

## Isolated points of the Zariski space

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ABSTRACT. Let  $D$  be an integral domain and  $L$  be a field containing  $D$ . We study the isolated points of the Zariski space  $\text{Zar}(L|D)$ , with respect to the constructible topology. In particular, we completely characterize when  $L$  (as a point) is isolated and, under the hypothesis that  $L$  is the quotient field of  $D$ , when a valuation domain of dimension 1 is isolated; as a consequence, we find all isolated points of  $\text{Zar}(D)$  when  $D$  is a Noetherian domain and, under the hypothesis that  $D$  and  $D'$  are Noetherian, local and countable, we characterize when  $\text{Zar}(D)$  and  $\text{Zar}(D')$  are homeomorphic. We also show that if  $V$  is a valuation domain and  $L$  is transcendental over  $V$  then the set of extensions of  $V$  to  $L$  has no isolated points.

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### 1. Introduction

Let  $D$  be an integral domain with quotient field  $K$ , and let  $L$  be a field containing  $K$ . The *Zariski space* of  $L$  over  $D$ , denoted by  $\text{Zar}(L|D)$ , is the set of all valuation rings containing  $D$  and having quotient field  $L$ . O. Zariski introduced this set (under the name *abstract Riemann surface*) and endowed it with a natural topology (later called the *Zariski topology*) during its study of resolution of singularities; in particular, he used the compactness of the Zariski space to reduce the problem of gluing infinitely many projective models to the gluing of only

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finitely many of them [30, 31]. Later on, it was showed that  $\text{Zar}(L|D)$  enjoys even deeper topological properties: in particular, it is a *spectral space*, meaning that there is always a ring  $R$  such that  $\text{Spec}(R)$  (endowed with the Zariski topology) is homeomorphic to  $\text{Zar}(L|D)$ , and an example of such an  $R$  can be found using the Kronecker function ring construction [5, 6, 8]. Beyond being a very natural example of a spectral space “occurring in nature”, the Zariski topology can also be used, for example, to study representation of integral domains as intersection of overrings [19, 20, 21], or in real and rigid algebraic geometry [15, 24].

As a spectral space, two other topologies can be constructed on  $\text{Zar}(L|D)$  starting from the Zariski topology: the *inverse* and the *constructible* (or *patch*) topology. Both of them give rise to spectral spaces (in particular, they are compact); furthermore, the constructible topology gains the property of being Hausdorff, and plays an important role in the topological characterization of spectral spaces (see for example Hochster’s article [14]). The constructible topology can also be studied through ultrafilters [7], and this point of view allows to give many examples of spectral spaces, for example by finding them inside other spectral spaces (see [21, Example 2.2(1)] for some very general constructions, [27] for examples in the overring case, and [10, 9] for examples in the setting of semistar operations).

In this paper, we want to study the points of  $\text{Zar}(L|D)$  that are isolated, with respect to the constructible topology. Our starting point is a new interpretation of a result about the compactness of spaces in the form  $\text{Zar}(K|D) \setminus \{V\}$  [26, Theorem 3.6], where  $K$  is the quotient field of  $D$ : in particular, we show that if  $V$  is isolated in  $\text{Zar}(L|D)$ , where  $L$  is a field containing  $V$ , then  $V$  is the integral closure of  $D[x_1, \dots, x_n]_M$  for some  $x_1, \dots, x_n \in L$ , where  $M$  is a maximal ideal of  $D[x_1, \dots, x_n]$  (Theorem 3.4). Through this result, we characterize when  $L$  is an isolated point of  $\text{Zar}(L|D)^{\text{cons}}$  (i.e.,  $\text{Zar}(L|D)$  endowed with the constructible topology; Proposition 4.1) and, under the hypothesis that  $L = K$  is the quotient field of  $D$ , when the one-dimensional valuation overrings are isolated (Theorem 5.2).

In Section 6, we study the isolated points of the constructible topology when  $D$  is a Noetherian domain and  $L = K$  is its quotient field. Theorem 6.3 gives a complete characterization:  $V \in \text{Zar}(K|D) = \text{Zar}(D)$  is isolated if and only if the center  $P$  of  $V$  on  $D$  has height at most 1 and  $P$  is contained in only finitely many prime ideals; in particular, this cannot happen if  $D$  is local and of dimension at least 3. In the countable case, we also give a complete characterization of when  $\text{Zar}(D)^{\text{cons}} \simeq \text{Zar}(D')^{\text{cons}}$  under the hypothesis that  $D$  and  $D'$  are Noetherian and local (Theorem 6.11).

The last two sections of the paper explore the case of extension of valuations. Section 7 studies the case where  $D$  itself is a field: in particular, we show that if the transcendence degree of  $L$  over  $D$  is at least 2 then  $\text{Zar}(L|D)^{\text{cons}}$  has no isolated points, improving [3, Theorem 4.45]. In Section 8, we show that if  $V$  is a valuation domain that is not a field and  $K(X)$  is the field of rational functions

in one indeterminate, then the set of extensions of  $V$  to  $K(X)$  has no isolated points (Theorem 8.2), and as a consequence we further extend [3, Theorem 4.45] to  $\text{Zar}(L|D)^{\text{cons}}$  when  $D$  is an arbitrary integral domain (Theorem 7.3 and Corollary 8.6).

## 2. Notation and preliminaries

Throughout the paper, all rings will be commutative, unitary and will have no zero-divisors (that is, they are integral domains). We usually denote by  $D$  such a domain and by  $K$  its quotient field; we use  $\bar{D}$  to denote the integral closure of  $D$  in  $K$ .

**2.1. Spectral spaces.** A topological space  $X$  is *spectral* if it is homeomorphic to the prime spectrum of a ring, endowed with the Zariski topology; spectral spaces can also be characterized in a purely topological way (see [14] and [4]). Among their properties, spectral spaces are always compact and have a basis of open and compact sets. If  $\Delta \subseteq X$ , we denote by  $\text{Cl}(\Delta)$  the closure of  $\Delta$ . The topology of  $X$  induces an order such that  $x \leq y$  if and only if  $y \in \text{Cl}(x)$ . If  $Y \subseteq X$ , the *closure under generization* of  $Y$  is the set  $Y^{\text{gen}} := \{x \in X \mid x \leq y \text{ for some } y \in Y\}$ , where  $\leq$  is the order induced by the topology, and  $Y$  is *closed by generizations* if  $Y = Y^{\text{gen}}$ .

If  $X$  is a spectral space, the *inverse topology* on  $X$  is the coarsest topology such that the open and compact subsets of  $X$  are closed. We denote by  $X^{\text{inv}}$  the space  $X$ , endowed with the inverse topology. A subset  $Y \subseteq X$  is closed in the inverse topology if and only if it is compact in the starting topology and closed by generizations; in particular, if  $Y$  is compact in the starting topology then its closure in the inverse topology is  $Y^{\text{gen}}$ .

If  $X$  is a spectral space, the *constructible topology* (or *patch topology*) on  $X$  is the coarsest topology such that the open and compact subsets of  $X$  are both open and closed. We denote by  $X^{\text{cons}}$  the space  $X$ , endowed with the constructible topology; if  $Y \subseteq X$ , we denote by  $Y^{\text{cons}}$  the subset  $Y$  considered with respect to the constructible topology, and by  $\text{Cl}^{\text{cons}}(Y)$  the closure of  $Y$  in  $X^{\text{cons}}$ . If  $Y = \text{Cl}^{\text{cons}}(Y)$ , then  $Y$  is compact in the starting topology; conversely, if  $Y$  is closed in the starting topology or in the inverse topology, then it is closed also in the constructible topology.

Both  $X^{\text{inv}}$  and  $X^{\text{cons}}$  are spectral spaces, and in particular compact spaces; moreover,  $X^{\text{cons}}$  is Hausdorff and zero-dimensional.

A map  $f : X \rightarrow Y$  of spectral spaces is a *spectral map* if  $f^{-1}(\Omega)$  is open and compact for every open and compact subset  $\Omega$  of  $Y$ ; in particular, a spectral map is continuous. If  $f$  is both spectral and closed, then it is also proper, and in particular  $f^{-1}(\Omega)$  is compact for every compact subset  $\Omega$  of  $Y$  [4, 5.3.7(i)]. If  $f : X \rightarrow Y$  is a spectral map, then it is spectral also when  $X$  and  $Y$  are both endowed with the inverse topology, and when they are both endowed with the constructible topology [4, Theorem 1.3.21]. In the latter case,  $f$  is also closed, since it is a continuous map between Hausdorff compact spaces.

**2.2. Isolated points.** If  $X$  is a topological space, a point  $p \in X$  is *isolated* in  $X$  if  $\{p\}$  is an open set. If  $X$  has no isolated points, then  $X$  is said to be *perfect*. The set of points that are not isolated in  $X$  is a closed set, called the *derived set* of  $X$ .

If  $\Omega \subseteq X$  and  $p \in \Omega$  is isolated in  $X$ , then  $p$  is also isolated in  $\Omega$ ; if  $\Omega$  is open, then  $p$  is isolated in  $X$  if and only if  $p$  is isolated in  $\Omega$ .

**2.3. Valuation domains.** A *valuation domain* is an integral domain  $V$  such that, for every  $x \neq 0$  in the quotient field of  $V$ , at least one of  $x$  and  $x^{-1}$  is in  $V$ . Any valuation domain is local; we denote the maximal ideal of  $V$  by  $\mathfrak{m}_V$ . If  $L$  is a field containing the quotient field  $K$  of  $V$ , an *extension* of  $V$  to  $L$  is a valuation domain  $W$  having quotient field  $L$  such that  $W \cap K = V$ . We denote the set of extension of  $V$  to  $L$  by  $\mathcal{E}(L|V)$ ; this set is always nonempty (see e.g. [12, Theorem 20.1]).

If  $D$  is an integral domain and  $L$  is a field containing  $D$ , the *Zariski space* (or *Zariski-Riemann space*) of  $L$  over  $D$ , denoted by  $\text{Zar}(L|D)$ , is the set of all valuation domains containing  $D$  and having quotient field  $L$ . The Zariski space  $\text{Zar}(L|D)$  is always nonempty. When  $L$  is the quotient field of  $D$ , we denote  $\text{Zar}(L|D)$  simply by  $\text{Zar}(D)$ , and we call its elements the *valuation overrings* of  $D$ .<sup>1</sup> If  $D'$  is the integral closure of  $D$  in  $L$ , then  $\text{Zar}(L|D) = \text{Zar}(L|D')$ ; in particular,  $\text{Zar}(D) = \text{Zar}(\overline{D})$ . A valuation ring in  $\text{Zar}(L|D)$  is *minimal* if it is minimal with respect to containment.

The Zariski-Riemann space  $\text{Zar}(L|D)$  can be endowed with a natural topology, called the *Zariski topology*, which is the topology generated by the basic open sets

$$\mathcal{B}(x_1, \dots, x_n) := \{V \in \text{Zar}(L|D) \mid x_1, \dots, x_n \in V\},$$

as  $x_1, \dots, x_n$  range among the elements of  $L$ ; we use the notation  $\mathcal{B}^L(x_1, \dots, x_n)$  if we need to underline the field  $L$ . Under this topology,  $\text{Zar}(L|D)$  is a spectral space whose order is the opposite of the containment order [6, 5]; in particular, the minimal valuation rings in  $\text{Zar}(L|D)$  are maximal with respect to the order induced by the Zariski topology. As a spectral space, we can define the inverse and the constructible topology on  $\text{Zar}(L|D)$ ; a set  $\Delta \subseteq \text{Zar}(L|D)$  is closed with respect to the inverse topology if and only if it is compact with respect to the Zariski topology and  $\Delta = \{W \in \text{Zar}(L|D) \mid W \supseteq V \text{ for some } V \in \Delta\}$  [8, Remark 2.2 and Proposition 2.6].

Since  $\mathcal{B}(z_1, \dots, z_n) = \mathcal{B}(z_1) \cap \dots \cap \mathcal{B}(z_n)$  for every  $z_1, \dots, z_n \in L$ , a basis of the constructible topology of  $\text{Zar}(L|D)$  is the family of the sets in the form  $\mathcal{B}(x_1, \dots, x_n) \cap \mathcal{B}(y_1)^c \cap \dots \cap \mathcal{B}(y_m)^c$ , as  $x_1, \dots, x_n, y_1, \dots, y_m$  range in  $L$ . In particular,  $V$  is isolated in  $\text{Zar}(L|D)^{\text{cons}}$  if and only if

$$\begin{aligned} \{V\} &= \mathcal{B}(x_1, \dots, x_n) \cap \mathcal{B}(y_1)^c \cap \dots \cap \mathcal{B}(y_m)^c = \\ &= \text{Zar}(L|D[x_1, \dots, x_n]) \cap \mathcal{B}(y_1)^c \cap \dots \cap \mathcal{B}(y_m)^c \end{aligned}$$

for some  $x_1, \dots, x_n, y_1, \dots, y_m \in L$ .

<sup>1</sup>An *overring* of  $D$  is, more generally, a ring contained between  $D$  and its quotient field.

If  $L' \subseteq L$  is a field extension and  $D \subseteq L'$ , we have a restriction map

$$\begin{aligned} \rho : \text{Zar}(L|D) &\longrightarrow \text{Zar}(L'|D), \\ V &\longmapsto V \cap L'. \end{aligned}$$

The map  $\rho$  is surjective due to Chevalley's extension theorem (see e.g. [1, Theorme 5.21] or [12, Theorem 19.5]), and is a spectral map since  $\rho^{-1}(\mathcal{B}^{L'}(x)) = \mathcal{B}^L(x)$ . Therefore, it is spectral and closed with respect to the constructible topology (on both sets). In particular, if  $V \in \text{Zar}(L'|D)$ , then  $\mathcal{E}(L|V) = \rho^{-1}(V)$ ; hence,  $\mathcal{E}(L|V)$  is always closed in  $\text{Zar}(L|D)^{\text{cons}}$ , and in particular it is compact both in the Zariski and the constructible topology.

Since, by definition, the spectrum  $\text{Spec}(D)$  is a spectral space (when endowed with the Zariski topology), we can define the inverse and the constructible topology also on  $\text{Spec}(D)$ . For every ideal  $I$  of  $D$ , set  $\mathcal{V}(I) := \{P \in \text{Spec}(D) \mid I \subseteq P\}$  and  $\mathcal{D}(I) := \text{Spec}(D) \setminus \mathcal{V}(I)$ : then, a basis of  $\text{Spec}(D)^{\text{cons}}$  is given by the intersections  $\mathcal{D}(aD) \cap \mathcal{V}(I)$ , as  $a$  ranges in  $D$  and  $I$  among the finitely generated ideals of  $D$  [4, Theorem 12.1.10(iv)].

For every field  $L$ , we can define a map

$$\begin{aligned} \gamma : \text{Zar}(L|D) &\longrightarrow \text{Spec}(D), \\ V &\longmapsto \mathfrak{m}_V \cap D, \end{aligned}$$

which is called the *center map*. When  $\text{Zar}(L|D)$  and  $\text{Spec}(D)$  are endowed with the Zariski topology,  $\gamma$  is spectral (in particular, continuous; see [32, Chapter VI, §17, Lemma 1] or [5, Theorem 4.1]), surjective (this follows, for example, from [1, Theorem 5.21] or [12, Theorem 19.6]) and closed [5, Theorem 2.5], so in particular it is proper. Therefore,  $\gamma$  is a spectral map also when  $\text{Zar}(L|D)$  and  $\text{Spec}(D)$  are endowed with their respective constructible topologies.

### 3. General results

We begin by establishing some general criteria to determine which valuation domains are isolated in  $\text{Zar}(D)$ .

Let  $D$  be an integral domain: a prime ideal is called *essential* if  $D_P$  is a valuation domain, and  $D_P$  is said to be an *essential valuation overring* of  $D$ . We shall need the following weaker notion: we say that a prime ideal  $P$  of  $D$  is *almost essential* if there is a unique valuation overring of  $D$  having center  $P$ ; equivalently,  $P$  is almost essential if and only if the integral closure of  $D_P$  is a valuation domain  $V$ . When this happens, we say that  $V$  is an *almost essential valuation overring* of  $D$ .

In the context of almost essential primes and valuation overrings, isolated valuation rings correspond to isolated prime ideals.

**Proposition 3.1.** *Let  $D$  be an integral domain, and let  $P$  be an almost essential prime ideal of  $D$ ; let  $V$  be the valuation overring with center  $P$ . Then,  $V$  is isolated in  $\text{Zar}(D)^{\text{cons}}$  if and only if  $P$  is isolated in  $\text{Spec}(D)^{\text{cons}}$ .*

**Proof.** Let  $\gamma : \text{Zar}(D) \longrightarrow \text{Spec}(D)$  be the center map. If  $P$  is isolated in  $\text{Spec}(D)^{\text{cons}}$ , then  $\{P\}$  is open and thus, as  $\gamma$  is continuous,  $\{V\} = \gamma^{-1}(\{P\})$  is open in  $\text{Zar}(D)^{\text{cons}}$ , i.e.,  $V$  is isolated. Conversely, if  $V$  is isolated then  $\text{Zar}(D) \setminus \{V\}$  is closed, with respect to the constructible topology, and thus  $\gamma(\text{Zar}(D) \setminus \{V\}) = \text{Spec}(D) \setminus \{P\}$  is closed in  $\text{Spec}(D)^{\text{cons}}$ . Hence,  $\{P\}$  is open and  $P$  is isolated in  $\text{Spec}(D)^{\text{cons}}$ , as claimed.  $\square$

**Corollary 3.2.** *Let  $D$  be a Prüfer domain, and let  $V$  be a valuation overring of  $D$  with center  $P$ . Then,  $V$  is isolated in  $\text{Zar}(D)^{\text{cons}}$  if and only if  $P$  is isolated in  $\text{Spec}(D)^{\text{cons}}$ . In particular,  $\text{Zar}(D)^{\text{cons}}$  is perfect if and only if  $\text{Spec}(D)^{\text{cons}}$  is perfect.*

**Proof.** Since  $D$  is a Prüfer domain, every valuation overring is essential. The claim follows from Proposition 3.1.  $\square$

In general, almost essential valuation overrings are rare; for example, if  $D$  is Noetherian, no prime ideal of height 2 or more can be almost essential. For this reason, we need more general results; the first step is connecting isolated valuation rings with compactness.

**Proposition 3.3.** *Let  $X$  be a spectral space, and let  $x$  be a maximal element with respect to the order induced by the topology. Then, the following are equivalent:*

- (i)  $x$  is isolated in  $X^{\text{cons}}$ ;
- (ii)  $X \setminus \{x\}$  is compact, with respect to the starting topology;
- (iii)  $X \setminus \{x\}$  is closed, with respect to the inverse topology.

**Proof.** Let  $Y := X \setminus \{x\}$ .

The equivalence of (ii) and (iii) follows from the fact that  $Y$  is closed by generalizations.

If (i) holds, then  $\{x\}$  is an open set in the constructible topology, and thus  $Y$  is closed; since  $X^{\text{cons}}$  is compact, it follows that  $Y$  is compact in the constructible topology and thus also in the Zariski topology (which is coarser). Thus, (ii) holds.

Conversely, if (iii) holds, then  $Y$  is closed also in the constructible topology; hence,  $\{x\}$  is open and  $x$  is isolated. Thus, (i) holds.  $\square$

In particular, the previous proposition applies when  $X = \text{Zar}(L|D)$  and  $V$  is a minimal element of  $\text{Zar}(L|D)$ , with respect to containment. In this case, the fact that  $\text{Zar}(L|D) \setminus \{V\}$  is compact has very strong consequences.

**Theorem 3.4.** *Let  $D$  be an integral domain and let  $V \in \text{Zar}(L|D)$ . Then, the following are equivalent.*

- (i)  $V$  is isolated in  $\text{Zar}(L|D)^{\text{cons}}$ ;
- (ii) there are  $x_1, \dots, x_n \in L$  and a maximal ideal  $M$  of  $D[x_1, \dots, x_n]$  such that  $V$  is the integral closure of  $D[x_1, \dots, x_n]_M$  and  $M$  is isolated in

$$\text{Spec}(D[x_1, \dots, x_n])^{\text{cons}};$$

(iii) there are  $x_1, \dots, x_n \in L$  and a prime ideal  $P$  of  $D[x_1, \dots, x_n]$  such that  $V$  is the integral closure of  $D[x_1, \dots, x_n]_P$  and  $P$  is isolated in

$$\text{Spec}(D[x_1, \dots, x_n])^{\text{cons}}.$$

**Proof.** Let  $X$  be an indeterminate over  $D$ , and let  $R := D + XL[[X]]$ . By the reasoning in the proof of [28, Proposition 3.3] (or by Lemma 4.2 below) the Zariski space  $\text{Zar}(L|D)^{\text{cons}}$  is homeomorphic to  $(\text{Zar}(R) \setminus \{L((X))\})^{\text{cons}}$ , which is open in  $\text{Zar}(R)^{\text{cons}}$ ; in particular, a  $W \in \text{Zar}(L|D)$  is isolated with respect to the constructible topology if and only if  $W + XL[[X]]$  is isolated in  $\text{Zar}(R)^{\text{cons}}$ . Therefore, without loss of generality we can suppose that  $L$  is the quotient field of  $D$ .

(i)  $\implies$  (iii) Since  $V$  is isolated, there are  $x_1, \dots, x_k, y_1, \dots, y_m \in L$  such that  $\{V\} = \text{Zar}(D[x_1, \dots, x_k]) \cap \mathcal{B}(y_1)^c \cap \dots \cap \mathcal{B}(y_m)^c$ . In particular,  $V$  is a minimal valuation overring of  $D[x_1, \dots, x_k]$ . By Proposition 3.3,  $\text{Zar}(D[x_1, \dots, x_k]) \setminus \{V\}$  is compact, with respect to the Zariski topology; therefore, by [26, Theorem 3.6], there are  $x_{k+1}, \dots, x_n \in L$  such that  $V$  is the integral closure of

$$D[x_1, \dots, x_k][x_{k+1}, \dots, x_n]_M = D[x_1, \dots, x_n]_M$$

for some maximal ideal  $M$  of  $D[x_1, \dots, x_n]$ . Hence,  $M$  is almost essential in  $D[x_1, \dots, x_n]$ , and by Proposition 3.1,  $M$  is isolated in  $\text{Spec}(D[x_1, \dots, x_n])^{\text{cons}}$ . Thus (ii) holds.

(ii)  $\implies$  (iii) is obvious.

(iii)  $\implies$  (i) The set  $\text{Zar}(D[x_1, \dots, x_n]) = \mathcal{B}(x_1, \dots, x_n)$  is open in the constructible topology, and thus  $V$  is isolated in  $\text{Zar}(D)^{\text{cons}}$  if and only if it is isolated in  $\text{Zar}(D[x_1, \dots, x_n])^{\text{cons}}$ . By hypothesis,  $P$  is almost essential for  $D[x_1, \dots, x_n]$ , and thus by Proposition 3.1 the integral closure  $V$  of  $D[x_1, \dots, x_n]_P$  is isolated, as claimed.  $\square$

#### 4. Dimension 0

In this section, we study when the field  $L$  is isolated in  $\text{Zar}(L|D)^{\text{cons}}$ . If  $L$  is the quotient field of  $D$ , then  $L$  is an essential valuation overring of  $D$ , and thus one can reason through Proposition 3.1; however, it is possible to use a more general approach.

A domain  $D$  with quotient field  $K$  is said to be a *Goldman domain* (or a *G-domain*) if  $K$  is a finitely generated  $D$ -algebra, or equivalently if  $K = D[u]$  for some  $u \in K$ .

**Proposition 4.1.** *Let  $D$  be an integral domain with quotient field  $K$ , and let  $L$  be a field extension of  $K$ . Then,  $L$  is isolated in  $\text{Zar}(L|D)^{\text{cons}}$  if and only if  $D$  is a Goldman domain and  $K \subseteq L$  is an algebraic extension.*

**Proof.** Suppose first that the two conditions hold. Then,  $K = D[u]$  for some  $u \in K$ ; since  $K \subseteq L$  is algebraic, it follows that  $\mathcal{B}(u) = \text{Zar}(L|K) = \{L\}$ . Hence,  $L$  is isolated in  $\text{Zar}(L|D)^{\text{cons}}$ .

Conversely, suppose that  $L$  is isolated. By Theorem 3.4, there are  $x_1, \dots, x_n \in L$  such that  $L$  is the integral closure of  $D[x_1, \dots, x_n]_M$  for some maximal ideal



$M$ ; since  $M$  must have height 0,  $F := D[x_1, \dots, x_n]$  must be a field such that  $F \subseteq L$  is algebraic.

Suppose that  $F$  is transcendental over  $K$ : then, we can take a transcendence basis  $y_1, \dots, y_k$  of  $F$  over  $K$ . By construction,  $F$  is algebraic over the quotient field of  $D[y_1, \dots, y_k]$ ; since  $F$  is a field, it is a Goldman domain, and thus by [16, Theorem 22] so should be  $D[y_1, \dots, y_k]$ , against [16, Theorem 21]. Thus,  $F$  is algebraic over  $K$ . Applying again [16, Theorem 22] to the extension  $D \subset F$ , we see that  $D$  is a Goldman domain; furthermore,  $L$  is algebraic over  $F$  and thus over  $K$ . The claim is proved.  $\square$

The previous result can be used to give some necessary conditions for  $V$  to be isolated. We premise a lemma.

**Lemma 4.2.** *Let  $D$  be an integral domain,  $L$  be a field containing  $D$ , and let  $W \in \text{Zar}(L|D)$ . Let  $\pi : W \rightarrow W/\mathfrak{m}_W$  be the quotient map. Then, the map*

$$\begin{aligned} \bar{\pi} : \{Z \in \text{Zar}(L|D) \mid Z \subseteq W\} &\longrightarrow \text{Zar}(W/\mathfrak{m}_W|D/(\mathfrak{m}_W \cap D)), \\ Z &\longmapsto \pi(Z) \end{aligned}$$

*is a homeomorphism, when both sets are endowed with either the Zariski or the constructible topology.*

**Proof.** Let  $Z \in \text{Zar}(L|D)$ : then,  $\ker \pi = \mathfrak{m}_W \subseteq Z$  since  $Z$  and  $W$  are valuation domains with the same quotient field and  $Z \subseteq W$ . Hence,  $\pi(Z) = Z/\mathfrak{m}_W$  is a valuation ring containing  $D/(\mathfrak{m}_W \cap D)$ ; moreover, since  $W$  is a localization of  $Z$ ,  $W/\mathfrak{m}_W$  is a localization of  $Z/\mathfrak{m}_W$  and thus  $W/\mathfrak{m}_W$  is the quotient field of  $\pi(Z)$ . Hence,  $\bar{\pi}$  is well-defined.

Moreover, if  $Z' \in \text{Zar}(W/\mathfrak{m}_W|D/(\mathfrak{m}_W \cap D))$ , then  $Z := \pi^{-1}(Z')$  is the pullback of  $Z'$  along the quotient  $W \rightarrow W/\mathfrak{m}_W$ . Thus,  $Z$  is a valuation domain by [11, Proposition 1.1.8(1)], and its quotient field is  $L$  by [11, Lemma 1.1.4(10)]. Hence  $\bar{\pi}$  is surjective. Furthermore, if  $Z \in \text{Zar}(L|D)$  and  $Z \subseteq W$ , then  $\ker \pi \subseteq Z$  and thus  $\pi^{-1}(\pi(Z)) = Z$ ; hence,  $\bar{\pi}$  is bijective.

Let now  $x \in W/\mathfrak{m}_W$ . Then,  $Z \in \bar{\pi}^{-1}(\mathcal{B}(x))$  if and only if  $x \in \pi(Z)$ . Since  $\ker \pi \subseteq Z$ , this happens if and only if  $Z$  contains all of  $\pi^{-1}(x)$ ; thus, for every  $y \in \pi^{-1}(x)$ , we have  $\bar{\pi}^{-1}(\mathcal{B}(x)) = \mathcal{B}(y)$ , and likewise  $\bar{\pi}(\mathcal{B}(x)) = \mathcal{B}(\pi(x))$  for every  $x \in L$ . Hence,  $\bar{\pi}$  is continuous and open when both  $\{Z \in \text{Zar}(L|D) \mid Z \subseteq W\}$  and  $\text{Zar}(W/\mathfrak{m}_W|D/(\mathfrak{m}_W \cap D))$  are endowed with the Zariski topology, and thus it is a homeomorphism. It follows that it is also a homeomorphism when both sets are endowed with the constructible topology, as claimed.  $\square$

**Proposition 4.3.** *Let  $V \in \text{Zar}(D)$  be a valuation domain with center  $P$  on  $D$ . If  $V$  is isolated in  $\text{Zar}(D)^{\text{cons}}$ , then the field extension  $D_P/PD_P \subseteq V/\mathfrak{m}_V$  is algebraic.*

**Proof.** Consider  $\Delta := \{W \in \text{Zar}(D) \mid W \subseteq V\}$ . Since  $\mathfrak{m}_V \cap D = P$ , by Lemma 4.2, the quotient map  $V \rightarrow V/\mathfrak{m}_V$  induces a homeomorphism between  $\Delta^{\text{cons}}$  and  $\text{Zar}(V/\mathfrak{m}_V|D/P)^{\text{cons}}$ , and thus  $V/\mathfrak{m}_V$  is isolated in  $\text{Zar}(V/\mathfrak{m}_V|D/P)^{\text{cons}}$ . Let  $F$  be the quotient field of  $D/P$ : then,  $F = (D/P)_{P/P} = D_P/PD_P$ . By Proposition 4.1,  $F \subseteq V/\mathfrak{m}_V$  must be algebraic, as claimed.  $\square$



**Corollary 4.4.** *Let  $D$  be an integral domain, let  $\gamma : \text{Zar}(D) \longrightarrow \text{Spec}(D)$  be the center map and let  $V \in \text{Zar}(D)$ . If  $V$  is isolated in  $\text{Zar}(D)^{\text{cons}}$ , then  $V$  is minimal in  $\gamma^{-1}(\gamma(V))$ .*

**Proof.** Let  $P := \gamma(V)$ . If  $V$  is not minimal, then  $V/\mathfrak{m}_V$  is not minimal in  $\text{Zar}(V/\mathfrak{m}_V|D_P/PD_P)$ ; hence, the extension  $D_P/PD_P \subseteq V/\mathfrak{m}_V$  cannot be algebraic, against Proposition 4.3.  $\square$

## 5. Dimension 1

We now analyze the case where the valuation ring  $V$  has (Krull) dimension 1; however, the methods we use only work when  $V$  is a valuation overring of  $D$ , i.e., only for the space  $\text{Zar}(D) = \text{Zar}(K|D)$ , where  $K$  is the quotient field of  $D$ . Unlike in the proof of Theorem 3.4, we cannot use [28, Proposition 3.3] to extend these results to arbitrary Zariski spaces  $\text{Zar}(L|D)$ , because that construction changes the dimension of the valuation domains involved.

The idea of this section is to study the maximal ideals of the finitely generated algebras  $D[x_1, \dots, x_n]$ .

**Proposition 5.1.** *Let  $(D, \mathfrak{m})$  be an integrally closed local domain, and let  $T \neq D$  be a finitely generated  $D$ -algebra contained in the quotient field  $K$  of  $D$ . If  $\mathfrak{m}T \neq T$ , then no maximal ideal of  $T$  above  $\mathfrak{m}$  has height 1.*

**Proof.** Let  $T := D[x_1, \dots, x_n]$ ; we proceed by induction on  $n$ .

Suppose  $n = 1$ , and let  $x := x_1$ ; then,  $x \notin D$ . If  $x^{-1} \in D$ , then  $x \in \mathfrak{m}$ , and thus  $\mathfrak{m}T = T$ , a contradiction. Hence,  $x, x^{-1} \notin D$ . By [25, Theorem 6], the ideal  $\mathfrak{p} := \mathfrak{m}T$  is prime but not maximal; since every maximal ideal of  $T$  above  $\mathfrak{m}$  must contain  $\mathfrak{p}$ , it follows that no such maximal ideal can have height 1.

Suppose that the claim holds up to  $n - 1$ ; let  $A := D[x_1, \dots, x_{n-1}]$ , so that  $T = A[x_n]$ ; without loss of generality,  $A \neq D$  and  $x_n \notin A$ . Let  $M$  be a maximal ideal of  $T$  above  $\mathfrak{m}$ . If  $x_n$  is integral over  $A$ , then  $T$  is integral over  $A$ , and thus the height of  $M$  is equal to the height of  $M \cap A$ , which is not equal to 1 by induction.

Suppose that  $x_n$  is not integral over  $A$ . Let  $A'$  be the integral closure of  $A$ ; then,  $T \subseteq A'[x_n]$  is an integral extension, and since  $x_n$  is not integral over  $A$  it follows that  $A' \subsetneq A'[x_n]$ . Take a maximal ideal  $M'$  of  $A'[x_n]$  above  $M$ . Let  $N := M' \cap A'$ ; then,  $N$  is a nonzero prime ideal of  $A'$ , and thus  $A'' := (A')_N$  is a local integrally closed domain with maximal ideal  $N(A')_N \neq (0)$ . Then, the ring  $A''[x_n]$  is the quotient ring of  $A'[x_n]$  with respect to the multiplicatively closed set  $A'[x_n] \setminus N$ , the set  $M'' := M'A''[x_n]$  is a maximal ideal, and  $N(A')_N \subseteq M''$ . Applying the case  $n = 1$  to  $A''$  and  $A''[x_n]$ , it follows that the height of  $M''$  is not 1; since the height of  $M''$  is the same of the height of  $M'$  and of  $M$ , it follows that the height of  $M$  is not 1, as claimed.  $\square$

**Theorem 5.2.** *Let  $D$  be an integral domain, and let  $V \in \text{Zar}(D)$  be a valuation overring of dimension 1. Then,  $V$  is isolated in  $\text{Zar}(D)^{\text{cons}}$  if and only if  $V$  is a localization of  $\overline{D}$  and its center on  $\overline{D}$  is isolated in  $\text{Spec}(\overline{D})^{\text{cons}}$ .*

**Proof.** Since  $\text{Zar}(D) = \text{Zar}(\overline{D})$ , we can suppose without loss of generality that  $D$  is integrally closed.

If the two conditions hold, then  $V$  is isolated by Proposition 3.1.

Suppose that  $V$  is isolated in  $\text{Zar}(D)^{\text{cons}}$ . Let  $P$  be the center of  $V$  on  $D$ , and suppose that  $V \neq D_P$ . Since  $V$  is also isolated in  $\text{Zar}(D_P)^{\text{cons}}$ , by Theorem 3.4 there are  $x_1, \dots, x_n \in K \setminus D_P$  such that  $V$  is the integral closure of  $D_P[x_1, \dots, x_n]_M$ , where  $M$  is a maximal ideal of  $D_P[x_1, \dots, x_n]$ . However,  $\mathfrak{m}_V \cap D_P[x_1, \dots, x_n] = M$ , and thus  $M \cap D_P = PD_P$ , so that  $PD_P \cdot D_P[x_1, \dots, x_n] \neq D_P[x_1, \dots, x_n]$ ; by Proposition 5.1,  $M$  cannot have height 1. However, the dimension of the integral closure of  $D_P[x_1, \dots, x_n]_M$  is exactly the height of  $M$ ; hence, this contradicts the fact that  $V$  has dimension 1. Thus,  $V = D_P$ . The fact that  $P$  is isolated in  $\text{Zar}(D)^{\text{cons}}$  now follows from Proposition 3.1.  $\square$

**Corollary 5.3.** *Let  $D$  be an integral domain, and let  $V \in \text{Zar}(D)$  be a minimal valuation overring of  $D$ . If  $\dim(V) = 1$  and  $V$  is isolated in  $\text{Zar}(D)^{\text{cons}}$ , then the center of  $V$  on  $D$  has height 1.*

**Proof.** The claim is a direct consequence of Theorem 5.2.  $\square$

Theorem 5.2 does not work when  $V$  has dimension 2 or more, as the next example shows.

**Example 5.4.** Let  $F$  be a field, take two independent indeterminates  $X$  and  $Y$ , and consider  $D := F + XF(Y)[[X]]$ , i.e.,  $D$  is the ring of all power series with coefficients in  $F(Y)$  such that the 0-degree coefficient belongs to  $F$ . Then,  $D$  is a one-dimensional local integrally closed domain (its maximal ideal is  $XF(Y)[[X]]$ ), and its valuation overrings are its quotient field,  $F(Y)[[X]]$  and the rings in the form  $W + XF(Y)[[X]]$ , where  $W$  belongs to  $\text{Zar}(F(Y)|F) \setminus \{F(Y)\}$ , i.e.,  $W$  is either  $F[Y]_{(f)}$  for some irreducible polynomial  $f \in F[Y]$  or  $W = F[Y^{-1}]_{(Y^{-1})}$ .

Each of these  $W + XF(Y)[[X]]$  is isolated in  $\text{Zar}(D)^{\text{cons}}$ , since each  $W$  is isolated in  $\text{Zar}(F(Y)|F)$  (this follows, for example, by applying Theorem 6.3 below to  $F[Y]$  or to  $F[Y^{-1}]$ ). However, since every  $W + XF(Y)[[X]]$  has dimension 2, it can't be a localization of  $D = \overline{D}$ .

## 6. The Noetherian case

In this section, we want to characterize the isolated points of  $\text{Zar}(D)^{\text{cons}}$  when  $D$  is a Noetherian domain. If  $D$  is integrally closed, this is a straightforward consequence of Theorem 5.2; to extend it to the non-integrally closed case, we need a few lemmas. (Note that the integral closure of a Noetherian domain is not necessarily Noetherian; see e.g. [18, Example 5, page 209].)

**Lemma 6.1.** *Let  $D$  be an integral domain. Let  $P$  be a prime ideal of  $D$  and let  $\Delta \subseteq \text{Spec}(D)$ . If  $P = \bigcap \{Q \mid Q \in \Delta\}$ , then  $P \in \text{Cl}^{\text{cons}}(\Delta)$ .*

**Proof.** Let  $\Omega = \mathcal{D}(aD) \cap \mathcal{V}(J)$  be a basic subset of  $\text{Spec}(D)^{\text{cons}}$  containing  $P$ , where  $a \in D$  and  $J$  is a finitely generated ideal. We claim that  $\Omega \cap \Delta \neq \emptyset$ .

Indeed,  $\Delta \subseteq \mathcal{V}(J)$  since  $J \subseteq P$  and  $P \subseteq Q$  for every  $Q \in \Delta$ . Moreover, since  $a \notin P$ , there must be a  $\bar{Q} \in \Delta$  such that  $a \notin \bar{Q}$ ; thus,  $\bar{Q} \in \mathcal{D}(aD) \cap \mathcal{V}(J) \cap \Delta = \Omega \cap \Delta$ . In particular,  $\Omega \cap \Delta \neq \emptyset$  and  $P \in \text{Cl}^{\text{cons}}(\Delta)$ .  $\square$

**Lemma 6.2.** *Let  $A \subseteq B$  be an integral extension, and let  $P \in \text{Spec}(A)$ ,  $Q \in \text{Spec}(B)$  be such that  $Q \cap A = P$ . If  $\bigcap \{P' \in \text{Spec}(A) \mid P' \supseteq P\} = P$ , then  $\bigcap \{Q' \in \text{Spec}(B) \mid Q' \supseteq Q\} = Q$ .*

**Proof.** Let  $I := \bigcap \{Q' \in \text{Spec}(B) \mid Q' \supseteq Q\}$ , and suppose  $I \neq Q$ ; then,  $Q \subsetneq I$  and  $\mathcal{V}(I) = \mathcal{V}(Q) \setminus \{Q\}$ . Consider the canonical map of spectra  $\phi : \text{Spec}(B) \rightarrow \text{Spec}(A)$ : then,  $\phi$  is closed (with respect to the Zariski topology) [2, Chapter V, §2, Remark (2)], and thus  $\phi(\mathcal{V}(I))$  is closed in  $\text{Spec}(A)$ .

By the lying over and the going up theorems, every  $P' \supseteq P$  belongs to  $\phi(\mathcal{V}(I))$ , while  $P \notin \phi(\mathcal{V}(I))$ ; hence,  $\phi(\mathcal{V}(I)) = \mathcal{V}(P) \setminus \{P\}$ . However, the condition  $\bigcap \{P' \in \text{Spec}(A) \mid P' \supseteq P\} = P$  shows that  $\mathcal{V}(P) \setminus \{P\}$  is not closed (its closure is  $\mathcal{V}(P)$ ), a contradiction. Hence,  $I = Q$ , as claimed.  $\square$

**Theorem 6.3.** *Let  $D$  be a Noetherian domain, and let  $V \in \text{Zar}(D)$ ; let  $P$  be the center of  $V$  on  $D$ . Then,  $V$  is isolated in  $\text{Zar}(D)^{\text{cons}}$  if and only if  $h(P) \leq 1$  and  $\mathcal{V}(P)$  is finite.*

**Proof.** Suppose first that  $V$  is isolated in  $\text{Zar}(D)^{\text{cons}}$ .

If  $\dim(V) > 1$ , then  $V$  is not Noetherian. By Theorem 3.4,  $V$  is the integral closure of  $D[x_1, \dots, x_n]_M$ , for some  $x_1, \dots, x_n \in V$  and some maximal ideal  $M$ . However,  $D[x_1, \dots, x_n]$  is Noetherian, and thus so is  $D[x_1, \dots, x_n]_M$ ; hence, its integral closure is a Krull domain, which can't be a non-Noetherian valuation domain, a contradiction.

If  $\dim(V) = 0$ , then  $V = K$ . By Proposition 4.1,  $D$  must be a Goldman domain; by [16, Theorem 146],  $\mathcal{V}(P)$  is finite.

If  $\dim(V) = 1$ , then by Theorem 5.2  $V$  is the localization of  $\bar{D}$  at a prime ideal of  $Q$  of height 1; hence,  $V$  is an essential prime ideal of  $\bar{D}$  and thus  $Q$  is isolated in  $\text{Spec}(\bar{D})^{\text{cons}}$  by Proposition 3.1.

Let  $P := Q \cap D$ . If  $\mathcal{V}(P)$  is infinite, then  $P$  is the intersection of all the prime ideals properly containing it (since  $D/P$  is not a Goldman domain); by Lemma 6.2, the same property holds for  $Q$ , and thus by Lemma 6.1,  $Q$  is not isolated in  $\text{Spec}(D)^{\text{cons}}$ . This is a contradiction, and thus  $\mathcal{V}(P)$  must be finite.

Conversely, suppose the two conditions hold and let  $\mathcal{V}(P) = \{P, Q_1, \dots, Q_n\}$ . For each  $i$ , let  $y_i \in Q_i \setminus P$  and let  $x_i := 1/y_i$ : then,  $A := D[x_1, \dots, x_n]$  is a Noetherian domain such that  $PA$  is a maximal ideal of  $A$  of height  $\leq 1$ ; moreover, since  $\mathfrak{m}_V \cap D = P$ , each  $x_i$  belongs to  $V$ , and thus  $V \in \text{Zar}(A)$  and  $\mathfrak{m}_V \cap A = PA$ .

The subspace  $\text{Zar}(A) = \mathcal{B}(x_1, \dots, x_n)$  is an open set of  $\text{Zar}(D)^{\text{cons}}$ : therefore, all isolated points of  $\text{Zar}(A)^{\text{cons}}$  are also isolated in  $\text{Zar}(D)^{\text{cons}}$ .

If  $P$  has height 0, then  $A = K = V$  and thus  $V$  is isolated. Suppose that  $h(P) = 1$ .

Since  $A$  is Noetherian,  $\{PA\} = \mathcal{V}(PA)$  is an open subset of  $\text{Spec}(A)^{\text{cons}}$ ; hence,  $\gamma_A^{-1}(PA)$  is an open subset of  $\text{Zar}(A)^{\text{cons}}$ , where  $\gamma_A : \text{Zar}(A) \rightarrow \text{Spec}(A)$

is the center map relative to  $A$ . However,  $\gamma_A^{-1}(PA)$  is the set of valuation over-rings of  $A_{PA} = D_P$  centered on  $(PA)A_{PA} = PD_P$ ; since  $P$  has height 1,  $D_P$  has dimension 1, and thus  $\gamma_A^{-1}(PA)$  is in bijective correspondence with the maximal ideals of the integral closure  $B$  of  $D_P$ , which is Noetherian by [16, Theorem 93]. The Jacobson radical of  $B$  is nonzero (since it contains  $P$ ), and thus  $B$  has only finitely many maximal ideal; thus,  $\gamma_A^{-1}(PA)$  is an open finite set of the Hausdorff space  $\text{Zar}(A)^{\text{cons}}$ , and so it is discrete. Since  $V \in \gamma_A^{-1}(PA)$ , we have that  $V$  is isolated in  $\text{Zar}(A)^{\text{cons}}$  and thus in  $\text{Zar}(D)^{\text{cons}}$ , as claimed.  $\square$

**Corollary 6.4.** *Let  $(D, \mathfrak{m})$  be a Noetherian local domain of dimension at least 3. Then,  $\text{Zar}(D)^{\text{cons}}$  is perfect.*

**Proof.** Suppose  $V$  is isolated in  $\text{Zar}(D)^{\text{cons}}$ . By Theorem 6.3, its center  $P$  must have height 1 and  $\mathcal{V}(P)$  must be finite. However, since  $P$  has height 1 and the maximal ideal  $\mathfrak{m}$  of  $D$  has height at least 3, there is at least one prime ideal between  $P$  and  $\mathfrak{m}$ , and since  $D$  is Noetherian there must be infinitely many of them [16, Theorem 144], a contradiction. Hence, no  $V$  can be isolated, and  $\text{Zar}(D)^{\text{cons}}$  is perfect.  $\square$

We now want to show that, when  $D$  is countable, there are few possible topological structures for  $\text{Zar}(D)^{\text{cons}}$ . The one-dimensional case is very easy.

**Proposition 6.5.** *Let  $(D, \mathfrak{m})$  and  $(D', \mathfrak{m}')$  be two Noetherian local domains of dimension 1. The following are equivalent:*

- (i)  $|\text{Max}(\overline{D})| = |\text{Max}(\overline{D}')|$ ;
- (ii)  $\text{Zar}(D) \simeq \text{Zar}(D')$ ;
- (iii)  $\text{Zar}(D)^{\text{cons}} \simeq \text{Zar}(D')^{\text{cons}}$ .

**Proof.** Since  $D$  is Noetherian and one-dimensional,  $\overline{D}$  is a principal ideal domain with finitely many maximal ideals; hence,  $\text{Zar}(D) = \text{Zar}(\overline{D}) \simeq \text{Spec}(\overline{D})$ , and the homeomorphism holds both in the Zariski and in the constructible topology.

Hence, if  $|\text{Max}(\overline{D})| = |\text{Max}(\overline{D}')|$  then  $\text{Spec}(\overline{D}) \simeq \text{Spec}(\overline{D}')$  and thus  $\text{Zar}(D)$  and  $\text{Zar}(D')$  are homeomorphic in both the Zariski and the constructible topology. Conversely, if  $\text{Zar}(D) \simeq \text{Zar}(D')$  (in any of the two topologies) then in particular they have the same cardinality, which is equal to  $|\text{Max}(\overline{D})| + 1 = |\text{Max}(\overline{D}')| + 1$ ; thus,  $|\text{Max}(\overline{D})| = |\text{Max}(\overline{D}')|$ . The claim is proved.  $\square$

For larger dimension, we need to join the previous theorems with the topological characterization of the Cantor set. We isolate a lemma.

**Lemma 6.6.** *Let  $D$  be a countable domain. Then,  $\text{Zar}(D)^{\text{cons}}$  is metrizable.*

**Proof.** The space  $\text{Zar}(D)^{\text{cons}}$  is compact and Hausdorff, hence normal [29, Theorem 17.10] and, in particular, regular. Furthermore, the family of sets  $\mathcal{B}(t)$  and  $\mathcal{B}(t)^c$  (as  $t$  ranges in the quotient field of  $D$ ) form a subbasis of  $\text{Zar}(D)^{\text{cons}}$ , and thus  $\text{Zar}(D)^{\text{cons}}$  is second countable. By Urysohn's metrization theorem (see e.g. [29, Theorem 23.1]),  $\text{Zar}(D)^{\text{cons}}$  is metrizable.  $\square$

**Proposition 6.7.** *Let  $(D, \mathfrak{m})$  and  $(D', \mathfrak{m}')$  be two countable Noetherian local domains of dimension at least 3. Then,  $\text{Zar}(D)^{\text{cons}} \simeq \text{Zar}(D')^{\text{cons}}$ .*

**Proof.** Both  $\text{Zar}(D)^{\text{cons}}$  and  $\text{Zar}(D')^{\text{cons}}$  are Boolean spaces, hence totally disconnected and compact; they are also perfect by Corollary 6.4 and metrizable by Lemma 6.6.

By [29, Theorem 30.3], any two spaces with these properties are homeomorphic; hence,  $\text{Zar}(D)^{\text{cons}} \simeq \text{Zar}(D')^{\text{cons}}$ .  $\square$

To study the case of dimension 2, we need two further lemmas.

**Lemma 6.8.** *Let  $(D, \mathfrak{m})$  be a local Noetherian domain with  $\dim(D) > 1$ . If  $D$  is countable, then  $D$  has exactly countably many prime ideals of height 1.*

**Proof.** By [16, Theorem 144], there are infinitely many prime ideals between  $(0)$  and  $\mathfrak{m}$ , and thus  $D$  has infinitely many prime ideals of height 1.

Moreover, every prime ideal is generated by a finite set, and thus the number of prime ideals of height 1 is at most equal to the number of finite subsets of  $D$ . Since  $D$  is countable, so is the set of its finite subsets; the claim is proved.  $\square$

**Lemma 6.9.** *Let  $(D, \mathfrak{m})$  be a local Noetherian domain of dimension 2 with quotient field  $K$ , and let  $X$  be the set of isolated points of  $\text{Zar}(D)^{\text{cons}}$ . Then:*

- (a) *a valuation overring of  $D$  belongs to  $X$  if and only if its center has height 1;*
- (b)  *$X$  is nonempty and compact, with respect to the Zariski topology;*
- (c) *if  $D$  is countable, then  $X$  is countable;*
- (d)  *$\text{Cl}^{\text{cons}}(X) = X \cup \{K\}$ ;*
- (e) *the only isolated point of  $(\text{Zar}(D) \setminus X)^{\text{cons}}$  is  $K$ ;*
- (f)  *$\text{Zar}(D) \setminus (X \cup \{K\})$  is closed and perfect, with respect to the constructible topology.*

**Proof.** (a) Let  $V \in \text{Zar}(D)$ . If  $V$  is isolated, then its center has height at most 1 by Theorem 6.3, but the height can't be 0 since  $\mathcal{V}((0))$  is infinite. Conversely, if  $P := \mathfrak{m}_V \cap D$  has height 1, then  $\mathcal{V}(P) = \{P, \mathfrak{m}\}$  is finite, and thus  $V \in X$ , by Theorem 6.3.

(b) Let  $X_1$  be the set of all height 1 prime ideals of  $D$ : by the previous point,  $X = \gamma^{-1}(X_1)$ . Since  $\gamma$  is surjective, and  $X_1$  is nonempty, also  $X$  is nonempty. Furthermore, since  $D$  is a Noetherian ring,  $\text{Spec}(D)$  is a Noetherian space with respect to the Zariski topology (i.e., all its subsets are compact; see [4, Theorem 12.4.3] or [1, Chapter 6, Exercises 5–8]). Since  $\gamma$  is a spectral closed map, it is proper, and thus the counterimage of any compact subset of  $\text{Spec}(D)$  is compact; therefore,  $X = \gamma^{-1}(X_1)$  is compact with respect to the Zariski topology, as claimed.

(c) By Lemma 6.8,  $X_1$  is countable; furthermore,  $\gamma^{-1}(P)$  is finite for every  $P \in X_1$ , since it is in bijective correspondence with the set of maximal ideals of the integral closure of  $D_P$ . Since  $X = \gamma^{-1}(X_1)$ , it follows that  $X$  is countable.

(d) Since  $X$  is compact, the set  $X^{\text{gen}} = \{W \in \text{Zar}(D) \mid W \supseteq V \text{ for some } V \in X\}$  is closed in the inverse topology, and thus in the constructible topology; since

every element of  $X$  is a one-dimensional valuation ring, furthermore,  $X^{\text{gen}} = X \cup \{K\}$ . Hence,  $\text{Cl}^{\text{cons}}(X) \subseteq X \cup \{K\}$ .

If they are not equal, then  $\text{Cl}^{\text{cons}}(X) = X$ . However,  $X$  is infinite (since  $X_1$  is infinite, by Lemma 6.8) and discrete (by definition, all its points are isolated) and thus it is not compact with respect to the constructible topology; this is a contradiction, since a closed set of a compact set is compact. Thus,  $\text{Cl}^{\text{cons}}(X) = X \cup \{K\}$ , as claimed.

(e) The set  $\text{Zar}(D) \setminus (X \cup \{K\})$  is open, with respect to the constructible topology (by part (d)), and its elements are not isolated in  $\text{Zar}(D)^{\text{cons}}$ ; therefore, none of its elements can be isolated in  $(\text{Zar}(D) \setminus X)^{\text{cons}}$ . On the other hand, let  $x \in \mathfrak{m}$ ,  $x \neq 0$ : then,  $D[x^{-1}]$  is a Noetherian domain of dimension 1, and its maximal ideals are extensions of prime ideals of  $D$  of height 1. Therefore, if  $V \in \text{Zar}(D[x^{-1}])$  has dimension 1 then the center of  $V$  on  $D$  has height 1, and thus it is an isolated point of  $\text{Zar}(D)$ , i.e.,  $\mathcal{B}(x^{-1}) = \text{Zar}(D[x^{-1}]) \subseteq X \cup \{K\}$ , and  $\mathcal{B}(x^{-1}) \cap (\text{Zar}(D) \setminus X) = \{K\}$ . Since  $\mathcal{B}(x^{-1})$  is open in  $\text{Zar}(D)^{\text{cons}}$ , it follows that  $K$  is isolated in  $(\text{Zar}(D) \setminus X)^{\text{cons}}$ .

(f) is a direct consequence of (e).  $\square$

Note that the set  $X$  of the previous proposition is *not* compact with respect to the constructible topology, as it is discrete and infinite.

**Proposition 6.10.** *Let  $(D, \mathfrak{m})$  and  $(D', \mathfrak{m}')$  be two countable Noetherian local domains of dimension 2. Then,  $\text{Zar}(D)^{\text{cons}} \simeq \text{Zar}(D')^{\text{cons}}$ .*

**Proof.** Denote by  $K, K'$  the quotient fields of  $D$  and  $D'$ , respectively.

Let  $X$  be the set of isolated points of  $\text{Zar}(D)^{\text{cons}}$ , and let  $C := \text{Zar}(D) \setminus (X \cup \{K\})$ : then,  $C$  is closed in  $\text{Zar}(D)^{\text{cons}}$ . Define in the same way  $X'$  and  $C'$  inside  $\text{Zar}(D')$ ; then,  $C'$  is closed.

As in the proof of Proposition 6.7, by Lemma 6.9(f)  $C^{\text{cons}}$  and  $(C')^{\text{cons}}$  are totally disconnected, perfect, compact and metrizable (with respect to the constructible topology), and thus they are homeomorphic. Let  $\phi_C : C^{\text{cons}} \rightarrow (C')^{\text{cons}}$  be a homeomorphism.

The set  $X$  is discrete and countable, and the unique nonisolated point of  $X \cup \{K\}$  is  $K$ ; since the same holds for  $X'$  and  $K'$ , any bijection  $X \rightarrow X'$  extends to a homeomorphism  $\phi_X : (X \cup \{K\})^{\text{cons}} \rightarrow (X' \cup \{K'\})^{\text{cons}}$  by setting  $\phi_X(K) = K'$ . Define

$$\begin{aligned} \phi : \text{Zar}(D)^{\text{cons}} &\longrightarrow \text{Zar}(D')^{\text{cons}}, \\ V &\longmapsto \begin{cases} \phi_C(V) & \text{if } V \in C, \\ \phi_X(V) & \text{if } V \in X \cup \{K\}. \end{cases} \end{aligned}$$

By construction,  $\phi$  is bijective, and  $\phi$  is a homeomorphism when restricted to  $C$  and to  $X \cup \{K\}$ . Since these two sets are closed, by [29, Theorem 7.6]  $\phi$  is a homeomorphism. In particular,  $\text{Zar}(D)^{\text{cons}} \simeq \text{Zar}(D')^{\text{cons}}$ .  $\square$

We summarize the previous results in the following theorem.



**Theorem 6.11.** *Let  $(D, \mathfrak{m})$  and  $(D', \mathfrak{m}')$  be two countable Noetherian local domains. Then,  $\text{Zar}(D)^{\text{cons}} \simeq \text{Zar}(D')^{\text{cons}}$  if and only if one of the following conditions hold:*

- (a)  $\dim(D) = \dim(D') = 1$  and  $|\text{Max}(\overline{D})| = |\text{Max}(\overline{D}')|$ ;
- (b)  $\dim(D) = \dim(D') = 2$ ;
- (c)  $\dim(D) \geq 3$  and  $\dim(D') \geq 3$ .

**Proof.** If  $D$  and  $D'$  satisfy one of the conditions, then  $\text{Zar}(D)^{\text{cons}} \simeq \text{Zar}(D')^{\text{cons}}$  by, respectively, Proposition 6.5, Proposition 6.10 and Proposition 6.7.

Suppose now that  $\text{Zar}(D)^{\text{cons}} \simeq \text{Zar}(D')^{\text{cons}}$ .

If  $\dim(D) = 1$ , then  $\text{Zar}(D)$  is finite, and thus so must be  $\text{Zar}(D')$ ; hence,  $\dim(D') = 1$ , and  $|\text{Max}(\overline{D})| = |\text{Max}(\overline{D}')|$  by Proposition 6.5.

Suppose  $\dim(D), \dim(D') \geq 2$ . By Corollary 6.4 and Lemma 6.8,  $\text{Zar}(D)^{\text{cons}}$  has isolated points if and only if  $\dim(D) = 2$ ; therefore,  $\dim(D) = 2$  if and only if  $\dim(D') = 2$ , and  $\dim(D) \geq 3$  if and only if  $\dim(D') \geq 3$ . The claim is proved.  $\square$

## 7. When $D$ is a field

In this section we analyze the isolated points Zariski space  $\text{Zar}(L|D)^{\text{cons}}$  when  $D = K$  is a field. Note that, if  $L$  is algebraic over  $K$ , then  $\text{Zar}(L|K)$  is just a point ( $L$  itself); thus, the only interesting case is when  $\text{trdeg}(L/K) \geq 1$ .

We start by connecting the isolated points of  $\text{Zar}(L|D)^{\text{cons}}$  and of  $\text{Zar}(L'|D)^{\text{cons}}$ , where  $L' \subseteq L$  is an algebraic extension.

**Proposition 7.1.** *Let  $V$  be a valuation domain, and  $L' \subseteq L$  be an algebraic extension such that  $V \subseteq L'$ . Let  $\rho : \text{Zar}(L|V) \rightarrow \text{Zar}(L'|V)$  be the restriction map, and let  $\mathcal{X} \subseteq \text{Zar}(L|V)$  be a subset such that  $\rho^{-1}(\rho(\mathcal{X})) = \mathcal{X}$ . Then, the following hold.*

- (a) *If  $W$  is isolated in  $\mathcal{X}^{\text{cons}}$ , then  $\rho(W)$  is isolated in  $\rho(\mathcal{X})^{\text{cons}}$ .*
- (b) *If  $\rho(\mathcal{X})$  is perfect and  $|\rho(\mathcal{X})| > 1$ , then  $\mathcal{X}$  is perfect.*

*In particular, the previous statements apply to  $\mathcal{X} = \text{Zar}(L|V)$  and  $\mathcal{X} = \mathcal{E}(L|V)$ .*

**Proof.** (a) Let  $W$  be an isolated point of  $\mathcal{X}^{\text{cons}}$ , and let  $W' := W \cap L' = \rho(W)$ .

Suppose first that  $L$  is finite and normal over  $L'$ . Let  $G$  be the group of  $L'$ -automorphisms of  $L$ : then, every  $\sigma \in G$  is continuous when seen as a map from  $\text{Zar}(L|V)^{\text{cons}}$  to itself. Moreover,  $\rho(\sigma(Z)) = \rho(Z)$  for every  $Z \in \text{Zar}(L|V)$ , and thus  $\sigma$  restricts to a self-homeomorphism of  $\mathcal{X}$ .

Since  $G$  acts transitively on  $\rho^{-1}(W')$  (see e.g. [12, Corollary 20.2]) and  $W \in \rho^{-1}(W')$  is isolated, all points of  $\rho^{-1}(W')$  are isolated in  $\mathcal{X}$ ; hence,  $\rho^{-1}(W')$  is open in  $\mathcal{X}^{\text{cons}}$ . Since  $\rho : \text{Zar}(L|V) \rightarrow \text{Zar}(L'|V)$  is a closed map (with respect to the constructible topology), it is also closed when seen as a map  $\mathcal{X} \rightarrow \rho(\mathcal{X})$ ; therefore,  $\rho(\mathcal{X} \setminus \rho^{-1}(W')) = \rho(\mathcal{X}) \setminus \{W'\}$  is closed in  $\rho(\mathcal{X})$ , with respect to the constructible topology, and thus  $W'$  is an isolated point of  $\rho(\mathcal{X})^{\text{cons}}$ , as claimed.

Suppose now that  $L$  is finite over  $L'$ , and let  $F$  be the normal closure of  $L'$ . Let  $\rho_0 : \text{Zar}(F|V) \rightarrow \text{Zar}(L|V)$  be the restriction map. Since  $W$  is isolated



in  $\mathcal{X}$ , the set  $\rho_0^{-1}(W)$  is open in  $\rho_0^{-1}(\mathcal{X})^{\text{cons}}$ ; moreover,  $\rho_0^{-1}(W)$  is finite since  $[F : L] < \infty$ . Therefore,  $\rho_0^{-1}(W)$  is a discrete subspace of  $\rho_0^{-1}(\mathcal{X})^{\text{cons}}$ , and in particular each  $Z \in \rho_0^{-1}(W)$  is isolated. Applying the previous part of the proof to the extension  $L' \subseteq F$  and to any such  $Z$ , we obtain that  $Z \cap L' = W \cap L' = \rho(W)$  is isolated, as claimed.

Suppose now that  $L' \subseteq L$  is arbitrary. Since  $W$  is isolated in  $\mathcal{X}$ , there are  $x_1, \dots, x_n, y_1, \dots, y_m \in L$  such that  $\{W\} = \mathcal{B}(x_1, \dots, x_n) \cap \mathcal{B}(y_1)^c \cap \dots \cap \mathcal{B}(y_m)^c \cap \mathcal{X}$ . Let  $F := L'(x_1, \dots, x_n, y_1, \dots, y_m)$ : then,  $W \cap F$  is isolated in  $\{Z \cap F \mid Z \in \mathcal{X}\}$ . Since  $[F : L'] < \infty$ , we can apply the previous part of the proof, obtaining that  $W \cap F \cap L' = W \cap L' = \rho(W)$  is isolated in  $\rho(\mathcal{X})^{\text{cons}}$ , as claimed.

(b) Suppose that  $\mathcal{X}$  is not perfect: then, there is a  $W \in \mathcal{X}$  that is isolated. By the previous part of the proof, it would follow that  $W \cap L'$  is isolated in  $\rho(\mathcal{X})^{\text{cons}}$ . Since  $\rho(\mathcal{X})$  has more than one point, this is impossible, and so  $\mathcal{X}$  is perfect.

The “in particular” statement follows from the fact that  $\text{Zar}(L|V)$  and  $\mathcal{E}(L|V)$  satisfy the hypothesis on  $\mathcal{X}$ .  $\square$

**Corollary 7.2.** *Let  $V$  be a valuation domain and  $L' \subseteq L$  be an algebraic extension; suppose that  $V \subseteq L'$  and that  $L'$  is transcendental over the quotient field of  $V$ . If  $\text{Zar}(L'|V)^{\text{cons}}$  (respectively,  $\mathcal{E}(L'|V)^{\text{cons}}$ ) is perfect, then  $\text{Zar}(L|V)^{\text{cons}}$  (resp.,  $\mathcal{E}(L|V)^{\text{cons}}$ ) is perfect.*

**Proof.** It is enough to apply Proposition 7.1(b) to  $\mathcal{X} = \text{Zar}(L|V)$  or  $\mathcal{X} = \mathcal{E}(L|V)$ , using the hypothesis that  $L'$  is transcendental over the quotient field of  $V$  to guarantee that  $|\text{Zar}(L'|V)| > 1$  and  $|\mathcal{E}(L'|V)| > 1$ .  $\square$

The following result completely settles the problem of finding the isolated points when  $\text{trdeg}(L/K) \geq 2$ , generalizing [3, Theorem 4.45] and solving the authors’ Conjecture A (in an even more general formulation). Note that the first case in the proof is exactly [3, Theorem 4.45], but we give a new proof of it using Theorem 6.3.

**Theorem 7.3.** *Let  $K \subseteq L$  be a field extension with  $\text{trdeg}(L/K) \geq 2$ . Then,  $\text{Zar}(L|K)^{\text{cons}}$  is perfect.*

**Proof.** Suppose first that  $L = K(x_1, \dots, x_n)$  is a finitely generated purely transcendental extension of  $K$ , with transcendence basis  $x_1, \dots, x_n$ . Suppose there exists an isolated point  $W$  of  $\text{Zar}(L|K)^{\text{cons}}$ . By Proposition 4.1,  $W \neq L$ .

For each  $i$ , at least one of  $x_i$  and  $x_i^{-1}$  belongs to  $W$ ; let it be  $t_i$ . Then,  $W \in \text{Zar}(K[t_1, \dots, t_n])$ , and so  $W$  is isolated in  $\text{Zar}(K[t_1, \dots, t_n])^{\text{cons}}$ . Let  $P$  be the center of  $W$  on  $K[t_1, \dots, t_n]$ ; since  $K[t_1, \dots, t_n]$  is Noetherian, by Theorem 6.3  $P$  has height 1 and  $\mathcal{V}(P)$  is finite.

Since  $K[t_1, \dots, t_n]$  is isomorphic to a polynomial ring, every maximal ideal of  $K[t_1, \dots, t_n]$  has height  $n > 1$  [16, Section 3.2, Exercise 3], and thus  $P$  is not maximal. However,  $K[t_1, \dots, t_n]$  is an Hilbert ring, and thus every non-maximal prime ideal is the intersection of the maximal ideals containing it [16, Theorem

147]; in particular, this happens for  $P$ , and thus  $\mathcal{V}(P)$  must be infinite. This is a contradiction, and so  $\text{Zar}(L|K)^{\text{cons}}$  is perfect.

Suppose now that  $L$  has finite transcendence degree over  $K$ , let  $x_1, \dots, x_n$  be a transcendence basis of  $L$  and let  $L' := K(x_1, \dots, x_n)$ . By the previous part of the proof,  $\text{Zar}(L'|K)^{\text{cons}}$  is perfect; since  $L' \subseteq L$  is algebraic, by Corollary 7.2 also  $\text{Zar}(L|K)^{\text{cons}}$  is perfect.

Take now any extension  $L$  of  $K$ , and suppose that  $W$  is an isolated point of  $\text{Zar}(L|K)^{\text{cons}}$ . Then, there are  $x_1, \dots, x_n, y_1, \dots, y_m \in L$  such that  $\{W\} = \mathcal{B}(x_1, \dots, x_n) \cap \mathcal{B}(y_1)^c \cap \dots \cap \mathcal{B}(y_m)^c$ . Take two elements  $a, b \in L$  that are algebraically independent over  $K$ , and let  $L' := K(a, b, x_1, \dots, x_n, y_1, \dots, y_m)$ : then,  $2 \leq \text{trdeg}(L'/K) < \infty$ . Set  $V := W \cap L'$ : then,  $\{V\} = \mathcal{B}^{L'}(x_1, \dots, x_n) \cap \mathcal{B}^{L'}(y_1)^c \cap \dots \cap \mathcal{B}^{L'}(y_m)^c$ , and thus  $V$  is isolated in  $\text{Zar}(L'|V)^{\text{cons}}$ . However, by the previous part of the proof,  $\text{Zar}(L'|V)^{\text{cons}}$  is perfect, a contradiction. Hence,  $\text{Zar}(L|K)^{\text{cons}}$  is perfect.  $\square$

When the transcendence degree of  $L$  over  $K$  is 1, the picture is very different, because it may even happen that all elements of  $\text{Zar}(L|K)^{\text{cons}}$  (except  $L$  itself) are isolated. Compare the next results with [26, Corollary 5.5(a)] and [28, Proposition 4.2].

**Proposition 7.4.** *Let  $K$  be a field. Then all points of  $\text{Zar}(K(X)|K)$ , except  $K(X)$ , are isolated with respect to the constructible topology.*

**Proof.** The points of  $\text{Zar}(K(X)|K)$  are  $K(X)$ ,  $K[X^{-1}]_{(X^{-1})}$ , and the rings  $K[X]_{(f(X))}$ , where  $f(X)$  is an irreducible polynomial of  $K[X]$ . The first one is not isolated by Proposition 4.1; on the other hand,  $\{K[X]_{(f(X))}\} = \mathcal{B}(f(X)^{-1})^c$  and  $\{K[X^{-1}]_{(X^{-1})}\} = \mathcal{B}(X)^c$ , and thus these domains are isolated, as claimed.  $\square$

**Lemma 7.5.** *Let  $D$  be an integral domain with quotient field  $K$ , and let  $L' \subseteq L$  be two extensions of  $K$ . Let  $V \in \text{Zar}(L'|D)$ . If  $V$  is isolated in  $\text{Zar}(L'|D)^{\text{cons}}$  and  $\mathcal{E}(L|V)$  is finite, then every  $W \in \mathcal{E}(L|V)$  is isolated in  $\text{Zar}(L|D)^{\text{cons}}$ .*

**Proof.** Let  $\rho : \text{Zar}(L|D) \rightarrow \text{Zar}(L'|D)$  be the restriction map. Then,  $\mathcal{E}(L|V) = \rho^{-1}(V)$  is open in  $\text{Zar}(L|D)^{\text{cons}}$  since  $V$  is isolated. Moreover,  $\mathcal{E}(L|V)$  is finite by hypothesis, and, since the constructible topology is Hausdorff, all its points are isolated in  $\text{Zar}(L|D)^{\text{cons}}$ .  $\square$

**Proposition 7.6.** *Let  $K$  be a field and let  $L$  be an extension of  $K$  with  $\text{trdeg}(L/K) = 1$ . Let  $V \in \text{Zar}(L|K)$ ,  $V \neq L$ . Then the following are equivalent:*

- (i)  $V$  is isolated in  $\text{Zar}(L|K)^{\text{cons}}$ ;
- (ii) there exists a finitely generated extension  $L'$  of  $K$  such that  $L' \subseteq L$  and  $\mathcal{E}(L|V \cap L') = \{V\}$ ;
- (iii) there exists a finitely generated extension  $L'$  of  $K$  such that  $L' \subseteq L$  and  $\mathcal{E}(L|V \cap L')$  is finite.

**Proof.** (i)  $\implies$  (ii) Since  $V$  is isolated, we have

$$\{V\} = \Omega := \mathcal{B}(x_1, \dots, x_n) \cap \mathcal{B}(y_1)^c \cap \dots \cap \mathcal{B}(y_m)^c$$

for some  $x_1, \dots, x_n, y_1, \dots, y_m \in L$ . Let  $L' = K(x_1, \dots, x_n, y_1, \dots, y_m)$ . Then every extension of  $V \cap L'$  to  $L$  belongs to  $\Omega$ , and thus it is equal to  $V$ . Hence,  $L'$  is the required field.

(ii)  $\implies$  (iii) is obvious.

(iii)  $\implies$  (i) Since  $\mathcal{E}(L|V \cap L')$  is finite,  $L' \subseteq L$  must be algebraic and so  $K \subseteq L'$  is transcendental; take any  $X \in L'$  that is transcendental over  $K$ . Since  $K \subseteq L'$  is finitely generated,  $K(X) \subseteq L'$  must be a finite extension.

Since  $V \neq L$ , we have  $V \cap K(X) \neq K(X)$ ; by Proposition 7.4,  $V \cap K(X)$  is isolated in  $\text{Zar}(K(X)|K)$ . Moreover, since  $K(X) \subseteq L'$  is a finite extension,  $\mathcal{E}(L'|V \cap K(X))$  is finite; by Lemma 7.5, all points of  $\mathcal{E}(L'|V \cap K(X))$  are isolated in  $\text{Zar}(L'|K)^{\text{cons}}$ , and in particular this happens for  $V \cap L'$ . We can now apply Lemma 7.5 to  $V \cap L'$  and  $L$ , obtaining that all elements of  $\mathcal{E}(L|V \cap L')$  (in particular,  $V$ ) are isolated in  $\text{Zar}(L|K)^{\text{cons}}$ .  $\square$

**Proposition 7.7.** *Let  $K$  be a field and let  $L$  be an extension of  $K$  with  $\text{trdeg}(L/K) = 1$ .*

1. *Let  $\mathcal{X} := \text{Zar}(L|K) \setminus \{L\}$ . Then, the following are equivalent:*

- (i) *all points of  $\mathcal{X}$  are isolated in  $\text{Zar}(L|K)^{\text{cons}}$ ;*
- (ii) *for every  $X \in L$ , transcendental over  $K$ , the set  $\mathcal{E}(L|V)$  is finite for every  $V \in \text{Zar}(K(X)|K)$ ;*
- (iii) *there is an  $X \in L$ , transcendental over  $K$ , such that the set  $\mathcal{E}(L|V)$  is finite for every  $V \in \text{Zar}(K(X)|K)$ .*

**Proof.** (i)  $\implies$  (ii) Take any  $X \in L$  that is transcendental over  $K$ , and let  $V \in \text{Zar}(K(X)|K)$ . The space  $\mathcal{E}(L|V)$  is closed in  $\text{Zar}(L|V)^{\text{cons}}$ , and thus it is compact. Since all its points are isolated, it is also discrete; hence,  $\mathcal{E}(L|V)$  is finite.

(ii)  $\implies$  (iii) is obvious.

(iii)  $\implies$  (i) Apply Proposition 7.6, (iii)  $\implies$  (i) with  $L' = K(X)$  to each  $V \in \mathcal{X}$ .  $\square$

**Corollary 7.8.** *Let  $K$  be a field and let  $L$  be a finitely generated extension of  $K$  such that  $\text{trdeg}(L/K) = 1$ . Then, all points of  $\text{Zar}(L|K) \setminus \{L\}$  are isolated in  $\text{Zar}(L|K)^{\text{cons}}$ .*

**Proof.** It is enough to apply Proposition 7.7.  $\square$

**Remark 7.9.** Let  $K \subseteq L$  be a transcendental extension of degree 1, and let  $V \in \text{Zar}(L|K)$ . Let  $X \in L$  be transcendental over  $K$ . By Proposition 7.6, if  $\mathcal{E}(L|V \cap K(X))$  is finite, then  $V$  is isolated in  $\text{Zar}(L|K)^{\text{cons}}$ ; however, unlike in Proposition 7.7, the converse does not hold, i.e.,  $\mathcal{E}(L|V \cap K(X))$  may be infinite even if  $V$  is isolated.

For example, let  $W = K[X]_{(X)}$  (or more generally, we can take any  $W \in \text{Zar}(K(X)|K)$ ,  $W \neq K(X)$ ). Since  $W$  is a discrete valuation ring, using [17] (see also [13, Section 3]), it is possible to construct a chain  $K(X) \subset F_0 \subset F_1 \subset \dots$  of extensions of  $K(X)$  such that:

- the extensions  $K(X) \subset F_0$  and  $F_i \subset F_{i+1}$  are finite, for each  $i > 0$ ;
- $W$  has two extensions to  $F_0$ , say  $W_1$  and  $W_2$ ;
- $W_1$  has only one extension to  $F_i$ , for each  $i > 0$ ;

- if  $W'$  is an extension of  $W_2$  to  $F_i$ , then  $W'$  has more than one extension to  $F_{i+1}$ .

Let  $L := \bigcup_{i \geq 0} F_i$ . Then,  $W_1$  has a unique extension  $V$  to  $L$ , while  $W_2$  has infinitely many extensions; in particular, the set  $\mathcal{X}$  of extensions of  $W$  to  $L$  is infinite. Let  $y \in W_1 \setminus W_2$ : then,  $\mathcal{B}(y) \cap \mathcal{X} = \{V\}$ , and thus  $\mathcal{B}(y) \cap \mathcal{B}(X^{-1})^c = \{V\}$ . Hence,  $V$  is isolated in  $\text{Zar}(L|K)^{\text{cons}}$ , despite  $V \cap K(X) = W$  having infinitely many extensions to  $L$ .

The reason why the proof of Proposition 7.7 fails in this context is that we are not requiring the *other* extensions of  $W$  to  $L$  to be isolated.

## 8. Extensions of valuations

In this section, we extend the results of the previous section from the case where  $D = K$  is a field to the case where  $D = V$  is a valuation domain. In particular, we want to study the set  $\mathcal{E}(L|V)$  of extensions of  $V$  to  $L$ .

The most important case is when  $L = K(X)$  is the field of rational functions. If  $V$  is a valuation domain with quotient field  $K$  and  $s \in K$ , we set

$$V_s := \{\phi \in K(X) \mid \phi(s) \in V\}.$$

Then,  $V_s$  is an extension of  $V$  to  $K(X)$ , and it is possible to analyze quite thoroughly its algebraic properties (see for example [22, Proposition 2.2] for a description when  $V$  has dimension 1).

The following lemma is a partial generalization of [22, Theorem 3.2], of which we follow the proof.

**Lemma 8.1.** *Let  $V$  be a valuation domain with quotient field  $K$ , and let  $U$  be an extension of  $V$  to the algebraic closure  $\bar{K}$ . Let  $s, t \in \bar{K}$ . Then,  $U_s \cap K(X) = U_t \cap K(X)$  if and only if  $s$  and  $t$  are conjugated over  $K$ .*

**Proof.** If  $s, t$  are conjugated, there is a  $K$ -automorphism  $\sigma$  of  $\bar{K}$  sending  $s$  to  $t$ . Setting  $\tilde{\sigma}(\sum_i a_i X^i) := \sum_i \sigma(a_i) X^i$ , we can extend  $\sigma$  to a  $K(X)$ -automorphism  $\tilde{\sigma}$  of  $\bar{K}(X)$  such that  $\tilde{\sigma}(\phi)(t) = \sigma(\phi(s))$  for every  $\phi \in \bar{K}(X)$ ; in particular, if  $\phi \in K(X)$  then  $\tilde{\sigma}(\phi) = \phi$  and thus  $\phi(s) \in V$  if and only if  $\phi(t) \in V$ , i.e.,  $\phi \in U_s \cap K(X)$  if and only if  $\phi \in U_t \cap K(X)$ . Therefore,  $U_s \cap K(X) = U_t \cap K(X)$ .

Conversely, suppose that  $s$  and  $t$  are not conjugate, and let  $p(X)$  be the minimal polynomial of  $s$  over  $K$ : then,  $p(t) \neq 0$ , and thus there is a  $c \in K$ ,  $c \neq 0$ , such that  $v(c) > u(p(t))$  (where  $v$  and  $u$  are, respectively, the valuations with respect to  $V$  and  $U$  and  $u|_K = v$ ). Then,  $q(X) := \frac{p(X)}{c} \in K(X)$  belongs to  $U_s$  (since  $q(s) = 0 \in V$ ) but not to  $U_t$  (since  $u(q(t)) = u(p(t)) - v(c) < 0$ ). Hence,  $U_s \cap K(X) \neq U_t \cap K(X)$ , as claimed.  $\square$

**Theorem 8.2.** *If  $V$  is a valuation domain that is not a field, then  $\mathcal{E}(K(X)|V)^{\text{cons}}$  is perfect.*

**Proof.** Suppose first that  $K$  is algebraically closed. By [23, Theorem 7.2], for all extensions  $W$  of  $V$  to  $K(X)$  there is a sequence  $E = \{s_\nu\}_{\nu \in \Lambda}$  (where  $\Lambda$  is a

well-ordered set without maximum) such that

$$W = V_E = \{\phi \in K(X) \mid \phi(s_\nu) \in V \text{ for all large } \nu\}$$

and  $W \neq V_{s_\nu}$  for every  $\nu$ . In particular, the elements  $\phi(s_\nu)$  are either eventually in  $V$  or eventually out of  $V$  (by [23, Proposition 3.2]; see also the proof of Theorem 3.4 therein). Take  $\psi \in K(X)$ : then, if  $W \in \mathcal{B}(\psi)$  then it must be  $\psi(s_\nu) \in V$  eventually, and thus  $\mathcal{B}(\psi)$  contains  $V_{s_\nu}$  for all large  $\nu$ ; on the other hand, if  $W \in \mathcal{B}(\psi)^c$  then  $\psi(s_\nu) \notin V$  eventually, and thus  $\mathcal{B}(\psi)^c$  contains  $V_{s_\nu}$  for all large  $\nu$ .

Now let

$$\Omega := \mathcal{B}(\psi_1, \dots, \psi_n) \cap \mathcal{B}(\phi_1)^c \cap \dots \cap \mathcal{B}(\phi_m)^c \cap \mathcal{E}(K(X)|V)$$

be a basic open set of  $\mathcal{E}(K(X)|V)^{\text{cons}}$  containing  $W$ . For every  $i$ , there is an index  $N_i$  such that  $\psi_i(s_\nu) \in V$  for all  $\nu \geq N_i$ ; likewise, for every  $j$  there is a  $M_j$  such that  $\phi_j(s_\nu) \notin V$  for all  $\nu \geq M_j$ . Therefore, for every

$$\nu \geq \max\{N_1, \dots, N_n, M_1, \dots, M_m\},$$

we have  $V_{s_\nu} \in \Omega$ . Hence  $W$  belongs to the closure of  $\{V_{s_\nu}\}_{\nu \in \Lambda} \subseteq \mathcal{E}(K(X)|V)$ , with respect to the constructible topology. It follows that  $W$  is not isolated in  $\mathcal{E}(K(X)|V)^{\text{cons}}$  and, since  $W$  was arbitrary,  $\mathcal{E}(K(X)|V)^{\text{cons}}$  is perfect.

Suppose now that  $K$  is any field. Let  $W \in \mathcal{E}(K(X)|V)$ , and suppose that  $W$  is isolated in  $\mathcal{E}(K(X)|V)^{\text{cons}}$ . Let  $\rho : \mathcal{E}(\overline{K}(X)|V) \rightarrow \mathcal{E}(K(X)|V)$  be the restriction map. Since  $W$  is isolated and  $\rho$  is continuous,  $\rho^{-1}(W)$  is open. Let  $W' \in \rho^{-1}(W)$  and let  $U := W' \cap \overline{K}$ : then,  $U$  is an extension of  $V$  to  $\overline{K}$ .

By the previous part of the proof, for every open neighborhood  $\Omega$  of  $W'$  there is an  $s \in \overline{K}$  such that  $U_s \neq W'$  and  $U_s \in \Omega$ ; since  $\rho^{-1}(W)$  is open, it follows that for every such  $\Omega$  there is a  $U_s \in \rho^{-1}(W)$  with these properties. Therefore, the set  $\Delta := \{U_s \in \rho^{-1}(W) \mid s \in \overline{K}\}$  is dense in  $\rho^{-1}(W)$ . Since  $U_s \cap K(X) = U_t \cap K(X) = W$  for every  $U_s, U_t \in \Delta$ , by Lemma 8.1  $\Delta$  is finite; since  $\mathcal{E}(K(X)|V)^{\text{cons}}$  is Hausdorff, it follows that  $\Delta = \rho^{-1}(W)$ , and in particular  $\rho^{-1}(W)$  is finite. Hence, all its points are isolated. However, this contradicts the fact that  $\mathcal{E}(\overline{K}(X)|V)^{\text{cons}}$  is perfect; thus, also  $\mathcal{E}(K(X)|V)^{\text{cons}}$  must be perfect.  $\square$

The theorem above allows to determine the isolated points of  $\text{Zar}(K(X)|D)^{\text{cons}}$  for every integral domain  $D$ .

**Proposition 8.3.** *Let  $D$  be an integral domain that is not a field, and let  $J$  be the intersection of the nonzero prime ideals of  $D$ .*

- (i) *If  $J = (0)$ , then  $\text{Zar}(K(X)|D)^{\text{cons}}$  is perfect.*
- (ii) *If  $J \neq (0)$ , then the only isolated points of  $\text{Zar}(K(X)|D)^{\text{cons}}$  are  $K[X]_{(f(X))}$  (where  $f(X)$  is an irreducible polynomial of  $K[X]$ ) and  $K[X]_{(X-1)}$ .*

**Proof.** Let  $W \in \text{Zar}(K(X)|D)$ . If  $W \cap K \neq K$ , then  $\mathcal{E}(K(X)|W \cap K)$  is perfect (when endowed with the constructible topology) by Theorem 8.2. Since  $W$  belongs to this set, it is not isolated in  $\text{Zar}(K(X)|D)^{\text{cons}}$ .

Suppose that  $W \cap K = K$ . If  $W = K(X)$ , then  $W$  is not isolated by Proposition 4.1, since  $K(X)$  is not algebraic over  $K$ . Thus let  $W \neq K(X)$ .

Suppose that  $J = (0)$ , and suppose that  $W$  is isolated in  $\text{Zar}(K(X)|D)^{\text{cons}}$ . Since  $K \subseteq W$ , we have  $\mathfrak{m}_W \cap D = (0)$ ; by Lemma 4.2, the quotient map of  $W$  onto its residue field induces a homeomorphism between the spaces  $\Delta := \{Z \in \text{Zar}(K(X)|D) \mid Z \subseteq W\}$  and  $\text{Zar}(W/\mathfrak{m}_W|D)$ , where  $W$  is sent to  $W/\mathfrak{m}_W$ . Since  $W$  is isolated in  $\text{Zar}(K(X)|D)^{\text{cons}}$ , it is also isolated in  $\Delta^{\text{cons}}$ , and thus  $W/\mathfrak{m}_W$  must be an isolated point of  $\text{Zar}(W/\mathfrak{m}_W|D)^{\text{cons}}$ . By Proposition 4.1,  $D$  must be a Goldman domain, against the hypothesis  $J = (0)$ . Therefore,  $W$  is not isolated and  $\text{Zar}(K(X)|D)^{\text{cons}}$  is perfect.

Suppose now that  $J \neq (0)$ , and let  $j \in J$ ,  $j \neq 0$ . Then,  $D[j^{-1}] = K$ , and thus  $\mathcal{B}(j^{-1}) = \mathcal{E}(K(X)|K) = \text{Zar}(K(X)|K)$  is a clopen subset of  $\text{Zar}(K(X)|D)^{\text{cons}}$ ; in particular,  $W \in \mathcal{E}(K(X)|K)$  is isolated in  $\text{Zar}(K(X)|D)^{\text{cons}}$  if and only if it is isolated in  $\text{Zar}(K(X)|K)^{\text{cons}}$ . The claim now follows from Proposition 7.4.  $\square$

To conclude the paper, we extend Theorem 7.3 to valuation domains.

**Theorem 8.4.** *Let  $V$  be a valuation domain with quotient field  $K$ , and let  $L$  be a field extension of  $K$  such that  $\text{trdeg}(L/K) \geq 2$ . Then,  $\mathcal{E}(L|V)^{\text{cons}}$  and  $\text{Zar}(L|V)^{\text{cons}}$  are perfect.*

**Proof.** We first show that  $\mathcal{E}(L|V)^{\text{cons}}$  is perfect: suppose that is not, and let  $W$  be an isolated point.

Suppose that  $L = K(x, z_2, \dots)$  is purely transcendental over  $K$ , where  $x, z_2, \dots$ , is a transcendence basis. Take an  $m \in \mathfrak{m}_V \subseteq \mathfrak{m}_W$ : then, at least one of  $mx$  and  $x^{-1}$  belongs to  $\mathfrak{m}_W$ . Let  $z_1$  be that element. Then,  $z_1, z_2, \dots$  is also a transcendence basis of  $L$ .

Let  $L' := K(z_1, z_3, \dots)$  be the extension of  $K$  obtained adjoining all the element of this basis except  $z_2$ . Then,  $z_1^{-1} \in L' \setminus W$ , and thus  $W \cap L' \neq L'$ ; since, by construction,  $L \simeq L'(X)$ , by Theorem 8.2  $\mathcal{E}(L|W \cap L')^{\text{cons}}$  is perfect. Since  $\mathcal{E}(L|W \cap L') \subseteq \mathcal{E}(L|W)$ , all the elements of  $\mathcal{E}(L|W \cap L')$  (in particular,  $W$ ) are not isolated in  $\mathcal{E}(L|V)^{\text{cons}}$ . This is a contradiction, and thus  $\mathcal{E}(L|V)^{\text{cons}}$  is perfect.

Suppose now that  $L$  is arbitrary: then, we can find a purely transcendental extension  $L'$  of  $K$  such that  $L' \subseteq L$  is algebraic. By the previous part of the proof,  $\mathcal{E}(L'|V)^{\text{cons}}$  is perfect; by Corollary 7.2, also  $\mathcal{E}(L|V)^{\text{cons}}$  is perfect. Therefore,  $\mathcal{E}(L|V)^{\text{cons}}$  is always perfect.

Finally,  $\text{Zar}(L|V)$  is the union of  $\mathcal{E}(L|V_0)$ , as  $V_0$  ranges among the valuation overrings of  $V$ ; since each of these is perfect with respect to the constructible topology (by the previous part of the proof), then also  $\text{Zar}(L|V)^{\text{cons}}$  is perfect, as claimed.  $\square$

**Corollary 8.5.** *Let  $V$  be a valuation domain with quotient field  $K$ , suppose  $V \neq K$ , and let  $L$  be a transcendental field extension of  $K$ . Then,  $\mathcal{E}(L|V)^{\text{cons}}$  is perfect.*

**Proof.** If  $\text{trdeg}(L/K) \geq 2$  the claim follows from Theorem 8.4. If  $\text{trdeg}(L/K) = 1$ , let  $X \in L$  be transcendental over  $K$ . By Theorem 8.2,  $\mathcal{E}(K(X)|V)^{\text{cons}}$  is perfect; by Corollary 7.2, also  $\mathcal{E}(L|V)^{\text{cons}}$  is perfect.  $\square$



**Corollary 8.6.** *Let  $D$  be an integral domain, and let  $L$  be a transcendental extension of the quotient field  $K$  of  $D$ . If  $\text{trdeg}(L/K) \geq 2$ , then  $\text{Zar}(L|D)^{\text{cons}}$  is perfect.*

**Proof.** Any  $W \in \text{Zar}(L|D)$  belongs to  $\mathcal{E}(L|V)$  for some  $V \in \text{Zar}(D)$ . By Theorem 8.4, all  $\mathcal{E}(L|V)^{\text{cons}}$  are perfect and thus no  $W$  is isolated. Hence,  $\text{Zar}(L|D)^{\text{cons}}$  is perfect.  $\square$

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