

Topological Frobenius reciprocity and invariant hermitian forms

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ABSTRACT. In his article *Unitary Representations and Complex Analysis*, David Vogan gives a characterization of the continuous invariant Hermitian forms defined on the compactly supported sheaf cohomology groups of certain homogeneous analytic sheaves defined on open orbits in generalized flag manifolds. In the last section of the manuscript, Vogan raises a question about the possibility of a topological Frobenius reciprocity for these representations. In this article we give a specific version of the topological reciprocity in the regular, antidominant case and use it to study the existence of continuous invariant hermitian forms on the sheaf cohomology. In particular, we obtain a natural relationship between invariant forms on the sheaf cohomology and invariant forms on the geometric fiber.

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1. Introduction

In the article [11], D. Vogan gives a characterization of the continuous invariant Hermitian forms that exist on some geometrically defined representations that were studied in [7] and [2]. In the last section of the manuscript, Vogan raises a question (Question 10.2) about the possibility of a topological Frobenius reciprocity for these representations. Motivated by Vogan’s question, in this manuscript we establish a specific version of the reciprocity, in the regular antidominant case, and use it to study the continuous invariant Hermitian forms on the nonzero compactly supported sheaf cohomology group of a homogeneous analytic sheaf. A natural application of the reciprocity is an analysis of the relationship between the invariant forms defined on sheaf cohomology

Received November 30, 2020.

2010 *Mathematics Subject Classification*. 22E45, 22E46, 14L30, 14M15.

Key words and phrases. Representations of reductive Lie groups, Frobenius reciprocity, localization of representations.

and invariant forms defined on the geometric fiber. In order to unpack this relationship, we need to understand a corresponding Lie algebra cohomology group defined on the conjugate point. A key ingredient in our analysis is a generalization of a duality theorem that appears in [14] and [5].

To treat our results in more detail, we introduce the basic set up. The proofs we give for reciprocity work for a reductive Lie group G_0 of Harish-Chandra class. However, for the sake of the introduction, we will assume that G_0 is a real form of a connected complex reductive group G . By this we mean: G_0 is a closed subgroup of G , G_0 has finitely many connected components, and the Lie algebra \mathfrak{g}_0 of G_0 is a real form of the Lie algebra, \mathfrak{g} of G . In general, $P \subseteq G$ will denote a parabolic subgroup of G and

$$Y = G/P$$

will be the corresponding generalized flag manifold. Since a parabolic subgroup is equal to its normalizer in G , points in Y are naturally identified with the parabolic subgroups conjugate to P . In general, we will use Y to indicate the complex analytic structure the generalized manifold has as a homogeneous holomorphic space for the complex analytic group G . However, in the process of our proof, we will also need to consider the compatible algebraic structure, Y^{alg} , that the complex manifold Y has, as an algebraic homogeneous space for the group G , when G is equipped with its structure as an affine algebraic group (we will refrain from utilizing the notation G^{alg}). Adopting Vogan’s nomenclature, a parabolic subgroup P will be called *nice* if it contains a Levi factor L such that

$$G_0 \cap P = G_0 \cap L.$$

In this case,

$$L_0 = G_0 \cap L$$

is a real form of L called the *real Levi subgroup*. A G_0 -orbit in Y is *nice* when it’s the orbit of a nice parabolic subgroup. Nice orbits are open in Y . We remark that every open orbit on the full flag manifold

$$X = G/B$$

of Borel subgroups, $B \subseteq G$, is nice and that sometimes, open orbits on generalized flag manifolds are not nice.

We recall a few facts about the representations of reductive Lie groups. Let $K_0 \subseteq G_0$ be a maximal compact subgroup and let $K \subseteq G$ be its complexification. A *Harish-Chandra module* M can be defined as a finite-length \mathfrak{g} -module with compatible, algebraic K -action. We remark that the structure of M as a (\mathfrak{g}_0, K_0) -module uniquely determines the \mathfrak{g} and K -actions, since both actions are obtained by complexification. A finite-length, admissible representation of G_0 on a complete, locally convex space is called a *globalization of M* if the subspace of K_0 -finite vectors in the representation is isomorphic to M . We use the notation M_{glob} to indicate the globalization of a Harish-Chandra module M . In

this manuscript, we will focus on two canonical globalizations: Schmid's minimal and maximal globalizations [9]. Both globalizations define exact functors and are related by the natural isomorphism

$$M_{\max} \cong ((M^\vee)_{\min})'$$

where M^\vee is the K -finite dual of the Harish-Chandra module M and the prime indicates the continuous dual with the strong topology. In general, there are continuous G_0 -equivariant inclusions

$$M_{\min} \rightarrow M_{\text{glob}} \rightarrow M_{\max}.$$

Suppose P is a nice parabolic subgroup and let $S = G_0 \cdot P$ be the G_0 -orbit of P in $Y = G/P$. There are certain points in S that work well with respect to some algebraic constructions involving the choice of maximal compact subgroup K_0 and its complexification K . Vogan calls these *the very nice parabolic subgroups* and they can be characterized as the points in S such that $K_0 \cap P = K_0 \cap L_0$ is a maximal compact subgroup of the real Levi factor. Both K_0 and its complexification K act transitively on the very nice parabolic subgroups and they form the unique K -orbit Q in Y that satisfies $Q \subseteq S$ [13]. The *vanishing number*, $q = q_S$, of S , is defined as the complex codimension of Q . *We want to emphasize that, throughout this manuscript, we will use the letter q exclusively in formulas to indicate the vanishing number of S , while running indices will be indicated with the letters p or n (or anything but q).*

Now let V_{\min} be a minimal globalization for L_0 . Then, generalizing the finite-dimensional case, when $V = V_{\min}$, one can define a corresponding G_0 -equivariant analytic sheaf $\mathcal{O}(P, V_{\min})$ on S [2, Section 6]. In the infinite-dimensional case it is not clear this sheaf is locally free over the structure sheaf of holomorphic functions, however, the geometric fiber is isomorphic to V_{\min} . It follows from the work in [2], that the compactly supported sheaf cohomology groups

$$H_c^p(S, \mathcal{O}(P, V_{\min}))$$

are minimal globalizations for G_0 and that they vanish if $p < q$. On the other hand, let \mathfrak{u} be the Lie algebra of the unipotent radical U of P . When W_{\max} is a maximal globalization for G_0 then it turns out the Lie algebra homology groups

$$H_n(\mathfrak{u}, W_{\max})$$

are maximal globalizations for L_0 [4]. In fact, when P is a very nice parabolic subgroup, then there is a natural isomorphism

$$H_n(\mathfrak{u}, W)_{\max} \cong H_n(\mathfrak{u}, W_{\max}).$$

Vogan's Question 10.2 asks about the relationship between

$$\text{Hom}_{G_0}(H_c^p(S, \mathcal{O}(P, V_{\min})), W_{\max}) \text{ and } \text{Hom}_{L_0}(V_{\min}, H^p(\mathfrak{u}, W_{\max}))$$

where $H^p(\mathfrak{u}, W_{\max})$ denotes the Lie algebra cohomology group (although it might be best to frame the question directly in terms of the \mathfrak{u} -homology). He poses a similar question for the dual case and goes on to conjecture the existence of a certain spectral sequence, relating Ext groups, that reduces to a sharp

identity of the corresponding spaces of homomorphisms, when enough items vanish. In this paper, we will directly prove the sharp identity in a key, and fundamentally important case.

In order to more precisely formulate our treatment of this relationship, we need to introduce the calculus of infinitesimal characters. Let $Z(\mathfrak{g})$ denote the center of the enveloping algebra $U(\mathfrak{g})$. A \mathfrak{g} -infinitesimal character is a morphism of algebras $\Theta : Z(\mathfrak{g}) \rightarrow \mathbb{C}$. A \mathfrak{g} -module is called *quasisimple* if $Z(\mathfrak{g})$ acts by an infinitesimal character. In general, a globalization, M_{glob} , of a Harish-Chandra module M , is called *quasisimple* if M is a quasisimple \mathfrak{g} -module. Since M_{min} and M_{max} are both smooth representations, they will be quasisimple \mathfrak{g} -modules if and only if M is. One key property of our geometric construction of representations is that it preserves the property of being quasisimple in a specific way. In particular, if V_{min} is a minimal globalization for L_0 with $Z(\mathfrak{l})$ -infinitesimal character $\sigma : Z(\mathfrak{l}) \rightarrow \mathbb{C}$, then the sheaf cohomology groups $H_c^p(S, \mathcal{O}(V_{\text{min}}))$ will have the infinitesimal character Θ given by composition of σ with the Harish-Chandra map

$$Z(\mathfrak{g}) \rightarrow Z(\mathfrak{l}).$$

In turn, if $\mathfrak{h} \subseteq \mathfrak{l}$ is a Cartan subalgebra, we have (unnormalized) Harish-Chandra maps $Z(\mathfrak{l}) \rightarrow U(\mathfrak{h})$ and $Z(\mathfrak{g}) \rightarrow U(\mathfrak{h})$, and a commutative diagram

$$\begin{array}{ccc} Z(\mathfrak{g}) & \longrightarrow & Z(\mathfrak{l}) \\ & \searrow & \downarrow \\ & & U(\mathfrak{h}) \end{array}$$

Let W be the Weyl group of \mathfrak{h} in \mathfrak{g} and let $W_{\mathfrak{l}}$ be the Weyl group of \mathfrak{h} in \mathfrak{l} . Then the respective Harish-Chandra maps identify Θ with a W -orbit and σ with a $W_{\mathfrak{l}}$ -orbit in the dual space \mathfrak{h}^* . The \mathfrak{l} -infinitesimal character σ is said to be *regular* (with respect to \mathfrak{g}) if W acts freely on the corresponding W -orbit in \mathfrak{h}^* . σ is called *antidominant* (with respect to parabolic subgroup P) if for any λ in the corresponding $W_{\mathfrak{l}}$ -orbit we have

$$\alpha^\vee(\lambda) \notin \{1, 2, 3, \dots\}$$

for every root α of \mathfrak{h} in \mathfrak{u} . In terms of our construction of representations we have the following:

- (1) When the infinitesimal character, σ of V_{min} is antidominant then

$$H_c^p(S, \mathcal{O}(P, V_{\text{min}})) = 0$$

unless $p = q$ is the vanishing number.

- (2) When the infinitesimal character of V_{min} is antidominant and regular then $H_c^q(S, \mathcal{O}(P, V_{\text{min}}))$ is irreducible when V_{min} is.

In particular, if σ is a regular, antidominant infinitesimal character for L_0 then

$$V_{\text{min}} \mapsto H_c^q(S, \mathcal{O}(P, V_{\text{min}}))$$

defines an exact (and, in fact, fully faithful) functor from the category of minimal globalizations for L_0 with infinitesimal character σ to the category of quasisimple finite length admissible representations for G_0 .

The *topological Frobenius reciprocity* is the fact that the geometric induction has a right adjoint given by the following construction. Let \mathfrak{u} be the nilradical of the Lie algebra of P and suppose M_{\max} is a quasisimple maximal globalization for G_0 . Let $H_q(\mathfrak{u}, M_{\max})_\sigma$ denote the corresponding $Z(\mathfrak{l})$ eigenspace in the Lie algebra homology group (q is the vanishing number). Then the right adjoint to the functor of geometric induction is given by

$$M_{\text{glob}} \mapsto H_q(\mathfrak{u}, M_{\max})_\sigma$$

where M_{glob} is a quasisimple finite length admissible representation for G_0 . That is:

$$\text{Hom}_{G_0}(H_c^q(S, \mathcal{O}(P, V_{\min})), M_{\text{glob}}) \cong \text{Hom}_{L_0}(V_{\min}, H_q(\mathfrak{u}, M_{\max}))$$

for every quasisimple finite length admissible representation M_{glob} (Theorem 3.1 of this manuscript). For any nice parabolic subgroup P , we obtain the natural isomorphism

$$\text{Hom}_{G_0}(H_c^q(S, \mathcal{O}(P, V_{\min})), M_{\max}) \cong \text{Hom}_{L_0}(V_{\min}, H_q(\mathfrak{u}, M_{\max})).$$

Dualizing, letting s be the complex dimension of Q and giving a completely formal definition to the representation $H^s(S, \mathcal{O}(\mathfrak{p}, W_{\max}))$ we obtain the natural isomorphism

$$\text{Hom}_{G_0}(M_{\min}, H^s(S, \mathcal{O}(\mathfrak{p}, W_{\max}))) \cong \text{Hom}_{L_0}(H_s(\mathfrak{u}, M_{\min}), W_{\max})$$

which is valid at least when M_{\min} is quasisimple and W_{\max} has an infinitesimal character that satisfies an appropriate regular dominant condition.

Our proof of the formula is based on the geometric construction of the Harish-Chandra module of $H_c^q(S, \mathcal{O}(P, V_{\min}))$ and depends on known properties of that construction, although we've never previously encountered them put together this way. The rest follows from results in [3] and [4]. We remark that this does give a very different proof than the one used by Vogan to derive analogous formulas for the cohomological parabolic induction in [12].

Ostensibly, the topological Frobenius reciprocity gives a relationship between the continuous, invariant Hermitian forms defined on the representation $H_c^q(S, \mathcal{O}(P, V_{\min}))$ and the continuous, invariant Hermitian forms defined on the geometric fiber. To explain this, let $(M_{\min})^h$ denote the Hermitian dual of a minimal globalization M_{\min} , as defined by Vogan in [11]. Then the space of continuous invariant sesquilinear forms on M_{\min} is naturally identified with the finite-dimensional complex vector space

$$\text{Hom}_{G_0}(M_{\min}, (M_{\min})^h).$$

The subspace of continuous invariant Hermitian forms is a real form. Since

$$H_p(\mathfrak{u}, (M_{\min})^h) \cong H^p(\mathfrak{u}^{\text{op}}, M_{\min})^h$$

for each p , where $H^p(\mathfrak{u}^{\text{op}}, M_{\min})$ is the corresponding Lie algebra cohomology group defined for the nilradical of the conjugate (or opposite) parabolic subalgebra, letting $I(P, V)_{\min}$ denote the representation $H_c^q(S, \mathcal{O}(P, V_{\min}))$, we obtain

$$\text{Hom}_{G_0}(I(P, V)_{\min}, (I(P, V)_{\min})^h) \cong \text{Hom}_{L_0}(V_{\min}, H_q(\mathfrak{u}, (I(P, V)_{\min})^h)) \cong$$

$$\text{Hom}_{L_0}(V_{\min}, H^q(\mathfrak{u}^{\text{op}}, I(P, V)_{\min})^h).$$

Thus, we would like to understand the \mathfrak{u}^{op} -cohomology group

$$H^q(\mathfrak{u}^{\text{op}}, H_c^q(S, \mathcal{O}(P, V_{\min}))).$$

Our strategy for this problem involves extending a duality theorem for standard modules that appears in [14] and [5]. This allows us to conclude that

$$H^q(\mathfrak{u}^{\text{op}}, H_c^q(S, \mathcal{O}(P, V_{\min})))_{\sigma} \cong V_{\min}$$

at least when V_{\min} is irreducible (our proof for duality, which is adapted from [5], depends on irreducibility). In this way, we provide a natural (and geometric) proof of the known result that a nondegenerate continuous invariant Hermitian form on V_{\min} induces a nondegenerate continuous invariant Hermitian form on $H_c^q(S, \mathcal{O}(P, V_{\min}))$ (Theorem 4.2 in the manuscript). Conversely, whether or not V_{\min} is irreducible, any nonzero continuous invariant Hermitian form on $H_c^q(S, \mathcal{O}(P, V_{\min}))$ induces a nonzero morphism between V_{\min} and its Hermitian dual in $H^q(\mathfrak{u}^{\text{op}}, H_c^q(S, \mathcal{O}(P, V_{\min})))^h$.

Our article is organized as follows. The first section is the introduction. In the second section, we give a geometric proof of the algebraic reciprocity. In the third section, we introduce the topological reciprocity and consider the effects of the Hermitian dual functor, introduced by Vogan in [11]. We also give a fairly detailed example of how things work when G_0 is a connected complex reductive group and S is the open orbit on a full flag space. In the final section, we generalize Chang’s duality theorem and use this to analyze the relationship between the continuous invariant Hermitian forms on the representation $H_c^q(S, \mathcal{O}(P, V_{\min}))$ and the continuous invariant Hermitian forms on the geometric fiber.

2. Algebraic Frobenius reciprocity

In this section, we give an argument for algebraic version of our reciprocity formula, using the localization theory to a generalized flag manifold. The algebraic reciprocity theorem is an adjointness property that holds, in general, for closed K -orbits in Y (i.e. does not depend on the assumption that P is nice.) The result follows from previously established facts about: (1) localization (the equivalence of categories given by localization and global sections), (2) the direct image for closed embeddings (Kashiwara’s equivalence of categories) and

(3) the identification of geometric fibers with \mathfrak{u} -homology. However, we are unaware of any version of this reciprocity in the published literature, so we provide a proof.

We remark that all objects in this section are treated in the algebraic sense. In particular, the generalized flag manifold $Y = Y^{alg}$ is a smooth algebraic variety with its accompanying structure sheaf of regular functions \mathcal{O}_Y . Since the objects and morphisms in this section live exclusively in the algebraic universe, we avoid writing superscripts to simplify notation.

A natural context for establishing our main result is to assume $K \subseteq G$ is an algebraic subgroup that has finitely many orbits on the full flag manifold (for example, this is the context used in [8]). We need to introduce the machinery of the localization theory and review the corresponding construction of the standard Harish-Chandra modules.

Let X be the full flag manifold of Borel subgroups of G , suppose $P \subseteq G$ is a parabolic subgroup and let $Y = G/P$. Since each Borel subgroup of P is conjugate in P , there is a canonical G -equivariant projection $\pi : X \rightarrow Y$.

For each $x \in X$, let \mathfrak{b}_x be the corresponding Borel subalgebra of \mathfrak{g} and let

$$\mathfrak{n}_x = [\mathfrak{b}_x, \mathfrak{b}_x]$$

be the nilradical of \mathfrak{b}_x . The adjoint action of the Borel subgroup B_x on \mathfrak{b}_x and \mathfrak{n}_x determines homogeneous algebraic vector bundles on X . We let \mathfrak{b}^\bullet and \mathfrak{n}^\bullet denote the corresponding sheaves of sections. Since B_x acts trivially on the quotient $\mathfrak{b}_x/\mathfrak{n}_x$, the sheaf $\mathfrak{b}^\bullet/\mathfrak{n}^\bullet$ is a free \mathcal{O}_X -module and the global sections

$$\mathfrak{h} = \Gamma(X, \mathfrak{b}^\bullet/\mathfrak{n}^\bullet)$$

are a finite-dimensional, abelian Lie algebra, isomorphic to a Cartan subalgebra \mathfrak{c} of \mathfrak{g} contained in \mathfrak{b}_x , via the *specialization to x* :

$$\mathfrak{c} \rightarrow \mathfrak{b}_x/\mathfrak{n}_x \leftarrow \mathfrak{h}.$$

We call \mathfrak{h} the *abstract Cartan subalgebra*. Let \mathfrak{h}^* be the complex dual of \mathfrak{h} . There is a set of *abstract roots* $\Sigma \subseteq \mathfrak{h}^*$ and a corresponding Weyl group W . The set of *positive roots* $\Sigma^+ \subseteq \Sigma$ correspond to the roots of a Cartan subalgebra \mathfrak{c} in \mathfrak{b}_x via the specialization to x .

Via the Harish-Chandra map, the Weyl group orbits in \mathfrak{h}^* parameterize the infinitesimal characters of $Z(\mathfrak{g})$. We will write $\lambda \in \Theta$ if the Weyl group orbit $W \cdot \lambda$ corresponds to the infinitesimal character Θ . Let U_Θ denote the quotient of $U(\mathfrak{g})$ by the ideal generated from the kernel of Θ . For each $\lambda \in \mathfrak{h}^*$, Beilinson and Bernstein define a twisted sheaf of differential operators \mathcal{D}_λ on X [1]. They show that \mathcal{D}_λ is acyclic for global sections and that

$$\Gamma(X, \mathcal{D}_\lambda) \cong U_\Theta$$

when $\lambda \in \Theta$. On the other hand, if M is a U_Θ -module, then for each $\lambda \in \mathfrak{h}^*$, there is a localization functor, Δ_λ , is defined by

$$\Delta_\lambda(M) = \mathcal{D}_\lambda \otimes_{U_\Theta} M.$$

Localization is left adjoint to the functor of global sections.

An infinitesimal character is called *regular* if the Weyl group acts freely on the corresponding orbit in \mathfrak{h}^* . An element $\lambda \in \mathfrak{h}^*$ is called *antidominant* if $\alpha^\vee(\lambda)$ is not a positive integer, for every positive coroot α^\vee . When λ is antidominant and regular then localization and global sections define an equivalence of categories between the category of Harish-Chandra modules with infinitesimal character Θ and the category of K -equivariant coherent sheaves of \mathcal{D}_λ -modules (called the *Harish-Chandra sheaves on X*).

We now consider how these results carry over to a generalized flag manifold, Y , in a way that is suitable for our purposes. The set up is the same as in [2]. For each $y \in Y$, \mathfrak{p}_y is the corresponding parabolic subalgebra and \mathfrak{u}_y will be the nilradical of \mathfrak{p}_y . The action of the parabolic subgroup P_y on the quotient

$$\mathfrak{l}_y = \mathfrak{p}_y / \mathfrak{u}_y$$

is not trivial in this more general setting. We introduce the G -equivariant sheaf of algebraic sections, $U(\mathfrak{l}^\bullet)$, corresponding to the P_y action on the universal enveloping algebra $U(\mathfrak{l}_y)$. This is a locally free sheaf of \mathcal{O}_Y -modules and a sheaf of algebras with the pointwise multiplication. Hence, the center $Z(\mathfrak{l}^\bullet)$ of $U(\mathfrak{l}^\bullet)$ is a free sheaf of \mathcal{O}_Y -modules and the global sections

$$Z(\mathfrak{l}) = \Gamma(Y, Z(\mathfrak{l}^\bullet))$$

will be isomorphic to the center, $Z(\mathfrak{l})$, of the enveloping algebra of a Levi factor $\mathfrak{l} \subseteq \mathfrak{p}_y$ via the specialization to a point y :

$$Z(\mathfrak{l}) \rightarrow Z(\mathfrak{l}_y) \leftarrow Z(\mathfrak{l}).$$

Since the natural projection identifies the Borel subalgebras of \mathfrak{g} contained in \mathfrak{p}_y with the Borel subalgebras of \mathfrak{l}_y , we can use specialization to identify a set of abstract roots of the Levi factor $\Sigma(\mathfrak{l}) \subseteq \Sigma$ and a corresponding set of positive roots $\Sigma(\mathfrak{l})^+ \subseteq \Sigma^+$. We let $W_{\mathfrak{l}} \subseteq W$ be the Weyl group of the root system $\Sigma(\mathfrak{l})$. Since $X_y = \pi^{-1}(\{y\})$ is the flag manifold for \mathfrak{l}_y , the restriction identifies \mathfrak{h} with the abstract Cartan of \mathfrak{l}_y . There is an unnormalized Harish-Chandra map $Z(\mathfrak{l}) \rightarrow U(\mathfrak{h})$. We parametrize the infinitesimal characters of $Z(\mathfrak{l})$ via the composition

$$Z(\mathfrak{l}) \rightarrow U(\mathfrak{h}) \xrightarrow{\lambda + \rho} \mathbb{C}$$

where $\lambda \in \mathfrak{h}^*$ and ρ denotes one-half the sum of the roots in Σ^+ . In this way, the $Z(\mathfrak{l})$ -infinitesimal characters are parameterized by $W_{\mathfrak{l}}$ -orbits in \mathfrak{h}^* . Occasionally we use the notation σ_λ to indicate the corresponding infinitesimal character, even though $\sigma_\lambda = \lambda + \rho$ when $\mathfrak{l}_y = \mathfrak{b}_x / \mathfrak{n}_x$. Since the unnormalized Harish-Chandra map sends $Z(\mathfrak{g})$ into $Z(\mathfrak{l})$, a \mathfrak{g} -infinitesimal character Θ , with $\lambda \in \Theta$, corresponds to the family of $Z(\mathfrak{l})$ -infinitesimal characters

$$\{\sigma_{[w]\lambda} : [w] \in W/W_{\mathfrak{l}}\}.$$

Each member of this family will be called *regular* if the corresponding $Z(\mathfrak{g})$ -infinitesimal character is regular. An element $\lambda \in \mathfrak{h}^*$ will be called *antidominant for Y* if there is an element in the orbit $W_{\mathfrak{l}} \cdot \lambda$ that is antidominant. Equivalently we could require that $\alpha^\vee(\lambda)$ not be a positive integer for each $\alpha \in$

$\Sigma^+ - \Sigma^+(\mathfrak{I})$. When this condition holds, we also refer to the $\mathcal{Z}(\mathfrak{I})$ -infinitesimal character σ_λ as being antidominant.

We consider the sheaf of algebras $\pi_* \mathcal{D}_\lambda$ on Y . One knows that D_λ is acyclic for π_* [5]. By the Leray spectral sequence, it follows that $\pi_* \mathcal{D}_\lambda$ is a acyclic for the global sections on Y . There is a morphism of sheaves of algebras $\mathcal{Z}(\mathfrak{I}^\bullet) \rightarrow \pi_* \mathcal{D}_\lambda$. Via the restriction map, $\mathcal{Z}(\mathfrak{I})$ acts on the geometric fiber of a sheaf of $\pi_* \mathcal{D}_\lambda$ -modules, by the infinitesimal character σ_λ and

$$\pi_* \mathcal{D}_\lambda \cong \pi_* \mathcal{D}_{w\lambda} \text{ for each } w \in W_{\mathfrak{I}}.$$

Given a U_Θ -module M and a $W_{\mathfrak{I}}$ -orbit $W_{\mathfrak{I}} \cdot \lambda$ for $\lambda \in \Theta$ (i.e. a $\mathcal{Z}(\mathfrak{I})$ -infinitesimal character σ_λ), we can define a localization functor

$$\Delta_Y(M) = \pi_* \mathcal{D}_\lambda \otimes_{U_\Theta} M.$$

Localization is left adjoint to the functor of global sections. When σ_λ is regular and antidominant, then localization and global sections define an equivalence of categories between the Harish-Chandra modules with infinitesimal character Θ and the K -equivariant coherent sheaves of $\pi_* \mathcal{D}_\lambda$ -modules (the Harish-Chandra sheaves on Y) [5].

We review the construction of the standard Harish-Chandra sheaves from [5] and used in [2]. Let

$$i : Q \rightarrow Y$$

be the inclusion and begin by assuming Q is an arbitrary K -orbit. Choose $y \in Q$ and suppose V is a Harish-Chandra module for (\mathfrak{I}_y, K_y) where K_y is the stabilizer of y in K and we also assume $\mathcal{Z}(\mathfrak{I})$ acts by an infinitesimal character σ . The Harish-Chandra module V determines a K -equivariant algebraic vector bundle over Q . The corresponding sheaf of sections \mathcal{V} is a locally free sheaf of \mathcal{O}_Q -modules. Let $U_Q(\mathfrak{I}^\bullet)$ be the K -equivariant locally free sheaf corresponding to the K_y -action on $U(\mathfrak{I}_y)$ and let \mathfrak{k} be the Lie algebra of K . The action of \mathfrak{I}_y on V determines a pointwise action of $U_Q(\mathfrak{I}^\bullet)$ on \mathcal{V} and, by differentiating the K -action, we obtain an action of $U(\mathfrak{k}^\bullet)$ on \mathcal{V} . Suppose $\lambda \in \sigma$. In turn, the actions of $U_Q(\mathfrak{I}^\bullet)$ and $U(\mathfrak{k}^\bullet)$ determine an action for a sheaf of algebras, $(\pi_* \mathcal{D}_\lambda)^i$, and there is a corresponding direct image functor

$$\mathcal{V} \mapsto i_+ \mathcal{V}$$

that sends Harish-Chandra sheaves for $((\pi_* (\mathcal{D}_\lambda)^i, K)$ to Harish-Chandra sheaves for $(\pi_* (\mathcal{D}_\lambda, K)$. When the orbit Q is affinely embedded then the direct image is exact. In particular, if the infinitesimal character σ is regular and antidominant for Y then the functor

$$V \mapsto \Gamma(Y, i_+ \mathcal{V})$$

is an exact functor from the category of Harish-Chandra modules, $\mathbf{M}_\sigma(\mathfrak{I}_y, K_y)$, with \mathfrak{I}_y -infinitesimal character σ , to the category of Harish-Chandra modules, $\mathbf{M}_\Theta(\mathfrak{g}, K)$, with \mathfrak{g} -infinitesimal character Θ .

We now introduce the \mathfrak{u}_y -homology groups $H_p(\mathfrak{u}_y, M)$ for a Harish-Chandra module M for (\mathfrak{g}, K) . These are Harish-Chandra modules for (\mathfrak{I}_y, K_y) (a geometric argument for this can be given as in [2, Section 7]). When M is in $\mathbf{M}_\Theta(\mathfrak{g}, K)$

with Θ regular, then the homology groups $H_p(\mathbf{u}_y, M)$ split as a direct sum of eigenspaces for the $\mathcal{Z}(\mathfrak{I})$ -infinitesimal characters coming from the compatible family mentioned previously [4]. Let

$$H_p(\mathbf{u}_y, M)_\sigma$$

denote the $\mathcal{Z}(\mathfrak{I})$ -eigenspace for the infinitesimal character σ . The algebraic Frobenius reciprocity is the following.

Theorem 2.1. *Maintain the previous notations. Assume Q is a closed orbit and let q be the codimension of Q in Y . Suppose σ is a $\mathcal{Z}(\mathfrak{I})$ -infinitesimal character that is regular and antidominant for Y . Then the exact functor*

$$V \mapsto \Gamma(Y, i_+ \mathcal{V})$$

from the category $\mathbf{M}_\sigma(\mathfrak{I}_y, K_y)$ to the category $\mathbf{M}_\Theta(\mathfrak{g}, K)$ has right adjoint

$$M \mapsto H_q(\mathbf{u}_y, M)_\sigma.$$

Proof. Suppose $\lambda \in \sigma$ and let

$$\Delta_Y(M) = \pi_* \mathcal{D}_\lambda \otimes_{U_\Theta} M$$

be the localization functor. If \mathcal{F} is a sheaf of \mathcal{O}_Y -modules then let

$$T_y \mathcal{F} = \mathbb{C} \otimes_{(\mathcal{O}_Y)_y} \mathcal{F}_y$$

denote the geometric fiber of \mathcal{F} at the point y . Since Θ is regular it follows [4] that for any M in $\mathbf{M}_\Theta(\mathfrak{g}, K)$ there is a natural isomorphism of Harish-Chandra modules

$$H_q(\mathbf{u}_y, M)_\sigma \cong L_q T_y \Delta_Y(M)$$

where $L_q T_y$ denotes the q -th derived functor of the geometric fiber. Since σ is regular and antidominant for Y it follows that global sections and localization are equivalences of categories. Therefore,

$$\mathrm{Hom}_{(\mathfrak{g}, K)}(\Gamma(Y, i_+ \mathcal{V}), M) \cong \mathrm{Hom}_{(\pi_* \mathcal{D}_\lambda, K)}(i_+ \mathcal{V}, \Delta_Y(M)).$$

Thus, it suffices to show

$$\mathrm{Hom}_{(\pi_* \mathcal{D}_\lambda, K)}(i_+ \mathcal{V}, \Delta_Y(M)) \cong \mathrm{Hom}_{(\mathfrak{I}_y, K_y)}(V, L_q T_y \Delta_Y(M)).$$

It turns out this last identity can be deduced from Kashiwara's equivalence of categories as follows. When \mathcal{F} is a sheaf of Harish-Chandra modules then the inverse image in the category category of \mathcal{O} -modules

$$i^* \mathcal{F} = \mathcal{O}_Q \otimes_{i^{-1} \mathcal{O}_Y} i^{-1} \mathcal{F}$$

is a Harish-Chandra sheaf for $((\pi_*(\mathcal{D}_\lambda)^i, K))$, as are the derived inverse images $L_p i^* \mathcal{F}$ which vanish for $p > q$. One knows

$$L_q i^* i_+ \mathcal{V} \cong \mathcal{V} \text{ and } i_+ L_q i^* \mathcal{F} \cong \Gamma_Q \mathcal{F}$$

where $\Gamma_Q \mathcal{F}$ denotes the sheaf of sections with support in Q . It follows from these formulas that the functor $L_q i^*$ is right adjoint to the direct image, so that

$$\mathrm{Hom}_{(\pi_* \mathcal{D}_\lambda, K)}(i_+ \mathcal{V}, \Delta_Y(M)) \cong \mathrm{Hom}_{((\pi_* \mathcal{D}_\lambda)^i, K)}(\mathcal{V}, L_q i^* \Delta_Y(M)).$$

Since $L_q i^* \Delta_Y(M)$ is a Harish-Chandra sheaf on a K -orbit, it is a locally free sheaf of \mathcal{O}_Q -modules so that morphisms are completely determined by morphisms of the geometric fiber. Thus,

$$\mathrm{Hom}_{((\pi_* \mathcal{D}_\lambda)^i, K)}(\mathcal{V}, L_q i^* \Delta_Y(M)) \cong \mathrm{Hom}_{(T_y, K_y)}(V, T_y^Q L_q i^* \Delta_Y(M))$$

where T_y^Q is the geometric fiber for a sheaf of \mathcal{O}_Q -modules. Thus it suffices to prove

$$T_y^Q L_q i^* \Delta_Y(M) \cong L_q T_y \Delta_Y(M).$$

But this last point follows from a standard homological algebra argument, since $T_y^Q \circ i^* \cong T_y$ and since each of the sheaves $L_p i^* \Delta_Y(M)$, for $0 \leq p \leq q$, is a Harish-Chandra sheaf of $(\pi_* (\mathcal{D}_\lambda)^i, K)$ -modules and thus a locally free sheaf of \mathcal{O}_Q -modules that is acyclic for the functor T_y^Q . \square

Remark 2.2. Maintaining the assumptions of the previous theorem, we obtain the formula

$$\mathrm{Hom}_{(\mathfrak{g}, K)}(\Gamma(Y, i_+ \mathcal{V}), M) \cong \mathrm{Hom}_{(T_y, K_y)}(V, H_q(\mathbf{u}_y, M))$$

for any quasisimple Harish-Chandra module M . For a general Harish-Chandra module M we can introduce the $Z(\mathfrak{g})$ -eigenspace M_Θ into the formula:

$$\begin{aligned} \mathrm{Hom}_{(\mathfrak{g}, K)}(\Gamma(Y, i_+ \mathcal{V}), M) &\cong \mathrm{Hom}_{(\mathfrak{g}, K)}(\Gamma(Y, i_+ \mathcal{V}), M_\Theta) \\ &\cong \mathrm{Hom}_{(T_y, K_y)}(V, H_q(\mathbf{u}_y, M_\Theta)). \end{aligned}$$

Remark 2.3. Now suppose that K is the complexification of a maximal compact subgroup K_0 of G_0 and P_y is a very nice parabolic subgroup with real Levi subgroup L_0 . We introduce the K_0 -finite dual

$$M \mapsto M^\vee$$

on the category of Harish-Chandra modules (we use the same notation for the $K_0 \cap L_0$ -finite dual). Then one knows [4]:

$$H_q(\mathbf{u}_y, M^\vee) \cong H^q(\mathbf{u}_y, M)^\vee$$

where $H^q(\mathbf{u}_y, M)$ denotes the q -th Lie algebra cohomology group. Using the duality theorem in [6] and the fact that the K_0 -finite dual is exact, one can transpose the result in Theorem 2.1 into a reciprocity theorem for the cohomological parabolic induction, where the $(n - q)$ -th Lie algebra homology group defines the left adjoint to the induction functor. We leave the details to the reader.

3. Topological reciprocity and the Hermitian dual

In this section, G_0 is a reductive group of Harish-Chandra class with maximal compact subgroup K_0 . We begin with the topological reciprocity.

Theorem 3.1. *Suppose \mathfrak{p} is a very nice parabolic subalgebra, L_0 is the real Levi subgroup, \mathfrak{u} is the nilradical of \mathfrak{p} , S is the G_0 -orbit of \mathfrak{p} in the corresponding flag manifold Y and q is the vanishing number of S . Let V_{\min} be a minimal globalization of L_0 that has a regular, antidominant infinitesimal character. Then for every quasisimple finite length admissible representation M_{glob} for G_0 there is a natural isomorphism*

$$\text{Hom}_{G_0}(H_c^q(S, \mathcal{O}(\mathfrak{p}, V_{\min})), M_{\text{glob}}) \cong \text{Hom}_{L_0}(V_{\min}, H_q(\mathfrak{u}, M_{\text{max}})).$$

Proof. Let $y \in Y$ be the point corresponding to the parabolic subalgebra \mathfrak{p} . Since $K_0 \cap L_0$ is a maximal compact subgroup of L_0 and since K is the complexification of K_0 , referring to the notation in the previous section, K_y is a parabolic subgroup of K with a Levi factor that is the complexification of $K_0 \cap L_0$. Thus, V_{\min} is the minimal globalization of a Harish-Chandra module, V for (I_y, K_y) , that has a regular antidominant infinitesimal character. Using the notation from the previous section, let $\Gamma(Y^{\text{alg}}, i_+ \mathcal{V})$ denote the corresponding standard Harish-Chandra module and let $I(\mathfrak{p}, V)$ denote the Harish-Chandra of $H_c^q(S, \mathcal{O}(\mathfrak{p}, V_{\min}))$. Then one of the main points of [2] is that

$$\Gamma(Y^{\text{alg}}, i_+ \mathcal{V}) \cong I(\mathfrak{p}, V).$$

On the other hand, the representation M_{glob} is the globalization of a quasisimple Harish-Chandra module M for (\mathfrak{g}, K) . Therefore,

$$\text{Hom}_{G_0}(H_c^q(S, \mathcal{O}(\mathfrak{p}, V_{\min})), M_{\text{glob}}) \cong \text{Hom}_{(\mathfrak{g}, K)}(I(\mathfrak{p}, V), M) \cong$$

$$\text{Hom}_{(\mathfrak{g}, K)}(\Gamma(Y^{\text{alg}}, i_+ \mathcal{V}), M) \cong \text{Hom}_{(I_y, K_y)}(V, H_q(\mathfrak{u}, M) \cong$$

$$\text{Hom}_{L_0}(V_{\min}, H_q(\mathfrak{u}, M)_{\text{max}}) \cong \text{Hom}_{L_0}(V_{\min}, H_q(\mathfrak{u}, M_{\text{max}}))$$

where the fact the maximal globalization commutes with the \mathfrak{u} -homology group is shown in [4]. □

If we replace M_{glob} with M_{max} in the above reciprocity formula, then it no longer depends on the identification of a Harish-Chandra module. Therefore, we obtain the following.

Corollary 3.2. *Suppose \mathfrak{p} is a nice parabolic subalgebra, L_0 is the real Levi subgroup, \mathfrak{u} is the nilradical of \mathfrak{p} , S is the G_0 -orbit of \mathfrak{p} in the corresponding flag manifold Y and q is the vanishing number of S . Let V_{\min} be a minimal globalization of L_0 that has a regular, antidominant infinitesimal character. Then for every quasisimple maximal globalization M_{max} , there is a natural isomorphism*

$$\text{Hom}_{G_0}(H_c^q(S, \mathcal{O}(\mathfrak{p}, V_{\min})), M_{\text{max}}) \cong \text{Hom}_{L_0}(V_{\min}, H_q(\mathfrak{u}, M_{\text{max}})).$$

Remark 3.3. The dual form of the topological Frobenius reciprocity, referred to in Vogan’s Question 10.2, is a formal consequence of the previous theorem. Indeed, since

$$\text{Hom}_{G_0}(M_{\min}, N_{\text{max}}) \cong \text{Hom}_{G_0}((N_{\text{max}})', (M_{\min})')$$

for an pair of representations, we have

$$\begin{aligned} \mathrm{Hom}_{G_0}((N_{\max})', H_c^q(S, \mathcal{O}(\mathfrak{p}, V_{\min}))') &\cong \mathrm{Hom}_{L_0}(H_q(\mathfrak{u}, N_{\max})', (V_{\min})') \cong \\ &\mathrm{Hom}_{L_0}(H^q(\mathfrak{u}, (N_{\max})'), (V_{\min})'). \end{aligned}$$

Replacing $(N_{\max})'$ with M_{\min} in the formula, and using the fact that

$$H^q(\mathfrak{u}, (N_{\max})') \cong H_s(\mathfrak{u}, (N_{\max})') \otimes \chi_{\mathfrak{u}}^{-1}$$

where $s = \dim_{\mathbb{C}}(Y) - q$ and $\chi_{\mathfrak{u}}$ is the determinant character for the adjoint action of L_0 on \mathfrak{u} , we obtain

$$\mathrm{Hom}_{G_0}(M_{\min}, H_c^q(S, \mathcal{O}(\mathfrak{p}, V_{\min}))') \cong \mathrm{Hom}_{L_0}(H_s(\mathfrak{u}, M_{\min}), (V_{\min})' \otimes \chi_{\mathfrak{u}}).$$

Introducing the perspicuous notation

$$H_c^q(S, \mathcal{O}(\mathfrak{p}, V_{\min}))' = H^s(S, \mathcal{O}(\mathfrak{p}, (V_{\min})') \otimes \chi_{\mathfrak{u}})$$

and replacing $(V_{\min})' \otimes \chi_{\mathfrak{u}}$ with W_{\max} , we obtain the reciprocity formula

$$\mathrm{Hom}_{G_0}(M_{\min}, H^s(S, \mathcal{O}(\mathfrak{p}, W_{\max}))) \cong \mathrm{Hom}_{L_0}(H_s(\mathfrak{u}, M_{\min}), W_{\max})$$

which is valid when M_{\min} is quasisimple and W_{\max} has an infinitesimal character that satisfies the corresponding regular, dominant condition. When W_{\max} is finite-dimensional then Serre duality [10] implies that the representation $H^s(S, \mathcal{O}(\mathfrak{p}, W_{\max}))$ is the sheaf cohomology of a corresponding G_0 -equivariant holomorphic vector bundle defined on S . However it is not obvious to us that this representation is realized as the sheaf cohomology group of some corresponding G_0 -equivariant sheaf on S , when W_{\max} is infinite-dimensional. Nevertheless, in the infinite-dimensional case, the representation $H^s(S, \mathcal{O}(\mathfrak{p}, W_{\max}))$ can be realized as the hypercohomology of the dual of the complex used to calculate the sheaf cohomology for $H_c^q(S, \mathcal{O}(\mathfrak{p}, V_{\min}))$.

We now introduce the Hermitian dual $(M_{\min})^h$ as defined by Vogan in [11]. In particular, $(M_{\min})^h$ is the space of continuous, conjugate-linear forms on M_{\min} with its natural topology as a maximal globalization, i.e. conjugation on \mathbb{C} defines an equivariant, conjugate-linear isomorphism between $(M_{\min})^h$ and the continuous dual $(M_{\min})' \cong (V^{\vee})_{\max}$. The contravariant functor

$$M \mapsto (M_{\min})^h$$

is exact. One can naturally extend the definition of Hermitian dual to maximal globalizations. In this way we obtain the identities

$$((M_{\min})^h)^h \cong M_{\min} \quad \text{and} \quad ((M_{\max})^h)^h \cong M_{\max}.$$

Proposition 3.4. *Suppose M_{\min} is a minimal globalization. Then the space of continuous invariant sesquilinear forms on M_{\min} is naturally isomorphic to*

$$\text{Hom}_{G_0}(M_{\min}, (M_{\min})^h)$$

and the corresponding space of continuous invariant Hermitian forms is a real form of $\text{Hom}_{G_0}(M_{\min}, (M_{\min})^h)$.

Proof. This is basically shown in Vogan’s manuscript, but we sketch some details for completeness.

If $\varphi \in \text{Hom}_{G_0}(M_{\min}, (M_{\min})^h)$ then

$$\langle m_1, m_2 \rangle = (\varphi(m_1))(m_2)$$

is a separately continuous invariant sesquilinear form and vice versa. The fact that separately continuous sesquilinear forms are continuous in this context follows from a theorem of topological vector spaces. On the other hand, for any continuous invariant sesquilinear form, $\langle \cdot, \cdot \rangle$, we can define the conjugate transpose $\langle \cdot, \cdot \rangle^\dagger$ by the equation

$$\langle v, w \rangle^\dagger = \overline{\langle w, v \rangle}.$$

This is another continuous invariant sesquilinear form and the original form is Hermitian if and only if

$$\langle \cdot, \cdot \rangle^\dagger = \langle \cdot, \cdot \rangle.$$

Now suppose μ is a continuous invariant sesquilinear form and define

$$\text{re}(\mu) = \frac{\mu + \mu^\dagger}{2} \text{ and } \text{im}(\mu) = \frac{i(\mu^\dagger - \mu)}{2}.$$

Then each of these forms is a continuous invariant hermitian form and $\mu = \text{re}(\mu) + i\text{im}(\mu)$. □

We now consider the effect of the Hermitian dual on the Lie algebra homology groups $H_p(\mathfrak{u}, (M_{\min}))$. Let $\tau : \mathfrak{g} \rightarrow \mathfrak{g}$ be the conjugation induced by the real form $\mathfrak{g}_0 \subseteq \mathfrak{g}$. We call $\tau(\mathfrak{p})$ the opposite parabolic subalgebra from \mathfrak{p} and write

$$\mathfrak{p}^{\text{op}} = \tau(\mathfrak{p}).$$

If \mathfrak{l} is the complexified Lie algebra of L_0 then

$$\mathfrak{p} \cap \tau(\mathfrak{p}) = \mathfrak{l}.$$

In particular,

$$\mathfrak{u}^{\text{op}} = \tau(\mathfrak{u})$$

is the nilradical of \mathfrak{p}^{op} .

Proposition 3.5. *Maintain the previous notations. Then there is a natural isomorphism of L_0 -modules:*

$$H_p(\mathfrak{u}, (M_{\min})^h) \cong H^p(\mathfrak{u}^{\text{op}}, M_{\min})^h$$

Proof. We claim that

$$\mathfrak{u}^h \cong \mathfrak{u}$$

as L_0 -modules. Let $B(\xi, \zeta)$ denote the Killing form on \mathfrak{g} . Then the Hermitian form on \mathfrak{u} defined by

$$\langle \xi, \zeta \rangle = B(\xi, \tau(\zeta))$$

is nondegenerate and L_0 -invariant, which proves the claim. Now an argument using the standard complex for computing the \mathfrak{u} -homology, as in the proof of Proposition 2.1 in [4], shows that

$$H_p(\mathfrak{u}, (M_{\min})^h) \cong H^p(\mathfrak{u}^*, M_{\min})^h$$

where \mathfrak{u}^* is the dual representation. Hence the result follows, as $\mathfrak{u}^* \cong \mathfrak{u}^{\text{op}}$. \square

Corollary 3.6. *Let $I(\mathfrak{p}, V)_{\min}$ denote the representation $H_c^q(S, \mathcal{O}(\mathfrak{p}, V_{\min}))$. Then*

$$\text{Hom}_{G_0}(I(\mathfrak{p}, V)_{\min}, (I(\mathfrak{p}, V)_{\min})^h) \cong \text{Hom}_{L_0}(V_{\min}, H^q(\mathfrak{u}^{\text{op}}, I(\mathfrak{p}, V)_{\min})^h).$$

Example 3.7. Suppose G_0 is a connected complex reductive group and let \mathfrak{b} be a nice Borel subalgebra. Let C_0 be the corresponding Cartan subgroup of G_0 and \mathfrak{n} the nilradical of \mathfrak{b} . Suppose

$$\chi_\mu : H_0 \rightarrow \mathbb{C}^*$$

is a continuous character with derivative $\mu_0 \in (\mathfrak{c}_0)^*$ and complexified derivative $\mu \in \mathfrak{c}^*$. Let ρ be one-half the sum of the roots of \mathfrak{c} in \mathfrak{b} and define $\lambda = \mu - \rho$. We assume λ is antidominant and regular. In this case it is not hard to explicitly calculate the C_0 -representation on the \mathfrak{n}^{op} -cohomology group

$$H^q(\mathfrak{n}^{\text{op}}, I(\mathfrak{b}, \mathbb{C}_\mu)_{\min}).$$

In fact,

$$H_q(\mathfrak{n}, I(\mathfrak{b}, \mathbb{C}_\mu)_{\min}) \cong H^q(\mathfrak{n}^{\text{op}}, I(\mathfrak{b}, \mathbb{C}_\mu)_{\min}).$$

We briefly elaborate. First, observe that since the maximal compact subgroup K_0 of G_0 is a compact real form, the complexification K is a complex group isomorphic to G , although the actions of the two groups on the full flag space X are distinct. Thus, $Q = K \cdot \mathfrak{b}$ is the full flag manifold of Borel subgroups of G_0 and this means the complex dimension s of Q and the vanishing number q coincide. Next, observe that the Weyl group, W_0 of G_0 acts simply transitively on the Borel subalgebras in S that contain \mathfrak{c} and that

$$w_l \cdot \mathfrak{b} = \mathfrak{b}^{\text{op}}$$

if w_l is the longest element in W_0 .

For each $w \in W_0$, let

$$\chi_{w\lambda+\rho} : H_0 \rightarrow \mathbb{C}^*$$

be the character with complexified derivative $w(\mu - \rho) + \rho$. We claim

$$H_q(\mathfrak{n}, I(\mathfrak{b}, \mathbb{C}_\mu)_{\min}) \cong \bigoplus_{w \in W_0} \mathbb{C}_{w\lambda+\rho}.$$

To establish this, let W be the Weyl group of \mathfrak{c} in \mathfrak{g} . Then, since the infinitesimal character of $I(\mathfrak{b}, \mathbb{C}_\mu)$ is regular, one knows

$$H_q(\mathfrak{n}, I(\mathfrak{b}, \mathbb{C}_\mu)_{\min}) = \bigoplus_{w \in W} H_q(\mathfrak{n}, I(\mathfrak{b}, \mathbb{C}_\mu)_{\min})_{w\lambda+\rho}$$

where the subindex indicates the corresponding eigenspace for the \mathfrak{c} -action. To calculate these eigenspaces, we use the following two ingredients: (1) The intertwining functor (see [7, Proposition 9.10]); (2) for each $w \in W_0$ there is a natural action on the standard complex for computing \mathfrak{n} -homology, which induces a linear isomorphism

$$H_q(\mathfrak{n}, I(\mathfrak{b}, \mathbb{C}_\mu)_{\min}) \rightarrow H_q(w^{-1}\mathfrak{n}, I(\mathfrak{b}, \mathbb{C}_\mu)_{\min})$$

that intertwines the C_0 action on $H_q(\mathfrak{n}, I(\mathfrak{b}, \mathbb{C}_\mu)_{\min})$ with the $w^{-1}C_0$ -action on $H_q(w^{-1}\mathfrak{n}, I(\mathfrak{b}, \mathbb{C}_\mu)_{\min})$.

Calculating with the intertwining functor, we can show

$$H_q(\mathfrak{n}, I(\mathfrak{b}, \mathbb{C}_\mu)_{\min})_{w\lambda+\rho} = \{0\} \text{ if } w \notin W_0$$

and

$$H_q(w^{-1}\mathfrak{n}, I(\mathfrak{b}, \mathbb{C}_\mu)_{\min})_{\lambda+w^{-1}\rho} \cong \mathbb{C}_{\mu-\rho+w^{-1}\rho} \text{ if } w \in W_0.$$

Hence,

$$H_q(\mathfrak{n}, I(\mathfrak{b}, \mathbb{C}_\mu)_{\min})_{w\lambda+\rho} \cong \mathbb{C}_{w\lambda+\rho} \text{ if } w \in W_0.$$

Now the result follows since

$$\begin{aligned} H^q(\mathfrak{n}^{\text{op}}, I(\mathfrak{b}, \mathbb{C}_\mu)_{\min}) &\cong H_q(\mathfrak{n}^{\text{op}}, I(\mathfrak{b}, \mathbb{C}_\mu)_{\min}) \otimes \mathbb{C}_{2\rho} = \\ &H_q(w_l\mathfrak{n}, I(\mathfrak{b}, \mathbb{C}_\mu)_{\min}) \otimes \mathbb{C}_{2\rho} \cong H_q(\mathfrak{n}, I(\mathfrak{b}, \mathbb{C}_\mu)_{\min}). \end{aligned}$$

It's worthwhile noting that the character

$$\chi_{\rho-w\rho} : C_0 \rightarrow S^1 \subseteq \mathbb{C}^*$$

is unitary. In particular, if α is a root of \mathfrak{c} in \mathfrak{b} , appearing in the sum of roots given by $\rho - w\rho$ then the root that takes the value $-\bar{\alpha}$ on \mathfrak{c}_0 also appears in the sum. Thus, in the corresponding product of characters of C_0 , we have χ_α and $\chi_{-\bar{\alpha}} = \chi_\alpha^h$. Thus,

$$|\chi_\alpha(g)\chi_{-\bar{\alpha}}(g)| = 1$$

for each $g \in C_0$.

Now we can use the Frobenius reciprocity. In particular, if χ_μ is a unitary character then there is a nondegenerate continuous invariant Hermitian form on the irreducible representation $I(\mathfrak{b}, \mathbb{C}_\mu)_{\min}$ and each of the characters $\chi_{w\lambda+\rho}$ is unitary. On the other hand, if there is a nonzero continuous invariant Hermitian form on $I(\mathfrak{b}, \mathbb{C}_\mu)_{\min}$, then since $I(\mathfrak{b}, \mathbb{C}_\mu)_{\min}$ is irreducible, the corresponding nonzero morphism

$$I(\mathfrak{b}, \mathbb{C}_\mu)_{\min} \rightarrow (I(\mathfrak{b}, \mathbb{C}_\mu)_{\min})^h$$

induces an isomorphism on the level of Harish-Chandra modules. Thus

$$H_q(\mathfrak{n}^{\text{op}}, I(\mathfrak{b}, \mathbb{C}_\mu)_{\min}) \cong H_q(\mathfrak{n}^{\text{op}}, (I(\mathfrak{b}, \mathbb{C}_\mu)_{\min})^h) \cong H^q(\mathfrak{n}^{\text{op}}, I(\mathfrak{b}, \mathbb{C}_\mu)_{\min})^h$$

and we have

$$H_q(\mathfrak{n}, I(\mathfrak{b}, \mathbb{C}_\mu)_{\min}) \cong H_q(\mathfrak{n}, I(\mathfrak{b}, \mathbb{C}_\mu)_{\min})^h.$$

In the next section, we give a general consideration of the L_0 -representation $H^q(\mathfrak{u}^{\text{op}}, H_c^q(S, \mathcal{O}(\mathfrak{p}, V_{\min})))^h$ and draw some conclusions.

4. Invariant Hermitian forms on representations

In order to complete our consideration of the continuous invariant Hermitian forms on $H_c^q(S, \mathcal{O}(\mathfrak{p}, V_{\min}))$, we will show in this section that

$$V_{\min} \cong H^q(\mathfrak{u}^{\text{op}}, H_c^q(S, \mathcal{O}(\mathfrak{p}, V_{\min})))_\sigma$$

at least when V_{\min} is irreducible and G_0 is a real form of a connected complex reductive group G . Our proof relies on the extension of a duality theorem, shown for discrete series in [14] and for open orbits in full flag manifolds in [5]. In particular, let $I(\mathfrak{p}, V)$ denote the Harish-Chandra module of $H_c^q(S, \mathcal{O}(\mathfrak{p}, V_{\min}))$ and let \mathfrak{p}^{op} be the parabolic subalgebra of \mathfrak{g} opposite to \mathfrak{p} . The duality theorem that interests us, is simply expressed as an isomorphism of Harish-Chandra modules

$$I(\mathfrak{p}, V)^\vee \cong I(\mathfrak{p}^{\text{op}}, V^\vee).$$

The proof utilized in [5] carries over in a rather straightforward manner to the context of a very nice parabolic subgroup and an irreducible representation. However, this demonstration doesn't produce a natural transformation of (exact, contravariant) functors. The proof in [14] does define a natural transformation of functors, but it is not clear to us how to generalize this.

We need to adapt our notation to the design of the argument in [5] and return to the conventions in Section 2. Let \mathfrak{h}^* denote the abstract Cartan dual and for $\lambda \in \mathfrak{h}^*$, let σ_λ denote the corresponding infinitesimal in $\mathcal{Z}(\mathfrak{l})$. Since the infinitesimal character σ for the L_0 -module V_{\min} is antidominant and regular with respect to the parabolic subalgebra \mathfrak{p} , there is a $\lambda \in \mathfrak{h}^*$ which is antidominant and regular, such that σ is the specialization of σ_λ to \mathfrak{l} in \mathfrak{p} . Let w_l be the longest element in the Weyl group of \mathfrak{h}^* . Then the abstract infinitesimal character $\sigma_{-w_l\lambda}$ is antidominant and regular. If σ^\vee denotes the infinitesimal character of V^\vee then the specialization of $\sigma_{-w_l\lambda}$ to \mathfrak{l} at \mathfrak{p}^{op} is σ^\vee . In general, if M is a \mathfrak{g} -module, we let

$$H_p(\mathfrak{u}_y, M)_{\sigma_{-w_l\lambda}}$$

denote the corresponding $Z(\mathfrak{l}_y)$ -eigenspace on the p -th \mathfrak{u}_y -homology group given by specialization. Let I denote the Harish-Chandra module $I(\mathfrak{p}, V)$. Since λ is a parameter for the infinitesimal character of I , it follows that $-w_l\lambda$ is a parameter for the infinitesimal character of I^\vee . The idea of the proof is to calculate the localization

$$\Delta_{-w_l\lambda}^{\text{alg}}(I^\vee)$$

to Y^{alg} with respect to $\sigma_{-w_l\lambda}$ by calculating the geometric fibers, which are given by the homology groups

$$H_p(\mathfrak{u}_y, I^\vee)_{\sigma_{-w_l\lambda}}.$$

Similarly, we can directly calculate the corresponding analytic localization of $(I^\vee)_{\min}$ to obtain the result. Actually, the calculation on the homology group $H_q(\mathfrak{u}^{\text{op}}, I^\vee)_{\sigma^\vee}$ is sufficient for our purposes, but the argument for the full duality theorem is a simple step more.

To calculate the homology groups, we use a construction which Chang credits to Vogan [5, Lemma 7.2]. In particular, suppose G_0 is a connected real form of a connected complex reductive Lie group G and K_0 is a maximal compact subgroup of G_0 . Let $C \subseteq G$ be a Cartan subgroup such of G that $G_0 \cap C = C_0$ is a real form of C and such that $K_0 \cap C_0 = T_0$ is a maximal torus of C_0 (C is called *stable*). Then there exists an involutive automorphism

$$\theta_C : G \rightarrow G$$

such that $\theta_C(K_0) = K_0$ and such that $\theta_C(h) = h^{-1}$ for every $h \in C$.

Given a Harish-Chandra module M for (\mathfrak{g}, K_0) , we can define a new Harish-Chandra module M^{θ_C} by applying first the automorphism and then the (\mathfrak{g}, K_0) -action on V . The property we use is that when M is irreducible, then

$$M^{\theta_C} \cong M^\vee.$$

To calculate the homology groups $H_p(\mathfrak{u}^{\text{op}}, I^\vee)_{\sigma^\vee}$, we let

$$L = P \cap P^{\text{op}}$$

and choose a stable Cartan subgroup $C \subseteq L$. Thus $\theta_C(L) = L$ and $\theta_C(K_0 \cap L_0) = K_0 \cap L_0$. Also note that the map $\theta_C : \mathfrak{u} \rightarrow \mathfrak{u}^{\text{op}}$ defines an isomorphism of L -modules

$$\mathfrak{u}^{\theta_C} \cong \mathfrak{u}^{\text{op}}.$$

Applying the automorphism θ_C to the standard complex for computing homology and using the fact that $\sigma \circ \theta_C = \sigma^\vee$, we obtain

$$H_p(\mathfrak{u}^{\text{op}}, I^\vee)_{\sigma^\vee} \cong H_p(\mathfrak{u}^{\theta_C}, I^{\theta_C})_{\sigma^\vee} \cong (H_p(\mathfrak{u}, I)_\sigma)^{\theta_C} \cong \begin{cases} \{0\} & p \neq q \\ V^{\theta_C} \cong V^\vee & p = q \end{cases}$$

since V is irreducible.

On the hand, suppose $\mathfrak{p}_y \in Y$ and $C \subseteq P_y$ is a stable Cartan subgroup such that $\theta_C(\mathfrak{p}_y) = \mathfrak{p}_y^{\text{op}} \notin S$. We choose a Levi factor L of P_y , that contains C and such that $K_0 \cap P_y \subseteq L$ and let \mathfrak{l} be the Lie algebra of L . Then $H_p(\mathfrak{u}_y, I^\vee)$ is a Harish-Chandra module (it has finite length and is admissible) for $(\mathfrak{l}, K_0 \cap L)$. As before, there is an isomorphism of L -modules

$$\mathfrak{u}_y^{\theta_C} \cong \mathfrak{u}_y^{\text{op}}$$

and the infinitesimal character for $Z(\mathfrak{l})$ given by the composition of the specialization of σ_{-w_λ} to y with θ_C is equal to the specialization of σ_λ to the opposite point. Thus we have

$$H_p(\mathfrak{u}_y, I^\vee)_{\sigma_{-w_\lambda}} \cong H_p((\mathfrak{u}_y^{\text{op}})^{\theta_C}, I^{\theta_C})_{\sigma_{-w_\lambda}} \cong (H_p(\mathfrak{u}_y^{\text{op}}, I)_{\sigma_\lambda})^{\theta_C} = 0.$$

Theorem 4.1. *Suppose G_0 is a real form of a connected complex reductive group G . Let P be very nice parabolic subgroup and suppose V_{\min} is an irreducible minimal globalization for L_0 with an antidominant and regular infinitesimal character. Let $I(P, V)$ be the Harish-Chandra module of the irreducible minimal globalization $H_c^q(S, \mathcal{O}(P, V_{\min}))$ and let $I(P^{\text{op}}, V^\vee)$ be the Harish-Chandra module of minimal globalization $H_c^q(S, \mathcal{O}(P^{\text{op}}, (V^\vee)_{\min}))$. Then*

$$I(P, V)^\vee \cong I(P^{\text{op}}, V^\vee).$$

Proof. When G_0 is connected, then the previous discussion establishes the formula in the following way. Letting $I = I(P, V)$, then the calculations on the Lie algebra homology groups determine the Harish-Chandra modules of the geometric fibers of the analytic localization $L\Delta_{Y, \sigma_{-w_l\lambda}}((I^\vee)_{\min})$ of $(I^\vee)_{\min}$ to Y with respect to the abstract infinitesimal character $\sigma_{-w_l\lambda}$. Hence

$$L\Delta_{Y, \sigma_{-w_l\lambda}}((I^\vee)_{\min}) \cong \mathcal{O}(P^{\text{op}}, (V^\vee)_{\min})|_{S^{\text{op}}} [q]$$

where $\mathcal{O}(P, (V^\vee)_{\min})|_{S^{\text{op}}}$ is the extension by zero of $\mathcal{O}(P, (V^\vee)_{\min})$ on S^{op} to all of Y . Thus the hypercohomology of the of the analytic localization provides the isomorphism

$$(I^\vee)_{\min} \cong H_c^q(S, \mathcal{O}(P^{\text{op}}, (V^\vee)_{\min})).$$

For the general case, let $(G_0)^e, (L_0)^e$ be the respective identity components for G_0 and L_0 . Then $(G_0)^e \cap P = (L_0)^e$ so that $(K_0)^e \cap P = (K_0 \cap L_0)^e$. In particular, we have a natural inclusion

$$K_0 \cap L_0 / (K_0 \cap L_0)^e \rightarrow K_0 / (K_0)^e.$$

The Harish-Chandra V is a finite sum of irreducible \mathfrak{I} -modules V_m so we can choose a (finite) set of representatives $\{k_j\}$ for the coclasses of $(K_0 \cap L_0)^e$ in $K_0 \cap L_0$ and

$$V = \bigoplus V_m$$

where the $K_0 \cap L_0$ -action is characterized by the $(K_0 \cap L_0)^e$ -action on the irreducible \mathfrak{I} -modules V_m and by the way the operators corresponding to the elements k_j permute the irreducible modules V_m . Thus

$$\Gamma(P, V)^\vee = \Gamma(P, \bigoplus V_m)^\vee \cong \bigoplus \Gamma(P, V_m)^\vee$$

where the K_0 -action on $\bigoplus \Gamma(P, V_m)^\vee$ is characterized by the $(K_0)^e$ -action on the irreducible \mathfrak{I} -modules $\Gamma(P, V_m)^\vee$ and by the way the operators k_j permute these \mathfrak{I} -modules.

Similarly,

$$\Gamma(P^{\text{op}}, V^\vee) = \Gamma(P^{\text{op}}, \bigoplus (V_m)^\vee) \cong \bigoplus \Gamma((P^{\text{op}}, (V_m)^\vee).$$

Hence the theorem is true, since

$$\Gamma((P^{\text{op}}, (V_m)^\vee) \cong \Gamma(P, V_m)^\vee. \quad \square$$

Theorem 4.2. *Suppose G_0 is a real form of a connected complex reductive group G . Let P be a nice parabolic subgroup and V_{\min} be an irreducible minimal globalization for L_0 with an antidominant and regular infinitesimal character. Then a nondegenerate continuous invariant Hermitian form on V_{\min} naturally induces a nondegenerate continuous invariant Hermitian form on $H_q(S, \mathcal{O}(P, V_{\min}))$.*

Proof. We can choose a maximal compact subgroup K_0 of G_0 such that P is a very nice parabolic subgroup. Let $I = I(P, V)$ be the Harish-Chandra module of $H_q(S, \mathcal{O}(P, V_{\min}))$. From our previous work we have

$$V^\vee \cong H_q(\mathfrak{u}^{\text{op}}, I(P^{\text{op}}, V^\vee))_{\sigma^\vee} \cong H_q(\mathfrak{u}^{\text{op}}, I^\vee)_{\sigma^\vee}.$$

Using the result in [4], we have

$$H_q(\mathfrak{u}^{\text{op}}, I^\vee) \cong H^q(\mathfrak{u}^{\text{op}}, I)^\vee \Rightarrow V^\vee \cong (H^q(\mathfrak{u}^{\text{op}}, I)_\sigma)^\vee.$$

Hence,

$$V \cong H^q(\mathfrak{u}^{\text{op}}, I)_\sigma$$

and therefore,

$$(V_{\min})^h \cong (H^q(\mathfrak{u}^{\text{op}}, I_{\min})_\sigma)^h \cong H_q(\mathfrak{u}, (I_{\min})^h)_{\sigma^h}.$$

It follows that a nondegenerate Hermitian form (in fact a nonzero Hermitian form) induces a nonzero morphism

$$V_{\min} \rightarrow H_q(\mathfrak{u}, (I_{\min})^h).$$

By the Frobenius reciprocity, we obtain a nonzero morphism

$$I_{\min} \rightarrow (I_{\min})^h$$

which is injective, since I_{\min} is irreducible. The result follows since the space of continuous invariant Hermitian forms on I_{\min} is a real form of the space of continuous invariant sesquilinear forms. \square

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This paper is available via <http://nyjm.albany.edu/j/2022/28-44.html>.