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von Neumann's inequality for the Hartogs triangle

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ABSTRACT. For a commuting pair T of bounded linear operators T_1 and T_2 on a Hilbert space $\mathcal{H}$, let $D_T = T_2^*T_2 - T_1^*T_1$. If $T_2^*D_TT_2 \leqslant D_T$ and the Taylor spectrum of T is contained in the Hartogs triangle $\triangle_H$, then for any bounded holomorphic function ϕ on $\triangle_H$, $\|\phi(T)\| \leqslant \|\phi\|_\infty$. We deduce this fact from an analogue of von Neumann's inequality for bounded domains in $\mathbb{C}^d$. The proof of the latter closely follows the model theory approach as developed in [1].

CONTENTS

1. Introduction

1.1. Hardy space of the Hartogs triangle. The *Hartogs triangle* $\triangle_H$ is given by

$$\triangle_H = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2| < 1\}.$$

Clearly,

$$\triangle_H = \{(z_1, z_2) \in \mathbb{D}^2 : (z_2\overline{z}_2 - z_1\overline{z}_1)(1 - z_2\overline{z}_2) > 0\}, \tag{1}$$

where $\mathbb{D}$ denotes the open unit disc. Moreover, $\triangle_H$ is bi-holomorphic to $\mathbb{D}\times\mathbb{D}^*$ via the map $(z_1, z_2) \to (\frac{z_1}{z_2}, z_2)$, where $\mathbb{D}^* = \mathbb{D}\backslash\{0\}$. Thus, $\triangle_H$ is a holomorphically convex domain. However, it is easy to see that $\triangle_H$ is not polynomially convex (refer to [19] for the basics of several complex variables and also to [20] for an exposition surveying classical and recent results on the Hartogs triangle).

Let $I = (0, 1)$. For $(s, t) \in I \times I$, let

$$\triangle_{H_{s,t}} = \Big\{(z_1, z_2) \in \mathbb{C}^2 : \frac{|z_1|}{s} < |z_2| < t\Big\}.$$

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The distinguished boundary $\partial_d(\triangle_{H_{s,t}})$ of $\triangle_{H_{s,t}}$ is given by

$$\partial_d(\triangle_{H_{s,t}}) = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = st, |z_2| = t\}.$$

Following [12, Section 3] (see also [9, Section 6]), we define the *Hardy space on the Hartogs triangle* $\triangle_H$ as

$$H^2(\triangle_H) = \Big\{f \in \mathrm{Hol}(\triangle_H) : \sup_{(s,t)\in I\times I} \frac{1}{4\pi^2} \int_{\partial_d(\triangle_{H_{s,t}})} |f|^2 d\sigma_{s,t} < \infty\Big\},$$

where $\sigma_{s,t}$ is the surface area measure on $\partial_d(\triangle_{H_{s,t}})$ induced by the Lebesgue measure on the unit torus $\mathbb{T}\times\mathbb{T}$. Then $H^2(\triangle_H)$ is a reproducing kernel Hilbert space endowed with the norm

$$\|f\|^2_{H^2(\triangle_H)} = \sup_{(s,t)\in I\times I} \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} |f(ste^{i\theta}, te^{i\gamma})|^2 st^2 d\theta d\gamma.$$

A *domain* Ω *in* $\mathbb{C}^d$ is a nonempty open subset of $\mathbb{C}^d$. For a bounded domain Ω in $\mathbb{C}^d$, let $H^\infty(\Omega)$ denote the Banach algebra of bounded holomorphic functions on Ω endowed with the sup norm $\|\cdot\|_{\infty,\Omega}$. Note that $H^\infty(\mathbb{D}^2) \subsetneq H^\infty(\triangle_H)$ (consider $f(z_1, z_2) = z_1/z_2$). However, any function holomorphic in a neighborhood of $\overline{\triangle_H}$ extends analytically to $\mathbb{D}^2$ (see [20, Proposition 1.1]). The multiplier algebra of a Hilbert space $\mathscr{H}$ of complex-valued holomorphic functions is denoted by $\mathrm{Mult}(\mathscr{H})$. If $\phi \in \mathrm{Mult}(\mathscr{H})$, then the operator M_ϕ of multiplication by ϕ defines a bounded linear operator. The multiplier norm of ϕ is the operator norm of M_ϕ.

Lemma 1.1. *The following statements are valid*:

(*i*) *The reproducing kernel for* $H^2(\triangle_H)$ *is given by*

$$\kappa_H(z, w) = \frac{1}{(z_2\overline{w}_2 - z_1\overline{w}_1)(1 - z_2\overline{w}_2)}.$$

(*ii*) *The multiplier algebra of* $H^2(\triangle_H)$ *is equal to* $H^\infty(\triangle_H)$ *with equality of norms.*

Proof. Note that (i) is precisely [12, Proposition 3.2]. To see (ii), note that the multiplier algebra of $H^2(\triangle_H)$ is contained in $H^\infty(\triangle_H)$ and $\|\phi\|_{\infty,\triangle_H} \leqslant \|M_\phi\|$ (see [14]). Also, for each $\phi \in H^\infty(\triangle_H)$,

$$\|\phi f\|_{H^2(\triangle_H)} \leqslant \|\phi\|_{\infty,\triangle_H}\|f\|_{H^2(\triangle_H)}, \quad f \in H^2(\triangle_H).$$

Thus, the multiplier norm of ϕ is equal to $\|\phi\|_{\infty,\triangle_H}$. □

1.2. Main theorem. Let $\mathcal{H}$ be a complex Hilbert space and let d be a positive integer. By a *commuting* d*-tuple* $T = (T_1, \dots, T_d)$ *on* $\mathcal{H}$, we understand the d-tuple of bounded linear operators $T_1, \dots, T_d$ on $\mathcal{H}$ such that $T_jT_k = T_kT_j$ for all integers $1 \leqslant j \neq k \leqslant d$. The Taylor spectrum and approximate point-spectrum of a commuting d-tuple T on $\mathcal{H}$ are denoted by $\sigma(T)$ and $\sigma_{\mathrm{ap}}(T)$, respectively (the reader is referred to [5] for definition and basic information on the various joint spectra of commuting tuples).

The purpose of this note is to prove von Neumann's inequality for the Hartogs triangle (cf. [22, Theorem 1.1]).

Theorem 1.2. *If $T = (T_1, T_2)$ is a commuting 2-tuple on $\mathcal{H}$ such that $\sigma(T) \subset \triangle_H$ and $T_2^*(T_2^*T_2 - T_1^*T_1)T_2 \leqslant T_2^*T_2 - T_1^*T_1$, then*

$$\|\phi(T)\| \leqslant \|\phi\|_{\infty,\triangle_H}, \quad \phi \in H^\infty(\triangle_H). \tag{2}$$

Remark 1.3. Set $D_T := T_2^*T_2 - T_1^*T_1$ and $E_T := D_T - T_2^*D_TT_2$.

(1) By the hereditary functional calculus (see [1]),

$$T_2^*D_TT_2 \leqslant D_T \Longleftrightarrow \frac{1}{\kappa_H}(T, T^*) \geqslant 0,$$

where $\frac{1}{\kappa_H}(z, w) = (z_2\overline{w}_2 - z_1\overline{w}_1)(1 - z_2\overline{w}_2)$ (cf. (1)).

(2) The condition $E_T \geqslant 0$ need not imply $\sigma(T) \subset \triangle_H$ (cf. [22, Lemma 4.1]). Indeed, any commuting subnormal 2-tuple T with minimal normal extension N such that

$$\sigma(N) \subset \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| > |z_2| > 1\}$$

satisfies $E_T \geqslant 0$ (consult the spectral theorem for commuting normal operators). However, since $\sigma(N) \subseteq \sigma(T)$ (see [17]) is a nonempty set (see [5]), $\sigma(T)$ intersects with the complement of $\triangle_H$. Also, by the spectral theorem for commuting normal operators,

$$T_1^*T_1 \geqslant T_2^*T_2 \geqslant I.$$

This example also shows that $E_T \geqslant 0$ need not imply

$$T_1^*T_1 \leqslant T_2^*T_2 \leqslant I.$$

(3) Any commuting 2-tuple T such that $E_T \geqslant 0$ satisfies

$$\sigma_{\mathrm{ap}}(T) \cap \overline{\mathbb{D}}^2 \subseteq \overline{\triangle_H}. \tag{3}$$

Indeed, if $(\lambda_1, \lambda_2) \in \sigma_{ap}(T)$, then there exists a sequence $\{x_n\}_{n=1}^\infty$ of unit vectors such that

$$\lim_{n\to\infty} \langle E_Tx_n,\, x_n\rangle = 0 \Longrightarrow (1 - |\lambda_2|^2)(|\lambda_2|^2 - |\lambda_1|^2) \geqslant 0,$$

and hence (1) yields the inclusion (3).

Finally, note that (2) implies that $\triangle_H$ is a spectral domain for T (see [1, Definition 1.46]).

Since $\triangle_H$ is not a P-ball in the sense [16] (indeed, $\triangle_H$ is a Reinhardt domain, which is not complete), [16, Corollary 5.4] is not applicable to the case of the Hartogs triangle. It turns out that an application of Lemma 1.1 together with an analogue of von Neumann's inequality (see Theorem 2.1) yields Theorem 1.2.

2. von Neumann's inequality for a bounded domain

Theorem 2.1. *Let Ω be a bounded domain in $\mathbb{C}^d$. Assume that there exists a reproducing kernel Hilbert space $\mathscr{H}_\kappa$ with reproducing kernel $\kappa : \Omega\times\Omega \to \mathbb{C}\backslash\{0\}$ such that $\frac{1}{\kappa}$ is a hereditary function on Ω. If T is a commuting d-tuple on $\mathcal{H}$ such that $\sigma(T) \subset \Omega$ and $\frac{1}{\kappa}(T,T^*) \geqslant 0$, then*

$$\|\phi(T)\| \leqslant \|M_\phi\|, \quad \phi \in \mathrm{Mult}(\mathscr{H}_\kappa). \tag{4}$$

If, in addition, the multiplier algebra $\mathrm{Mult}(\mathscr{H}_\kappa)$ *of* $\mathscr{H}_\kappa$ *is equal to* $H^\infty(\Omega)$ (*with equality of norms*), *then*

$$\|\phi(T)\| \leqslant \|\phi\|_{\infty,\Omega}, \quad \phi \in H^\infty(\Omega). \tag{5}$$

Remark 2.2. The above result is applicable to a number of domains Ω in $\mathbb{C}^d$ allowing us to recover variants of several known results ([3, 1, 8, 13, 16, 22]):

- Letting $\Omega = \mathbb{D}^d$ and $\kappa(z,w) = \frac{1}{\prod_{j=1}^d (1-\overline{w}_j z_j)}$, $z,w \in \mathbb{D}^d$, we get (5) for any commuting d-tuple T satisfying $\sigma(T) \subset \mathbb{D}^d$ and
$$\sum_{\substack{\alpha_j\in\{0,1\} \\ j=1,\dots,d}} (-1)^{|\alpha|} T^{*\alpha}T^\alpha \geqslant 0,$$
where $T^\alpha = \prod_{j=1}^d T_j^{\alpha_j}$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$, $\alpha = (\alpha_1,\dots,\alpha_d)$. This recovers a variant of von Neumann inequality for the polydisc (cf. [6, Theorem 4.7] and [21, Theorem 1.9.1]).
- Letting $\Omega = \mathbb{B}^d$ and $\kappa(z,w) = \frac{1}{1-\langle z,w\rangle}$, $z,w \in \mathbb{B}^d$, we get (4) for any commuting d-tuple T satisfying $\sigma(T) \subset \mathbb{B}^d$ and $\sum_{j=1}^d T_j^* T_j \leqslant I$. This recovers Drury's theorem (see [8, pp 300-301]).
- Letting $\Omega = \mathbb{B}^d$ and $\kappa(z,w) = \frac{1}{(1-\langle z,w\rangle)^d}$, $z,w \in \mathbb{B}^d$, we get (5) for any commuting d-tuple T satisfying $\sigma(T) \subset \mathbb{B}^d$ and
$$\sum_{0\leqslant k\leqslant d} (-1)^k \binom{d}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} T^{*\alpha}T^\alpha \geqslant 0.$$
This is a consequence of [13, Theorem 11].
- For $0 < r < 1$, letting $\Omega = \mathbb{A}_r = \{z \in \mathbb{C} : r < |z| < 1\}$ and $\kappa(z,w) = \frac{1-r^2}{(1-z\overline{w})(1-r^2/z\overline{w})}$, $z,w \in \mathbb{A}_r$, we get (4) for any bounded linear operator T satisfying $\sigma(T) \subset \mathbb{A}_r$ and
$$r^2 I - (1+r^2)T^*T + T^{*2}T^2 \leqslant 0.$$
This recovers a part of [22, Theorem 1.1].

Recall from [2] (see also [1, Chapter 4]) that h is a *hereditary function on* Ω if h is a mapping from $\Omega\times\Omega$ to $\mathbb{C}$ such that $(z,w) \to h(z,\overline{w})$ is holomorphic function on $\Omega\times\Omega$. A *dyad* in $\mathrm{Her}(\Omega)$ is a function of the form $(z,w) \to \overline{\psi(w)}\phi(z)$,

$z, w \in \Omega$, for some functions ϕ and ψ in $\mathrm{Hol}(\Omega)$. Here $\mathrm{Her}(\Omega)$ denotes the set of hereditary functions on Ω.

It has been recorded in [1, Remark 4.15] that for any domain Ω in $\mathbb{C}^d$, the hereditary functional calculus is well-defined and satisfies the properties as listed in [1, Lemma 4.13]. Alternatively, following [4, Section 1], we may use the iterated Cauchy-Weil integral (see [5, Corollary 5.4]) to define

$$h(T) = \frac{1}{(2\pi i)^{2d}} \int_{\partial\Omega'} \int_{\partial\Omega'} h(z, \overline{w}) M(\overline{w} - T^*) M(z - T) \wedge dz \wedge d\overline{w}$$

for every $h \in \mathrm{Her}(\Omega)$, where M is the Martinelli kernel (see [5]) and Ω' is an open domain with C^1-boundary such that $\sigma(T) \subset \Omega' \subset \overline{\Omega'} \subset \Omega$. In the proof of Theorem 2.1, we need the continuity of the hereditary functional calculus:

Lemma 2.3. *Let Ω be a bounded domain in $\mathbb{C}^d$. If $\{h_n\}_{n=1}^{\infty}$ is a sequence in* $\mathrm{Her}(\Omega)$*, $h \in \mathrm{Her}(\Omega)$, and $h_n \to h$ uniformly on compact subsets of $\Omega \times \Omega$, then $h_n(T) \to h(T)$ with respect to the operator norm, whenever T is a commuting d-tuple with $\sigma(T) \subset \Omega$.*

Proof. Let T be a commuting d-tuple satisfying $\sigma(T) \subset \Omega$. As pointed out in [1, Remark 4.15], the proof of this fact relies on the nuclearity of $\mathrm{Hol}(\Omega)$ (see [10, Theorem V.4.1], [7, Theorem 3.64]) and [15, Proposition, p 113]. Alternatively, one may adapt the proof of [1, Proposition 2.87] to the present situation. Since $\sigma(T)$ is contained in Ω' and

$$\lambda \mapsto \Big(\sum_{j=1}^{d} (T_j - \lambda_j)^* (T_j - \lambda_j) \Big)^{-1}$$

is bounded on $\partial\Omega'$ (since the approximate point-spectrum is a closed subset of Ω' and the inverse map on invertible bounded linear operators is continuous), by [5, Lemma 5.11], the map $(z, w) \mapsto M(\overline{w} - T^*) M(z - T)$ is bounded on $\partial\Omega'$. It follows that

$$h_n(z, w) M(\overline{w} - T^*) M(z - T) \to h(z, w) M(\overline{w} - T^*) M(z - T)$$

in the operator norm, uniformly for $z, w \in \partial\Omega'$. Hence, $h_n(T) \to h(T)$. □

If Ω is a Reinhardt domain in $\mathbb{C}^d$, then by [11, Proposition 1.7.1(b)&(c)], $(z, w) \to h(z, \overline{w})$ has a Laurent series expansion converging compactly on $\Omega \times \Omega$, and hence for some complex numbers $a_{\alpha,\beta}$,

$$h(T) = \sum_{\alpha,\beta \in \mathbb{Z}^d} a_{\alpha,\beta} T^{*\beta} T^{\alpha},$$

where $\mathbb{Z}$ denotes the set of integers. This observation is applicable to the Hartogs triangle (see [20, Eq (1.1)]).

The following is a counter-part of [1, Theorem 4.4] for any bounded domain in $\mathbb{C}^d$.

Lemma 2.4. *Let Ω be a bounded domain in $\mathbb{C}^d$. If A is a positive semi-definite hereditary function on Ω, then there exists a sequence $\{f_n\}_{n=1}^{\infty}$ in* $\mathrm{Hol}(\Omega)$ *such that*

$$A(z,w) = \sum_{n=1}^{\infty} \overline{f_n(w)} f_n(z), \quad z, w \in \Omega, \tag{6}$$

where the series in (6) *converges uniformly on compact subsets of* $\Omega \times \Omega$.

Proof. We closely follow the proof of [1, Theorem 4.4]. Since A is positive semi-definite on Ω, by Moore's theorem (see [1, Theorem 2.5]), there exist a Hilbert space $\mathcal{M}$ and a function $u : \Omega \to \mathcal{M}$ such that

$$\vee\{u(z) : z \in \Omega\} = \mathcal{M}, \quad A(z,w) = \langle u(z), u(w)\rangle_{\mathcal{M}}, \ z, w \in \Omega.$$

By the reproducing property of $\mathcal{M}$, u is easily seen to be a weakly holomorphic function, and hence by [1, Lemma 2.90], u is holomorphic. Also, since Ω is separable, so is $\mathcal{M}$. Let $\{e_n\}_{n=1}^{N}$ be an orthonormal basis for $\mathcal{M}$, where N is a positive integer or $N = \infty$. By Parseval's identity and the continuity of the inner-product, we have

$$A(z,w) = \langle u(z), u(w)\rangle_{\mathcal{M}} = \sum_{n=1}^{N} \overline{\langle u(w), e_n\rangle_{\mathcal{M}}} \langle u(z), e_n\rangle_{\mathcal{M}}, \quad z, w \in \Omega.$$

Therefore, if we define $f_n = \langle u(\cdot), e_n\rangle_{\mathcal{M}}$, $n = 1, \ldots, N$, then $f_n \in \mathrm{Hol}(\Omega)$ and (6) holds (if $N < \infty$, then let $f_n = 0$ for $n > N$). Letting $z = w$ in (6), we may conclude from Dini's theorem (see [18, Theorem 7.13]) that $\sum_{n=1}^{\infty} |f_n(\cdot)|^2$ converges uniformly on compact subsets of Ω. We obtain the desired conclusion now from the Cauchy-Schwarz inequality. □

Proof of Theorem 2.1. Assume that $\sigma(T) \subset \Omega$ and $\frac{1}{\kappa}(T, T^*) \geqslant 0$. Let $\phi \in \mathrm{Mult}(\mathscr{H}_\kappa)$. After multiplying by a scalar, if required, we may assume that ϕ belongs to the closed unit ball of $\mathrm{Mult}(\mathscr{H}_\kappa)$. By [14, Theorem 5.21], there exists a positive semi-definite kernel $A : \Omega \times \Omega \to \mathbb{C}$ such that

$$(1 - \overline{\phi(w)}\phi(z))\kappa(z,w) = A(z,w), \quad z, w \in \Omega. \tag{7}$$

Thus, we have the model formula

$$1 - \overline{\phi(w)}\phi(z) = \frac{A(z,w)}{\kappa(z,w)}, \quad z, w \in \Omega.$$

Since A is a positive semi-definite element of $\mathrm{Her}(\Omega)$, by Lemma 2.4, there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of elements in $\mathrm{Hol}(\Omega)$ such that

$$1 - \phi(z)\overline{\phi(w)} = \sum_{n=1}^{\infty} \overline{f_n(w)} \Big(\frac{1}{\kappa}(z,w)\Big) f_n(z)$$

converges uniformly on compact subsets of $\Omega \times \Omega$. Since $\sigma(T) \subset \Omega$, by the continuity of the hereditary functional calculus on Ω (see Lemma 2.3),

$$I - \phi(T)^*\phi(T) = \sum_{n=1}^{\infty} f_n(T)^*\Big(\frac{1}{\kappa}(T, T^*)\Big) f_n(T).$$

Hence, by the assumption $\frac{1}{\kappa}(T, T^*) \geqslant 0$, we conclude that $I - \phi(T)^*\phi(T) \geqslant 0$ or equivalently, $\|\phi(T)\| \leqslant 1$ completing the proof of the first part. The remaining part is now immediate. □

Proof of Theorem 1.2. An application of Lemma 1.1 to the choice $\kappa = \kappa_H$ together with Theorem 2.1 and Remark 1.3(1) yields (2). □

Let us discuss some consequences of Theorem 1.2.

Corollary 2.5. *If $T = (T_1, T_2)$ is a commuting 2-tuple on $\mathcal{H}$ such that $\sigma(T) \subset \triangle_H$ and $T_2^*(T_2^*T_2 - T_1^*T_1)T_2 \leqslant T_2^*T_2 - T_1^*T_1$, then*

$$T_1^*T_1 \leqslant T_2^*T_2 \leqslant I. \tag{8}$$

Proof. Letting $\phi(z_1, z_2) = z_2$ in (2), we conclude that $T_2^*T_2 \leqslant I$. To see the remaining inequality, by the spectral mapping property (see [5, Theorem 5.19]), T_2 is invertible. Applying (2) to the bounded holomorphic function $\phi(z_1, z_2) = z_1/z_2$ on $\triangle_H$, we get $\|T_1T_2^{-1}\| \leqslant 1$, or equivalently, $(T_1T_2^{-1})^*(T_1T_2^{-1}) \leqslant I$. Thus, $T_1^*T_1 \leqslant T_2^*T_2$ completing the proof. □

Remark 2.6. Assume that (8) holds and T_2 on $\mathcal{H}$ is invertible. Note that 2-tuple $(T_1T_2^{-1}, T_2)$ is a commuting pair of contractions. By Ando's dilation theorem (see [3]), there exists a unitary commuting 2-tuple $U = (U_1, U_2)$ on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ such that

$$T_1^mT_2^{n-m} = (T_1T_2^{-1})^mT_2^n = P_HU_1^mU_2^n, \quad m, n \geqslant 0,$$

where $P_{\mathcal{H}}$ denotes the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. Finally, it has been pointed out by the anonymous referee that (8) can also be deduced from a couple of applications of the Gelfand spectral radius formula.

For certain Taylor invertible 2-tuples, the inclusion $\sigma(T) \subset \overline{\triangle_H}$ appearing in Theorem 1.2 may be replaced by the weaker condition $\sigma(T) \subseteq \overline{\triangle_H}$.

Corollary 2.7. *Let $T = (T_1, T_2)$ be a commuting 2-tuple on $\mathcal{H}$ such that $\sigma(T) \subseteq \overline{\triangle_H}$ and*

$$E_T := T_2^*T_2 - T_1^*T_1 - T_2^*(T_2^*T_2 - T_1^*T_1)T_2 \geqslant 0.$$

If T is Taylor-invertible and $T_1^*T_1 \leqslant T_2^*T_2 \leqslant I$, then for every bounded rational function ϕ with poles off $\overline{\triangle_H}$, $\|\phi(T)\| \leqslant \|\phi\|_{\infty, \triangle_H}$.

Proof. Assume that T is Taylor-invertible and $T_1^*T_1 \leqslant T_2^*T_2 \leqslant I$. For $0 < s, t < 1$, let $T_{s,t}$ denote the commuting pair (stT_1, tT_2). Note that

$$E_{T_{s,t}} = t^2\big(E_T + (1 - t^2)T_2^{*2}T_2^2 + (1 - s^2)T_1^*T_1 + (s^2t^2 - 1)T_2^*T_1^*T_1T_2\big).$$

Thus, if $T_1^*T_1 \leqslant T_2^*T_2 \leqslant I$, then

$$\begin{aligned} E_{T_{s,t}} &\geqslant t^2(E_T) + t^2(1-t^2+1-s^2+s^2t^2-1)T_2^*T_1^*T_1T_2 \\ &= t^2(E_T) + t^2(1-t^2)(1-s^2)T_2^*T_1^*T_1T_2, \end{aligned}$$

which is a positive operator by assumption. Since $(0,0) \notin \sigma(T)$, by the spectral mapping property (see [5, Theorem 5.19]), $\sigma(T_{s,t}) \subset \triangle_H$. One may now apply Theorem 1.2 to the commuting 2-tuple $T_{s,t}$ and any bounded rational function ϕ with poles off $\triangle_H$, and let s, t tend to 1. □

We do not know whether the assumption of the Taylor invertibility of T can be relaxed from Corollary 2.7. This constraint may be related to the fact that the origin is a singularity of the boundary of the Hartogs triangle.

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