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Essential properties for rings of integer-valued polynomials

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Abstract. Let D be an integral domain with quotient field K. We consider the ring of integer-valued polynomials over D, namely, $\text{Int}(D) := \{f \in K[X];\ f(D) \subseteq D\}$. In this paper we investigate when $\text{Int}(D)$ has essential-type properties. In particular, we give a complete characterization of when $\text{Int}(D)$ is locally essential, locally PvMD, locally UFD, locally GCD, Krull-type or generalized Krull.

Contents

Introduction and preliminaries

Throughout this paper D will denote an integral domain with quotient field K.

We first review some definitions and notation.

The polynomials with coefficients in K that take values from D into D form a commutative D-algebra denoted by $\text{Int}(D)$. More precisely $\text{Int}(D) := \{f \in K[X];\ f(D) \subseteq D\}$ is called *the ring of integer-valued polynomials over D*. Obviously, $D[X] \subseteq \text{Int}(D) \subseteq K[X]$, and if $\text{Int}(D) = D[X]$ we say that $\text{Int}(D)$ is *trivial*.

Let $\mathcal{F}(D)$ be the set of nonzero fractional ideals of D. For $I \in \mathcal{F}(D)$ it is set $I^{-1} := \{x \in K;\ xI \subseteq D\}$. The *$v$-operation* is defined on $\mathcal{F}(D)$ by $I_v := (I^{-1})^{-1}$ and the *t-operation* is defined by $I_t := \bigcup J_v$, where J ranges over the set of all nonzero finitely generated ideals contained in I. An ideal $I \in \mathcal{F}(D)$ is a *v-ideal* (or divisorial) (resp., *t-ideal*) if $I_v = I$ (resp., $I_t = I$). A prime ideal that is also a t-ideal is called *t-prime* and an ideal maximal among integral t-ideals is

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called *t-maximal* (and it is a prime ideal). We let t-Max(D) denote the set of all t-maximal ideals of D. It is well known that $D = \cap_{\mathfrak{p}\in t\text{-Max}(D)} D_{\mathfrak{p}}$ for any integral domain D.

In 1977 S. Glaz and W. Vasconcelos ([16]) gave the notion of *semi-divisorial ideal* of an integral domain D in order to study divisibility properties of finitely generated flat ideals. Later, in [30], W. Fanggui and L.R. McCasland renominated these ideals *Glaz-Vasconcelos ideals* (GV-ideals) and used them to define a closure operation on the ideals of D called *w-operation* ([30]).

An ideal J of D is a *Glaz-Vasconcelos ideal* if J is finitely generated and $J_v = J_t = D$. This set of ideals is denoted by $GV(D)$. Given a nonzero fractional ideal I of D, the *w-closure* of I is the ideal $I_w = \{x \in K,\ xJ \subseteq I \text{ for some } J \in GV(D)\}$. A nonzero ideal I of D is *w-ideal* if $I_w = I$.

It is straightforward that $I \subseteq I_w \subseteq I_t \subseteq I_v$.

Following [4], a prime ideal $\mathfrak{p}$ of D is called an *associated prime of a principal ideal aD* of D if $\mathfrak{p}$ is minimal over $(aD : bD)$ for some $b \in D \setminus aD$. For the sake of brevity, we call such ideal $\mathfrak{p}$ an *associated prime* of D and we denote by Ass(D) the set of all associated prime ideals of D. From [4, Proposition 4] we have that $D = \cap_{\mathfrak{p}\in \text{Ass}(D)} D_{\mathfrak{p}}$ and any associated prime ideal is a t-prime because it is minimal over the t-ideal $(aD : bD)$ for some $a, b \in D$.

Given a subset $\mathcal{P}$ of Spec(D), we say that D is an *essential domain with defining family $\mathcal{P}$* if $D = \cap_{\mathfrak{p}\in\mathcal{P}} D_{\mathfrak{p}}$ and each $D_{\mathfrak{p}}$ is a valuation domain.

In [22] J. Mott and M. Zafrullah introduced the notion of a *P-domain* as an integral domain D such that $D_{\mathfrak{p}}$ is a valuation domain for each associated prime ideal $\mathfrak{p}$ of D. In particular, a P-domain D is essential with defining family Ass(D). The authors also showed that P-domains are exactly the integral domains such that their rings of fractions are essential domains (equivalently, $D_{\mathfrak{p}}$ is essential for all prime ideals $\mathfrak{p}$ of D). Thus, these domains are also called *locally essential domains*.

An integral domain is a *Prüfer v-multiplication domain* (for short, PvMD) (resp., *t-almost Dedekind*, *almost Dedekind*, *almost Krull*) if it is t-locally valuation - that is $D_{\mathfrak{p}}$ is a valuation domain for each t-prime ideal $\mathfrak{p}$ (resp., t-locally DVR, locally DVR, locally Krull) (by a DVR we mean a rank-one discrete valuation domain). It is immediate that Krull domains and almost Dedekind domains are t-almost Dedekind and t-almost Dedekind domains are PvMDs. On the contrary, if we take a non-Noetherian almost Dedekind domain D (see, for example, [6, Examples VI.4.15 and VI.4.16]) then $D[X]$ is a t-almost Dedekind domain that is neither Krull nor almost Dedekind ([20, Remark, page 167]). Also, $\mathbb{Z}[X]$ is a Krull domain, hence it is t-almost Dedekind but, since it is two-dimensional, it is not almost Dedekind. Moreover, Int$(\mathbb{Z})$ or $\mathbb{Z} + X\mathbb{Q}[X]$ are Prüfer domains, so they are PvMDs, but they are not t-almost Dedekind. In fact, any Prüfer t-almost Dedekind domain is almost Dedekind, so it is one-dimensional, and Int$(\mathbb{Z})$ or $\mathbb{Z} + X\mathbb{Q}[X]$ are two-dimensional. Finally, almost

Dedekind domains and Krull domains are almost Krull, while an almost Krull needs neither be Krull nor almost Dedekind (see, for example, [3]).

An integral domain is called *locally UFD* (resp., *locally GCD*, *locally PvMD*) if its localizations at prime ideals are UFDs (resp., GCDs, PvMDs). Obviously, almost Dedekind domains are locally UFDs and locally UFDs are almost Krull and locally GCD domains.

For any integral domain D we denote by $X^1(D)$ the set of all height-one prime ideals of D. We recall the following definition of locally finite intersection of integral domains. Let $\{D_\alpha\}_{\alpha\in\Lambda}$ be a family of integral domains having the same quotient field. The intersection $\cap_{\alpha\in\Lambda} D_\alpha =: D$ is said to be *locally finite* if every nonzero element of D is a unit in D_α for all but finitely many $\alpha \in \Lambda$. In particular, if each D_α is local with maximal ideal $\mathfrak{m}_\alpha$, the above intersection is locally finite if and only if each nonzero element of D belongs to only finitely many ideals $\mathfrak{m}_\alpha$.

For instance, *Krull-type* domains are domains D for which $D = \cap_{\mathfrak{p}\in\mathcal{P}} D_\mathfrak{p}$, where $\mathcal{P} \subseteq \mathrm{Spec}(D)$, $D_\mathfrak{p}$ is a valuation domain for each $\mathfrak{p} \in \mathcal{P}$ and the intersection is locally finite (see, for instance, [17]).

Another interesting class of (locally) essential domains are the *generalized Krull* domains, described by R. Gilmer in [15, Section 43]. These are integral domains D such that $D = \cap_{\mathfrak{p}\in X^1(D)} D_\mathfrak{p}$, where the intersection is locally finite and $D_\mathfrak{p}$ is a valuation domain for each $\mathfrak{p} \in X^1(D)$. In particular, Krull domains are of this type and generalized Krull domains are a subclass of Krull-type domains. In [25, page 439] E.M. Pirtle pointed out that the integral domain D constructed in [14, Example 1, page 338] is generalized Krull but not Krull. Moreover a valuation domain of dimension greater than one is Krull-type but not generalized Krull.

We also consider integral domains for which $D_\mathfrak{p}$ is a DVR for each $\mathfrak{p} \in \mathrm{Ass}(D)$. Following [21], we call these domains MZ-DVRs because their definition is inspired by the P-domains of J. Mott and M. Zafrullah. These domains are locally essential. We notice that if we take a rank-one valuation domain that is not a DVR, then this is locally essential but not MZ-DVR.

We have that almost Krull domains and t-almost Dedekind domains are MZ-DVRs. In [3, page 52] it is given an example of an almost Krull domain that is not PvMD, whence it is a MZ-DVR but not t-almost Dedekind. Moreover, in [26, Section 3], E.M. Pirtle constructs an example of a K-domain that is not almost Krull (we recall that an integral domain D is a *K-domain* [26] if $D = \cap_{\mathfrak{p}\in X^1(D)} D_\mathfrak{p}$, where $D_\mathfrak{p}$ is a DVR and $\mathfrak{p}$ is divisorial for each $\mathfrak{p} \in X^1(D)$). Notice that any K-domain is a MZ-DVR; indeed, $D = \cap_{\mathfrak{p}\in X^1(D)} D_\mathfrak{p}$ where each $D_\mathfrak{p}$ is a DVR, and using [15, Exercise 22, page 52] we infer that $\mathrm{Ass}(D) = X^1(D)$.

In the following we represent a diagram of implications among different subclasses of essential domains involved in this paper. Any implication in the diagram is not reversible.

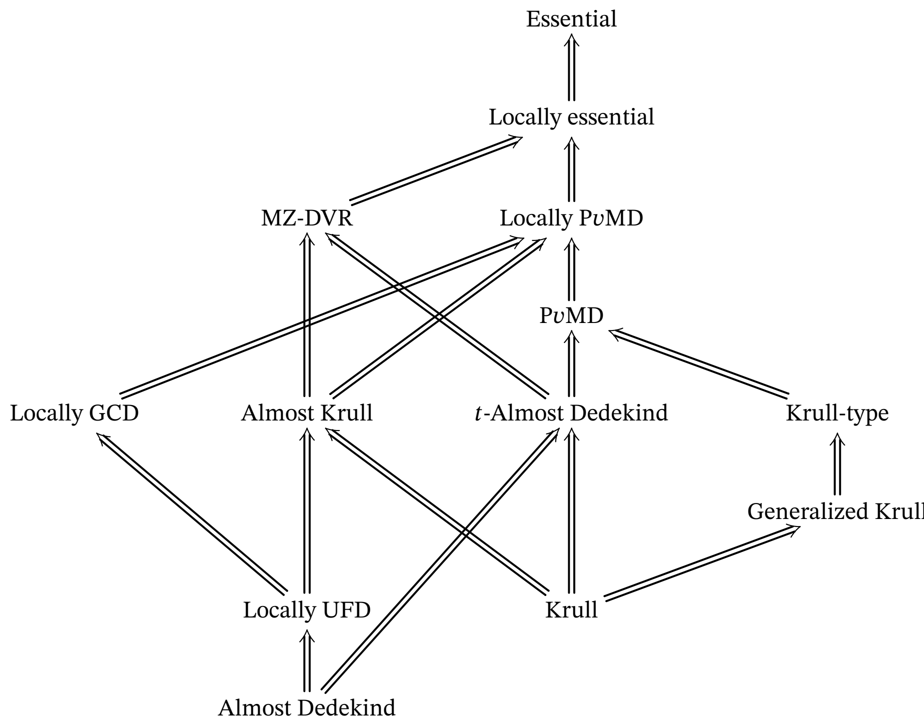

FIGURE 1. Essential properties for domains

Among the properties indicated in this diagram, PvMD is the only one for which, at the moment, there is a non-trivial complete characterization for Int(D) (that is, a characterization for integral domains D such that $D[X] \neq \mathrm{Int}(D)$) given in [8, Theorem 3.4]. In [7, Corollary 2.7] the authors showed that Int(D) is Krull if and only if D is Krull and $\mathrm{Int}(D) = D[X]$. An analogous result has been recently showed in [10] for almost Krull domains (Int(D) is almost Krull if and only if D is almost Krull and $\mathrm{Int}(D) = D[X]$). We will see that the same holds for Krull-type domains (so including generalized Krull domains), almost Krull, t-almost Dedekind and locally UFDs (Corollary 2.36).

In Section 1 we will discuss an example, arising from rings of integer-valued polynomials, of an essential domain that is not a PvMD. We will see, in particular, that this ring is locally PvMD, hence it is a locally essential domain.

In Section 2 we will investigate the transfer to rings of integer-valued polynomials of many of the essential properties described in Figure 1.

1. An example of locally essential ring of integer-valued polynomials

In [22, Example 2.1] J. Mott and M. Zafrullah gave an example of a locally essential domain that is not PvMD. This example involves polynomial rings with many indeterminates.

In this section we show that [8, Example 5.1] is another example of locally essential domain that is not PvMD. This example motivated the following investigation about essential properties of Int(D).

In [8, Example 5.1] it is stated that Int(D) is an essential domain that is not PvMD. While the fact that Int(D) is not PvMD is accurately explained, the essential property for Int(D) is not explicitly proved. We are going to see that Int(D) is locally PvMD, thus it is locally essential and finally it is essential.

We fix some terminology and notation.

Let be given an integral domain D with a prime ideal $\mathfrak{p}$. An ideal $\mathfrak{Q}$ of a ring B between $D[X]$ and $K[X]$ is called *upper to* $\mathfrak{p}$ if $\mathfrak{Q}$ is maximal among the ideals of B that contract to $\mathfrak{p}$ (in this case $\mathfrak{Q}$ is also a prime ideal of B).

A prime ideal $\mathfrak{p}$ of D is called *int prime* if $\mathrm{Int}(D) \nsubseteq D_{\mathfrak{p}}[X]$ and it is called *polynomial prime* if $\mathrm{Int}(D) \subseteq D_{\mathfrak{p}}[X]$ (notice that (0) is a polynomial prime since $\mathrm{Int}(D)_{D\setminus(0)} = K[X]$). If $\mathfrak{p}$ is a polynomial prime we also have that $\mathrm{Int}(D)_{\mathfrak{p}} = D_{\mathfrak{p}}[X]$ (where $\mathrm{Int}(D)_{\mathfrak{p}} := \mathrm{Int}(D)_{D\setminus\mathfrak{p}}$).

Let Δ_0 be the set of int prime ideals of D and Δ_1 be the set of polynomial prime ideals of D. Then $\Delta_0 \cap \Delta_1 = \emptyset$, $\Delta_0 \cup \Delta_1 = \mathrm{Spec}(D)$ and each prime in Δ_0 is maximal (because if a prime $\mathfrak{p}$ has infinite residue field, then it is a polynomial prime by [6, Proposition I.3.4]).

Setting $D_0 := \bigcap_{\mathfrak{p}\in\Delta_0} D_{\mathfrak{p}}$ and $D_1 := \bigcap_{\mathfrak{p}\in\Delta_1} D_{\mathfrak{p}}$, we have that $D = D_0 \cap D_1$. By [8, Lemma 4.1], $\mathrm{Int}(D) = \mathrm{Int}(D_0) \cap D_1[X]$ (where $D_1[X] = \mathrm{Int}(D_1)$), $\mathrm{Int}(D_0)$ is Prüfer, $D_1[X]$ is PvMD.

Let $\mathfrak{P}$ be an associated prime ideal of Int(D) and $\mathfrak{P} \cap D =: \mathfrak{p}$.

If $\mathfrak{p} \in \Delta_0$, then $\mathrm{Int}(D)_{\mathfrak{P}}$ is a valuation domain by [8, Lemma 3.1] (since D is almost Dedekind).

For the case $\mathfrak{p} \in \Delta_1$ we first need to recall the definition of Nagata ring (that we will use also in Section 2). Let X be an indeterminate over an integral domain D. For each polynomial $f \in D[X]$, $c(f)$ is the *content* of f, i.e. the ideal of D generated by the coefficients of f. The set $S := \{f \in D[X];\ c(f) = D\}$ is a multiplicatively closed subset of $D[X]$ and the ring of fractions $D(X) := D[X]_S$ is called the *Nagata ring* of D. It is well known that $D(X) \cap K = D$. In particular, if $\mathfrak{p} \in \mathrm{Spec}(D)$, we have that $D_{\mathfrak{p}}[X]_{\mathfrak{p}D_{\mathfrak{p}}[X]} = D_{\mathfrak{p}}(X)$ which is the Nagata ring of $D_{\mathfrak{p}}$. It is well known that if $D_{\mathfrak{p}}$ is a valuation domain then $D_{\mathfrak{p}}(X)$ is a valuation domain too.

Take now $\mathfrak{p} \in \Delta_1$. Then $\mathfrak{P}\mathrm{Int}(D)_{\mathfrak{p}} = \mathfrak{P}D_{\mathfrak{p}}[X]$ is an associated prime of $D_{\mathfrak{p}}[X]$ by [4, Lemma 1], whence it is t-prime. Since $D_{\mathfrak{p}}$ is integrally closed, by

[18, Lemma 4.5], the t-primes of $D_{\mathfrak{p}}[X]$ are the upper to zero ideals and the extended ideals of t-primes of $D_{\mathfrak{p}}$ (i.e. $\mathfrak{q}[X]$ where $\mathfrak{q}$ is a t-prime of $D_{\mathfrak{p}}$). Thus $\mathfrak{P}D_{\mathfrak{p}}[X] = \mathfrak{p}D_{\mathfrak{p}}[X]$ and $\mathrm{Int}(D)_{\mathfrak{P}} = D_{\mathfrak{p}}(X)$ which is a valuation domain (since $D_{\mathfrak{p}}$ is a valuation domain). It follows that $\mathrm{Int}(D)$ is locally essential.

We can see that it is locally PvMD too.

Take $\mathfrak{P} \in \mathrm{Spec}(\mathrm{Int}(D))$. If $\mathfrak{P} \cap D \in \Delta_0$ then, as above, $\mathrm{Int}(D)_{\mathfrak{P}}$ is a valuation domain, hence a PvMD.

If $\mathfrak{p} := \mathfrak{P} \cap D \in \Delta_1$ then $\mathrm{Int}(D)_{\mathfrak{P}}$ is a localization of $D_{\mathfrak{p}}[X]$ that is a PvMD, so $\mathrm{Int}(D)_{\mathfrak{P}}$ is a PvMD. The thesis follows.

For the sake of clarity, in the following we will refer to this example of integer-valued polynomial ring as *Example CLT*.

2. Essential type properties for Int(D)

In this section we investigate the problem of when $\mathrm{Int}(D)$ has some essential properties among those indicated in Figure 1. In particular we will give a complete characterization of when $\mathrm{Int}(D)$ is locally essential (Theorem 2.9), locally PvMD (Proposition 2.21), locally UFD, locally GCD (Proposition 2.27), Krull-type (Theorem 2.30), generalized Krull (Theorem 2.31), MZ-DVR (Proposition 2.35) and we will give partial results for the essential case (Theorem 2.18).

We will freely use the following fact: if $D = \cap_{\mathfrak{p}\in\mathcal{P}} D_{\mathfrak{p}}$, where $\mathcal{P} \subseteq \mathrm{Spec}(D)$, then $\mathrm{Int}(D) = \cap_{\mathfrak{p}\in\mathcal{P}} \mathrm{Int}(D)_{\mathfrak{p}} = \cap_{\mathfrak{p}\in\mathcal{P}} \mathrm{Int}(D_{\mathfrak{p}})$ ([5, Corollaires (3), page 303]).

We say that an integral domain D has *t-dimension one* if each t-prime ideal of D is height-one; we then write $t\text{-}\dim(D) = 1$. Notice that generalized Krull domains and t-almost Dedekind domains are PvMDs of t-dimension one (see [2, Lemma 2.1(1)] and [20, Theorem 4.5] , respectively), and if $t\text{-}\dim(D) = 1$ then $\mathrm{Ass}(D) = t\text{-}\mathrm{Max}(D) = X^1(D)$.

Proposition 2.1. *Let D be an integral domain that is not a field. Then each of the following statements implies the next:*

(a) $t\text{-}\dim(\mathrm{Int}(D)) = 1$*;*
(b) $t\text{-}\dim(D) = 1$*;*
(c) $D = \cap_{\mathfrak{p}\in X^1(D)} D_{\mathfrak{p}}$ *(equivalently,* $\mathrm{Ass}(D) = X^1(D)$ [15, Exercise 22, page 52]*);*
(d) each int prime ideal of D is height-one.

Proof. (a) $\Rightarrow$ (b) Suppose that $t\text{-}\dim(D) > 1$. Then there exist at least two nonzero t-primes of D such that $(0) \subsetneq \mathfrak{q} \subsetneq \mathfrak{p}$. The ideal $\mathfrak{q}$ is obviously a polynomial prime since it is not maximal. Thus $\mathfrak{q}D_{\mathfrak{q}}[X] \cap \mathrm{Int}(D)$ is a t-prime of $\mathrm{Int}(D)$, because it is the contraction of the t-prime $\mathfrak{q}\mathrm{Int}(D)_{\mathfrak{q}}$ (where $\mathrm{Int}(D)_{\mathfrak{q}} = D_{\mathfrak{q}}[X]$). If $\mathfrak{p}$ is a polynomial prime too, then we argue similarly as done for $\mathfrak{q}$ and we have the chain of t-primes in $\mathrm{Int}(D)$: $(0) \subsetneq \mathfrak{q}D_{\mathfrak{q}}[X] \cap \mathrm{Int}(D) \subsetneq \mathfrak{p}D_{\mathfrak{p}}[X] \cap \mathrm{Int}(D)$. If $\mathfrak{p}$ is an int prime, then $\mathfrak{P}_0 := \{f \in \mathrm{Int}(D); f(0) \in \mathfrak{p}\}$ contains $\mathrm{Int}(D, \mathfrak{p}) = \{f \in \mathrm{Int}(D); f(D) \subseteq \mathfrak{p}\}$ and from [8, Proposition 1.4], $\mathfrak{P}_0$ is an int prime, so it is t-maximal as asserted [11, Proposition 3.3]. Thus, since $t\text{-}\dim(\mathrm{Int}(D)) = 1$, the height of $\mathfrak{P}_0$ is one. On the other hand, we have $\mathfrak{q}D_{\mathfrak{q}}[X] \cap \mathrm{Int}(D) \subseteq \mathrm{Int}(D, \mathfrak{q}) \subsetneq \mathrm{Int}(D, \mathfrak{p}) \subsetneq \mathfrak{P}_0$, whence the chain of t-primes $(0) \subsetneq \mathfrak{q}D_{\mathfrak{q}}[X] \cap \mathrm{Int}(D) \subsetneq \mathfrak{P}_0$ is of length 2, which contradicts the fact that $\mathfrak{P}_0$ is height-one.

(b) $\Rightarrow$ (c) If t-dim$(D) = 1$, then $\mathrm{Ass}(D) = X^1(D)$ and hence it follows from [4, Proposition 4].

(c) $\Rightarrow$ (d) By [15, Exercise 22, page 52], we have the following: $\mathrm{Ass}(D) = X^1(D)$ if and only if $D = \cap_{\mathfrak{p} \in X^1(D)} D_{\mathfrak{p}}$, and so the conclusion follows from the fact that each int prime ideal of D is associated prime (cf. [11, Proposition 3.3]). □

Proposition 2.2. *Let $D = \cap_{\mathfrak{p} \in \mathcal{P}} D_{\mathfrak{p}}$, where $\mathcal{P} \subseteq \mathrm{Spec}(D)$, be an integral domain such that $\cap_{\mathfrak{p} \in \mathcal{P}} D_{\mathfrak{p}}$ is a locally finite intersection. If $\mathfrak{m}$ is an int prime of D then $\mathfrak{m} \in \mathcal{P}$.*

Proof. By way of contradiction assume that $\mathfrak{m} \notin \mathcal{P}$. By the local finiteness of the intersection $\cap_{\mathfrak{p} \in \mathcal{P}} D_{\mathfrak{p}}$ it follows from [27, Lemma 1.5] that $D_{\mathfrak{m}} = \cap_{\mathfrak{p} \in \mathcal{P}} (D_{\mathfrak{p}})_{\mathfrak{m}}$. On the other hand, for each $\mathfrak{p} \in \mathcal{P}$ we have that $\mathfrak{m} \neq \mathfrak{p}$ and then $\mathrm{Int}((D_{\mathfrak{p}})_{\mathfrak{m}}) = (D_{\mathfrak{p}})_{\mathfrak{m}}[X]$. Indeed, in both cases $\mathfrak{p} \subsetneq \mathfrak{m}$ and $\mathfrak{p} \not\subseteq \mathfrak{m}$, we have that $(D_{\mathfrak{p}})_{\mathfrak{m}}$ is an intersection of localizations of D at nonmaximal prime ideals which are polynomial primes. Thus

$$\mathrm{Int}(D_{\mathfrak{m}}) = \bigcap_{\mathfrak{p} \in \mathcal{P}} \mathrm{Int}((D_{\mathfrak{p}})_{\mathfrak{m}}) = \bigcap_{\mathfrak{p} \in \mathcal{P}} (D_{\mathfrak{p}})_{\mathfrak{m}}[X] = D_{\mathfrak{m}}[X].$$

Therefore $\mathrm{Int}(D)_{\mathfrak{m}} = \mathrm{Int}(D_{\mathfrak{m}}) = D_{\mathfrak{m}}[X]$, which contradicts the hypothesis that $\mathfrak{m}$ is an int prime. □

The following result can be found in [19, Lemma 2] but, for the sake of completeness and for a better comprehension of Remark 2.4, we include a detailed proof of it.

Proposition 2.3. *Let D be a locally essential domain. Then, for each int prime ideal $\mathfrak{m}$ of D, the integral domain $D_{\mathfrak{m}}$ is a valuation domain with maximal principal ideal.*

Proof. Let $\mathcal{P}$ be the defining family of D, that is $D = \cap_{\mathfrak{p} \in \mathcal{P}} D_{\mathfrak{p}}$. We first need to show that $D_{\mathfrak{m}} = \cap_{\mathfrak{p} \in \mathcal{P}, \mathfrak{p} \subsetneq \mathfrak{m}} D_{\mathfrak{p}}$, for each maximal ideal $\mathfrak{m}$ of D which is not in $\mathcal{P}$.

The inclusion $\subseteq$ follows immediately (also if D is not locally essential). In fact

$$D_{\mathfrak{m}} = (\cap_{\mathfrak{p} \in \mathcal{P}} D_{\mathfrak{p}})_{\mathfrak{m}} \subseteq \cap_{\mathfrak{p} \in \mathcal{P}} (D_{\mathfrak{p}})_{\mathfrak{m}} = \cap_{\mathfrak{p} \in \mathcal{P}, \mathfrak{p} \subsetneq \mathfrak{m}} D_{\mathfrak{p}}.$$

For the reverse inclusion, since D is locally essential then $D_{\mathfrak{m}}$ is essential, whence $D_{\mathfrak{m}}$ is an intersection of essential valuation overrings of itself and these are also essential valuation overrings of D (because $D_{\mathfrak{m}}$ is a localization of D). The essential valuation overrings of D are of the type $D_{\mathfrak{p}}$, for $\mathfrak{p} \in \mathcal{P}$. Hence $D_{\mathfrak{m}} = \cap_{\mathfrak{p} \in \mathcal{P}'} D_{\mathfrak{p}}$ where $\mathcal{P}' = \{\mathfrak{p} \in \mathcal{P}; \mathfrak{p} \subsetneq \mathfrak{m}\}$. In fact $D_{\mathfrak{p}} \supseteq D_{\mathfrak{m}}$ if and only if $\mathfrak{p} \subseteq \mathfrak{m}$. But $\mathfrak{m} \notin \mathcal{P}$ and so we have that $\mathfrak{p} \subsetneq \mathfrak{m}$.

Now, let $\mathfrak{m}$ be an int prime ideal of D. We claim that $\mathfrak{m} \in \mathcal{P}$. If not, by the previous argument we can write $D_{\mathfrak{m}} = \cap_{\mathfrak{p} \in \mathcal{P}, \mathfrak{p} \subsetneq \mathfrak{m}} D_{\mathfrak{p}}$. Each prime $\mathfrak{p} \in \mathcal{P}$ with $\mathfrak{p} \subsetneq \mathfrak{m}$ is nonmaximal, hence it has infinite residue field and then $\mathrm{Int}(D_{\mathfrak{p}}) =$

$D_{\mathfrak{p}}[X]$ (because $D_{\mathfrak{p}}$ is a local domain with infinite residue field, [6, Corollary I.3.7]). Hence

$$\mathrm{Int}(D_{\mathfrak{m}}) = \cap_{\mathfrak{p}\in\mathcal{P},\mathfrak{p}\subsetneq\mathfrak{m}}\mathrm{Int}(D_{\mathfrak{p}}) = \cap_{\mathfrak{p}\in\mathcal{P},\mathfrak{p}\subsetneq\mathfrak{m}}D_{\mathfrak{p}}[X] = D_{\mathfrak{m}}[X],$$

which is a contradiction. Therefore $\mathfrak{m} \in \mathcal{P}$ and so $D_{\mathfrak{m}}$ is a valuation domain. Since $\mathrm{Int}(D)_{\mathfrak{m}} \neq D_{\mathfrak{m}}[X]$ and $D_{\mathfrak{m}}[X] \subseteq \mathrm{Int}(D)_{\mathfrak{m}} \subseteq \mathrm{Int}(D_{\mathfrak{m}})$, we have that $\mathrm{Int}(D_{\mathfrak{m}}) \neq D_{\mathfrak{m}}[X]$. Hence, by [6, Proposition I.3.16], $\mathfrak{m}D_{\mathfrak{m}}$ is a principal ideal with finite residue field and the proof is complete. □

Remark 2.4. The argument used in the previous proof strongly needs that, for any int prime ideal $\mathfrak{m}$ of D, $D_{\mathfrak{m}}$ is essential and so $D_{\mathfrak{m}} = \cap D_{\mathfrak{p}}$ where this intersection is taken over a subset of the defining family $\mathcal{P}$ of D. This condition may not be in general satisfied for an essential domain that is not locally essential.

Thus, it is an open question whether Proposition 2.3 may work for essential domains using a different proof. This is crucial because the consistence of Proposition 2.3 for essential domains would allow us to complete the partial characterization of when $\mathrm{Int}(D)$ is essential given in Theorem 2.18.

We start with the investigation on the locally essential property.

Lemma 2.5. *Let D be an integral domain. Then D is locally essential if and only if $D[X]$ is locally essential.*

Proof. ($\Rightarrow$) It follows from [22, Corollary 1.2].

($\Leftarrow$) Assume that $D[X]$ is locally essential and let $\mathfrak{p} \in \mathrm{Ass}(D)$. Notice that if $\mathfrak{p}$ is minimal over $(aD :_D bD)$ for some $a, b \in D$ then $\mathfrak{p}D[X]$ is minimal over $(aD : bD)D[X] = (aD[X] : bD[X])$ (this follows from flatness of $D[X]$ over D) and hence it is an associated prime of $D[X]$. Then $D[X]_{\mathfrak{p}D[X]} = D_{\mathfrak{p}}(X)$ is a valuation domain and hence $D_{\mathfrak{p}} = D_{\mathfrak{p}}(X) \cap K$ is also a valuation domain. Therefore D is a locally essential domain. □

Proposition 2.6. *Let D be an integral domain. If* $\mathrm{Int}(D)$ *is a locally essential domain then the following statements hold:*

(a) D is locally essential;
(b) for each int prime ideal $\mathfrak{p}$ of D, $D_{\mathfrak{p}}$ is a DVR with finite residue field.

Proof. (a) Assume that $\mathrm{Int}(D)$ is a locally essential domain and let $\mathfrak{p}$ be a prime ideal of D. Set $\mathfrak{P}_{\mathfrak{p},0} := \{f \in \mathrm{Int}(D);\ f(0) \in \mathfrak{p}\}$. So, we have the following possible cases:

Case 1: $\mathrm{Int}(D)_{\mathfrak{p}} \neq D_{\mathfrak{p}}[X]$; that is, $\mathfrak{p}$ is an int prime of D. Since $\mathfrak{P}_{\mathfrak{p},0}$ contains $\mathrm{Int}(D, \mathfrak{p})$, it follows from [8, Proposition 1.4] that $\mathfrak{P}_{\mathfrak{p},0}$ is an int prime of $\mathrm{Int}(D)$ and then, by [11, Proposition 3.3], $\mathfrak{P}_{\mathfrak{p},0}$ is an associated prime of $\mathrm{Int}(D)$. Hence, $\mathrm{Int}(D)_{\mathfrak{P}_{\mathfrak{p},0}}$ is a valuation domain. We claim that $\mathrm{Int}(D)_{\mathfrak{P}_{\mathfrak{p},0}} \cap K = D_{\mathfrak{p}}$. The inclusion $\supseteq$ is obvious. For the inclusion $\subseteq$, let $\alpha = \frac{f(X)}{g(X)} \in \mathrm{Int}(D)_{\mathfrak{P}_{\mathfrak{p},0}} \cap K$, where $f(X) \in \mathrm{Int}(D)$ and $g(X) \in$

$\mathrm{Int}(D)\setminus\mathfrak{P}_{\mathfrak{p},0}$. Since $g(X)\in \mathrm{Int}(D)\setminus\mathfrak{P}_{\mathfrak{p},0}$ it follows that $g(X)\in\mathrm{Int}(D)\setminus \mathrm{Int}(D,\mathfrak{p})$. Thus there exists $a\in D$ such that $g(a)\in D\setminus\mathfrak{p}$. By evaluating at a, we get that $\alpha=\frac{f(a)}{g(a)}\in D_{\mathfrak{p}}$. Thus $D_{\mathfrak{p}}=\mathrm{Int}(D)_{\mathfrak{P}_{\mathfrak{p},0}}\cap K$ is a valuation domain.

Case 2: $\mathrm{Int}(D)_{\mathfrak{p}}=D_{\mathfrak{p}}[X]$; that is, $\mathfrak{p}$ is a polynomial prime of D. As $\mathrm{Int}(D)$ is locally essential, it follows from [22, Corollary 1.2] that $D_{\mathfrak{p}}[X]$ is locally essential and hence, by Lemma 2.5, so is $D_{\mathfrak{p}}$.

Therefore D is a locally essential domain by [22, Corollary 1.3].

(b) From statement (a) and Proposition 2.3, we deduce that $D_{\mathfrak{p}}$ is a valuation domain with maximal principal ideal, for each int prime $\mathfrak{p}$ of D. On the other hand, inspired from the proof of [8, Proposition 1.7], we prove that each int prime of D is height-one. So, let $\mathfrak{m}$ be an int prime of D and set $\mathfrak{P}_{\mathfrak{m},0}:=\{f\in\mathrm{Int}(D);\ f(0)\in\mathfrak{m}\}$. Then, as showed in **Case 1** above, $\mathrm{Int}(D)_{\mathfrak{P}_{\mathfrak{m},0}}$ is a valuation domain. Assume, by way of contradiction, that $\mathfrak{m}$ is of height at least 2. Then there is some nonzero prime ideal $\mathfrak{p}$ of D contained in $\mathfrak{m}$. Hence, $\mathfrak{P}_{\mathfrak{p},0}:=\{f\in\mathrm{Int}(D);\ f(0)\in\mathfrak{p}\}$ contained in $\mathfrak{P}_{\mathfrak{m},0}$ and thus, $\mathrm{Int}(D)_{\mathfrak{P}_{\mathfrak{p},0}}$ is also a valuation domain (it is an overring of a $\mathrm{Int}(D)_{\mathfrak{P}_{\mathfrak{m},0}}$). Since $\mathrm{Int}(D)\subseteq D_{\mathfrak{p}}[X]$, $\mathrm{Int}(D)_{\mathfrak{P}_{\mathfrak{p},0}}=D[X]_{(\mathfrak{p},X)}$ and then the contradiction follows from the fact that $D[X]_{(\mathfrak{p},X)}$ is never a valuation domain [8, Lemma 1.6]. Thus $\mathfrak{m}$ is height-one and the proof is complete.

□

The D-module $\mathrm{Int}(D)$ is said to be *locally free* if $\mathrm{Int}(D)_{\mathfrak{m}}$ is free as a $D_{\mathfrak{m}}$-module for each maximal ideal $\mathfrak{m}$ of D.

Corollary 2.7. *Let D be an integral domain. If* $\mathrm{Int}(D)$ *is a locally essential domain then it is a locally free D-module.*

Proof. It follows from Proposition 2.6 and [19, Corollary 1(1)]. □

The following lemma is a re-arrangement of [8, Lemma 3.1] for integral domains D such that D is integrally closed and $D_{\mathfrak{p}}$ is a DVR for each int prime ideal $\mathfrak{p}$ of D.

Lemma 2.8. *Let D be an integrally closed domain and suppose that for each int prime ideal $\mathfrak{p}$ of D, $D_{\mathfrak{p}}$ is a DVR. Then for each prime ideal $\mathfrak{P}$ of* $\mathrm{Int}(D)$ *above an int prime of D we have that* $\mathrm{Int}(D)_{\mathfrak{P}}$ *is a valuation domain.*

Proof. The proof replicates exactly the arguments used in [8, Lemma 3.1]. Indeed in [8, Lemma 3.1] the hypothesis is that D is a PvMD, but the properties of D really needed by the arguments of the proof are that D is an integrally closed domain and that $D_{\mathfrak{p}}$ is a DVR for each int prime ideal $\mathfrak{p}$ of D. □

The following result gives a characterization of integral domains D such that $\mathrm{Int}(D)$ is locally essential.

Theorem 2.9. *Let D be an integral domain. Then* $\mathrm{Int}(D)$ *is locally essential if and only if the following conditions hold:*

(a) D is locally essential;
(b) each int prime ideal of D is height-one.

Proof. ($\Rightarrow$) From Proposition 2.6 we have that conditions (a) and (b) are necessary.

($\Leftarrow$) By Proposition 2.3, we have that $D_{\mathfrak{p}}$ is a valuation domain with principal maximal ideal, for each int prime $\mathfrak{p}$ of D. Then, since int primes are supposed to be height-one, we have that $D_{\mathfrak{p}}$ is a DVR. Hence we are in the hypothesis of Lemma 2.8.

Let $\mathfrak{P}$ be an associated prime ideal of $\mathrm{Int}(D)$ and see that $\mathrm{Int}(D)_{\mathfrak{P}}$ is a valuation domain. We set $\mathfrak{p} := \mathfrak{P} \cap D$.

If $\mathfrak{p}$ is an int prime ideal, the conclusion follows from Lemma 2.8.

If $\mathfrak{p}$ is a polynomial prime then $\mathrm{Int}(D)_{\mathfrak{p}} = D_{\mathfrak{p}}[X]$. By [4, Lemma 1] we have that $\mathfrak{P}D_{\mathfrak{p}}[X]$ is an associated prime of $D_{\mathfrak{p}}[X]$ and so it is a t-prime. Since $D_{\mathfrak{p}}$ is integrally closed, the t-primes of $D_{\mathfrak{p}}[X]$ are the uppers to zero and the extended ideals of t-primes of $D_{\mathfrak{p}}$ ([18, Lemma 4.5]). Thus $\mathfrak{P}D_{\mathfrak{p}} = \mathfrak{p}D_{\mathfrak{p}}[X]$ and $\mathfrak{P} = \mathfrak{p}D_{\mathfrak{p}}[X] \cap \mathrm{Int}(D)$. Then $\mathrm{Int}(D)_{\mathfrak{P}} = D_{\mathfrak{p}}[X]_{\mathfrak{p}D_{\mathfrak{p}}[X]} = D_{\mathfrak{p}}(X)$ which is the Nagata ring of $D_{\mathfrak{p}}$. We claim that $D_{\mathfrak{p}}$ is a valuation domain. In fact we have just showed that $\mathfrak{p}D_{\mathfrak{p}}[X] = \mathfrak{P}D_{\mathfrak{p}}$ is an associated prime of $D_{\mathfrak{p}}[X]$ and, by [4, Corollary 8], $\mathfrak{p}D_{\mathfrak{p}}$ is an associated prime of $D_{\mathfrak{p}}$. Moreover, by [4, Lemma 1] $\mathfrak{p}$ is an associated prime of D and so $D_{\mathfrak{p}}$ is a valuation domain. Hence $D_{\mathfrak{p}}(X)$ is a valuation domain and the thesis follows. □

Corollary 2.10. *Let D be an integral domain with* $\mathrm{Ass}(D) = X^1(D)$. *Then* $\mathrm{Int}(D)$ *is locally essential if and only if so is D.*

Proof. By Proposition 2.1, if $\mathrm{Ass}(D) = X^1(D)$ then each int prime ideal of D is height-one. The thesis follows from Theorem 2.9. □

Corollary 2.11. *For any integral domain D that is either t-almost Dedekind or almost Krull,* $\mathrm{Int}(D)$ *is locally essential.*

Proof. If D satisfies the hypothesis, then $D = \cap_{\mathfrak{p} \in X^1(D)} D_{\mathfrak{p}}$. This is immediate if D is t-almost Dedekind and it follows from [24, Proposition 6] if D is almost Krull. Now, by Proposition 2.1(c), $\mathrm{Ass}(D) = X^1(D)$ and the thesis follows from Corollay 2.10 since D is locally essential. □

Example CLT shows that $\mathrm{Int}(D)$ can be locally essential but not PvMD.

Corollary 2.11 allows to construct other examples of locally essential domains that are not PvMD. For example, if D is an almost Krull domain that is not PvMD (for such example see [3, Example, page 52]) then $\mathrm{Int}(D)$ is locally essential but not PvMD.

In the following we investigate relations between the notions "PvMD" and "locally essential" for Int(D) and show that they coincide if D is a Krull-type, strong Mori or a valuation domain.

First let us recall that a *Krull-type domain* is a PvMD of t-finite character, that is, each nonzero non-unit element of D is contained in only finitely many t-maximal ideals ([17, Proposition 4, Theorems 5 and 7]).

Corollary 2.12. *Let D be a Krull-type domain. Then the following statements are equivalent.*

(1) Int(D) *is a PvMD;*
(2) Int(D) *is a locally essential domain;*
(3) *$D_{\mathfrak{p}}$ is a DVR, for each int prime ideal $\mathfrak{p}$ of D;*
(4) Int(D_0) *is a Prüfer domain, where D_0 is the domain constructed in Section* 1.

Proof. (1) ⇒ (2) It is straightforward.
(2) ⇒ (3) It follows from Proposition 2.6.
(3) ⇒ (1) It follows from [28, Theorem 3.2].
(1) ⇔ (4) It follows from [12, Theorem 3.1]. □

We recall that an integral domain D is *strong Mori* if it satisfies the ascending chain condition (a.c.c.) on integral w-ideals (see [30, 31]). Thus, the class of strong Mori domains includes that of Noetherian domains. For one dimensional domains the property of being Noetherian is equivalent to that of being strong Mori but, in general, strong Mori domains may not be Noetherian; to see this, take a field K and the polynomial ring in infinite indeterminates $K[x_1, \dots, x_n, \dots]$ (cf. [23]).

Corollary 2.13. *Let D be a strong Mori domain. Then the following statements are equivalent.*

(1) Int(D) *is a PvMD;*
(2) Int(D) *is a locally essential domain;*
(3) *D is an integrally closed domain (i.e. a Krull domain).*

Proof. (1) ⇒ (2) This is straightforward.
(2) ⇒ (3) If Int(D) is a locally essential domain, then, by Proposition 2.6, so D is and hence it is integrally closed. Then D is Krull because any integrally closed strong Mori domain is Krull ([31, Theorem 2.8]).
(3) ⇒ (1) It follows from Corollary 2.12. □

Corollary 2.14. *Let V be a valuation domain. Then the following statements are equivalent.*

(1) Int(V) *is a PvMD;*
(2) Int(V) *is a locally essential domain;*
(3) Int(V) $= V[X]$ *or V is a DVR with finite residue field. In this last case,* Int(V) *is Prüfer.*

Proof. (1) ⇒ (2) This is straightforward.

(2) ⇒ (3) Assume that $\mathrm{Int}(V)$ is locally essential. If $\mathfrak{m}$ is the maximal ideal of V, we have the following possible cases:

Case 1: $\mathfrak{m}$ is an int prime of V. In this case, it follows from Proposition 2.6 that V is a DVR with finite residue field.

Case 2: $\mathfrak{m}$ is a polynomial prime of V. In this case, $\mathrm{Int}(V) = \mathrm{Int}(V)_{\mathfrak{m}} = V[X]$.

(3) ⇒ (1) If $\mathrm{Int}(V) = V[X]$, then $\mathrm{Int}(V)$ is a PvMD.

If V is a DVR with finite residue field $\mathrm{Int}(V)$ is a Prüfer domain ([6, Theorem VI.1.7]) and hence it is a PvMD. □

Remark 2.15. Let V be a valuation domain such that $\mathrm{Int}(V) \neq V[X]$ (i.e. V has principal maximal ideal and finite residue field by [6, Proposition I.3.16]). Then, we have the two (opposite) following cases:

— If $\dim(V) = 1$, then V is a DVR with finite residue field and hence $\mathrm{Int}(V)$ is a Prüfer domain, so it is locally essential.
— If $\dim(V) \geqslant 2$, then $\mathrm{Int}(V)$ is never a locally essential domain.

Proposition 2.16. *Let D be a locally essential domain. If t-$\dim(\mathrm{Int}(D)) = 1$ then $\mathrm{Int}(D)$ is a PvMD.*

Proof. Assume that t-$\dim(\mathrm{Int}(D)) = 1$. Notice first that the result remains true if D is a field. So, we may assume that D is not a field. Then, by Proposition 2.1, t-$\dim(D) = 1$ and hence each int prime ideal of D is height-one. Thus, it follows from Theorem 2.9 that $\mathrm{Int}(D)$ is locally essential. As t-$\dim(\mathrm{Int}(D)) = 1$ and in this case the notions “locally essential” and “PvMD” coincide, we deduce that $\mathrm{Int}(D)$ is a PvMD. □

Remark 2.17. (a) We observe that under the conditions of Proposition 2.16 $\mathrm{Int}(D)$ is also completely integrally closed. Indeed it is a PvMD of t-dimension one, whence it can be represented as an intersection of one-dimensional valuation domains and any one-dimensional valuation domain is completely integrally closed.

(b) The converse of Proposition 2.16 is not, in general, true. Indeed, $\mathrm{Int}(\mathbb{Z})$ is a two-dimensional Prüfer domain and hence it is of t-dimension two (since all ideals of a Prüfer domain are t-ideals).

(c) Example CLT is an almost Dedekind domain D such that $\mathrm{Int}(D)$ is not a PvMD and, by Proposition 2.16, it is of t-dimension at least two.

We can construct another example of integer-valued polynomial ring of t-dimension greater than one we can consider a valuation domain V of dimension at least two such that $\mathrm{Int}(V) \neq V[X]$. Then, by Remark 2.15, $\mathrm{Int}(V)$ is not locally essential, and hence, by Proposition 2.16 $\mathrm{Int}(V)$ is of t-dimension at least two.

We now consider the more general problem of characterizing integral domains D for which $\mathrm{Int}(D)$ is essential. We give a partial answer to this question.

Theorem 2.18. *Let D be an integral domain such that $D_{\mathfrak{p}}$ is a DVR for each int prime ideal $\mathfrak{p}$ of D. Then* Int(D) *is essential if and only if D is essential.*

Proof. ($\Rightarrow$) Suppose that Int(D) is essential and let $\mathrm{Int}(D) = \cap_{\mathfrak{P}\in\mathcal{P}}\mathrm{Int}(D)_{\mathfrak{P}}$, where $\mathcal{P} \subseteq \mathrm{Spec}(\mathrm{Int}(D))$ and $\mathrm{Int}(D)_{\mathfrak{P}}$ is a valuation domain for each $\mathfrak{P} \in \mathcal{P}$. Obviously $D = \mathrm{Int}(D) \cap K = \cap_{\mathfrak{P}\in\mathcal{P}}(\mathrm{Int}(D)_{\mathfrak{P}} \cap K)$.

If $\mathfrak{p} = \mathfrak{P} \cap D$ is a polynomial prime, then $\mathfrak{P} = \mathfrak{p}D_{\mathfrak{p}}[X] \cap \mathrm{Int}(D)$ or $\mathfrak{P} = \mathfrak{Q}D_{\mathfrak{p}}[X] \cap \mathrm{Int}(D)$, where $\mathfrak{Q}$ is a prime ideal of $D[X]$ upper to $\mathfrak{p}$.

In the first case, $\mathrm{Int}(D)_{\mathfrak{P}} = D_{\mathfrak{p}}[X]_{\mathfrak{p}D_{\mathfrak{p}}[X]} = D_{\mathfrak{p}}(X)$ and this is a valuation domain if and only if $D_{\mathfrak{p}}$ is a valuation domain. Moreover $D_{\mathfrak{p}}(X) \cap K = D_{\mathfrak{p}}$.

In the second case we have the inclusion $\mathrm{Int}(D)_{\mathfrak{P}} \subsetneq \mathrm{Int}(D)_{\mathfrak{p}D_{\mathfrak{p}}[X]\cap\mathrm{Int}(D)} = D_{\mathfrak{p}}(X)$ and so $D_{\mathfrak{p}}(X)$ is a valuation domain. It follows that $D_{\mathfrak{p}}$ is a valuation domain.

If $\mathfrak{p} = \mathfrak{P} \cap D$ is an int prime, then $D_{\mathfrak{p}}$ is a DVR by hypothesis.

Thus D is essential.

($\Leftarrow$) We have that D is essential and

$$\mathrm{Int}(D) = \cap_{\mathfrak{p}\in\mathcal{P}}\mathrm{Int}(D)_{\mathfrak{p}} = (\cap_{(\mathfrak{p}\in\mathcal{P},\,\text{polynomial})}\mathrm{Int}(D)_{\mathfrak{p}}) \cap (\cap_{(\mathfrak{p}\in\mathcal{P},\ \text{int prime})}\mathrm{Int}(D)_{\mathfrak{p}}).$$

Let $\mathfrak{P} \in \mathrm{Spec}(\mathrm{Int}(D))$ and suppose that for $\mathfrak{p} := \mathfrak{P} \cap D$, the integral domain $D_{\mathfrak{p}}$ is valuation (for instance, this holds if $\mathfrak{p} \in \mathcal{P}$).

If $\mathfrak{p}$ is an int prime, then $D_{\mathfrak{p}}$ is a DVR with finite residue field and by Lemma 2.8 $\mathrm{Int}(D)_{\mathfrak{P}}$ is a valuation domain.

If $\mathfrak{p}$ is a polynomial prime, then $\mathrm{Int}(D)_{\mathfrak{p}} = D_{\mathfrak{p}}[X] = D_{\mathfrak{p}}(X) \cap K[X]$. Now $D_{\mathfrak{p}}(X) = \mathrm{Int}(D)_{\mathfrak{p}D_{\mathfrak{p}}[X]\cap\mathrm{Int}(D)}$ is an essential valuation overring of Int(D) and $K[X]$ is obviously an intersection of essential valuations overrings of Int(D) (take the localizations at the uppers to zero).

We also observe that if $\mathfrak{p} \in \mathcal{P}$, each $\mathfrak{q} \subseteq \mathfrak{p}$ is such that $D_{\mathfrak{q}}$ is a valuation domain (because $D_{\mathfrak{p}} \subseteq D_{\mathfrak{q}}$).

Then $\mathrm{Int}(D)_{\mathfrak{p}} = \cap_{(\mathfrak{P}\in\mathrm{Spec}(\mathrm{Int}(D)),\ \mathfrak{P}\cap D\subseteq\mathfrak{p})}\mathrm{Int}(D)_{\mathfrak{P}}$ is an intersection of valuation overrings of Int(D) that are essential. Thus Int(D) is essential. □

In Section 1 we have seen that the ring Int(D) of Example CLT is locally PvMD. We give a characterization for D in order to get that Int(D) is locally PvMD. We recall that locally PvMDs are not necessarily PvMDs.

We need the following two technical lemmas.

Lemma 2.19. *For any integral domain D, the following statements are equivalent:*

(1) D is a locally PvMD;
(2) D_S is a locally PvMD, for any multiplicative set S of D;
(3) $D_{\mathfrak{p}}$ is a locally PvMD, for each prime ideal $\mathfrak{p}$ of D.

Proof. (1) $\Rightarrow$ (2) Let $\mathfrak{q}$ be a prime of D_S. Then there is a prime $\mathfrak{p}$ of D such that $\mathfrak{p}\cap S = \emptyset$ and $\mathfrak{q} = \mathfrak{p}D_S$. Hence, as localization of a PvMD, $(D_S)_{\mathfrak{q}} = (D_S)_{\mathfrak{p}D_S} = D_{\mathfrak{p}}$ is a PvMD [20, Theorem 3.11], and thus D_S is a locally PvMD.

(2) ⇒ (3) It is straightforward.

(3) ⇒ (1) Let $\mathfrak{p}$ be a prime ideal of D. Then, since $D_{\mathfrak{p}}$ is locally PvMD and it is local, $D_{\mathfrak{p}}$ is a PvMD and thus D is a locally PvMD. □

Lemma 2.20. *Let D be an integral domain. Then D is a locally PvMD if and only if* $D[X]$ *is a locally PvMD.*

Proof. (⇒) Assume that D is a locally PvMD. Let $\mathfrak{P}$ be a prime ideal of $D[X]$ and set $\mathfrak{p} := \mathfrak{P} \cap D$. Then, since $\mathfrak{p}$ is a prime ideal of D, $D_{\mathfrak{p}}$ is a PvMD and then so is $D_{\mathfrak{p}}[X]$ (cf. [20, Theorem 3.7]). Thus $D[X]_{\mathfrak{P}} = D_{\mathfrak{p}}[X]_{\mathfrak{P}D_{\mathfrak{p}}[X]}$ is also a PvMD. Therefore $D[X]$ is a locally PvMD.

(⇐) Let $\mathfrak{m}$ be a maximal ideal of D. Then $\mathfrak{m}D[X]$ is a prime ideal of $D[X]$ and hence $D[X]_{\mathfrak{m}D[X]} = D_{\mathfrak{m}}(X)$ is a PvMD. Thus, by [1, Lemma 2.9(1)], $D_{\mathfrak{m}}$ is a PvMD. Therefore D is a locally PvMD. □

Proposition 2.21. *Let D be an integral domain. Then* Int(D) *is locally PvMD if and only if the following conditions hold:*

(a) D is a locally PvMD;

(b) each int prime ideal of D is height-one.

Proof. (⇒) From Proposition 2.6 we have that condition (b) is necessary.

To prove (a), let $\mathfrak{p}$ be a prime ideal of D.

If $\mathfrak{p}$ is an int prime, then $D_{\mathfrak{p}}$ is a DVR (Proposition 2.6) and so it is a PvMD.

If $\mathfrak{p}$ is a polynomial prime, since Int(D) is a locally PvMD, it follows that $\text{Int}(D)_{\mathfrak{p}} = D_{\mathfrak{p}}[X]$ is a locally PvMD (Lemma 2.19) and $D_{\mathfrak{p}}$ is also a locally PvMD (Lemma 2.20).

Therefore D is a locally PvMD by Lemma 2.19.

(⇐) Let $\mathfrak{P}$ be a prime ideal of Int(D) and $\mathfrak{p} := \mathfrak{P} \cap D$. Since D is a locally PvMD, we have that $D_{\mathfrak{p}}$ is PvMD and then $D_{\mathfrak{p}}[X]$ is also PvMD.

If $\mathfrak{p}$ is an int prime ideal, then $\mathfrak{p}$ is height-one and the conclusion follows from Proposition 2.3 and Lemma 2.8.

Suppose now that $\mathfrak{p}$ is a polynomial prime. Then $\mathfrak{P} = \mathfrak{Q}D_{\mathfrak{p}}[X] \cap \text{Int}(D)$, where $\mathfrak{Q}$ is a prime ideal of $D[X]$ above $\mathfrak{p}$. Thus $\text{Int}(D)_{\mathfrak{P}} = (\text{Int}(D)_{\mathfrak{p}})_{\mathfrak{P}} = D_{\mathfrak{p}}[X]_{\mathfrak{Q}D_{\mathfrak{p}}[X]}$ is a PvMD.

Therefore Int(D) is a locally PvMD. □

Corollary 2.22. *For any integral domain D that is either almost Krull or t-almost Dedekind,* Int(D) *is a locally PvMD.*

From Proposition 2.1 and Proposition 2.21, we deduce the following:

Corollary 2.23. *For any integral domain D that is a PvMD of t-dimension one, we have that* Int(D) *is locally PvMD.*

Example 2.24. In [3, Example, page 52], the authors construct an almost Krull domain D that is not a PvMD. It follows from Corollary 2.22 and [28, Proposition 3.1] that Int(D) is locally PvMD but not PvMD.

We recall that an integral domain D is called *GCD domain* if the intersection of two principal ideals of D is principal (this is equivalent to ask that any couple of not both zero elements of D admits GCD). Notice that valuation domains and UFDs are GCD domains and it is well known that GCD domains are PvMDs ([29, Proposition 5.1.30, and Theorems 5.1.20 and 7.6.4(1)]).

In [13] the authors gave necessary and sufficient conditions on D for which Int(D) is a GCD domain. In particular, they showed that for any integral domain D with t-finite character, Int(D) is a GCD domain if and only if D is a GCD domain and $\text{Int}(D) = D[X]$ ([13, Theorem 2.14]). The case in which D has not the t-finite character was left open and so also the question about the existence of a non-trivial integer-valued polynomial ring Int(D) that is GCD. In the following we focus on locally GCD domains, i.e. integral domains that are locally GCD, and show that there exist non-trivial integer-valued polynomial rings Int(D) that are locally GCD (see Remark 2.29).

Lemma 2.25. *For any integral domain D, the following statements are equivalent:*

(1) D is locally GCD;
(2) D_S is locally GCD, for any multiplicative set S of D;
(3) $D_{\mathfrak{p}}$ is locally GCD, for each prime ideal $\mathfrak{p}$ of D.

Proof. (1) $\Rightarrow$ (2) It follows from [9, Corollary 2.2].

(2) $\Rightarrow$ (3) This is straightforward.

(3) $\Rightarrow$ (1) Let $\mathfrak{p}$ be a prime ideal of D. Then, since $D_{\mathfrak{p}}$ is locally GCD and it is local then $D_{\mathfrak{p}}$ is a GCD domain. It follows that D is a locally GCD domain. □

Lemma 2.26 ([9, Lemma 2.4])**.** *Let D be an integral domain. Then D is locally GCD if and only if $D[X]$ is locally GCD.*

Proposition 2.27. *Let D be an integral domain. Then* Int(D) *is locally GCD if and only if the following conditions hold:*

(a) D is locally GCD;
(b) each int prime ideal of D is a height-one prime ideal.

Proof. The proof of this theorem is similar to that of Proposition 2.21.

($\Rightarrow$) From Proposition 2.6 the condition (b) is necessary. For proving (a) take a prime ideal $\mathfrak{p}$ of D.

If $\mathfrak{p}$ is an int prime, then it follows from Proposition 2.6 that $D_{\mathfrak{p}}$ is a DVR and hence it is a GCD domain.

If $\mathfrak{p}$ is polynomial, then, since Int(D) is locally GCD, it follows from Lemma 2.25 that $\text{Int}(D)_{\mathfrak{p}} = D_{\mathfrak{p}}[X]$ is locally GCD and hence, by Lemma 2.26, $D_{\mathfrak{p}}$ is also locally GCD.

Therefore, by Lemma 2.25, D is a locally GCD domain.

($\Leftarrow$) Let $\mathfrak{P}$ be a prime ideal of Int(D) and set $\mathfrak{p} := \mathfrak{P} \cap D$. Since D is locally GCD, we have $D_{\mathfrak{p}}$ is GCD and then $D_{\mathfrak{p}}[X]$ is also GCD [15, Theorem 34.10].

If $\mathfrak{p}$ is an int prime ideal, then $\mathfrak{p}$ is height-one and the conclusion follows from Proposition 2.3 and Lemma 2.8.

Suppose now that $\mathfrak{p}$ is a polynomial prime. Then $\mathfrak{P} = \mathfrak{Q}D_{\mathfrak{p}}[X] \cap \mathrm{Int}(D)$, where $\mathfrak{Q}$ is a prime ideal of $D[X]$ above $\mathfrak{p}$. Thus $\mathrm{Int}(D)_{\mathfrak{P}} = (\mathrm{Int}(D)_{\mathfrak{p}})_{\mathfrak{P}} = D_{\mathfrak{p}}[X]_{\mathfrak{Q}D_{\mathfrak{p}}[X]}$ is GCD since D is locally GCD.

Then $\mathrm{Int}(D)$ is locally GCD. □

Corollary 2.28. *Let D be an integral domain. If D is locally UFD, then* $\mathrm{Int}(D)$ *is locally GCD.*

Proof. Since D is locally UFD, then it is locally GCD. Let $\mathfrak{p}$ be an int prime ideal of D. By Proposition 2.3 $D_{\mathfrak{p}}$ is a valuation domain and by hypothesis it is UFD. Thus $D_{\mathfrak{p}}$ is DVR and $\mathfrak{p}$ is height-one. The thesis follows from Proposition 2.27. □

Remark 2.29. (a) From the previous corollary we deduce that we can construct non-trivial rings $\mathrm{Int}(D)$ that are locally GCD by taking D almost Dedekind such that $\mathrm{Int}(D) \neq D[X]$. For instance, Example CLT verifies this condition.

(b) In [22, Example 2.1] Mott and Zafrullah give an example of a locally essential domain D that is not PvMD. Moreover, as noticed in [9, Remark 2.10(2)], D is a locally UFD and so it follows from Corollary 2.28 and [28, Proposition 3.1] that $\mathrm{Int}(D)$ is a locally GCD domain that is not PvMD.

Among the classes of essential domains considered in this paper, Krull-type and generalized Krull domains have the property of being defined by a locally finite intersection of a family of essential overrings (we have recalled in the introduction that this means that there is a family of prime ideals $\mathcal{P} \subseteq \mathrm{Spec}(D)$ such that $D = \cap_{\mathfrak{p}\in\mathcal{P}} D_{\mathfrak{p}}$ and each nonzero element of D belongs to finitely many ideals of $\mathcal{P}$).

When the family $\mathcal{P}$ in the above definition consists of all maximal ideals (resp. t-maximal ideals) of D, D is said to have the *finite character on maximal (resp., t-maximal) ideals*. We observe that the finite (or t-finite) character for an integral domain D may not transfer to $\mathrm{Int}(D)$. Indeed we recalled in the introduction that $\mathrm{Int}(\mathbb{Z})$ is Prüfer and it is well known that $\mathrm{Int}(\mathbb{Z}) \neq \mathbb{Z}[X]$ (in fact, $\binom{X}{n} \in \mathrm{Int}(\mathbb{Z})\setminus\mathbb{Z}[X]$, for $n \geq 2$). Thus $\mathrm{Int}(\mathbb{Z})$ is PvMD (because it is Prüfer) but not Krull-type (by Theorem 2.30). Then $\mathrm{Int}(\mathbb{Z})$ has not the (t-)finite character while $\mathbb{Z}$ has got it.

In the following we investigate how $\mathrm{Int}(D)$ relates to Krull-type and generalized Krull properties. We start dealing with the question of when $\mathrm{Int}(D)$ is Krull-type.

Theorem 2.30. *Let D be an integral domain. Then* $\mathrm{Int}(D)$ *is Krull-type if and only if* $\mathrm{Int}(D) = D[X]$ *and D is Krull-type.*

Proof. ($\Leftarrow$) If D is Krull-type and $\mathrm{Int}(D) = D[X]$ the conclusion follows from [15, Exercise 1, page 537].

($\Rightarrow$) Suppose that $\mathrm{Int}(D)$ is Krull-type. Then $\mathrm{Int}(D)$ is a PvMD, whence D is a PvMD. We claim that D is Krull-type. Indeed, if $\mathrm{Int}(D) = \cap_{\mathfrak{P}\in\mathcal{P}} \mathrm{Int}(D)_{\mathfrak{P}}$, where $\mathcal{P} \subseteq \mathrm{Spec}(\mathrm{Int}(D))$, is the representation of Int(D) as a Krull-type domain, then $D = \mathrm{Int}(D) \cap K = \cap_{\mathfrak{P}\in\mathcal{P}}(\mathrm{Int}(D)_{\mathfrak{P}} \cap K) \supseteq \cap_{(\mathfrak{p}=\mathfrak{P}\cap D,\mathfrak{P}\in\mathcal{P})} D_{\mathfrak{p}} \supseteq D$ (in fact, $\mathrm{Int}(D)_{\mathfrak{P}} \cap K \supseteq D_{\mathfrak{p}}$). Thus $D = \cap_{(\mathfrak{p}=\mathfrak{P}\cap D,\mathfrak{P}\in\mathcal{P})} D_{\mathfrak{p}}$. Obviously this intersection is locally finite because the intersection $\cap_{\mathfrak{P}\in\mathcal{P}}\mathrm{Int}(D)_{\mathfrak{P}}$ is locally finite. We also observe that each $\mathfrak{P} \in \mathcal{P}$ is a t-prime because Int(D) is a PvMD and $\mathrm{Int}(D)_{\mathfrak{P}}$ is a valuation domain.

If $\mathfrak{p}$ is an int prime, then $D_{\mathfrak{p}}$ is a valuation domain from [8, Corollary 1.3]. If $\mathfrak{p}$ is polynomial, by [8, Lemma 3.3 and Theorem 3.4], the uppers to $\mathfrak{p}$ of Int(D) are not t-primes. Then $\mathfrak{P} = \mathfrak{p}D_{\mathfrak{p}}[X]$ and $\mathrm{Int}(D)_{\mathfrak{P}} = D_{\mathfrak{p}}[X]_{\mathfrak{p}D_{\mathfrak{p}}[X]} = D_{\mathfrak{p}}(X)$ is a valuation domain, whence $D_{\mathfrak{p}}$ is a valuation domain. Thus D is Krull-type.

Now, we see that $\mathrm{Int}(D) = D[X]$. Since Int(D) is a PvMD, for each int prime $\mathfrak{p}$ of D, $D_{\mathfrak{p}}$ is a DVR with finite residue field (Corollary 2.12). We set $\mathcal{P}' := \{\mathfrak{P} \cap D; \mathfrak{P} \in \mathcal{P}\} \subseteq \mathrm{Spec}(D)$. Since D is Krull-type, we have that $\mathrm{Int}(D)_{\mathfrak{p}} = \mathrm{Int}(D_{\mathfrak{p}})$, for each $\mathfrak{p} \in \mathcal{P}'$ ([28, Proposition 2.3]). Moreover, localizations of Krull-type domains are Krull-type by [15, Exercise 1, page 537]. Then, for $\mathfrak{p} \in \mathcal{P}'$, $\mathrm{Int}(D_{\mathfrak{p}})$ is Krull-type. Suppose there exists $\mathfrak{p} \in \mathcal{P}'$ that is int-prime. Then $D_{\mathfrak{p}}$ is a DVR with finite residue field, and so $\mathrm{Int}(D_{\mathfrak{p}})$ is Prüfer and Krull-type. It follows that $\mathrm{Int}(D_{\mathfrak{p}})$ has the finite character on maximal ideals. But this is not true because $\mathfrak{p}$ is obviously contained in any maximal ideal of Int(D) above $\mathfrak{p}$ itself, and these maximal ideals are infinitely many because they are in one-to-one correspondence with the elements of the $\mathfrak{p}$-adic completion $\widehat{D_{\mathfrak{p}}}$ (see [6, Proposition V.2.3]). Thus, each prime ideal in $\mathcal{P}'$ is polynomial and $\mathrm{Int}(D) = \cap_{\mathfrak{p}\in\mathcal{P}'}\mathrm{Int}(D_{\mathfrak{p}}) = \cap_{\mathfrak{p}\in\mathcal{P}'} D_{\mathfrak{p}}[X] = D[X]$. □

In the following proposition we characterize integral domains D such that Int(D) is generalized Krull.

Proposition 2.31. *For any integral domain D, we have that:*

(a) *if D is generalized Krull, then* Int(D) *is a PvMD;*

(b) Int(D) *is generalized Krull if and only if D is generalized Krull and* $\mathrm{Int}(D) = D[X]$.

Proof. (a) If D is generalized Krull, it follows from Proposition 2.2 that each int prime of D is height-one and hence $D_{\mathfrak{p}}$ is a DVR. Moreover, generalized Krull domains are also Krull-type, thus the thesis follows from Corollary 2.12.

(b) ($\Rightarrow$) We first note that the implication is true if D is a field and so we suppose that D is not a field. If Int(D) is generalized Krull then it is Krull-type of t-dimension one. From Theorem 2.30 we have that $\mathrm{Int}(D) = D[X]$ and D is Krull-type. Then, since $t\text{-dim}(\mathrm{Int}(D)) = 1$ by Proposition 2.1 it follows that $t\text{-dim}(D) = 1$ and hence D is generalized Krull.

($\Leftarrow$) This follows from [15, Theorem 43.11(3)].

□

Example 2.32. In [14, Example 1, page 338] R. Gilmer constructs a one-dimensional Prüfer domain D which is not almost Dedekind. Successively, in [25, page 439] E.M. Pirtle observes that this domain is generalized Krull (but not Krull). By Proposition 2.31 Int(D) is a PvMD and it is not Prüfer since D is not almost Dedekind (which is necessary in order to have that Int(D) is Prüfer, by [6, Proposition VI.1.5]).

Finally we turn our attention to MZ-DVRs. They are a subclass of locally essential domains (in particular, they are completely integrally closed) and almost Krull domains are MZ-DVR.

In [10, Theorem 2.6] the authors show that Int(D) is almost Krull if and only if D is almost Krull and Int(D) is trivial. In the following we give a similar result for MZ-DVRs.

The following lemmas can be found in [21].

Lemma 2.33. *For any integral domain D, the following statements are equivalent:*

(1) D is MZ-*DVR;*
(2) D_S is MZ-*DVR, for any multiplicative set S of D;*
(3) $D_{\mathfrak{p}}$ is MZ-*DVR, for each prime ideal $\mathfrak{p}$ of D;*
(4) $D_{\mathfrak{p}}$ is MZ-*DVR, for each t-prime ideal $\mathfrak{p}$ of D.*

Proof. It follows from [21, Propositions 1.1(2) and 1.4]. □

Lemma 2.34. *Let D be an integral domain. Then D is* MZ-*DVR if and only if $D[X]$ is* MZ-*DVR.*

Proof. It follows from [21, Proposition 1.1(2), Theorem 2.1 and Corollary 2.8]. □

Proposition 2.35. *For any integral domain D, we have that:*

(a) if D is MZ-*DVR, then* Int(D) *is a locally essential domain;*
(b) Int(D) *is* MZ-*DVR if and only if D is* MZ-*DVR and* Int(D) $= D[X]$.

Proof. (a) The thesis follows from Theorem 2.9 since any MZ-DVR D is locally essential with Ass(D) $= X^1(D)$.

(b) (⇒) Assume that Int(D) is a MZ-DVR and let $\mathfrak{p}$ be a prime ideal of D.

If $\mathfrak{p}$ is an int prime, we consider $\mathfrak{P}_{\mathfrak{p},0} := \{f \in \mathrm{Int}(D);\ f(0) \in \mathfrak{p}\} \in \mathrm{Spec}(\mathrm{Int}(D))$. Since $\mathfrak{P}_{\mathfrak{p},0}$ contains Int($D, \mathfrak{p}$), then $\mathfrak{P}_{\mathfrak{p},0}$ is an associated prime of Int(D) (for the same argument used in the proof of Proposition 2.6). Hence, $\mathrm{Int}(D)_{\mathfrak{P}_{\mathfrak{p},0}}$ is a DVR. Arguing as in the proof of Proposition 2.6, $D_{\mathfrak{p}} = \mathrm{Int}(D)_{\mathfrak{P}_{\mathfrak{p},0}} \cap K$ is a DVR.

If $\mathfrak{p}$ is a polynomial prime, then $D_{\mathfrak{p}}[X] = \mathrm{Int}(D)_{\mathfrak{p}}$ is MZ-DVR (see Lemma 2.33), so is $D_{\mathfrak{p}}$ (see Lemma 2.34).

Therefore, by Lemma 2.33, D is MZ-DVR.

To show that Int(D) $= D[X]$ it is sufficient to see that D has no int primes. If $\mathfrak{p} \in \mathrm{Spec}(D)$ is an int prime, by [11, Proposition 3.3], $\mathfrak{p}$ is an associated prime of D. Thus, $D_{\mathfrak{p}}$ is a DVR (with finite residue field). Consider the ideal $\mathfrak{P}_{\mathfrak{p},0} := \{f \in \mathrm{Int}(D);\ f(0) \in \mathfrak{p}\}$. Then, as seen above, $\mathfrak{P}_{\mathfrak{p},0}$ is an associated prime of Int(D) and therefore $\mathrm{Int}(D)_{\mathfrak{P}_{\mathfrak{p},0}}$ is a DVR. But $\mathfrak{P}_{\mathfrak{p},0}$ has height at least two because it contains the nonzero prime ideal $\{f \in \mathrm{Int}(D);\ f(0) = 0\}$ and this is a contradiction.

($\Leftarrow$) This follows from Lemma 2.34.

□

We conclude by collecting the various cases in which some of the essential-type properties considered in this paper transfer to Int(D) only when it is trivial.

Corollary 2.36. *Let* ($\mathcal{P}$) *denote one of the following properties for integral domains: almost Krull, t-almost Dedekind, locally UFD, Krull-type, generalized Krull and* MZ-*DVR. Then* Int(D) *has the property* ($\mathcal{P}$) *if and only if D has the same property and* $\mathrm{Int}(D) = D[X]$.

Proof. For Krull-type and MZ-DVR domains the thesis follows from Theorem 2.30 and Proposition 2.35. We notice that almost Krull, t-almost Dedekind and locally UFD domains are MZ-DVRs. Then if Int(D) has one of these properties, it is MZ-DVR and so $\mathrm{Int}(D) = D[X]$. From [24, Theorem 2.11] and [20, Theorem 4.2] we have that D is respectively almost Krull or t-almost Dedekind. Finally, as UFD domains are exactly Krull GCD domais, $D[X]$ is locally UFD if and only if it is both almost Krull and locally GCD, and hence we infer the desired conclusion from Lemma 2.26 and [10, Proposition 1.5]. □

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