Nonmonogenity of number fields defined by trinomials

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Abstract. Let \( f(x) = x^n - ax^m - b \) be a monic irreducible polynomial of degree \( n \) having integer coefficients. Let \( K = \mathbb{Q}(\theta) \) be an algebraic number field with \( \theta \) a root of \( f(x) \). In this paper, we provide some explicit conditions involving only \( a, b, m, n \) for which \( K \) is not monogenic. Further, as an application, in a special case, we show that if \( p \) is a prime number of the form \( 32k + 1, k \in \mathbb{Z} \) and \( \theta \) is a root of a monic polynomial \( x^{12n} - 64ax^m - p \) with \( 2 \nmid n, p \nmid a \), then \( \mathbb{Q}(\theta) \) is not monogenic.

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1. INTRODUCTION AND STATEMENT OF THE RESULT

For a given algebraic number field \( K \), it is a classical problem in Algebraic Number Theory whether \( K \) is monogenic or not. There are many results in the literature for testing the monogenity of number fields using different approaches (cf. [1], [3], [5], [6], [7], [8], [9], [12], [16], [2]). Let \( \mathbb{Z}_K \) denote the ring of algebraic integers of an algebraic number field \( K = \mathbb{Q}(\theta) \) where \( \theta \) is a root of a monic irreducible polynomial \( f(x) \) of degree \( n \) having coefficients from the ring \( \mathbb{Z} \) of integers. It is well-known that \( \mathbb{Z}_K \) is a free abelian group of rank \( n \). Let \( \text{ind} \theta \) denote the index of the subgroup \( \mathbb{Z}[\theta] \) in \( \mathbb{Z}_K \). The index \( i(K) \) of the field \( K \) is defined as

\[
i(K) = \gcd\{\text{ind} \alpha \mid \alpha \in \mathbb{Z}_K \text{ generates the field extension } K/\mathbb{Q}\}.
\]

A prime number \( p \) dividing \( i(K) \) is called a prime common index divisor of \( K \). A number field \( K \) is called monogenic if there exists an element \( \alpha \in \mathbb{Z}_K \) such that \( \{1, \alpha, \ldots, \alpha^{n-1}\} \) is an integral basis of \( K \); if no such \( \alpha \) exists, then we say that \( K \) is not monogenic. In 2016, Ahmad, Nakahara, and Husnine [1] proved that
the sextic number field generated by $b^\frac{1}{6}$ is not monogenic if $b \equiv 1 \mod 4$ and $b \not\equiv \pm 1 \mod 9$. In 2017, Gaál and Remete [9] provided some new results on monogeneity of number fields generated by $b^\frac{1}{n}$ with $b$ a square free integer and $3 \leq n \leq 9$ by applying the explicit form of the index equation. In 2021, Yakkou and Fadil [2] studied the monogenity of number fields generated by $b^{\frac{1}{p}}$, where $b$ is a square free integer and $p$ be a prime number. In this paper, using the splitting of primes in $\mathbb{Z}_K$, we prove some results regarding the non-monogenity of a number field $K$ defined by an irreducible trinomial of the type $x^n - ax^m - b$ having integer coefficients. As an application of our results, we provide a class of non-monogenic number fields defined by irreducible trinomials (see Example 1.3).

For a prime number $q$ and a non-zero $a$ belonging to the ring $\mathbb{Z}_q$ of $q$-adic integers, $\nu_q(a)$ will be the highest power of $q$ dividing $a$ and $\nu_q(a) = \infty$ when $a = 0$. Let $\mathbb{F}_q$ denote the field with $q$ elements and $N(q, \ell)$ denote the number of irreducible polynomials of degree $\ell$ over $\mathbb{F}_q$. It is well known that

$$N(q, \ell) = \frac{1}{\ell} \sum_{k|\ell} \mu(k)q^\frac{\ell}{k},$$

where $\mu$ is the Möbius function. Observe that

$$N(q, 1) = q, \quad N(q, 2) = \frac{q(q - 1)}{2}, \quad N(q, 3) = \frac{q(q^2 - 1)}{3}.$$  

We now state our main result.

**Theorem 1.1.** Let $K = \mathbb{Q}(\theta)$ be an algebraic number field with $\theta$ a root of a monic irreducible polynomial $f(x) = x^n - ax^m - b$ of degree $n$ having integer coefficients. Let $q$ be a prime factor of $n$ with $n = q^r u$, $q \not| u$. Assume that $q^{r+1}$ divides $a$ and $q \not| b$. Suppose $\phi(x)$ is a monic irreducible factor of degree $\ell$ of the polynomial $x^u - b$ over $\mathbb{F}_q$ and $N(q, \ell)$ is as above. If $r_1$ stands for the integer $\nu_q(b^{q^{r-1}} - 1)$, then in the following cases $q$ divides $i(K)$.

1. $q \neq 2$ and $N(q, \ell) < r_1 \leq r$.
2. $q = 2$ and $N(2, \ell) + 2 < r_1 \leq r$.
3. $N(q, \ell) + 1 < r < r_1$.

In the special case when $\ell = 1$, the following corollary is an immediate consequence of the above theorem.

**Corollary 1.2.** Let $K = \mathbb{Q}(\theta)$, $f(x) = x^n - ax^m - b$, and $r_1$ be as in Theorem 1.1. If $q^{r+1}$ divides $a$, $b \equiv 1 \mod q$ and $\min\{r, r_1\} > q + 2$, then $K$ is not monogenic.

It may be pointed out that if we have $b = 1$ in the above corollary, then $K$ is not monogenic for $r > q + 2$.

As an application, we provide a class of non-monogenic number fields defined by irreducible trinomials.
Example 1.3. Let \( p \) be a prime number\(^1\) of the form \( 32k + 1 \) with \( k \in \mathbb{Z} \). Consider a monic polynomial \( f(x) = x^n - ax^m - p \in \mathbb{Z}[x] \) with \( v_2(n) = 5 \) and \( 64p \) divides \( a \). Note that \( f(x) \) is irreducible over \( \mathbb{Q} \) as \( f(x) \) satisfies Eisenstein criterion with respect to \( p \). If \( \theta \) is a root of \( f(x) \) and \( K = \mathbb{Q}(\theta) \), then as in the notations of Corollary 1.2, for \( q = 2 \) we have \( r = 5 \) and \( r_1 \geq 5 \). Therefore \( K \) is not monogenic in view of Corollary 1.2.

2. PRELIMINARY RESULTS

Let \( K = \mathbb{Q}(\theta) \) be an algebraic number field with \( \theta \) a root of an irreducible polynomial \( f(x) \) having integer coefficients and \( \mathbb{Z}_K \) denote the ring of algebraic integers of \( K \). Let \( q \) be a prime number. If \( q \) does not divide \( \text{ind} \theta \), then Dedekind [4] proved a significant theorem in 1878 which relates the decomposition of \( f(x) \) modulo \( q \) with the factorization of \( q\mathbb{Z}_K \) into a product of prime ideals of \( \mathbb{Z}_K \). Precisely, he proved the following.

**Dedekind Theorem.** Let \( K = \mathbb{Q}(\theta) \) be an algebraic number field of degree \( n \) with \( \theta \) an algebraic integer. Let \( f(x) \) be the minimal polynomial of \( \theta \) over \( \mathbb{Q} \) and \( q \) be a rational prime not dividing \( \text{ind} \theta \). Let \( \overline{f}(x) = \overline{g}_1(x)^{e_1} \cdots \overline{g}_t(x)^{e_t} \) be the factorization of \( f(x) \) into powers of distinct irreducible polynomials over \( \mathbb{Z}/q\mathbb{Z} \), where each \( g_i(x) \in \mathbb{Z}[x] \) is monic. Then \( \mathfrak{p}_i = (g_i(\theta), q) \) for \( 1 \leq i \leq t \) are distinct prime ideals of \( \mathbb{Z}_K \) and \( q\mathbb{Z}_K = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_t^{e_t} \), moreover the norm of \( \mathfrak{p}_i \) is \( q^{\deg g_i(x)} \) for \( 1 \leq i \leq t \).

The following lemma is an immediate consequence of Dedekind’s theorem. It plays a key role in the proof of Theorem 1.1. We shall denote by \( \mathbb{F}_q \) the field with \( q \) elements.

**Lemma 2.1.** Let \( K \) be a number field and \( q \) be a prime number. For every positive integer \( f \), let \( N(q, f) \) denote the number of irreducible polynomials of \( \mathbb{F}_q[x] \) of degree \( f \) and \( P(q, f) \) denote the number of distinct prime ideals of \( \mathbb{Z}_K \) lying above \( q \) having residual degree \( f \). If \( P(q, f) > N(q, f) \) for some \( f \), then for every algebraic integer \( \alpha \) generating the field extension \( K/\mathbb{Q} \), the prime \( q \) divides \( \text{ind} \alpha \).

When Dedekind’s theorem fails, i.e., \( q \) divides \( \text{i}(K) \), then Ore developed an alternative approach in 1928 for obtaining the prime ideal factorization of the rational primes in a number field \( K \) by using Newton polygons (cf. [14], [15]).

We now introduce the notion of Gauss valuation which is required for defining the \( \phi \)-Newton polygon of a polynomial, where \( \phi(x) \) belonging to \( \mathbb{Z}_q[x] \) is a monic polynomial with \( \phi(x) \) irreducible over \( \mathbb{F}_q \).

We shall denote by \( v_{q,x} \) the Gauss valuation of the field \( \mathbb{Q}_q(x) \) of rational functions in an indeterminate \( x \) which extends the valuation \( v_q \) of \( \mathbb{Q}_q \) and is defined on \( \mathbb{Q}_q[x] \) by

\[
v_{q,x}(\sum_i b_i x^i) = \min_i \{v_q(b_i)\}, \quad b_i \in \mathbb{Q}_q.
\]  

\((2.1)\)

\(^1\)It is known that there exists infinitely many primes of the form \( 32k + 1, k \in \mathbb{Z} \).
Now we define the notion of $\phi$-Newton polygon with respect to some prime $q$.

**Definition 2.2.** Let $q$ be a prime number and $\phi(x) \in \mathbb{Z}_q[x]$ be a monic polynomial which is irreducible modulo $q$. Let $f(x) \in \mathbb{Z}_q[x]$ be a monic polynomial not divisible by $\phi(x)$ with $\phi$-expansion $\phi(x)^n + a_{n-1}(x)\phi(x)^{n-1} + \cdots + a_0(x)$. Suppose that the $\phi$-Newton polygon of $f(x)$ with respect to $q$ consists of a single edge, say $S$ having positive slope denoted by $\frac{d}{e}$ with $d, e$ coprime, i.e.,

$$\min\left\{ \frac{v_{q,x}(a_{n-i}(x))}{i} \mid 1 \leq i \leq n \right\} = \frac{v_{q,x}(a_0(x))}{n} = \frac{d}{e}$$

so that $n$ is divisible by $e$, say $n = et$ and $v_{q,x}(a_{n-ej}(x)) \geq dj$ for $1 \leq j \leq t$.

Thus the polynomial $\frac{a_{n-ej}(x)}{q^{dj}} = b_j(x)$ (say) has coefficients in $\mathbb{Z}_q$ and hence
$b_j(\alpha) \in \mathbb{Z}_q[\alpha]$ for $1 \leq j \leq t$. The polynomial $T(y)$ in an indeterminate $y$ defined by $T(y) = y^t + \sum_{j=1}^{t} b_j(\alpha)y^{t-j}$ having coefficients in $\mathbb{F}_q[\alpha]$ is said to be the polynomial associated to $f(x)$ with respect to $(\phi, S)$; here the field $\mathbb{F}_q[\alpha]$ is isomorphic to the field $\mathbb{F}_q[x]/(\phi(x))$.

**Example 2.5.** Consider $f(x) = (x + 5)^d - 5$. Then, as in Example 2.3, the $x$-Newton polygon of $f(x)$ with respect to prime 2 consists of only one edge joining the points $(0, 0)$ and $(4, 2)$ with the lattice point $(2, 1)$ lying on it. With notations as in the above definition, we see that $e = 2$, $d = 1$ and the polynomial associated to $f(x)$ with respect to $(x, S)$ is $T(y) = y^2 + y + \bar{1}$ belonging to $\mathbb{F}_2[y]$.

We now state a weaker version of Theorem 1.2 of [13].

**Theorem 2.6.** Let $L = \mathbb{Q}(\eta)$ be an algebraic number field with $\eta$ satisfying a monic irreducible polynomial $g(x) \in \mathbb{Z}[x]$ and $q$ be a prime number. Let $\phi_1(x)^{e_1} \cdots \phi_k(x)^{e_k}$ be the factorization of $g(x)$ modulo $q$ into a product of powers of distinct irreducible polynomials over $\mathbb{F}_q$ with each $\phi_i(x) \neq g(x)$ belonging to $\mathbb{Z}[x]$ monic. Assume that, for a fixed $i$, the $\phi_i$-Newton polygon of $g(x)$ has $k$ edges, say $S_j$ having positive slopes $\lambda_j = \frac{d_j}{e_j}$ with $\gcd(d_j, e_j) = 1$ for $1 \leq j \leq k$.

If the polynomial $T_j(y)$ associated to $f(x)$ with respect to $(\phi_i, S_j)$ is linear for $k_1$ edges with $1 \leq j \leq k_1 \leq k$, then there are at least $k_1$ distinct prime ideals of $\mathbb{Z}_L$ having residual degree $\deg \phi_i(x)$.

In [10], Guàrdia, Montes, and Nart introduced the notion of $\phi$-admissible expansion, which is used in order to treat some special cases when the $\phi$-expansion of a polynomial $g(x)$ is not obvious.

Let $q$ be a prime number and $f(x) \in \mathbb{Z}_q[x]$ be a monic polynomial not divisible by $\phi(x)$ with $\phi(x)$-development $\sum_{j=0}^{n} a_j'(x)\phi(x)^j$, $a_j'(x) \in \mathbb{Z}_q[x]$; here $\deg a_j'(x)$ can be greater than or equal to $\deg \phi(x)$. Analogous to the definition of $\phi$-Newton polygon of $f(x)$ with respect to $q$, to each non-zero term $a_j'(x)\phi(x)^k$, we associate the point $(n-k, v_{q,x}(a_j'(x)))$ and the polygonal path formed by the lower edges along the convex hull of the points of $\{(k, v_{q,x}(a_{n-k}')) \mid 0 \leq k \leq n, a_{n-k}'(x) \neq 0\}$ defines the $\phi$-development Newton polygon of $f(x)$ with respect to $q$ in this case. Now as in Definition 2.4, suppose that the $\phi$-development Newton polygon of $f(x)$ with respect to $q$ consists of a single edge, say $S'$ having positive slope denoted by $\frac{d}{e}$ with $d, e$ coprime, i.e.,

$$\min\{\frac{v_{q,x}(a_{n-k}'(x))}{i} \mid 1 \leq i \leq n\} = \frac{v_{q,x}(a_0'(x))}{n} = \frac{d}{e}.$$
so that $n$ is divisible by $e$, say $n = et$ and $v_{q,x}(a'_{n-ej}(x)) \geq dj$ for $1 \leq j \leq t$. Let $a'_{n-ej}(x) \mod q^{dj}$ be denoted by $b'_j(x)$. We define the polynomial $T'(y)$ in an indeterminate $y$ by $T'(y) = y^t + \sum_{j=1}^{t} b'_j(x)y^{t-j}$ having coefficients in $\mathbb{F}_q[\alpha](\cong \mathbb{F}_q[\alpha])$.

$T'(y)$ is said to be the polynomial associated to $f(x)$ with respect to $(\phi, S')$. We say that a $\phi$-development of $f(x)$ is called admissible with respect to $(\phi, S')$ if and only if $\phi$ does not divide $b'_j(x)$ for each $j$. If the $\phi$-development Newton polygon of a polynomial $f(x)$ has $\ell$ many edges $S_i$ having positive slopes, then $\phi$-development of $f(x)$ is called admissible when $\phi$-development of $f(x)$ is admissible with respect to $(\phi, S_i)$ for each $i$, $1 \leq i \leq \ell$. It is proved in [10] that if a $\phi$-development of $f(x)$ is admissible, then the principal $\phi$-Newton polygon of $f(x)$ with respect to $q$ will be the same as $\phi$-development Newton polygon of $f(x)$ with respect to prime $q$ for edges having positive slopes; in particular, for any edge $S$ having positive slope of the $\phi$-Newton polygon of $f(x)$, we have $T(y) = T'(y)$.

3. PROOF OF THEOREM 1.1

Proof of Theorem 1.1. Keeping in mind that $q^r - 1 = (q - 1)m$ with $m \equiv 1 \mod q$ and $b^{q-1} \equiv 1 \mod q$, one can quickly verify that $v_q(b^{q-1} - 1) = v_q(b^{q-1} - 1) = r_1$.

Since $q | a$ and $q \nmid b$, we have $f(x) \equiv x^n - b \mod q$. Using Fermat's little theorem and the fact that $n = q^ru$, $q \nmid u$, it follows that $f(x) \equiv (x^u - b)^q \mod q$. Since $q$ does not divide $ab$, the monic polynomial $x^u - b$ is separable in $\mathbb{F}_q[x]$. Let $\phi_1(\cdots \phi_t(\phi(x) = \phi(x)$ be the factorization of $x^u - b$ into a product of monic irreducible polynomials in $\mathbb{F}_q[x]$, then $f(x) \equiv (\phi_1(\cdots \phi_t(x))^q \mod q$. Now we fix an irreducible factor $\phi(\cdots \phi_t(x) = \phi(x)$ of the polynomial $\phi(x)$ in $\mathbb{F}_q[x]$. Write $x^u - b = \phi(\cdots \phi_t(x) + q^{k_1}h_1(x) = \phi(x)g_1(x) + q^{k_1}h_1(x)$, where $g_1(x) = \prod_{j=1,j \neq i}^{t} \phi_j(x), h_1(x) \in \mathbb{Z}[x]$ and $k_1 \geq 1$ is an integer such that $\hat{h}_1(x) \neq 0$. Note that $\phi(x) \nmid g_1(x)$. Now we observe that there exists $g(x)$ and $h(x)$ such that $\phi(x) \nmid g(x)h(x)$ and $x^u - b = \phi(x)g(x) + q^k h(x)$ for some $k \geq 1$. Because if $\phi(x)$ divides $h_1(x)$, we can write $h_1(x) = \phi(x)g_2(x)$ such that $e \geq 1$ and $\phi(x) \nmid g_2(x)$. So we have $h_1(x) = \phi(x)^e g_2(x) + q^k h_2(x)$ and $k_2$ is a positive integer such that $\hat{h}_2(x) \neq 0$. If $\phi(x) \nmid h_2(x)$, then we set $g(x) = g_1(x) + q^k \phi(x)^{e-1} g_2(x)$ and $h(x) = h_2(x)$ with $k = k_1 + k_2$. If $\phi(x)$ divides $h_2(x)$, then we can repeat this process. Therefore, let $g(x), h(x) \in \mathbb{Z}[x]$ be such that

$$x^u - b = \phi(x)g(x) + q^k h(x) \text{ with } k \geq 1, \phi(x) \nmid g(x)h(x). \quad (3.1)$$

Applying the binomial theorem, we see that

$$f(x) = (x^u - b + b)^q - ax^m - b = (\phi(x)g(x) + q^k h(x) + b)^q - ax^m - b$$
can be written as

\[ f(x) = \sum_{j=0}^{q^r} \binom{q^r}{j}(q^k h(x) + b)q^r-j g(x)^j \phi(x)^j + (q^k h(x) + b)q^r - ax^m - b. \]

Let \(d(x) \in \mathbb{Z}[x]\) be a polynomial such that

\[ (q^k h(x) + b)q^r - b q^r = q^{r+k}d(x). \]

Then

\[ d(x) = b q^{r-1} h(x) + \frac{1}{q^{r+k}} \sum_{j=0}^{q^r-2} \binom{q^r}{j} b^j (q^k h(x)) q^{r-j}. \]

It follows that

\[ f(x) = \phi(x) g(x))^q^r \]

\[ + \sum_{j=1}^{q^r-1} \binom{q^r}{j}(q^k h(x) + b)q^r-j g(x) ^j \phi(x)^j + q^{r+k}d(x) - ax^m + b q^r - b. \]

Thus \( f(x) = \sum_{j=0}^{q^r} a'_j(x) \phi(x)^j \) is the \(\phi\)-development of \(f(x)\), where

\[ a'_0(x) = q^{r+k}d(x) - ax^m + b q^r - b. \]

\[ a'_j(x) = \sum_{j=1}^{q^r} \binom{q^r}{j}(q^k h(x) + b)q^r-j g(x)^j. \]

Note that

\[ v_{q,x}(\binom{q^r}{j}(q^k h(x) + b)q^r-j g(x)^j) = v_q(\binom{q^r}{j}) \text{ for every } j = 1, 2, \ldots, q^r. \quad (3.3) \]

We now divide our proof into two cases.

**Case (1).** Suppose \( r_1 \leq r \). Keeping in mind that \(q^{r+1}\) divides \(a\), one can easily verify that the successive vertices of the \(\phi\)-development Newton polygon of \(f(x)\) with respect to an odd prime \(q\) is given by the set \{(0,0), (q^r - q^{r-1}, 1), \ldots, (q^r - q^{r-r_1}, r_1 - 1), (q^r, r_1)\} having \(r_1\) edges \(S'_i\) with slopes \(\lambda_i = \frac{1}{q^{r_i+1} - q^{r_i}}\) for \(1 \leq i \leq r_1 - 1\) and \(\lambda_{r_1} = \frac{1}{q^{r_1+1} - 1}\). Since \(q \nmid b\) and \(\phi(x) \nmid g(x) h(x)\), one can see that the \(\phi\)-development of \(f(x)\) is admissible with respect to \((\phi, S'_i)\) for each \(i\), and hence \(\phi\)-development of \(f(x)\) is admissible. Further, the polynomial associated to \(f(x)\) with respect to \((\phi, S'_i)\) is linear for \(1 \leq i \leq r_1\). Therefore, the \(\phi\)-Newton polygon of \(f(x)\) has \(r_1\) edges and the polynomials associated to \(f(x)\) with respect to these edges are linear. Hence by Theorem 2.6, there are at least \(r_1\) distinct prime ideals of \(\mathbb{Z}_K\) lying above \(q\) having residual degree \(\deg \phi(x) = \ell\). It is known [11] that the number of monic irreducible polynomials of degree \(\ell\) over \(\mathbb{F}_q\) are \(N(q, \ell)\). Therefore, if \(r_1 > N(q, \ell)\), then applying Lemma 2.1 it follows that \(q\) divides \(i(K)\). We now consider the situation when \(q = 2\). In this situation, the successive vertices of the \(\phi\)-development Newton polygon of \(f(x)\) with respect
to 2 is given by the set \{(0, 0), (2^r - 2^{r-1}, 1), \cdots, (2^r - 2^{r-r+2}, r_1 - 2), (2^r, r_1)\} having \( r_1 - 1 \) edges \( S'_i \) with slopes \( \lambda_i = \frac{1}{2^r - i - 1} \) for \( 1 \leq i \leq r_1 - 2 \) and \( \lambda_{r_1-1} = \frac{1}{2^{r-r+1}} \). The polynomial associated to \( f(x) \) with respect to \( \phi, S'_i \) is linear for \( 1 \leq i \leq r_1 - 2 \) and the polynomial associated to \( f(x) \) with respect to \( \phi, S'_{r_1-1} \) is a second degree irreducible polynomial \( y^2 + y + 1 \) over \( \mathbb{F}_2 \). Since \( q \nmid b \) and \( \phi(x) \nmid g(x)h(x) \), \( \phi \)-development of \( f(x) \) is admissible. Hence, the \( \phi \)-Newton polygon of \( f(x) \) has \( r_1 - 2 \) edges such that the polynomials associated to \( f(x) \) with respect to these edges are linear. Therefore, by Theorem 2.6, there are at least \( r_1 - 2 \) distinct prime ideals of \( \mathbb{Z}_K \) lying above 2 having residual degree \( \ell \). So, if \( r_1 - 2 > N(2, \ell) \), then applying Lemma 2.1 it follows that \( 2 \) divides \( i(K) \).

Case (2). Suppose \( r_1 > r \). Keeping in mind that \( q^{i+1} \) divides \( a \), one can easily verify that the successive vertices of the \( \phi \)-development Newton polygon of \( f(x) \) with respect to an odd prime \( q \) are given by the set \{(0, 0), (q^r - q^{r-1}, 1), \cdots, (q^r - q, r - 1), (q^r - q, r - 1), (q^r, z)\} having \( r + 1 \) edges \( S'_i \) with \( z \geq r + 1 \) and slopes \( \lambda_i = \frac{1}{q^r - i - 1} \) for \( 1 \leq i \leq r \), \( \lambda_{r+1} = z - r - 1 \). Also, if \( v_q(x(a'_i(x))) = r + 1 \), then the successive vertices of the \( \phi \)-development Newton polygon of \( f(x) \) with respect to 2 is given by the set \{(0, 0), (2^r - 2^{r-1}, 1), \cdots, (2^r - 2, r - 1), (2^r, r + 1)\} having \( r \) edges \( S'_i \) with slopes \( \lambda_i = \frac{1}{2^r - i - 1} \) for \( 1 \leq i \leq r - 1 \) and \( \lambda_r = 1 \). Arguing exactly as in the above case, we see that there are at least \( r - 1 \) distinct prime ideals of \( \mathbb{Z}_K \) lying above \( q \) having residual degree \( \ell \). So, if \( r - 1 > N(q, \ell) \), then applying Lemma 2.1 we see that \( q \) divides \( i(K) \). This completes the proof of the theorem.

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