,\) we write \(\mathcal{B}(H, H) = \mathcal{B}(H)\). If \(X\) is a subset of \(\mathcal{B}(H, K)\) and \(Y\) is a subset of \(\mathcal{B}(K, L)\), then we denote by \([XY]\) the norm-closure of the linear span of the set

\[
\{y x \in \mathcal{B}(H, L) \mid y \in Y, \; x \in X\}.
\]

Similarly, if \(Z\) is a subset of \(\mathcal{B}(L, R),\) we define the space \([ZYX]\).

**Definition 1.2.** (i) A linear subspace \(M \subseteq \mathcal{B}(H, K)\) is called a ternary ring of operators (TRO) if \(MM^* \subseteq M.\)

(ii) A norm closed ternary ring of operators \(M\) is called a \(\sigma\)-TRO if there exist sequences \(\{m_i \in M \mid i \in \mathbb{N}\}\) and \(\{n_j \in M \mid j \in \mathbb{N}\}\) such that

\[
\lim_{n} \sum_{i=1}^{n} m_i m_i^* m = m, \; \lim_{t} \sum_{j=1}^{t} m^* n_j n_j = m, \forall m \in M
\]

and

\[
\left\| \sum_{i=1}^{n} m_i m_i^* \right\| \leq 1, \; \left\| \sum_{j=1}^{t} n_j n_j^* \right\| \leq 1, \forall n, t \in \mathbb{N}.
\]

A norm closed TRO \(M\) is a \(\sigma\)-TRO if and only if the \(\mathcal{C}^*\)-algebras \([M^* M]\) and \([MM^*]\) have \(\sigma\)-units, [6].
If $X$ is an operator space, then the spatial tensor product $X \otimes \mathcal{K}$ is completely isometrically isomorphic with the space $K_{\infty}(X)$, which is the norm closure of the finitely supported matrices in $M_{\infty}(X)$. Here, $M_{\infty}(X)$ is the space of $\infty \times \infty$ matrices with entries in $X$ which define bounded operators. Also, for another operator space $Y$, we denote by $X \otimes_{h}^{\Delta} Y$ the Haagerup tensor product of $X$ and $Y$. If $A$ is an operator algebra, $X$ is a right $A$-module, and $Y$ is a left $A$-module, then we denote by $X \otimes_{A}^{h} Y$ the balanced Haagerup tensor product of $X$ and $Y$ over $A$, see [4]. We now give two basic definitions.

**Definition 1.3.** Let $X \subseteq \mathcal{B}(H,K)$, $Y \subseteq \mathcal{B}(L,R)$ be operator spaces. We call them $\sigma$-TRO equivalent if there exist $\sigma$-TROs $M_{1} \subseteq \mathcal{B}(H,L)$, $M_{2} \subseteq \mathcal{B}(K,R)$ such that

$$X = [M_{2}^{*} Y M_{1}], \quad Y = [M_{2} X M_{1}^{*}].$$

In this case we write $X \sim_{\sigma \text{TRO}} Y$.

**Definition 1.4.** Let $X$, $Y$ be operator spaces. We call them $\sigma \Delta$ equivalent if there exist completely isometric maps $\phi : X \to \mathcal{B}(H,K)$, $\psi : Y \to \mathcal{B}(L,R)$ such that $\phi(X) \sim_{\sigma \text{TRO}} \psi(Y)$. We write $X \sim_{\sigma \Delta} Y$.

If $A$, $B$ are abstract or concrete operator algebras, we say that they are $\sigma \Delta$ equivalent and we write $A \sim_{\sigma \Delta} B$ if there exist completely isometric representations $\alpha : A \to \alpha(A) \subseteq \mathcal{B}(H)$, $\beta : B \to \beta(B) \subseteq \mathcal{B}(K)$ and a $\sigma$-TRO $M \subseteq \mathcal{B}(H,K)$ such that

$$\alpha(A) = [M^{*} \beta(B) M], \quad \beta(B) = [M a(A) M^{*}].$$

For further details about the notion of $\sigma \Delta$ equivalence of operator algebras and operator spaces, we refer the reader to [7, 8, 9, 10]. If $X$, $Y$ are operator spaces, then $X \sim_{\sigma \Delta} Y$ if and only if $X$ and $Y$ are stably isomorphic, that is, $K_{\infty}(X) \cong K_{\infty}(Y)$ (similarly for operator algebras). We present a lemma which will be used in some of the proofs in the following sections.

**Lemma 1.5.** Suppose that $A$, $B$ are operator algebras and $D \subseteq B$ is a $C^{*}$-algebra such that $[\overline{D} B] = [\overline{B} D] = B$. Let $M \subseteq \mathcal{B}(H,K)$ be a $\sigma$-TRO such that $[M^{*} M] \cong D$ (as $C^{*}$-algebras) and assume that $A \cong M \otimes_{D}^{h} B \otimes_{D}^{h} M^{*}$. Then, $A \sim_{\sigma \Delta} B$.

A proof of this lemma can be found in [10, Lemma 2.2].

2. **Orthogonally complemented modules and $\sigma \Delta$-rigged modules**

Let $A$ be an approximately unital operator algebra and $P : C_{\infty}(A) \to C_{\infty}(A)$ be a left multiplier of $C_{\infty}(A)$ (that is, $P \in M_{e}(C_{\infty}(A))$) such that $P$ is contractive and $P^{2} = P$. Then the space $W = P(C_{\infty}(A))$ is said to be orthogonally complemented in $C_{\infty}(A)$. In this section we characterize the orthogonally complemented modules in the terms of ternary rings of operators. A dual version of the results obtained here is in [2].
Definition 2.1. Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be an approximately unital operator algebra and $M \subseteq \mathcal{B}(H,K)$ be a $\sigma$-TRO such that $M^* M \mathcal{A} \subseteq \mathcal{A}$. The operator space $Y_0 = [M \mathcal{A}] \subseteq \mathcal{B}(H,K)$ is called a $\sigma$-TRO-$\mathcal{A}$-rigged module.

We recall that $Y_0$ is a right $\mathcal{A}$-operator module with action

$$(ma) \cdot x = m(a \cdot x), \quad m \in M, \quad a, x \in \mathcal{A}.$$ 

Definition 2.2. Let $\mathcal{A}$ be an abstract approximately unital operator algebra and let $Y$ be an abstract right $\mathcal{A}$-module. We call $Y$ a $\sigma\Delta$-$\mathcal{A}$-rigged module if there exists a completely isometric homomorphism $a : \mathcal{A} \to a(\mathcal{A})$ and there exist a $\sigma$-TRO-$a(\mathcal{A})$-rigged module $Y_0$ and a complete surjective isometry $\rho : Y \to Y_0$ which is also a right $\mathcal{A}$-module map. In case $\mathcal{A}$ is a $C^\ast$-algebra we call $Y$ a $\sigma\Delta$-$\mathcal{A}$-Hilbert module.

Proposition 2.3. Let $\mathcal{A}$ be an approximately unital operator algebra. Every $\sigma\Delta$-$\mathcal{A}$-rigged module is a right rigged module over $\mathcal{A}$ in the sense of Definition 1.1.

Proof. Let $Y$ be a right $\sigma\Delta$-$\mathcal{A}$-rigged module. Then there exist a completely isometric homomorphism $a : \mathcal{A} \to a(\mathcal{A}) \subseteq \mathcal{B}(H)$, a $\sigma$-TRO $M \subseteq \mathcal{B}(H,K)$ and a complete surjective isometry $\rho : Y \to Y_0 = [M a(\mathcal{A})]$ which is also a right $\mathcal{A}$-module map. So, if we choose a $\{\Phi_b, \Psi_b \mid b \in B\}$ for the module $Y_0$, then we define for each $b \in B$ the maps $\Phi'_b = \Phi_b \circ \rho$, $\Psi'_b = \rho^{-1} \circ \Psi_b$ and we can see that the $\{\Phi'_b, \Psi'_b \mid b \in B\}$ satisfy the conditions of Definition 1.1. So, $Y$ becomes a right $\mathcal{A}$-rigged module. Therefore, it suffices to prove the proposition when $Y = [M a(\mathcal{A})] \subseteq \mathcal{B}(H,K)$. Since $M$ is a $\sigma$-TRO, there exists a sequence $\{m_i \in M \mid i \in \mathbb{N}\}$ such that $||\{m_i\}_{i \in \mathbb{N}}|| \leq 1$ and

$$\sum_{i=1}^{\infty} m_i m_i^* m = m, \quad \forall m \in M.$$ 

Since $Y = [M a(\mathcal{A})]$, it follows that

$$\sum_{i=1}^{\infty} m_i m_i^* y = y, \quad \forall y \in Y.$$ 

For $n \in \mathbb{N}$ we define

$$\Phi_n : Y \to C_n(\mathcal{A}), \quad \Phi_n(y) = \begin{pmatrix} m_1^* y \\ \vdots \\ m_n^* y \end{pmatrix},$$

which is linear and a completely contractive right $\mathcal{A}$-module map. We also define the linear, completely contractive and right $\mathcal{A}$-module map

$$\Psi_n : C_n(\mathcal{A}) \to Y, \quad \Psi_n \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \sum_{i=1}^{n} m_i a_i.$$
For all \( y \in Y \), it holds that
\[
\Psi_n \circ \Phi_n(y) = \Psi_n \left( \begin{pmatrix} m_1^* y \\ \vdots \\ m_n^* y \end{pmatrix} \right) = \sum_{i=1}^{n} m_i m_i^* y \rightarrow y = I_d Y(y)
\]
and we conclude that \( \Psi_n \circ \Phi_n \rightarrow I_d Y \) strongly on \( Y \). The next step is to prove that \( \Psi_n, n \in \mathbb{N} \), is a right \( A \)-essential map. To this end, let \( (e_i)_{i \in I} \) be a contractive approximate identity of \( A \). We have that
\[
\left\| \Psi_n e_i \left( \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \right) - \Psi_n \left( \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \right) \right\| = \left\| \Psi_n \left( \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \right) e_i - \Psi_n \left( \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \right) \right\|
\]
\[
= \left\| \sum_{j=1}^{n} (m_j a_j) e_i - \sum_{j=1}^{n} m_j a_j \right\|
\]
\[
= \left\| \sum_{j=1}^{n} m_j (a_j e_i - a_j) \right\|
\]
\[
\leq \sum_{j=1}^{n} |m_j| |a_j e_i - a_j|
\]
where
\[
\lim_i |a_j e_i - a_j| = 0
\]
for all \( j = 1, \ldots, n \), so
\[
\lim_i \left\| \Psi_n e_i \left( \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \right) - \Psi_n \left( \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \right) \right\| = 0.
\]
Finally, let \( r \in \mathbb{N} \). We shall show that
\[
\lim_n \|\Phi_r \circ \Psi_n \circ \Phi_n - \Phi_r\| = 0.
\]
We denote by \( s_n \) the operators
\[
s_n = \sum_{i=1}^{n} m_i m_i^*, n \in \mathbb{N}.
\]
Hence, if \( y \in Y \), we have that
\[
\left\| \Phi_r \circ \Psi_n \circ \Phi_n(y) - \Phi_r(y) \right\| = \left\| \Phi_r(\Psi_n \circ \Phi_n(y) - y) \right\|
\]
\[
= \left\| \begin{pmatrix} m_1^* s_n - m_1^* \\ \vdots \\ m_n^* s_n - m_n^* \end{pmatrix} y \right\|
\]
\[
\leq \left\| \begin{pmatrix} m_1^* s_n - m_1^* \\ \vdots \\ m_n^* s_n - m_n^* \end{pmatrix} \right\| |y|
\]
Therefore, $||\Phi_r \circ \Psi_n \circ \Phi_n - \Phi_r|| \leq \begin{pmatrix} m_1^* s_n - m_1^* \\ \vdots \\ m_r^* s_n - m_r^* \end{pmatrix}$

Since

$$\lim_{n} ||m_i^* s_n - m_i^*|| = 0, \forall i = 1, ..., r,$$

we have that

$$\lim_{n} ||\Phi_r \circ \Psi_n \circ \Phi_n - \Phi_r|| = 0.$$

We conclude that $Y$ is a right $\mathcal{A}$-rigged module in the sense of Definition 1.1.

**Theorem 2.4.** Let $\mathcal{A}$ be an approximately unital operator algebra and $Y$ be a right $\mathcal{A}$-operator module. Then the following are equivalent:

(i) $Y$ is a right $\sigma \Delta$-$\mathcal{A}$-rigged module.

(ii) $Y$ is orthogonally complemented in $C_\infty(\mathcal{A})$.

**Proof.** (i) $\implies$ (ii) Let $\alpha : \mathcal{A} \to \mathcal{A}$ be a completely isometric representation of $\mathcal{A}$ on $H$ and assume there is a $\sigma$-TRO $M \subseteq \mathbb{B}(H, K)$ such that $M^* M \alpha(\mathcal{A}) \subseteq \alpha(\mathcal{A})$. Consider the $\sigma \Delta$-$\mathcal{A}$-rigged module $Y_0 = [M \mathcal{A}(\mathcal{A})] \subseteq \mathbb{B}(H, K)$ and a complete surjective isometry $\phi : Y \to Y_0$ which is also a right $\mathcal{A}$-module map. Let $\{m_i \in M \mid i \in \mathbb{N}\}$ be a sequence of elements of $M$ having the property

$$\left\| \sum_{i=1}^{n} m_i m_i^* \right\| \leq 1, \forall n \in \mathbb{N}, \sum_{i=1}^{\infty} m_i m_i^* m = m, \forall m \in M.$$

It follows that

$$\sum_{i=1}^{\infty} m_i m_i^* y = y, \forall y \in Y_0.$$

We define the map $f : Y_0 \to C_\infty(\mathcal{A})$ by $f(y) = (m_i^* y)_{i \in \mathbb{N}}$, which is linear and a $\mathcal{A}$-module map. Also,

$$||f(y)||^2 = \left\| \sum_{i=1}^{\infty} (m_i^* y)^* m_i^* y \right\| = \left\| \sum_{i=1}^{\infty} y^* m_i m_i^* y \right\| = ||y^* y|| = ||y||^2,$$

so $f$ is an isometry. We also define

$$g : C_\infty(\mathcal{A}) \to Y_0, g((\alpha(x_i))_{i \in \mathbb{N}}) = \sum_{i=1}^{\infty} m_i \alpha(x_i),$$

which is linear and a contractive $\mathcal{A}$-right module map. We see that

$$(g \circ f)(y) = g((m_i^* y)_{i \in \mathbb{N}}) = \sum_{i=1}^{\infty} m_i m_i^* y = y, \forall y \in Y_0,$$

that is, $g \circ f = Id_{Y_0}$. We now define $P = f \circ g : C_\infty(\mathcal{A}) \to C_\infty(\mathcal{A})$. Clearly $P$ is a contractive map satisfying $P^2 = P$. We shall prove that $P \in M_e(C_\infty(\mathcal{A}))$. 

For all \( x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} \in C_\infty(a(A)) \) we have that

\[
P(x) = \left( m_i^* \sum_{j=1}^{\infty} m_j x_j \right)_{i \in \mathbb{N}} = s x,
\]

where \( s = (m_i^* m_j)_{i,j=1}^{\infty} \in M_\infty(\mathbb{B}(H)) \). Observe that \( s = \begin{pmatrix} m_1^* \\ \vdots \\ m_2^* \end{pmatrix} \begin{pmatrix} m_1 \\ \vdots \\ m_2 \end{pmatrix} \) and due to the fact that \( |(m_1, m_2, \ldots)| \leq 1 \) we get \( ||s|| \leq 1 \). We define the map

\[
\tau_P : C_2(C_\infty(a(A))) \to C_2(C_\infty(a(A))), \quad \tau_P \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} P(x) \\ y \end{pmatrix} = \begin{pmatrix} s x \\ y \end{pmatrix}
\]

and for all \( \begin{pmatrix} x \\ y \end{pmatrix} \in C_2(C_\infty(a(A))) \) holds that

\[
\left\| \tau_P \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) \right\| = \left\| \begin{pmatrix} s x \\ y \end{pmatrix} \right\| = \left\| \begin{pmatrix} s & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|
\]

so \( \tau_P \) is a contraction. Similarly, we can prove that \( \tau_P \) is completely contractive. Therefore by [3, Theorem 4.5.2], \( P \) is a left multiplier of \( C_\infty(a(A)) \). It is easy to see that \( f(Y_0) = P(C_\infty(\alpha(A))) \) and thus \( Y \approx P(C_\infty(\alpha(A))) \).

(ii) \( \implies \) (i) Suppose that \( A \subseteq A^{**} \subseteq \mathbb{B}(H) \). Let \( P : C_\infty(A) \to C_\infty(A) \) be a left multiplier of \( C_\infty(A) \) which is a right \( A \)-module map with \( ||P||_{cb} \leq 1 \) and such that \( P^2 = P \). By [5, Appendix B], there is an extension \( \hat{P} : C_\infty w(A^{**}) \to C_\infty w(A^{**}) \) of \( P \). The operator \( \hat{P} \) lies in the diagonal of \( M_1(C_\infty w(A^{**})) \), which is contained in \( M_\infty(A^{**}) \). Therefore, \( \hat{P} = (\hat{p}_{i,j})_{i,j \in \mathbb{N}} \) where \( \hat{p}_{i,j} \in A^{**} \), \( \forall i, j \in \mathbb{N} \). Thus,

\[
\hat{P}(u) = (\hat{p}_{i,j})_{i,j \in \mathbb{N}} \cdot u, \quad \forall u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \end{pmatrix} \in C_\infty(A).
\]

In what follows we identify \( \hat{P} \) and \( (\hat{p}_{i,j})_{i,j} \). We have that \( Y \cong P(C_\infty(A)) = \hat{P}(C_\infty(A)) \) and \( \hat{P}^2 = \hat{P} = \hat{P}_0 \). Let \( N_2 = [\hat{P}], D \) be the \( C^* \)-algebra generated by \( \hat{P} \) and \( \mathcal{K}_\infty \) and let \( N_1 = C_\infty \). By [8, Lemma 2.5], \( M = [N_2 D N_1] = [\hat{P} D C_\infty] \) is a \( \sigma \)-TRO. We claim that \( D C_\infty(A) \subseteq C_\infty(A) \). Indeed,

\[
\hat{P}(C_\infty(A)) = P(C_\infty(A)) \subseteq C_\infty(A)
\]

and \( C_\infty R_\infty C_\infty(A) \subseteq C_\infty(A) \). Due to the fact that \( \mathcal{K}_\infty = C_\infty R_\infty \), we have that

\[
\mathcal{K}_\infty C_\infty(A) \subseteq C_\infty(A).
\]

But since \( D \) is generated by \( \hat{P} \), \( \mathcal{K}_\infty \) by (2.1) and (2.2) we have that \( D C_\infty(A) \subseteq C_\infty(A) \). Now,

\[
P(C_\infty(A)) \subseteq \hat{P} D C_\infty(A) = [M A].
\]
On the other hand,
\[ [M \mathcal{A}] = [\bar{P} DC_\infty \cdot \mathcal{A}] \subseteq \bar{P} (C_\infty (\mathcal{A})) = P(C_\infty (\mathcal{A})) \]
so, \[ [M \mathcal{A}] = P(C_\infty (\mathcal{A})) \].

Finally,
\[ M^* M \mathcal{A} \subseteq M^* P(C_\infty (\mathcal{A})) = R_\infty D \bar{P} (C_\infty (\mathcal{A})) \]
\[ \subseteq R_\infty D C_\infty (\mathcal{A}) \subseteq R_\infty C_\infty (\mathcal{A}) \]
\[ = R_\infty C_\infty \cdot \mathcal{A} = \mathcal{A} \]
so \( Y \) is a right \( \sigma \Delta \mathcal{A} \)-rigged module.

There is a category of rigged modules, the so-called countably column generated and approximately projective modules. We are going to examine whether there is a connection between them and the \( \sigma \Delta \)-rigged modules.

**Definition 2.5.** [1].

Let \( \mathcal{A} \) be an approximately unital operator algebra. A right \( \mathcal{A} \) operator module \( Y \) is called countably column generated and approximately projective (CCGP for short) if there are completely contractive right \( \mathcal{A} \)-module maps \( \phi : Y \to C_\infty (\mathcal{A}) \) and \( \psi : C_\infty (\mathcal{A}) \to Y \) with \( \psi \) finitely \( \mathcal{A} \)-essential (that is, for all \( n \in \mathbb{N} \) the restriction map of \( \psi \) to \( C_n (\mathcal{A}) \subseteq C_\infty (\mathcal{A}) \) is right \( \mathcal{A} \)-essential) and also \( \psi \circ \phi = Id_Y \).

**Remark 2.6.** From [1, Theorem 8.3] and Theorem 2.4, it is obvious that a CCGP module is a \( \sigma \Delta \mathcal{A} \)-rigged module. The converse is not true. Indeed, by [1, Theorem 8.2], we have that the CCGP modules over \( C^* \)-algebras are precisely the countably generated right Hilbert modules, but there exist \( \sigma \Delta \)-Hilbert modules which are not countably generated. For example if \( \mathcal{A} \) a \( C^* \)-algebra without \( \sigma \)-unit, since \( C \mathcal{A} = \mathcal{A} \) then \( \mathcal{A} \) is a \( \sigma \Delta \)-Hilbert module over itself, but clearly is not countably generated, so it is not a CCGP module.

**3. Doubly \( \sigma \Delta \)-rigged modules**

In this section we introduce a subcategory of \( \sigma \Delta \)-rigged modules, the doubly \( \sigma \Delta \)-rigged modules and we prove that these modules implement stable isomorphism between the corresponding operator algebras.

**Definition 3.1.** Let \( Y \) be a right \( \mathcal{A} \)-operator module over the approximately unital operator algebra \( \mathcal{A} \). We call \( Y \) a BMP equivalence bimodule if there exist an operator algebra \( \mathcal{B} \) such that \( Y \) is a left \( \mathcal{B} \)-operator module and a \( \mathcal{B} \)-\( \mathcal{A} \)-operator module \( X \) such that
\[ \mathcal{B} \cong Y \otimes^h \mathcal{A}, \quad \mathcal{A} \cong X \otimes^h \mathcal{B} \]
In this case we call \( X \) and \( Y \) bimodules of BMP-Morita equivalence.

We note that every \( \mathcal{B} \)-\( \mathcal{A} \)-bimodule of Morita equivalence is a right \( \mathcal{A} \)-rigged module. We now introduce the notion of a doubly \( \sigma \Delta \)-rigged module.
Definition 3.2. Let \( \mathcal{A} \subseteq \mathcal{B}(H) \) be an approximately unital operator algebra and \( M \subseteq \mathcal{B}(H, K) \) be a \( \sigma \)-TRO such that
\[
M^* M \mathcal{A} \subseteq \mathcal{A}, \quad [M^* M \mathcal{A}] = [\mathcal{A} M^* M].
\]
We call the operator space \( Y = [M \mathcal{A}] \subseteq \mathcal{B}(H, K) \) a doubly \( \sigma \)-TRO-\( \mathcal{A} \)-rigged module.

We note that every doubly \( \sigma \)-TRO-\( \mathcal{A} \)-rigged module is also a \( \sigma \)-TRO-\( \mathcal{A} \)-rigged module in the sense of Definition 2.1.

Definition 3.3. Let \( \mathcal{A} \) be an abstract approximately unital operator algebra and \( Y \) be an abstract right \( \mathcal{A} \)-module. We call \( Y \) a doubly \( \sigma \Delta \)-\( \mathcal{A} \)-rigged module if there exists a completely isometric homomorphism \( a : \mathcal{A} \rightarrow a(\mathcal{A}) \) and also there exists a \( \sigma \)-TRO-\( a(\mathcal{A}) \)-rigged module \( Y_0 \) and a complete onto isometry \( \phi : Y \rightarrow Y_0 \) which is a right \( \mathcal{A} \)-module map.

Definition 3.4. Let \( \mathcal{A} \) be an approximately unital operator algebra and \( Y \) be a \( \sigma \Delta \)-\( \mathcal{A} \)-rigged module. There exist \( a : \mathcal{A} \rightarrow a(\mathcal{A}) \subseteq \mathcal{B}(H) \), a completely isometric representation of \( \mathcal{A} \) on \( H \) and a \( \sigma \)-TRO \( M \subseteq \mathcal{B}(H, K) \) such that \( M^* M a(\mathcal{A}) \subseteq a(\mathcal{A}) \) and \( Y \cong Y_0 = [M a(\mathcal{A})] \). Then the operator space \( Z = [Y_0 M^* M] \subseteq \mathcal{B}(H, K) \) is called the restriction of \( Y \) over \( \mathcal{A} \). Observe that \( Z \) is a right module over the operator algebra \( [a(\mathcal{A}) M^* M] \).

In the following theorem we prove that the notions of \( \sigma \Delta \) right Hilbert modules and of doubly \( \sigma \Delta \) right Hilbert modules coincide:

Theorem 3.5. Let \( \mathcal{A} \) be a \( C^* \)-algebra and let \( Y \) be a right Hilbert module over \( \mathcal{A} \). The following are equivalent:

(i) \( Y \) is orthogonally complemented in \( C_\infty(\mathcal{A}) \).
(ii) \( Y \) is a \( \sigma \Delta \) right Hilbert module over \( \mathcal{A} \).
(iii) \( Y \) is a doubly \( \sigma \Delta \) right Hilbert module over \( \mathcal{A} \).

Proof. (i) \( \implies \) (iii) Let \( P : C_\infty(\mathcal{A}) \rightarrow C_\infty(\mathcal{A}) \) be an adjointable map such that \( P = P^2 = P^* \) and \( Y \cong P(C_\infty(\mathcal{A})) \). Since \( P \in M_f(C_\infty(\mathcal{A})) \), where \( M_f(C_\infty(\mathcal{A})) \) is the left multiplier algebra of \( C_\infty(\mathcal{A}) \), \( P \) can be extended to a multiplier of \( C^w_\infty(\mathcal{A}^{**}) \). Here \( \mathcal{A}^{**} \) is the second dual of \( \mathcal{A} \) and \( C^w_\infty(\mathcal{A}^{**}) \) is the space of columns with entries in \( \mathcal{A}^{**} \) which define bounded operators. The algebra of left multipliers of \( C^w_\infty(\mathcal{A}^{**}) \) is isomorphic to \( M_{\mathcal{A}^{**}}(\mathcal{A}^{**}) \) (we refer the reader to [5]). Therefore, we may assume that there exist \( a_{i,j} \in \mathcal{A}^{**}, i, j \in \mathbb{N} \) such that
\[
P(u) = (a_{i,j}) \cdot u, \ \forall u \in C_\infty(\mathcal{A}).
\]
In what follows we identify \( P \) with the matrix \( (a_{i,j}) \). We also may consider a Hilbert space \( K \) such that \( \mathcal{A} \subseteq \mathcal{A}^{**} \subseteq \mathcal{B}(K) \) and also \( I_K \subseteq \mathcal{A}^{**} \).

Let \( N_2 \) be the linear span of the element \( P \). Since \( P^2 = P = P^* \) we get that \( N_2 \) is a \( \sigma \)-TRO. Let \( \mathcal{A}^1 = [\mathcal{A} + C I_K] \) and \( N_1 = C_\infty(\mathcal{A}^1) \). Clearly \( N_1 \) is a \( \sigma \)-TRO. If \( D \) is the \( C^* \)-algebra generated by \( P \) and \( K_\infty(\mathcal{A}^1) \), then \( M = [N_2 D N_1] \) is a \( \sigma \)-TRO, [8, Lemma 2.5].
We note that
\[ M^* M A = [N^*_1 D N^*_2 N_2 D N_1 A] = [N^*_1 D N^*_2 C^\infty_0(A^1) A] \subseteq A. \]
If \( Y_0 = [M A] \), then
\[ Y_0 = [N_2 D N_1 A] = [N_2 D C^\infty_0(A)] = [P D C^\infty_0(A)] = P(C^\infty_0(A)). \]
We have that
\[ M^* M A = [M^* P(C^\infty_0(A))] = [N^*_1 D N^*_2 P C^\infty_0(A)] \]
\[ = [N^*_1 D P C^\infty_0(A)] = [R^\infty_0(A^1) P(C^\infty_0(A))] \]
\[ = [R^\infty_0(A) P(C^\infty_0(A))] \]
and therefore
\[ ([M^* M A])^* = ([R^\infty_0(A) P(C^\infty_0(A))])^*, \]
that is
\[ [A M^* M] = [R^\infty_0(A) P(C^\infty_0(A))] = [M^* M A], \]
which implies that
\[ [M^* M A] = [A M^* M] \subseteq A. \]
Since also
\[ Y \equiv P(C^\infty_0(A)) = Y_0 = [M A], \]
we conclude that \( Y \) is a doubly \( \sigma \Delta \) Hilbert module.

(iii) \( \Rightarrow \) (ii) This is obvious.

(ii) \( \Rightarrow \) (i) This is a consequence of Theorem 2.4.

At this point, we prove a Lemma which will be very useful for what follows.

**Lemma 3.6.** Let \( A \) be an operator algebra with \( \text{cai} (a_k)_{k \in K} \) and \( \mathcal{E} \) be a \( C^* \)-algebra with \( \text{cai} (c_i)_{i \in I} \). Assume that \( \mathcal{E} A \subseteq A \), \( A \mathcal{E} \subseteq A \). We define \( A_0 = [\mathcal{E} A \mathcal{E}] \subseteq A \).

Then \( A_0 \) is an operator algebra with a two-sided approximate identity
\[ x_{i(k)} = c_i a_k c_i^*, \quad i \in I, \quad k \in K. \]

**Proof.** The space \( A_0 \) is a closed subspace of \( A \) and is an algebra since
\[ A_0 A_0 \subseteq [\mathcal{E} A \mathcal{E}] \subseteq [\mathcal{E} A \mathcal{E} \mathcal{E}] \subseteq [\mathcal{E} A \mathcal{A} \mathcal{E}] \subseteq [\mathcal{E} \mathcal{A} \mathcal{E}] = A_0. \]

It is obvious that \( x_{i(k)} = c_i a_k c_i^* \in A_0, \quad i \in I, \quad k \in K \) and \( A_0 \subseteq A \). Now, if \( a \in A_0 \), then \( c_i a \to a \) and \( a_k a \to a \). For all \( i \in I, \quad k \in K \) we have that
\[
||x_{i(k)} a - a|| = ||c_i a_k c_i a - a|| \\
\leq ||c_i a_k c_i a - c_i a|| + ||c_i a - a|| \\
\leq ||a_k c_i a - a|| + ||c_i a - a|| \\
\leq ||a_k c_i a - a_k a|| + ||a_k a - a|| + ||c_i a - a|| \\
\leq ||c_i a - a|| + ||a_k a - a|| + ||c_i a - a|| \\
= 2 ||c_i a - a|| + ||a_k a - a||
\]
Thus, \[ \lim_{(i,k)} x_{i(k)} a = a. \]

Similarly, we can prove that \[ \lim_{(i,k)} a x_{i(k)} = a. \]

**Lemma 3.7.** Let \( \mathcal{A} \subseteq \mathbb{B}(H) \) be an approximately unital operator algebra and \( M \subseteq \mathbb{B}(H,K) \) be a \( \sigma \)-TRO such that \( M^* \mathcal{A} \subseteq \mathcal{A} \). We also assume that \( \mathcal{A} M^* M \subseteq \mathcal{A} \). We define \( \mathcal{B} = [M \mathcal{A} M^*] \subseteq \mathbb{B}(K) \) and also \( A_0 = [M^* \mathcal{B} M] \subseteq \mathbb{B}(H) \). Then \( A_0 \) and \( \mathcal{B} \) are approximately unital operator algebras and \( A_0 \sim_{\sigma \text{TRO}} \mathcal{B} \).

**Proof.** It suffices to prove that \( A_0, \mathcal{B} \) are closed under multiplication and that \( A_0 \sim_{\sigma \text{TRO}} \mathcal{B} \). Indeed, \[
\mathcal{B} \mathcal{B} \subseteq [M \mathcal{A} M^* M \mathcal{A} M^*] \subseteq [M \mathcal{A} \mathcal{A} M^*] = [M \mathcal{A} M^*] = \mathcal{B}
\]
so \( \mathcal{B} \) is an operator algebra. Now, we observe that \( MM^* \mathcal{B} \subseteq \mathcal{B} \) and then \[
A_0 A_0 \subseteq [M^* \mathcal{B} M M^* \mathcal{B} M] \subseteq [M^* \mathcal{B} \mathcal{B} M] \subseteq [M^* \mathcal{B} M] = A_0
\]
which means that \( A_0 \) is an operator algebra. We have that \( A_0 = [M^* \mathcal{B} M] = [M^* M \mathcal{A} M^* M] \). If \( C \) is the \( C^* \)-algebra \( [M^* M] \), then \( C \mathcal{A} \subseteq \mathcal{A} \), \( C \mathcal{C} \subseteq \mathcal{A} \). By Lemma 3.6, the operator algebra \( A_0 \) has a cai. Also, since \( A_0 = [M^* \mathcal{B} M] \) and on the other hand \[
[M A_0 M^*] = [M M^* \mathcal{B} MM^*] = [M^* M \mathcal{A} M^* M M^*] = [M \mathcal{A} M^*] = \mathcal{B}
\]
we deduce that \( A_0 \sim_{\sigma \text{TRO}} \mathcal{B} \). Since \( A_0 \) has a cai, we have that \( \mathcal{B} \) has also a cai.

**Theorem 3.8.** Let \( \mathcal{A} \) be an approximately unital operator algebra and \( Y \) be a doubly \( \sigma \Delta \)-\( \mathcal{A} \)-rigged module. Then, there exist operator algebras \( A_0,\mathcal{B} \) with cai’s such that \( A_0 \sim_{\sigma \text{TRO}} \mathcal{B} \) and also \( \mathcal{B} \sim_{\sigma \text{TRO}} Y \). In case \( \mathcal{A} \) is a \( C^* \)-algebra and \( Y \) is a \( \sigma \Delta \)-\( \mathcal{A} \)-Hilbert module then \( A_0 \simeq I_\mathcal{A}(Y) \), \( \mathcal{B} \simeq K_\mathcal{A}(Y) \).

**Proof.** Let \( H \) be a Hilbert space, \( a : \mathcal{A} \rightarrow a(\mathcal{A}) \subseteq \mathbb{B}(H) \) be a completely isometric representation of \( \mathcal{A} \) on \( H \) and let \( M \subseteq \mathbb{B}(H,K) \) be a \( \sigma \)-TRO such that \( M^* M a(\mathcal{A}) \subseteq a(\mathcal{A}) \) and also \[
[M^* M a(\mathcal{A})] = [a(\mathcal{A}) M^* M]\tag{3.1}
\]
Consider now a complete surjective isometry \( \phi : Y \rightarrow Y_0 = [M a(\mathcal{A})] \subseteq \mathbb{B}(H,K) \)
which is a right \( \mathcal{A} \)-module map. We define the spaces \( \mathcal{B} = [M a(\mathcal{A}) M^*] \subseteq \mathbb{B}(K) \) and \( A_0 = [M^* \mathcal{B} M] \subseteq \mathbb{B}(H) \). Now by Lemma 3.7, \( A_0,\mathcal{B} \) are operator algebras with cai’s such that \( A_0 \sim_{\sigma \text{TRO}} \mathcal{B} \). It remains to prove that \( \mathcal{B} \sim_{\sigma \text{TRO}} Y \).
Set $M_1 = M^* \subseteq \mathcal{B}(K, H)$ and $M_2 = [M M^*] \subseteq \mathcal{B}(K)$. Then, $M_1$, $M_2$ are $\sigma$-TRO’s and we have that

$$[M_2^* \phi(Y) M_1] = [M M^* M a(A) M^*] = [M a(A) M^*] = \mathcal{B}$$

and

$$[M_2 \mathcal{B} M_1^*] = [M a(A) M^* M] = [M M^* M a(A)] = [M a(A) M^*] = \phi(Y).$$

From Definition 1.4 we get that $\mathcal{B} \sim_{\sigma \text{TRO}} Y$.

The other assertions follow easily. \hfill \Box

**Theorem 3.9.** Let $A$ be an approximately unital operator algebra, $a : A \to \mathcal{B}(H)$ be a completely isometric homomorphism and $M \subseteq \mathcal{B}(H, K)$ be a $\sigma$-TRO such that $M^* M a(A) \subseteq a(A)$. We define the $\sigma \Delta$-TRO module $Y = [M a(A)]$. Then there exist operator algebras $A_0$, $\mathcal{B}$ with cai’s and a restriction $Z$ of $Y$ such that $Z$ is a doubly $\sigma \Delta$-TRO module and $A_0 \sim_{\sigma \text{TRO}} \mathcal{B} \sim_{\sigma \text{TRO}} Z$.

**Proof.** We define the restriction $Z = [YM^* M] = [M a(A) M^* M]$ of $Y$. Let $A_0 = [M^* M a(A) M^* M]$. Then $A_0$ is an operator algebra and

$$[M A_0] = [M M^* M a(A) M^* M] = [M a(A) M^* M] = Z$$

such that

$$\begin{align*}
[M^* A_0] &= [M^* M^* M a(A) M^* M] = [M^* M a(A) M^* M] = A_0 \\
\end{align*}$$

which means that $[M^* A_0] = [A_0 M^* M]$, that is, $Z = [M A_0]$ is a doubly $\sigma \Delta$-TRO module. If we define $\mathcal{B} = [M a(A) M^*]$ then by Lemma 3.6, $A_0$ and $\mathcal{B}$ have cai’s and by Lemma 3.7, we get that $A_0 \sim_{\sigma \text{TRO}} \mathcal{B}$.

Finally, $\mathcal{B} \sim_{\sigma \text{TRO}} Z$. Indeed, if we consider the $\sigma$-TRO’s $M_1 = M$ and $M_2 = [M M^*]$, then

$$\begin{align*}
[M_2 Z M_1^*] &= [M M^* M a(A) M^* M M^*] = [M A_0 M^*] = \mathcal{B} \\
[M_2^* \mathcal{B} M_1] &= [M M^* M a(A) M^* M] = [M M^* \mathcal{B} M] = [M A_0] = Z.
\end{align*}$$

\hfill \Box

**Corollary 3.10.** Every $\sigma \Delta$-TRO module $Y$ over an approximately unital operator algebra $A$ has a restriction which is a bimodule of BMP equivalence, which actually implements a stable isomorphism over the operator algebras $A_0$ and $\mathcal{B}$ defined as in Theorem 3.9.

**Corollary 3.11.** Every orthogonally complemented module over an approximately unital operator algebra $A$ has a restriction which is a bimodule of BMP equivalence between operator algebras which are stably isomorphic.
Proof. If $Y$ is an orthogonally complemented module over the operator algebra $\mathcal{A}$, then according to Theorem 2.4, $Y$ is a $\sigma\Delta$-$\mathcal{A}$-rigged module and due to the previous corollary, $Y$ has a restriction which is a bimodule of BMP equivalence between operator algebras which are stably isomorphic.

Another interesting category of rigged modules is the category of column stable generator modules. We prove that the restriction of a $\sigma\Delta$-rigged module over $\mathcal{A}$ is a column stable generated module (maybe over another operator algebra than $\mathcal{A}$). We refer the reader to [1, Section 8] for facts about column stable generated modules.

Definition 3.12. [1].
A right $\mathcal{A}$-rigged module $Y$ is called a column stable generator (CSG for short) if there exist completely contractive right $\mathcal{A}$-module maps $\sigma : \mathcal{A} \to C_\infty(Y)$ and $\tau : C_\infty(Y) \to \mathcal{A}$ such that $\tau \circ \sigma = \text{Id}_\mathcal{A}$.

Proposition 3.13. Let $\mathcal{A}$ be an approximately unital operator algebra, $a : \mathcal{A} \to a(\mathcal{A}) \subseteq \mathfrak{B}(H)$ be a completely isometric homomorphism and suppose there is a $\sigma$-$\text{TRO}$ $M \subseteq \mathfrak{B}(H, K)$ such that

$$M^* M a(\mathcal{A}) \subseteq a(\mathcal{A}), \quad a(\mathcal{A}) M^* M \subseteq a(\mathcal{A}).$$

Consider the $\sigma\Delta$-$\mathcal{A}$-rigged module $Y = \overline{[M a(\mathcal{A})]}$. Then, there exist operator algebras $\mathcal{A}_0$ and $\mathcal{B}$ and a restriction $Z$ of $Y$ over $\mathcal{A}_0$ such that $Z$ is a CSG module over $\mathcal{A}_0$.

Proof. Since $M$ is a $\sigma$-$\text{TRO}$, we fix a sequence $\{m_i \in M \mid i \in \mathbb{N}\} \subseteq M$ such that

$$\left\| \sum_{i=1}^{n} m_i^* m_i \right\| \leq 1, \forall n \in \mathbb{N}, \sum_{i=1}^{\infty} m_i^* m_i m^* = m^*, \forall m \in M. \quad (3.2)$$

We define the operator algebras $\mathcal{B} = \overline{[M a(\mathcal{A}) M^*]} \subseteq \mathfrak{B}(K), \mathcal{A}_0 = \overline{[M^* \mathcal{B} M]} \subseteq \mathfrak{B}(H)$ and also $Z = \overline{[YM^* M]} = \overline{[BM]}$, which is a restriction of $Y$, and is also a doubly $\sigma\Delta$-$\mathcal{A}_0$-rigged module (Theorem 3.9). Since

$$\overline{[M \mathcal{A}_0]} = \overline{[M M^* M a(\mathcal{A}) M^* M]} = \overline{[M a(\mathcal{A}) M^* M]} = \overline{[BM]} = \overline{Z}$$

and $\overline{[M^* Z]} = \overline{[M^* \mathcal{B} M]} = \mathcal{A}_0$, the maps

$$\sigma : \mathcal{A}_0 \to C_\infty(Z), \quad \sigma(a) = (m_i a)_{i \in \mathbb{N}}$$

and

$$\tau : C_\infty(Z) \to \mathcal{A}_0, \quad \tau((z_i)_{i \in \mathbb{N}}) = \sum_{i=1}^{\infty} m_i^* z_i$$

are well defined and also completely contractive right $\mathcal{A}_0$-module maps. For all $m^* b n \in M^* \mathcal{B} M \subseteq \mathcal{A}_0$ we have that

$$(\tau \circ \sigma)(m^* b n) = \tau((m_i m^* b n)_{i \in \mathbb{N}}) = \sum_{i=1}^{\infty} m_i^* m_i m^* b n \quad (3.2)$$

It follows that $$(\tau \circ \sigma)(a) = \text{Id}_{\mathcal{A}_0}(a), \forall a \in \mathcal{A}_0 \quad \Rightarrow \quad \tau \circ \sigma = \text{Id}_{\mathcal{A}_0},$$

$\square$
Theorem 3.14. Let $\mathcal{A}$, $\mathcal{B}$ be approximately unital operator algebras such that $\mathcal{A}$, $\mathcal{B}$ are stably isomorphic. Then, there exists a doubly $\sigma\Delta$-$\mathcal{A}$-rigged module $Y$ which is also an $\mathcal{A}$-$\mathcal{B}$-operator module and there exists a $\mathcal{B}$-$\mathcal{A}$-operator module $X$ such that $\mathcal{B} \cong Y \otimes^h_{\mathcal{A}} X$ and $\mathcal{A} \cong X \otimes^h_{\mathcal{B}} Y$. Furthermore, $\mathcal{A}$, $\mathcal{B}$, $X$, $Y$ are all stably isomorphic.

Proof. Since $\mathcal{A}$ and $\mathcal{B}$ are stably isomorphic, we have that they are also $\sigma\Delta$ equivalent, that is, $\mathcal{A} \sim_{\sigma\Delta} \mathcal{B}$, [8, Theorem 3.3]. So, there exist Hilbert spaces $H, K$ and completely isometric homomorphisms $a : \mathcal{A} \to \mathbb{B}(H)$ and $\beta : \mathcal{B} \to \mathbb{B}(K)$ and also a $\sigma$-TRO $M \subseteq \mathbb{B}(H, K)$ such that $a(\mathcal{A}) = \overline{[M^* \beta(\mathcal{B}) M]}$, $\beta(\mathcal{B}) = \overline{[M a(\mathcal{A}) M^*]}$. We have that

$$[a(\mathcal{A}) M^* M] = a(\mathcal{A}) = [M^* M a(\mathcal{A})]$$

and so $Y = \overline{[M a(\mathcal{A})]} \subseteq \mathbb{B}(H, K)$ is a doubly $\sigma\Delta$-$\mathcal{A}$-rigged module which is also a left $\mathcal{B}$-operator module since

$$\beta(\mathcal{B}) Y \subseteq [M a(\mathcal{A}) M^* M a(\mathcal{A})] \subseteq [M a(\mathcal{A}) a(\mathcal{A})] \subseteq [M a(\mathcal{A})] = Y.$$  

We also define $X = \overline{[a(\mathcal{A}) M^*]} \subseteq \mathbb{B}(K, H)$ which is a left $\mathcal{A}$-operator module via the module action

$$a(x) \cdot (a(y) m^*) = a(x y) m^*, \ x, y \in \mathcal{A}, m \in M.$$  

Furthermore $X$ is a right $\mathcal{B}$-operator module since

$$X \beta(\mathcal{B}) \subseteq [a(\mathcal{A}) M^* M a(\mathcal{A}) M^*] = [a(\mathcal{A}) a(\mathcal{A}) M^* M M^*] \subseteq [a(\mathcal{A}) M^*] = X.$$  

By Lemma 1.5, if $D_1 = \overline{[M^* M]}$, then

$$Y \otimes^h_{a(\mathcal{A})} X = \overline{[M a(\mathcal{A})]} \otimes^h_{a(\mathcal{A})} [a(\mathcal{A}) M^*]$$

$$\cong \left( M \otimes^h_{D_1} a(\mathcal{A}) \otimes^h_{a(\mathcal{A})} (a(\mathcal{A}) \otimes^h_{D_1} M^*) \right)$$

$$\cong M \otimes^h_{D_1} a(\mathcal{A}) \otimes^h_{D_1} M^*$$

$$(1.5) \cong [M a(\mathcal{A}) M^*] = \beta(\mathcal{B})$$
and also, due to the fact that $Y = [a(A)M^*] = [M^* \hat{\beta}(B)]$, if $D_2 = [M M^*]$ we have that

$$
X \otimes^h_{\hat{\beta}(B)} Y = [M^* \hat{\beta}(B)] \otimes^h_{\hat{\beta}(B)} [M M^* \hat{\beta}(B) M]
$$

By the same arguments, we obtain the following corollary:

**Corollary 3.15.** Let $\mathcal{A}$, $\mathcal{B}$ be stably isomorphic $C^*$-algebras. There exists a $\sigma\Delta$-Hilbert module $Y$ over a $C^*$-algebra $\mathcal{D}$ such that

$$
\mathcal{A} \cong K_2(Y), \mathcal{B} \cong I_2(Y).
$$

Furthermore $\mathcal{A}$, $\mathcal{B}$ and $Y$ are all stably isomorphic.

### 4. Morita equivalence of rigged modules

**Definition 4.1.** [1]. Let $\mathcal{A}$ be an approximately unital operator algebra and let $Y$ be a right $\mathcal{A}$-rigged module. If $\{\Phi_b, \Psi_b \mid b \in B\}$ is a choice for $Y$ as in Definition 1.1, then we write $E_b$ for the map $E_b = \Psi_b \circ \Phi_b : Y \to Y$, $b \in B$. We define

$$
\hat{Y} = \{f \in CB_\mathcal{A}(Y, \mathcal{A}) \mid f \circ E_b \to f \text{ uniformly}\}
$$

and $\mathcal{K}(Y)$ to be the closure in $CB_\mathcal{A}(Y, Y)$ of the set of finite rank operators

$$
T_{y,f} : Y \to Y, \;
T_{y,f}(y') = y f(y')
$$

where $y \in Y$, $f \in \hat{Y}$.

For further details we refer the reader to [1, Section 3]. We also note that $\mathcal{K}(Y)$ and $\hat{Y}$ are actually independent of the particular directed set and nets $\{\Phi_b, \Psi_b \mid b \in B\}$. In the following lemma we use the notion of a complete quotient map. For further details we refer the reader to [4].
Lemma 4.2. Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be an approximately unital operator algebra, $M \subseteq \mathcal{B}(H, K)$ be a $\sigma$-TRO and $Y = [M \mathcal{A}] \subseteq \mathcal{B}(H, K)$. Assume that $M^* M \mathcal{A} \subseteq \mathcal{A}, \mathcal{A} M^* M \subseteq \mathcal{A}$ (thus $Y$ is a $\sigma\Delta\mathcal{A}$-rigged module). Then $\bar{Y} \cong [\mathcal{A} M^*]$ and $\kappa(Y) \cong [M \mathcal{A} M^*]$.

Proof. We define $\mathcal{B} = [M \mathcal{A} M^*]$. Clearly, $\mathcal{B}$ is an operator algebra. By Lemma 3.6, the algebra $\mathcal{A}_0 = [M^* M \mathcal{A} M^*]$ has a cai. By Lemma 3.7, the algebra $\mathcal{B}$ has also cai and obviously the algebras $\mathcal{A}_0$ and $\mathcal{B}$ are $\sigma$-TRO equivalent. If $X = [\mathcal{A} M^*]$, then we define the completely contractive maps

\[(\cdot, \cdot): X \times Y \to \mathcal{A}, (x, y) \mapsto (x, y) = x y\]
\n\[[\cdot, \cdot]: Y \times X \to \mathcal{B}, (y, x) \mapsto [y, x] = y x.\]

These maps satisfy

\[(x, y) x' = x [y, x'], (x, y) = [y, x] y', \forall x, x' \in X, y, y' \in Y.\]

The map $[\cdot, \cdot]$ induces a complete quotient map $Y \otimes^h X \to \mathcal{B}, y \otimes x \to y x$. Indeed, by making the same calculations as those of the proof of Theorem 3.14, we have that $Y \otimes^h X \cong [M^* M \mathcal{A}^*] = \mathcal{B}$. Furthermore, the map $\phi: Y \otimes^h X \to Y \otimes^h \mathcal{A}, y \otimes x \to y \otimes \mathcal{A} x$ is a complete quotient since the map $\phi: (Y \otimes^h X)/\ker(\phi) \to Y \otimes^h \mathcal{A}$ is a complete surjective isometry. From [1, Theorem 5.1] it follows that $\bar{Y} \cong [\mathcal{A} M^*]$ and $\kappa(Y) \cong [M \mathcal{A} M^*]$. \hfill $\square$

Theorem 4.3. If $\mathcal{A}$ is an approximately unital operator algebra and $Y$ is a doubly $\sigma\Delta\mathcal{A}$-rigged module, then there exist approximately unital operator algebras $\mathcal{A}_0 \subseteq \mathcal{A}$ and $\mathcal{B}$ such that

(i) $\mathcal{B} \cong Y \otimes^h_{\mathcal{A}_0} \bar{Y}$

(ii) $\mathcal{A}_0 \cong \bar{Y} \otimes^h_{\mathcal{B}} Y$

(iii) $\mathcal{A}_0 \sim_{\sigma\Delta} \mathcal{B}, \mathcal{A}_0 \sim_{\sigma\Delta} Y, Y \sim_{\sigma\Delta} \bar{Y}$.

Proof. It suffices to prove the above assertions for the case of a doubly $\sigma$-TRO-$\mathcal{A}$-module $Y = [M \mathcal{A}]$ where $\mathcal{A} \subseteq \mathcal{B}(H), M \subseteq \mathcal{B}(H, K)$ is a $\sigma$-TRO such that $M^* M \mathcal{A} \subseteq \mathcal{A}$ and

\[M^* M \mathcal{A} = [M \mathcal{A} M^*]. \tag{4.1}\]

We set $\mathcal{A}_0 = [M \mathcal{A} M^*] \subseteq \mathcal{A}$. Clearly $\mathcal{A}_0$ is an approximately unital operator algebra.

(i) By Lemma 4.2, $\bar{Y} \cong [\mathcal{A} M^*]$, and so
\n\[\mathcal{A}_0 \cong [M^* M M^*] = [M M^* M] \cong [\mathcal{A} M^*] = \bar{Y}.\]

On the other hand

\[\bar{\mathcal{M} \mathcal{A}} = [M^* M \mathcal{A} M^*] \cong [M M^* M \mathcal{A}] = [M \mathcal{A}] = Y.\]

Using Lemma 1.5 and making the same calculations as in the proof of Theorem 3.14 we have that $Y \otimes^h_{\mathcal{A}_0} \bar{Y} \cong [M \mathcal{A} M^*]$. If we define $\mathcal{B} = [M \mathcal{A} M^*]$, then $\mathcal{B}$ is
an approximately unital operator algebra such that $\mathcal{B} \cong Y \otimes^{h}_{A_{0}} \hat{Y}$.

(ii) It is true that $\hat{Y} \cong [A M^{*}] = [M^{*} \mathcal{B}]$, so if $D_{1} = [M^{*} M]$ and $D_{2} = [M M^{*}]$, it follows that

$$
\hat{Y} \otimes^{h}_{\mathcal{B}} Y = [M^{*} \mathcal{B}] \otimes^{h}_{\mathcal{B}} [M A] \\
\cong (M^{*} \otimes^{h}_{D_{2}} \mathcal{B}) \otimes^{h}_{\mathcal{B}} Y \\
\cong M^{*} \otimes^{h}_{D_{2}} [\mathcal{B} \otimes^{h}_{\mathcal{B}} (M \otimes^{h}_{D_{1}} A)] \\
\cong M^{*} \otimes^{h}_{D_{2}} M \otimes^{h}_{D_{1}} A \\
\cong [M^{*} M] \otimes^{h}_{D_{1}} A \\
\cong [M^{*} M] \\
= [A M^{*} M] = A_{0}.
$$

(iii) Consider the $\sigma$-TROs $M_{1} = M^{*} \subseteq \mathbb{B}(K,H)$ and $M_{2} = M \subseteq \mathbb{B}(H,K)$. Then

$$
[M_{2}^{*} Y M_{1}] = [M^{*} M A M^{*}] = [A M^{*} M M^{*}] = [A M^{*}] = \hat{Y}
$$

and

$$
[M_{2} \hat{Y} M_{1}^{*}] = [M A M^{*} M] = [M M^{*} M A] = [M A] = Y
$$

so $Y \sim_{\sigma\text{TRO}} \hat{Y}$. By Theorem 3.8, we also have that $\mathcal{B} \sim_{\sigma\text{TRO}} Y$ and $\mathcal{B} \sim_{\sigma\Delta} \mathcal{A}_{0}$.

**Definition 4.4.** Let $\mathcal{A}$, $\mathcal{B}$ be approximately unital operator algebras, $E$ be a right $\mathcal{B}$-rigged module and $F$ be a right $\mathcal{A}$-rigged module. We call $E$ and $F$ Morita equivalent if there exists a right $\mathcal{A}$-rigged module $Y$ such that

(i) $\mathcal{A} \cong Y \otimes^{h}_{\mathcal{B}} Y$ as operator algebras,

(ii) $\mathcal{B} \cong Y \otimes^{h}_{\mathcal{B}} \hat{Y}$ as operator algebras,

(iii) $F \cong E \otimes^{h}_{\mathcal{B}} Y$ as right $\mathcal{A}$-rigged modules.

In this case we write $E \sim_{M} F$.

**Remark 4.5.** If $\mathcal{A}$, $\mathcal{B}$, $E$ and $F$ are as above, then by [1, Theorem 6.1] we get that

$$
\mathcal{K}(F) \cong \mathcal{K}(E \otimes^{h}_{\mathcal{B}} Y) \cong \mathcal{K}(E).
$$

**Definition 4.6.** Let $\mathcal{A}$, $\mathcal{B}$ be approximately unital operator algebras, $E$ be a right $\mathcal{B}$-rigged module and $F$ be a right $\mathcal{A}$-rigged module. We call $E$ and $F$ $\sigma$-Morita equivalent if there exists a doubly $\sigma\Delta\mathcal{A}$-rigged module $Y$ such that

(i) $\mathcal{A} \cong Y \otimes^{h}_{\mathcal{B}} Y$ as operator algebras,

(ii) $\mathcal{B} \cong Y \otimes^{h}_{\mathcal{B}} \hat{Y}$ as operator algebras,

(iii) $F \cong E \otimes^{h}_{\mathcal{B}} Y$ as right $\mathcal{A}$-rigged modules.

In this case we write $E \sim_{\sigma M} F$. 
Remark 4.7. Other notions of Morita equivalence for the subcategory of Hilbert modules exist in [11, 17].

Proposition 4.8. If $E \sim_{\sigma_M} F$, then $\mathcal{K}(E) \cong \mathcal{K}(F)$.

Proof. If $E \sim_{\sigma_M} F$, then $E \sim_M F$ and the conclusion comes from Remark 4.5. □

Lemma 4.9. Let $M$ be a $\sigma$-TRO, $D_1 = [MM^*], D_2 = [M^*M]$, $E$ be a right $D_1$-module and $F$ be a right $D_2$-module such that $F \cong E \otimes_{D_1}^h M$. Then $E \sim_{\sigma\Delta} F$.

Proof. By [10, Theorem 3.8], it suffices to prove that $E$ and $F$ are stably isomorphic. We may assume that $F = E \otimes_{D_1}^h M$. Hence,

$$F \otimes_{D_2}^h M^* = \left( E \otimes_{D_1}^h M \right) \otimes_{D_2}^h M^* \cong E \otimes_{D_1}^h \left( M \otimes_{D_2}^h M^* \right) \cong E \otimes_{D_1}^h D_1 \cong E.$$ 

We can also assume that there exists a complete onto isometry

$$a : F \otimes_{D_2}^h M^* \to E$$

such that

$$a((e \otimes_{D_1} m) \otimes_{D_2} n^*) = e m n^*, \forall e \in E, m, n \in M. \quad (4.2)$$

There exists a sequence $\{m_i \in M \mid i \in \mathbb{N}\}$ such that

$$\left\| \sum_{i=1}^n m_i^* m_i \right\| \leq 1, \forall n \in \mathbb{N}$$

and also

$$\sum_{i=1}^\infty m m_i^* m_i = m, \forall m \in M.$$ 

We observe that for all $e \in E$ and $m \in M$ we have that

$$\sum_{i=1}^\infty a((e \otimes_{D_1} m) \otimes_{D_2} m_i^*) \otimes_{D_1} m_i \overset{(4.2)}{=} \sum_{i=1}^\infty e \otimes_{D_1} m_i^* \otimes_{D_1} m_i = \sum_{i=1}^\infty e \otimes_{D_1} m m_i^* m_i = e \otimes_{D_1} m.$$ 

Thus,

$$\sum_{i=1}^\infty a(f \otimes_{D_2} m_i^*) \otimes_{D_1} m_i = f, \forall f \in F. \quad (4.3)$$

We define the completely contractive maps

$$\Phi : F \to R_{\alpha}(E), \Phi(f) = (a(f \otimes_{D_2} m_i^*))_{i \in \mathbb{N}}.$$
\[\Psi : R_\infty(E) \to F, \ \Psi((e_i)_{i \in \mathbb{N}}) = \sum_{i=1}^{\infty} e_i \otimes_{D_1} m_i.\]

Using (4.3) we have that
\[
(\Psi \circ \Phi)(f) = \sum_{i=1}^{\infty} a(f \otimes_{D_2} m_i^*) \otimes_{D_1} m_i = f, \ \forall f \in F.
\]

So, \(\Phi\) is a complete isometry and \(P = \Phi \circ \Psi : R_\infty(E) \to R_\infty(E)\) is a projection such that \(\Phi(F) = \text{Ran}(P)\). Now we employ the usual arguments, see for example the proof of [3, Corollary 8.2.6]:

\[
R_\infty(E) \cong \text{Ran}(P) \oplus_r \text{Ran}(I - P) \cong \Phi(F) \oplus_r \text{Ran}(I - P) \cong F \oplus_r \text{Ran}(I - P)
\]

where \(I = I_{R_\infty(E)}\). Thus,

\[
R_\infty(E) \cong R_\infty(R_\infty(E))
\]

\[
\cong (F \oplus_r \text{Ran}(I - P)) \oplus_r (F \oplus_r \text{Ran}(I - P)) \oplus_r ...
\]

\[
\cong F \oplus_r (\text{Ran}(I - P) \oplus_r F) \oplus_r (\text{Ran}(I - P) \oplus_r F) \oplus_r ...
\]

\[
\cong F \oplus_r R_\infty(E).
\]

Therefore, \(R_\infty(E) \cong R_\infty(R_\infty(E)) \cong R_\infty(F) \oplus_r R_\infty(E)\). By symmetry, \(R_\infty(F) \cong R_\infty(E) \oplus_r R_\infty(F)\), so \(R_\infty(E) \cong R_\infty(F)\) which implies that \(K_\infty(E) \cong K_\infty(F)\).

\[\square\]

**Theorem 4.10.** Let \(A, B\) be approximately unital operator algebras, \(E\) be a right \(B\)-rigged module and \(F\) be a right \(A\)-rigged module such that \(E \sim_{\sigma} M F\). Then \(E \sim_{\sigma \Delta} F\).

**Proof.** Let \(a : A \to B(H)\) be a completely-isometric representation of \(A\) on \(H\) and \(M \subseteq B(H, K)\) be a \(\sigma\)-TRO such that \(M^* M a(A) \subseteq a(A)\) and also \(M^* M a(A) = [a(A) M^* M]\). Consider also the doubly \(\sigma \Delta\)-rigged module \(Y = [a(A)]\) such that \(a(A) \cong \tilde{Y} \otimes^h_B Y, B \cong Y \otimes^h_A \tilde{Y} \cong [M a(A) M^*]\) and also \(F \cong E \otimes^h_B Y\). We define \(D_1 = [M M^*]\) and we have that \(B M M^* \subseteq B\). So

\[
E = \left[ E \right] \supseteq \left[ E \right] B M M^* \right] = [E M M^*]
\]

which means that \(E\) is a right \(D_1\)-module. Therefore, since \(Y = [a(A)] = [B M]\), it holds that

\[
F \cong E \otimes^h_B Y = E \otimes^h_B [B M] \cong E \otimes^h_B (B \otimes^h_{D_1} M) \cong (E \otimes^h_B B) \otimes^h_{D_1} M \cong E \otimes^h_{D_1} M.
\]

Observe that if \(D_2 = [M^* M]\), then \(F = [F A] \supseteq [F A M^* M] = [F M^* M]\) which means that \(F\) is a right \(D_2\)-module. From Lemma 4.9, we get that \(E \sim_{\sigma \Delta} F\). \[\square\]
References


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