

Global obstructions to conformally Einstein metrics in dimension six

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ABSTRACT. We present a global conformal invariant on closed six-manifolds which obstructs the existence of a conformally Einstein metric. We show that this obstruction is nontrivial and, up to multiplication by a constant, is the unique such invariant. This also gives rise to a (possibly trivial) diffeomorphism invariant which obstructs the existence of an Einstein metric. We also discuss global conformal invariants which obstruct the existence of a conformally Einstein metric on closed six-manifolds with infinite fundamental group.

CONTENTS

1. Introduction	1085
2. Global conformal invariants in dimension six	1089
3. Some computations on locally symmetric spaces	1091
4. Proofs of main results	1093
5. Some examples	1095
Acknowledgments	1096
References	1097

1. Introduction

A fundamental problem in Riemannian geometry is to determine whether a given closed manifold admits an **Einstein metric**; i.e. a Riemannian metric g for which the Ricci curvature satisfies $\text{Ric}_g = \lambda g$ for some constant $\lambda \in \mathbb{R}$. Much is known about this problem in dimension at most four: Every closed surface admits an Einstein metric, while there are topological obstructions to the existence of an Einstein metric on a closed three- or four-manifold [4]. There are no known obstructions in higher dimension, and indeed there are many constructions of Einstein metrics in higher dimensions [4, 5, 6, 9, 10, 11, 23, 24].

A natural approach to finding obstructions, if they exist, is to first seek global obstructions to the existence of an Einstein metric in a given conformal class. One reason for this is that the space of **global conformal invariants** — i.e.

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conformally invariant integrals of complete contractions of the metric, its inverse, the Riemannian volume form, and covariant derivatives of the Riemann curvature tensor — is classified [1]. Another reason is that the Berger [4, Paragraph 6.32] and Hitchin–Thorpe [18, 22] inequalities are quickly derived by considering global conformal invariants in dimension four:

The space of global conformal invariants of a closed, oriented Riemannian four-manifold is three-dimensional and spanned by

$$\int_M Q_4^g \, d\text{vol}_g \quad \text{and} \quad \int_M |W_g^\pm|_g^2 \, d\text{vol}_g,$$

where $Q_4 := -\frac{1}{6}\Delta R + \frac{1}{24}R^2 - \frac{1}{2}|E|^2$ is the fourth-order Q -curvature [7], $E := \text{Ric} - \frac{R}{4}g$ is the trace-free part of the Ricci tensor, and W^+ (resp. W^-) is the self-dual (resp. anti-self-dual) part of the Weyl tensor. Each of $\int Q_4$, $\int |W^+|^2$, and $\int |W^-|^2$ is nonnegative if (M^4, g) is Einstein. In particular, if the diffeomorphism invariant $\mathcal{Q}(M) := \sup_g \int Q_4^g \, d\text{vol}_g$ is negative, then M does not admit an Einstein metric. Moreover, the Gauss–Bonnet–Chern and Hirzebruch signature formulas,

$$\begin{aligned} 8\pi^2\chi(M) &= \frac{1}{4} \int |W_g^+|_g^2 \, d\text{vol}_g + \frac{1}{4} \int |W_g^-|_g^2 \, d\text{vol}_g + \int Q_4^g \, d\text{vol}_g, \\ 12\pi^2\tau(M) &= \frac{1}{4} \int |W_g^+|_g^2 \, d\text{vol}_g - \frac{1}{4} \int |W_g^-|_g^2 \, d\text{vol}_g, \end{aligned}$$

respectively, imply that $\mathcal{Q}(M) \leq 8\pi^2\chi(M)$ and $\mathcal{Q}(M) \leq 8\pi^2(\chi(M) \pm \frac{3}{2}\tau(M))$. The first estimate yields Berger’s inequality: If $\chi(M) \leq 0$, then M admits an Einstein metric if and only if it is, up to taking a finite cover, a four-torus. The second estimate yields the Hitchin–Thorpe inequality: If $\chi(M) \leq \frac{3}{2}|\tau(M)|$, then M admits an Einstein metric if and only if it is, up to taking a finite cover and reversing the orientation, a four-torus or a K3 surface. Note that there are infinitely many closed, simply-connected four-manifolds with $\chi(M) > \frac{3}{2}|\tau(M)|$ which do not admit Einstein metrics [19].

We now specialize to closed six-manifolds. In this case, the space of global conformal invariants is four-dimensional [1, 2, 3, 8, 13, 14] and spanned by the total integrals of

$$\begin{aligned} Q_6 &:= -\Delta^2 J + 4\Delta J^2 - 8\delta(P(\nabla J)) + 4\Delta|P|^2 \\ &\quad - 8J^3 + 24J|P|^2 - 16 \text{tr} P^3 - 8\langle B, P \rangle, \end{aligned} \tag{1.1a}$$

$$\begin{aligned} L_1 &:= 2\Delta|W|^2 - 48\nabla^l(W_{ijkl}C^{ijk}) \\ &\quad + |\nabla W|^2 + 48P_t^s W_{sijk} W^{tijk} - 8J|W|^2 - 64|C|^2, \end{aligned} \tag{1.1b}$$

$$L_2 := \frac{1}{2}\Delta|W|^2 - 8\nabla^l(W_{ijkl}C^{ijk}) + 8P_t^s W_{sijk} W^{tijk} - 2J|W|^2 - 8|C|^2, \tag{1.1c}$$

$$L_3 := W_{ij}{}^{kl}W_{kl}{}^{st}W_{st}{}^{ij}, \tag{1.1d}$$

where P_{ij} is the Schouten tensor, J is its trace, C_{ijk} is the Cotton tensor, W_{ijkl} is the Weyl tensor, and B_{ij} is the Bach tensor; see Section 2 for their definitions. Note that Q_6 is (the negative of) the critical **Q-curvature** [7] in dimension six. We denote by $\mathcal{Q}(M, g) := \int Q^g \, \text{dvol}_g$ and $\mathcal{L}_j(M, g) := \int L_j^g \, \text{dvol}_g$, $j \in \{1, 2, 3\}$, the corresponding global conformal invariants.

We have chosen the basis (1.1) to illustrate the signs taken by evaluating global conformal invariants on conformal classes of closed, Einstein manifolds of dimension 6. For example, Equation (1.1b) implies that the negativity of $\mathcal{L}_1(M, g)$ obstructs the existence of an Einstein metric conformal to g . Indeed, this is the only such obstruction, up to multiplication by a positive constant.

Theorem 1.1. *Let (M^6, g) be a closed Riemannian 6-manifold. Then*

$$\mathcal{L}_1(M, g) = \int_M \left(|\nabla W|^2 + 12E_s^t W_{tijk} W^{sjk} - 64|C|^2 \right) \text{dvol} \tag{1.2}$$

is conformally invariant, where $E := \text{Ric} - \frac{R}{6}g$ is the trace-free part of the Ricci tensor. If $\hat{g} := e^{2u}g$ is Einstein, then $\mathcal{L}_1(M, g) \geq 0$ with equality if and only if \hat{g} is locally symmetric. Moreover, if J is a global conformal invariant which is nonnegative at every Einstein metric, then $J = a\mathcal{L}_1$ for some constant $a \geq 0$.

The proof of Theorem 1.1 is relatively simple. We have already noted that (1.2) is a global conformal invariant which is nonnegative at conformally Einstein metrics. The uniqueness of \mathcal{L}_1 follows by considering simple examples; see Section 4 for details.

Importantly, \mathcal{L}_1 is a *nontrivial* obstruction to the existence of an Einstein metric in a given conformal class.

Proposition 1.2. *Let $(T^2 \times S^4, g := dx^2 + d\theta^2)$ be the Riemannian product of a flat two-torus and a round four-sphere of constant sectional curvature one. Then*

$$\mathcal{L}_1(T^2 \times S^4, g) = -96\pi^2 \text{Vol}_{dx^2}(T^2).$$

In particular, g is not conformally Einstein.

Proposition 1.2 follows by direct computation; see Proposition 5.2 for a more general statement. Work of Gover and Nurowski [15] implies that g is not even locally conformally Einstein.

Theorem 1.1 motivates the introduction of the diffeomorphism invariant

$$\mathcal{L}_1(M^6) := \sup \{ \mathcal{L}_1(M, g) : g \in \text{Met}(M) \},$$

where $\text{Met}(M)$ denotes the space of Riemannian metrics on a given closed six-manifold M . This invariant gives a topological obstruction to the existence of an Einstein metric.

Theorem 1.3. *Let M^6 be a closed six-manifold with $\mathcal{L}_1(M) \leq 0$. If $\mathcal{L}_1(M) = 0$, assume additionally that $|\pi_1(M)| = \infty$ and the universal cover of M is not diffeomorphic to \mathbb{R}^6 . Then M does not admit an Einstein metric.*

We are not presently aware of a closed six-manifold M for which $\mathcal{L}_1(M)$ is finite, let alone nonpositive.

The fact that the product metric on $S^1 \times S^5$ is locally conformally flat implies that $\mathcal{L}_1(S^1 \times S^5) \geq 0$. A scaling argument and Proposition 1.2 together imply that $\mathcal{L}_1(T^2 \times S^4) \geq 0$. If equality holds in either case, then Theorem 1.3 implies that the relevant manifold does not admit an Einstein metric. Note that, by Berger's inequality, neither $S^1 \times S^3$ nor $T^2 \times S^2$ admits an Einstein metric.

The global conformal invariant $\mathcal{L}_1(M, g)$ depends on the choice of Einstein metric on a given closed six-manifold, if one exists. Additionally, Einstein metrics are not generally critical points of the functional $g \mapsto \mathcal{L}_1(M, g)$. See Section 5 for details. These observations limit the applicability of \mathcal{L}_1 to other questions involving Einstein manifolds.

The total integrals of (1.1) give additional obstructions to the existence of an Einstein metric with nonpositive scalar curvature in a given conformal class.

Theorem 1.4. *Let (M^6, g) be a closed Riemannian six-manifold. Then*

$$\begin{aligned} \mathcal{Q}(M, g) &:= \int_M \left(-\frac{1}{225}R^3 + \frac{1}{10}R|E|^2 - \frac{1}{4}\operatorname{tr} E^3 - 2\langle B, E \rangle \right) \operatorname{dvol}_g, \\ \mathcal{L}_1(M, g) &:= \int_M \left(|\nabla W|^2 + 12E_s^t W_{tijk} W^{sjk} - 64|C|^2 \right) \operatorname{dvol}_g, \\ \mathcal{L}_2(M, g) &:= \int_M \left(2E_s^t W_{tijk} W^{sjk} - \frac{1}{15}R|W|^2 - 8|C|^2 \right) \operatorname{dvol}_g \end{aligned} \quad (1.3)$$

are conformally invariant. Suppose that $\hat{g} := e^{2u}g$ is an Einstein metric with nonpositive curvature. Then

- (i) $\mathcal{Q}(M, g) \geq 0$ with equality if and only if $\mathcal{Y}(M, g) = 0$;
- (ii) $\mathcal{L}_1(M, g) \geq 0$ with equality if and only if \hat{g} is locally symmetric; and
- (iii) $\mathcal{L}_2(M, g) \geq 0$ with equality if and only if $\mathcal{Y}(M, g) = 0$ or g is locally conformally flat.

Here $\mathcal{Y}(M, g)$ is the **Yamabe constant** [20]; i.e. the infimum of the total scalar curvature over all unit-volume metrics conformal to g .

Any convex linear combination of $\mathcal{Q}(M, g)$, $\mathcal{L}_1(M, g)$, and $\mathcal{L}_2(M, g)$ is also nonnegative if (M, g) is conformal to an Einstein manifold with nonpositive scalar curvature. We do not presently know whether these are the only such invariants.

The motivation for Theorem 1.4 is that, by Myers' Theorem [21], the global conformal invariants \mathcal{Q} , \mathcal{L}_1 , and \mathcal{L}_2 obstruct the existence of an Einstein metric on a closed conformal six-manifold with infinite fundamental group.

Corollary 1.5. *Let (M^6, g) be a closed Einstein six-manifold with $|\pi_1(M)| = \infty$. Then*

- (i) $\mathcal{Q}(M, g) \geq 0$ with equality if and only if $\mathcal{Y}(M, g) = 0$;
- (ii) $\mathcal{L}_1(M, g) \geq 0$ with equality if and only if g is locally symmetric; and

(iii) $\mathcal{L}_2(M, g) \geq 0$ with equality if and only if $\mathcal{Y}(M, g) = 0$ or g is locally conformally flat.

Theorem 1.4 and Corollary 1.5 are simple consequences of the formulas (1.1). In Section 5 we compute the invariants \mathcal{Q} , \mathcal{L}_1 , and \mathcal{L}_2 on Riemannian products of spaceforms to show that they all give nontrivial obstructions.

Theorem 1.4 motivates the introduction of the diffeomorphism invariants

$$\begin{aligned} \mathcal{Q}^{\leq 0}(M) &:= \sup \{ \mathcal{Q}(M, g) : g \in \text{Met}(M), \mathcal{Y}(M, g) \leq 0 \}, \\ \mathcal{L}_1^{\leq 0}(M) &:= \sup \{ \mathcal{L}_1(M, g) : g \in \text{Met}(M), \mathcal{Y}(M, g) \leq 0 \}, \\ \mathcal{L}_2^{\leq 0}(M) &:= \sup \{ \mathcal{L}_2(M, g) : g \in \text{Met}(M), \mathcal{Y}(M, g) \leq 0 \}. \end{aligned}$$

One can formulate an analogue of Theorem 1.3 for the invariants $\mathcal{Q}^{\leq 0}(M)$, $\mathcal{L}_1^{\leq 0}(M)$, and $\mathcal{L}_2^{\leq 0}(M)$ on a closed six-manifold with infinite fundamental group. We presently cannot show that any of these invariants is finite, let alone non-positive.

Unlike the situation in dimension four, the space of global conformal invariants in dimension six does *not* admit a basis of invariants which are nonnegative at Einstein six-manifolds with infinite fundamental group, let alone Einstein six-manifolds in general. However, the Gauss–Bonnet–Chern Theorem [12] implies that any global conformal invariant is a linear combination of the Euler characteristic and the three linearly independent invariants of Theorem 1.4.

Theorem 1.6. *Let (M^6, g) be a closed six-manifold. Then*

$$64\pi^3 \chi(M) = -\mathcal{Q}(M, g) - \frac{1}{3} \mathcal{L}_1(M, g) + \frac{7}{6} \mathcal{L}_2(M, g) + \mathcal{L}_3(M, g).$$

Theorem 1.6 follows readily from a known expression [16] of the Gauss–Bonnet–Chern formula in terms of global conformal invariants. It is presently unclear whether one can combine Corollary 1.5 and Theorem 1.6 to find a closed six-manifold which does not admit an Einstein metric.

This article is organized as follows.

In Section 2 we give a detailed account of the global conformal invariants determined by (1.1). We also prove Theorem 1.6.

In Section 3 we compute the scalar invariants (1.1) at products of spaceforms.

In Section 4 we prove Theorems 1.1, 1.3 and 1.4 and Corollary 1.5.

In Section 5 we discuss some examples, and in particular prove Proposition 1.2.

2. Global conformal invariants in dimension six

Let (M^n, g) be a Riemannian n -manifold. The **Schouten tensor** P_{ij} , the **Weyl tensor** W_{ijkl} , the **Cotton tensor** C_{ijk} , and the **Bach tensor** B_{ij} are

$$\begin{aligned} P_{ij} &:= \frac{1}{n-2} (R_{ij} - Jg_{ij}), \\ W_{ijkl} &:= R_{ijkl} - P_{ik}g_{jl} - P_{jl}g_{ik} + P_{il}g_{jk} + P_{jk}g_{il}, \end{aligned}$$

$$\begin{aligned} C_{ijk} &:= \nabla_i P_{jk} - \nabla_j P_{ik}, \\ B_{ij} &:= \nabla^s C_{sij} + W_{isjt} P^{st}, \end{aligned}$$

respectively, where R_{ij} is the Ricci tensor, R_{ijkl} is the Riemann curvature tensor, and $J := P_s^s$. We use Penrose's abstract index notation throughout this article.

The importance of these tensors stems from their symmetries and conformal transformation properties. The Weyl tensor is conformally invariant, and hence the Schouten tensor determines the behavior of the Riemann curvature tensor under conformal transformation. Under conformal change of metric, the Cotton and Bach tensors depend only on the one-jet of the conformal factor [13]. The relevant algebraic symmetries of these tensors are

$$\begin{aligned} P_{[ij]} &= 0, \\ W_{ijkl} &= W_{[ij][kl]} = W_{[kl][ij]}, & W_{[ijk]l} &= 0, \\ C_{ijk} &= C_{[ij]k}, & C_{[ijk]} &= 0, \\ B_{[ij]} &= 0, \end{aligned}$$

where square brackets indicate skew symmetrization. All traces of these tensors can be computed from these symmetries and the identities

$$J = P_s^s, \quad W_{isj}^s = 0, \quad C_{si}^s = 0.$$

We also require the consequence

$$\nabla_{[i} W_{jk]}^{lm} = -2C_{[ij}^{[l} g_k]^{m]} \quad (2.1)$$

of the second Bianchi identity.

We now verify that the total integrals of (1.1) form a basis for the space of global conformal invariants in dimension six.

Proposition 2.1. *The space of global conformal invariants is four-dimensional and spanned by the total integrals of (1.1).*

Proof. This follows from the following three observations. First, any global conformal invariant is a linear combination of $\int Q^g \, \text{dvol}_g$ and an integral of a local conformal invariant [1]. Second, in dimension six,

$$\begin{aligned} I_1 &:= |\nabla W|^2 - 16\nabla^l(W_{ijkl}C^{ijk}) + 16P_t^s W_{sijk} W^{tijk} - 32|C|^2, \\ I_2 &:= W_i^k j^l W_k^s l^t W_s^i t^j, \\ I_3 &:= W_{ij}^{kl} W_{kl}^{st} W_{st}^{ij}. \end{aligned}$$

give a basis for the space of local conformal invariants [2, 3, 13, 14]. Third, (2.1) implies that, in dimension six,

$$\begin{aligned} \frac{1}{2}\Delta|W|^2 &= |\nabla W|^2 - 8\nabla^l(W_{ijkl}C^{ijk}) + 2J|W|^2 + 8P_t^s W_{sijk} W^{tijk} \\ &\quad - 24|C|^2 - 4I_2 - I_3. \quad \square \end{aligned}$$

We conclude with the Gauss–Bonnet–Chern theorem in dimension six.

Proof of Theorem 1.6. It is known [16, pg. 38] that

$$64\pi^3 \chi(M) = \int_M \left(-Q_6^g - \frac{1}{2}I_1 + \frac{2}{3}I_2 + \frac{7}{6}I_3 \right) \text{dvol}_g.$$

The conclusion readily follows. □

3. Some computations on locally symmetric spaces

In this section we collect some basic computations on locally symmetric spaces, and especially products of spaceforms.

We begin by discussing contractions of certain algebraic curvature operators. To that end, given elements G_{ij} and H_{ij} of $S^2T_p^*M$, denote by

$$\begin{aligned} (G \otimes H)_{ijkl} &:= G_{ik}H_{jl} + G_{jl}H_{ik} - G_{il}H_{jk} - G_{jk}H_{il}, \\ (G \circ H)_{ij} &:= G_i^u H_{uj}, \end{aligned}$$

the Kulkarni–Nomizu product and composition with respect to the identification $S^2T_p^*M \cong \text{End}(T_pM)$, respectively. Given elements U_{ijkl} and V_{ijkl} of $S^2\Lambda^2T_p^*M$ — so that $U_{ijkl} = U_{[ij][kl]} = U_{[kl][ij]}$ — denote by

$$(U \circ V)_{ijkl} := \frac{1}{2} U_{ij}^{uv} V_{uvkl}$$

the composition with respect to the identification $S^2\Lambda^2T_p^*M \cong \text{End}(\Lambda^2T_p^*M)$. We have the following identities for compositions of certain Kulkarni–Nomizu products.

Lemma 3.1. *Let (M^n, g) be a Riemannian manifold and let $G, H \in S^2T_p^*M$ be such that $G \circ H = 0$. Then*

$$\begin{aligned} (G \otimes G) \circ (G \otimes G) &= 2G^2 \otimes G^2, \\ (G \otimes G) \circ (G \otimes H) &= 0, \\ (G \otimes G) \circ (H \otimes H) &= 0, \\ (G \otimes H) \circ (G \otimes H) &= G^2 \otimes H^2. \end{aligned}$$

Proof. Given $G_1, G_2, G_3, G_4 \in S^2T_p^*M$, it is straightforward to check that

$$(G_1 \otimes G_2) \circ (G_3 \otimes G_4) = (G_1 \circ G_3) \otimes (G_2 \circ G_4) + (G_1 \circ G_4) \otimes (G_2 \circ G_3).$$

The conclusion readily follows. □

Define the partial trace $\text{tr} : S^2\Lambda^2T_p^*M \rightarrow S^2T_p^*M$ by

$$(\text{tr } U)_{ij} := U_{iuj}^u.$$

We have the following identities for the trace of the Kulkarni–Nomizu products of elements of $S^2T_p^*M$.

Lemma 3.2. *Let (M^n, g) be a Riemannian manifold and let $G, H \in S^2T_p^*M$ be such that $G \circ H = 0$. Then*

$$\text{tr}(G \otimes G) = 2(\text{tr } G)G - 2G^2,$$

$$\text{tr}(G \otimes H) = (\text{tr } G)H + (\text{tr } H)G.$$

Proof. Given $G, H \in S^2T_p^*M$, we compute that

$$\text{tr}(G \otimes H) = (\text{tr } G)H + (\text{tr } H)G - G \circ H - H \circ G.$$

The conclusion readily follows. □

Lemmas 3.1 and 3.2 give simple formulas for the curvature of a Riemannian product of spaceforms.

Lemma 3.3. *Let (M^m, g) and (N^n, h) be Riemannian spaceforms of constant sectional curvature μ and ν , respectively. Set $(X^{m+n}, \bar{g}) := (M \times N, g + h)$. Then*

$$R_{\bar{g}} = m(m - 1)\mu + n(n - 1)\nu, \tag{3.1}$$

$$E_{\bar{g}} = \frac{(m - 1)\mu - (n - 1)\nu}{m + n} (ng - mh), \tag{3.2}$$

$$W_{\bar{g}} = \frac{\mu + \nu}{2(m + n - 1)(m + n - 2)} (n(n - 1)g \otimes g - 2(n - 1)(m - 1)g \otimes h + m(m - 1)h \otimes h), \tag{3.3}$$

where $E_{\bar{g}} := \text{Ric}_{\bar{g}} - \frac{R_{\bar{g}}}{m+n}\bar{g}$ is the trace-free part of the Ricci tensor of \bar{g} .

Proof. Observe that $\text{Ric}_{\bar{g}} = (m - 1)\mu g + (n - 1)\nu h$. This readily yields (3.1) and (3.2). Observe also that

$$W_{\bar{g}} = \frac{\mu}{2}g \otimes g + \frac{\nu}{2}h \otimes h - \frac{1}{m + n - 2}E_{\bar{g}} \otimes \bar{g} - \frac{1}{2(m + n)(m + n - 1)}R_{\bar{g}}\bar{g} \otimes \bar{g}.$$

Combining this with (3.1) and (3.2) yields (3.3). □

Recall that a Riemannian manifold (M^n, g) is **locally symmetric** if its Riemann curvature tensor is parallel. In this case, the local invariants (1.1) simplify.

Lemma 3.4. *Let (M^6, g) be locally symmetric. Then*

$$Q = -\frac{1}{225}R^3 + \frac{1}{10}R|E|^2 - \frac{1}{4}\text{tr } E^3 - \frac{1}{2}W_{ijkl}E^{ik}E^{jl},$$

$$L_1 = 24\langle E, \text{tr } W^2 \rangle,$$

$$L_2 = 4\langle E, \text{tr } W^2 \rangle - \frac{2}{15}R \text{tr}^2 W^2,$$

$$L_3 = 4 \text{tr}^2 W^3,$$

where $W^2 := W \circ W$ and $W^3 := W \circ W \circ W$.

Proof. The assumption that (M^6, g) is locally symmetric implies that $\nabla W = 0$ and $\nabla P = 0$. Hence $B_{ij} = \frac{1}{4}W_{isjt}E^{st}$. The conclusion readily follows. □

Specializing Lemma 3.4 to the Riemannian product of two three-dimensional spaceforms and to the Riemannian product of a two- and a four-dimensional spaceform yields enough examples for the results of this article.

Proposition 3.5. *Let (M^3, g) and (N^3, h) be closed three-manifolds with constant sectional curvature μ and ν , respectively. Then $(\overline{M}^6, \overline{g}) := (M \times N, g + h)$ satisfies*

$$\begin{aligned} Q &= -\frac{24}{25}(\mu + \nu)^3, \\ L_1 &= 0, \\ L_2 &= -\frac{36}{25}(\mu + \nu)^3, \\ L_3 &= \frac{18}{25}(\mu + \nu)^3. \end{aligned}$$

Proof. Lemma 3.3 implies that

$$\begin{aligned} R_{\overline{g}} &= 6(\mu + \nu), \\ E_{\overline{g}} &= (\mu - \nu)(g - h), \\ W_{\overline{g}} &= \frac{\mu + \nu}{20} (3g \otimes g - 4g \otimes h + 3h \otimes h). \end{aligned}$$

The conclusion follows from Lemmas 3.1, 3.2 and 3.4. □

Proposition 3.6. *Let (M^2, g) be a closed surface with constant sectional curvature μ and let (N^4, h) be a closed four-manifold with constant sectional curvature ν . Then $(\overline{M}^6, \overline{g}) := (M \times N, g + h)$ satisfies*

$$\begin{aligned} Q &= -\frac{12}{25}(\mu^3 - 7\mu^2\nu + 33\mu\nu^2 - 9\nu^3), \\ L_1 &= 12(\mu + \nu)^2(\mu - 3\nu), \\ L_2 &= \frac{6}{25}(\mu + \nu)^2(7\mu - 33\nu), \\ L_3 &= \frac{39}{25}(\mu + \nu)^3. \end{aligned}$$

Proof. Lemma 3.3 implies that

$$\begin{aligned} R_{\overline{g}} &= 2(\mu + 6\nu), \\ E_{\overline{g}} &= \frac{\mu - 3\nu}{3} (2g - h), \\ W_{\overline{g}} &= \frac{\mu + \nu}{20} (6g \otimes g - 3g \otimes h + h \otimes h). \end{aligned}$$

The conclusion follows from Lemmas 3.1, 3.2 and 3.4. □

4. Proofs of main results

In this section we prove our main results, Theorems 1.1 and 1.4 and Corollary 1.5.

First we consider the obstruction for a conformally Einstein metric.

Proof of Theorem 1.1. Proposition 2.1 implies that $\mathcal{L}_1(M, g)$ is conformally invariant. Integrating (1.1b) yields (1.2).

Let g be Einstein. Equation (1.2) implies that

$$\mathcal{L}_1(M, g) = \int_M |\nabla W|^2 \, d\text{vol}.$$

Thus $\mathcal{L}_1(M, g) \geq 0$ with equality if and only if $\nabla W = 0$. Since $\nabla W = \nabla \text{Rm}$, we see that equality holds if and only if g is locally symmetric.

Suppose now that \mathcal{J} is a global conformal invariant which is nonnegative at every Einstein metric. Proposition 2.1 implies that there are constants $a, b, c, e \in \mathbb{R}$ such that

$$\mathcal{J}(M, g) = a\mathcal{L}_1(M, g) + \int_M (bQ_6^g + cL_2^g + eL_3^g) \, d\text{vol}_g. \quad (4.1)$$

We prove that $b = c = e = 0$. The conclusion then follows from the existence [4] of closed Einstein manifolds which are not locally symmetric.

First, let (M^6, g) be a closed spaceform with constant sectional curvature λ . Lemma 3.4 implies that $\mathcal{J}(M, g) = -120b\lambda^3 \text{Vol}_g(M)$. Since λ is arbitrary, $b = 0$.

Second, let (M^3, g) and (N^3, h) be closed spaceforms with constant sectional curvature λ . Lemma 3.3 and Proposition 3.5 imply that $(M \times N, g + h)$ is Einstein with

$$\begin{aligned} L_1 &= 0, \\ L_2 &= -\frac{288}{25}\lambda^3, \\ L_3 &= \frac{144}{25}\lambda^3. \end{aligned}$$

Recalling that $b = 0$, we deduce that

$$\mathcal{J}(M \times N, g + h) = \frac{144}{25}(e - 2c)\lambda^3 \text{Vol}_g(M) \text{Vol}_h(N).$$

Since λ is arbitrary, $e = 2c$.

Third, let (M^2, g) be a closed surface with constant sectional curvature λ and let (N^4, h) be a closed four-manifold with constant sectional curvature $\lambda/3$. Lemma 3.3 and Proposition 3.6 imply that $(M \times N, g + h)$ is Einstein with

$$\begin{aligned} L_1 &= 0, \\ L_2 &= -\frac{128}{75}\lambda^3, \\ L_3 &= \frac{832}{225}\lambda^3. \end{aligned}$$

Recalling that $b = 0$ and $e = 2c$, we deduce that

$$\mathcal{J}(M \times N, g + h) = \frac{256}{45}c\lambda^3 \text{Vol}_g(M) \text{Vol}_h(N).$$

Since λ is arbitrary, $c = e = 0$. □

Next we prove that the nonpositivity of the diffeomorphism invariant $\mathcal{L}_1(M)$ obstructs the existence of an Einstein metric on a given closed six-manifold M^6 .

Proof of Theorem 1.3. Suppose that (M^6, g) is a closed Einstein six-manifold with $\mathcal{L}_1(M) \leq 0$. Theorem 1.1 implies that $\mathcal{L}_1(M) = 0$ and g is locally symmetric. Hence its universal cover (\tilde{M}, \tilde{g}) is a symmetric Einstein manifold [17, Corollary IV.5.7].

If g has positive scalar curvature, then Myers’ Theorem implies that \tilde{M} is compact. Hence $|\pi_1(M)| < \infty$.

If g has nonpositive scalar curvature, then (\tilde{M}, \tilde{g}) has nonpositive sectional curvature [17, Theorem V.3.1 and Proposition V.4.2]. The Cartan–Hadamard Theorem then implies that \tilde{M} is diffeomorphic to \mathbb{R}^n . □

We conclude this section by discussing global conformal invariants which obstruct the existence of an Einstein metric with nonpositive scalar curvature, and hence the existence of an Einstein metric on a closed six-manifold with infinite fundamental group.

Proof of Theorem 1.4. Proposition 2.1 implies that $\mathcal{Q}(M, g)$, $\mathcal{L}_1(M, g)$, and $\mathcal{L}_2(M, g)$ are conformally invariant. Integrating (1.1) yields (1.3).

Suppose that (M^6, g) is a closed Einstein six-manifold with $\text{Ric} = 5\lambda g \leq 0$. Then

$$\begin{aligned} \mathcal{Q}(M, g) &= -120\lambda^3 \text{Vol}_g(M), \\ \mathcal{L}_1(M, g) &= \int_M |\nabla^g W^g|_g^2 \, d\text{vol}_g, \\ \mathcal{L}_2(M, g) &= -2\lambda \int_M |W^g|_g^2 \, d\text{vol}_g. \end{aligned}$$

The conclusion readily follows. □

Proof of Corollary 1.5. Let M^6 be a closed six-manifold with $|\pi_1(M)| = \infty$. Suppose that g is an Einstein metric on M . Myers’ Theorem implies that $\text{Ric}_g \leq 0$. The conclusion follows from Theorem 1.4. □

5. Some examples

We conclude by discussing additional properties of the global conformal invariants of Theorems 1.1 and 1.4 through the context of some examples.

Since there are closed six-manifolds which admit geometrically distinct Einstein metrics [5], one might ask whether $\mathcal{L}_1(M, g)$ is independent of the choice of Einstein metric. This is not the case.

Proposition 5.1. *There is an Einstein metric g on S^6 with $\mathcal{L}_1(S^6, g) > 0$. In particular, $\mathcal{L}_1(M, g)$ depends on the choice of Einstein metric on M .*

Proof. Let g be one of the nontrivial cohomogeneity one Einstein metrics on S^6 constructed by Böhm [5, Theorem 3.6]. Then (S^6, g) is not a global Riemannian symmetric space. Since S^6 is simply connected, (S^6, g) is not locally symmetric. We conclude from Theorem 1.1 that $\mathcal{L}_1(S^6, g) > 0$. The final conclusion follows from the fact that the round metric $d\theta^2$ on S^6 is such that $\mathcal{L}_1(S^6, d\theta^2) = 0$. □

Next we show that \mathcal{Q} , \mathcal{L}_1 , and \mathcal{L}_2 give nontrivial obstructions, in the sense that for each invariant one can find a closed Riemannian six-manifold with infinite fundamental group for which the given invariant is negative.

Proposition 5.2. *Let (T^k, dx^2) and $(S^k, d\theta^2)$ denote a k -dimensional flat torus and a round k -sphere of constant sectional curvature one, respectively. Then*

$$\begin{aligned}\mathcal{Q}(S^2 \times T^4, d\theta^2 + dx^2) &= -\frac{48\pi}{25} \text{Vol}_{dx^2}(T^4), \\ \mathcal{L}_1(T^2 \times S^4, dx^2 + d\theta^2) &= -96\pi^2 \text{Vol}_{dx^2}(T^2), \\ \mathcal{L}_2(T^2 \times S^4, dx^2 + d\theta^2) &= -\frac{528\pi^2}{25} \text{Vol}_{dx^2}(T^2).\end{aligned}$$

In particular, each of \mathcal{Q} , \mathcal{L}_1 , and \mathcal{L}_2 gives a nontrivial obstruction to the existence of an Einstein metric on a closed six-manifold with infinite fundamental group.

Proof. Proposition 3.6 implies that

$$\begin{aligned}\mathcal{L}_1(T^2 \times S^4, dx^2 + d\theta^2) &= -36 \text{Vol}_{dx^2}(T^2) \text{Vol}_{d\theta^2}(S^4), \\ \mathcal{L}_2(T^2 \times S^4, dx^2 + d\theta^2) &= -\frac{198}{25} \text{Vol}_{dx^2}(T^2) \text{Vol}_{d\theta^2}(S^4).\end{aligned}$$

and

$$\mathcal{Q}(S^2 \times T^4, d\theta^2 + dx^2) = -\frac{12}{25} \text{Vol}_{d\theta^2}(S^2) \text{Vol}_{dx^2}(T^4).$$

The conclusion follows from the formulas $\text{Vol}_{d\theta^2}(S^2) = 4\pi$ and $\text{Vol}_{d\theta^2}(S^4) = \frac{8}{3}\pi^2$. \square

We conclude by pointing out that Einstein metrics are not critical points of \mathcal{L}_1 in general.

Proposition 5.3. *Let (S^2, g) be the closed two-sphere of constant sectional curvature 3 and let (S^4, h) be a closed four-sphere of constant sectional curvature 1. Then*

$$\mathcal{L}_1(S^2 \times S^4, g + c^2h) = 128\pi^3 c^{-1}(3c + 1)^2(c - 1) \quad (5.1)$$

for all $c > 0$. In particular, the Einstein manifold $(S^2 \times S^4, g + h)$ is not a critical point of \mathcal{L}_1 in the space of Riemannian metrics on $S^2 \times S^4$.

Proof. Equation (5.1) follows immediately from Proposition 3.6 and the fact that (S^4, ch) is a spaceform with constant sectional curvature c^{-1} . It is clear that $(S^2 \times S^4, g + h)$ is Einstein, while (5.1) implies that

$$\left. \frac{d}{dc} \right|_{c=1} \mathcal{J}(S^2 \times S^4, g + ch) \neq 0. \quad \square$$

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