Direct products of null semigroups and rectangular bands in $\beta\mathbb{N}$

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Abstract. We show that, for every $m \in \mathbb{N}$, the direct product of the $m$-element null semigroup and the $2^r \times 2^r$ rectangular band has copies in $\beta\mathbb{N}$. In particular, the direct product of the 2-element null semigroup and the $2 \times 2$ rectangular band has copies in $\beta\mathbb{N}$. We also point out a Ramsey theoretic consequence of the latter fact.

The addition of the discrete semigroup $\mathbb{N}$ of natural numbers extends to the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ so that for each $a \in \mathbb{N}$, the left translation $\beta\mathbb{N} \ni x \mapsto a + x \in \beta\mathbb{N}$ is continuous, and for each $q \in \beta\mathbb{N}$, the right translation $\beta\mathbb{N} \ni x \mapsto x + q \in \beta\mathbb{N}$ is continuous.

We take the points of $\beta\mathbb{N}$ to be the ultrafilters on $\mathbb{N}$, identifying the principal ultrafilters with the points of $\mathbb{N}$. For every $A \subseteq \mathbb{N}$, $\overline{A} = \{p \in \beta\mathbb{N} : A \in p\}$ and $A^* = \overline{A} \setminus A$. The subsets $\overline{A}$, where $A \subseteq \mathbb{N}$, form a base for the topology of $\beta\mathbb{N}$, and $\overline{A}$ is the closure of $A$. For $p, q \in \beta\mathbb{N}$, the ultrafilter $p + q$ has a base consisting of subsets of the form $\bigcup_{x \in A} (x + B_x)$, where $A \in p$ and for each $x \in A$, $B_x \in q$.

Being a compact Hausdorff right topological semigroup, $\beta\mathbb{N}$ has a smallest two sided ideal $K(\beta\mathbb{N})$ which is a disjoint union of minimal right ideals and a disjoint union of minimal left ideals. Every right (left) ideal of $\beta\mathbb{N}$ contains a minimal right (left) ideal, the intersection of a minimal right ideal and a minimal left ideal is a group, and the idempotents in a minimal right (left) ideal form a right (left) zero semigroup, that is, $x + y = y$ ($x + y = x$) for all $x, y$.

The semigroup $\beta\mathbb{N}$ is interesting both for its own sake and for its applications to Ramsey theory and to topological dynamics. The first application to Ramsey theory was the proof of Hindman’s theorem: whenever $\mathbb{N}$ is finitely colored, there is an infinite subset all of whose sums are monochrome. An elementary introduction to $\beta\mathbb{N}$ can be found in [1].

In [3] D. Strauss showed that if $\varphi$ is a continuous homomorphism from $\beta\mathbb{N}$ to $\mathbb{N}^*$, then $\varphi(\beta\mathbb{N})$ is finite and $\varphi(\mathbb{N}^*)$ is a group. In 1996 the author proved that $\beta\mathbb{N}$ contains no nontrivial finite groups (see [1, Theorem 7.17]). In contrast, it does contain bands (= semigroups of idempotents). For example, apart from

Received June 1, 2021.
2010 Mathematics Subject Classification. Primary 22A15, 05D10; Secondary 22A30, 54D80.
Key words and phrases. Stone-Čech compactification, idempotent, right cancelable ultrafilter, null semigroup, rectangular band, Ramsey theory.
mentioned already left (right) zero semigroups, it contains chains of idempotents \((x \leq y\) if and only if \(x + y = y + x = x\)). A large enough class of finite bands that have copies in \(\beta\mathbb{N}\) was constructed in [4]. It includes, in particular, all finite rectangular bands (= direct products of a left zero semigroup and a right zero semigroup, so \((x_1, y_1) + (x_2, y_2) = (x_1, y_2)\)). In [2] it was shown that \(\beta\mathbb{N}\) contains even \(2^r \times 2^s\) rectangular bands. The question of whether there are finite semigroups in \(\beta\mathbb{N}\) distinct from bands is equivalent to asking whether there exist nontrivial continuous homomorphisms from \(\beta\mathbb{N}\) to \(\mathbb{N}^+\) and it was an open problem since 1992. It was solved in [6] by constructing a 2-element null semigroup \((x + x = y + y = x + y + x = y)\) in \(\beta\mathbb{N}\). In [7] it was shown that all finite null semigroups have copies in \(\beta\mathbb{N}\) and a connection of finite semigroups in \(\beta\mathbb{N}\) with Ramsey theory was established.

In this paper we modify construction in [7] and show that

**Theorem 1.** For every \(m \in \mathbb{N}\), the direct product of the \(m\)-element null semigroup and the \(2^r \times 2^s\) rectangular band has copies in \(\beta\mathbb{N}\).

In particular, by Theorem 1, the direct product of the 2-element null semigroup and the 2x2 rectangular band has copies in \(\beta\mathbb{N}\). This fact and [7, Theorem 4.4] give us the following Ramsey theoretic consequence.

Define \(r: \mathbb{N} \to \{1, 2, 3, 4\}\) by \(n \equiv r(n) \pmod{4}\).

**Corollary 2.** There exists a partition \(\{A_1, \ldots, A_8\}\) of \(\mathbb{N}\) with the following property: for any finite partitions \(B_1\) of \(A_i\) there exist \(B_i \in B_1\) and a sequence \((x_n)_{n=1}^{\infty}\) such that \(x_n \in B_{r(n)} \cap 2^n \mathbb{N}\) for each \(n \in \mathbb{N}\) and for each finite \(F \subseteq \mathbb{N}\) with \(|F| \geq 2\), if \(j = r(\min F)\) and \(k = r(\max F)\), then \(\sum_{n \in F} x_n \in B_i\), where

\[
i = \begin{cases} 
5 & \text{if } j \in \{1, 4\} \text{ and } k \in \{1, 2\} \\
6 & \text{if } j \in \{2, 3\} \text{ and } k \in \{1, 2\} \\
7 & \text{if } j \in \{2, 3\} \text{ and } k \in \{3, 4\} \\
8 & \text{if } j \in \{1, 4\} \text{ and } k \in \{3, 4\}.
\end{cases}
\]

**Proof.** Let \(S\) be a subsemigroup of \(\mathbb{N}^+\) splitting into the direct product of a null semigroup \(\{a_1, a_0\}\) and a rectangular band \(\{b_{10}, b_{00}, b_{01}, b_{11}\}\). Enumerate \(S\) as

\[
S = \{q_1, \ldots, q_8\} = \{a_1 b_{10}, a_1 b_{00}, a_1 b_{01}, a_1 b_{11}, a_0 b_{10}, a_0 b_{00}, a_0 b_{01}, a_0 b_{11}\}.
\]

Then for any \(j, k \in \{1, 2, 3, 4\}\), one has \(q_j + q_k = q_i\), where \(i\) is as in statement of Corollary 2. Pick a partition \(\{A_1, \ldots, A_8\}\) of \(\mathbb{N}\) such that \(A_i \subseteq q_i\). Let \((z_n)_{n=1}^{\infty}\) be a sequence guaranteed by [7, Theorem 4.4] and take the subsequence

\[
z_1, z_2, z_3, z_4, z_9, z_{10}, z_{11}, z_{12}, z_{17}, z_{18}, z_{19}, z_{20}, \ldots
\]

as \((x_n)_{n=1}^{\infty}\).

Notice that it is not true that if each of two finite semigroups has copies in \(\beta\mathbb{N}\), so does their direct product. Indeed, the direct product of the 2-element chain of idempotents with itself contains a 3-element semilattice which has no copy in \(\beta\mathbb{N}\) [5, Lemma 3].
In the rest of the paper we prove Theorem 1. In fact, we prove a bit stronger result.

Any \( x \in \mathbb{N} \) can be uniquely written as \( x = \sum_{n \in F} 2^n \) for some finite nonempty \( F \subseteq \omega \). Let \( \text{supp } x = F \), \( \phi(x) = \max F \), and \( \vartheta(x) = \min F \). We shall need the continuous extension \( \beta \mathbb{N} \to \beta \omega \) of the function \( \phi \) and we denote it by the same letter \( \phi \). If \( x, y \in \mathbb{N} \) and \( \phi(x) < \vartheta(y) \), then \( \phi(x + y) = \phi(y) \). If \( \phi(x) \leq \phi(y) \), then \( \phi(x + y) \in \{ \phi(y), \phi(y) + 1 \} \), and if \( \phi(x) + 1 < \vartheta(y) \), then \( \phi(y - x) \in \{ \phi(y), \phi(y) - 1 \} \). It then follows that for any \( \nu \in \mathbb{N}^* \) and \( \omega \in \beta \mathbb{Z} \), \( \phi(\omega + \nu) \in \{ \phi(\nu) - 1, \phi(\nu), \phi(\nu) + 1 \} \), and if \( \nu \in \mathbb{H} \), where \( \mathbb{H} = \bigcap_{n \in \omega} 2^n \mathbb{N} \), and \( \omega \in \beta \mathbb{N} \), then \( \phi(\omega + \nu) = \phi(\nu) \) (see [1, Lemma 6.8 and Lemma 13.4]). The last statement implies that for every \( u \in \omega^* \), \( \phi^{-1}(u) \cap \mathbb{H} \) is a left ideal of \( \mathbb{H} \) (since \( \phi(2^n) = n \), \( \phi(\mathbb{H}) = \omega^* \)).

Pick an increasing sequence \( U_0 \subseteq U_1 \subseteq \cdots \subseteq U_m = \omega \) of infinite subsets of \( \omega \) such that \( U_{i+1} \setminus U_i \) is infinite for each \( i \in \{ 0, \ldots, m-1 \} \). Define a function \( h \) from \( \mathbb{N} \) onto the decreasing \( (m + 1) \)-element chain \( 0 > 1 > \cdots > m \) of idempotents (with the operation \( i \wedge j = \max\{i, j\} \)) by

\[
  h(x) = \min\{i \in \{0, 1, \ldots, m\} : \text{supp } x \subseteq U_i\}
\]

(here \( \max \) and \( \min \) refer to the usual order, and \( \wedge \) is the operation induced by the order \( 0 > 1 > \cdots > m \) and let the same letter \( h \) denote its continuous extension \( \beta \mathbb{N} \to \{0, 1, \ldots, m\} \). If \( x, y \in \mathbb{N} \) and \( \phi(x) < \vartheta(y) \), then \( h(x + y) = h(x) \wedge h(y) \). Consequently, for any \( \nu \in \mathbb{H} \) and \( \omega \in \beta \mathbb{N} \), \( h(\omega + \nu) = h(\omega) \wedge h(\nu) \), in particular, the restriction of \( h \) to \( \mathbb{H} \) is a homomorphism. For each \( i \in \{0, 1, \ldots, m\} \), let \( T_i = h^{-1}(\{0, \ldots, i\}) \cap \mathbb{H} \).

**Lemma 3.** For each \( i \in \{0, 1, \ldots, m\} \), \( h(K(T_i)) = \{i\} \), and \( K(T_m) = K(\beta \mathbb{N}) \cap T_m \).

**Proof.** This is [7, Lemma 3.1].

We thus have that \( T_0 \subseteq T_1 \subseteq \cdots \subseteq T_m = \mathbb{H} \) is an increasing sequence of closed subsemigroups of \( \mathbb{H} \) such that \( T_i \cap K(T_{i+1}) = \emptyset \) for each \( i \in \{0, \ldots, m-1\} \) and \( K(T_m) = K(\beta \mathbb{N}) \cap T_m \), and for every \( u \in U_0^* \), \( \phi^{-1}(u) \cap T_0 \) is a left ideal of \( T_0 \).

Pick an injective sequence \( (u_n)_{n<\omega} \) in \( U_0^* \). Choose a minimal right ideal \( R_0 \) of \( T_0 \), and for every \( n < \omega \), a minimal left ideal \( L(n) \) of \( T_0 \) contained in \( \phi^{-1}(u_n) \cap T_0 \), and let \( p(n) \) be the identity of the group \( R_0 \cap L(n) \). Then \( \{ p(n) : n < \omega \} \) is a right zero semigroup. Let \( p_0 = p(0) \).

Enumerate \( \{ 2^n : n \in U_1 \setminus U_0^* \} \) without repetitions as \( \{ r_\alpha : \alpha < 2^c \} \).

**Lemma 4.** \( (p_0 + r_\alpha + T_m) \cap (p_0 + r_\beta + T_m) = \emptyset \) if \( \alpha \neq \beta \).

**Proof.** This is [7, Lemma 3.2].

For every \( \alpha < 2^c \), choose a minimal right ideal \( R_{1,\alpha} \) of \( T_1 \) contained in \( p_0 + r_\alpha + T_1 \), and choose a minimal left ideal \( L_1 \) of \( T_1 \) contained in \( T_1 + p_0 \), and let \( p_{1,\alpha} \) denote the identity of the group \( R_{1,\alpha} \cap L_1 \) and \( p_1 = p_{1,0} \). Then by Lemma 4, \( p_{1,\alpha} \neq p_{1,\beta} \) if \( \alpha \neq \beta \), \( p_{1,\alpha} + p_0 = p_0 + p_{1,\alpha} = p_{1,\alpha} \), and \( \{ p_{1,\alpha} : \alpha < 2^c \} \) is a left zero semigroup.
Inductively, for each $i \in \{2, \ldots, m\}$, choose a minimal right ideal $R_i$ of $T_i$ contained in $p_i + T_i$ and a minimal left ideal $L_i$ of $T_i$ contained in $T_i + p_i$, let $p_i$ denote the identity of the group $R_i \cap L_i$, and for every $\alpha < 2^i$, let $p_{i,\alpha} = p_{1,\alpha} + p_i$. Then $p_i + p_i + p_{i-1} + p_{i-1} + \cdots + p_1 = p_0 > p_1 > \ldots > p_i$ is a chain, and $p_i = p_i^0 = p_i$. By Lemma 4, $p_{i,\alpha} \neq p_{i,\beta}$ if $\alpha \neq \beta$, and since $p_{i,\alpha} \in K(T_i)$, it follows that all elements $p_{i,\alpha}$, where $i \in \{1, \ldots, m\}$ and $\alpha < 2^i$, are distinct.

We then obtain that $p_{i,\alpha} + p_0 = p_0 + p_{i,\alpha} = p_{i,\alpha}$ and

\[
p_{i,\alpha} + p_{j,\beta} = p_{1,\alpha} + p_i + p_{1,\beta} + p_j = p_{1,\alpha} + (p_i + p_1) + p_{1,\beta} + p_j = p_{1,\alpha} + p_i + p_1 + p_{1,\beta} + p_j = p_{1,\alpha} + p_i + p_1 + p_j = p_{1,\alpha} + p_i + p_j = p_{1,\alpha} + p_i + j
\]

= $p_{i,\alpha} + p_i + (p_1 + p_{1,\beta}) + p_j = p_{1,\alpha} + p_i + p_1 + p_j = p_{1,\alpha} + p_i + p_j$.

For every $i \in \{1, \ldots, m\}$ and $\alpha < 2^i$, let $D_{i,\alpha} = \{p_{i,\alpha} + p(n) : n < \omega\}$ and pick $q_{i,\alpha} \in \overline{D_{i,\alpha}} \setminus D_{i,\alpha}$. Notice that $\phi(p_{i,\alpha} + p(n)) = \phi(p(n)) = u_n$. (It is easy to see, although it is not directly important to us, that $D_{i,\alpha}$ is a right zero semigroup.)

**Lemma 5.** $q_{i,\alpha} + p_0 = p_{i,\alpha}$, and so $q_{i,\alpha} + p_{j,\beta} = p_{i,\alpha}$.

**Proof.** Since the right translation by $p_0$ is continuous and

\[
(p_{i,\alpha} + p(n)) + p_0 = p_{i,\alpha} + (p(n) + p_0) = p_{i,\alpha} + p_0 = p_{i,\alpha},
\]

one has $q_{i,\alpha} + p_0 = p_{i,\alpha}$. Then

\[
q_{i,\alpha} + p_{j,\beta} = q_{i,\alpha} + (p_0 + p_{j,\beta}) = (q_{i,\alpha} + p_0) + p_{j,\beta} = p_{i,\alpha} + p_{j,\beta} = p_{i,\alpha} + p_{j,\beta} = p_{i,\alpha} + p_{j,\beta} = p_{i,\alpha}.
\]

Define $Q \subseteq \mathbb{N}^\ast$ by

\[
Q = \{p_{i,\alpha} + q_{j,\beta} : i, j \in \{1, \ldots, m\} \text{ and } \alpha, \beta < 2^i\}.
\]

Using Lemma 5, we obtain that

\[
(p_{i,\alpha} + q_{j,\beta}) + (p_{k,\gamma} + q_{l,\delta}) = p_{i,\alpha} + (q_{j,\beta} + p_{k,\gamma}) + q_{l,\delta} = p_{i,\alpha} + p_{j,\gamma} + q_{l,\delta} = p_{i,\alpha} + p_{j,\gamma} + q_{l,\delta} = p_{i,\alpha} + p_{j,\gamma} + q_{l,\delta}.
\]

Now we shall show that all elements $p_{i,\alpha} + q_{j,\beta}$ of the semigroup $Q$ are distinct.

An ultrafilter $p \in Z^\ast$ is

(i) **prime** if $p \not\in Z^\ast + Z^\ast$, and

(ii) **right cancelable** if the right translation of $\beta \mathbb{Z}$ by $p$ is injective.

An ultrafilter $p \in Z^\ast$ is right cancelable if and only if $p \not\in Z^\ast + p$ (see [7, Lemma 3.5]). Thus, every prime ultrafilter is right cancelable.

**Lemma 6.** Let $D$ be a countable subset of $\mathbb{H}$ and suppose that $\phi$ is injective on $D$. Then every $q \in \overline{D \setminus D}$ is prime.

**Proof.** Assume the contrary. Then $q \in Z^\ast + v$ for some $v \in Z^\ast$. Since $-N^\ast$ is a left ideal of $\beta \mathbb{Z}$, one has $v \in N^\ast$. Let $Z = \{n \in Z : n + v \not\in \mathbb{H}\}$ and let $D' = \{p \in D : \phi(p) \not\in \phi(v), \phi(v), \phi(v) + 1\}$. Notice that $|Z \setminus Z| = 1$ and $|D \setminus D'| \leq 3$. We then have that $q \in \overline{D' \cap Z + v}$, so by [1, Theorem 3.40], either
Lemma 7.  (1) All subsets $\overline{D}_i,\alpha$, where $i \in \{1, \ldots, m\}$ and $\alpha < 2^i$, are pairwise disjoint.  
(2) All elements $q_{i,\alpha}$, where $i \in \{1, \ldots, m\}$ and $\alpha < 2^i$, are distinct.  
(3) All elements $p_{i,\alpha} + q_{j,\beta}$, where $i, j \in \{1, \ldots, m\}$ and $\alpha, \beta < 2^i$, are distinct.

Proof. (1) Assume on the contrary that $\overline{D}_i \cap \overline{D}_j \neq \emptyset$ for some $(i, \alpha) \neq (j, \beta)$. Then either $D_i \cap D_j \neq \emptyset$ or $\overline{D}_i \cap D_j = \emptyset$. It suffices to consider the first case. Since $D_i \cap D_j = \emptyset$, it follows that $p_{i,\alpha} + p(n) = q$ for some $n < \omega$ and $q \in \overline{D}_j \setminus D_j$. But by Lemma 6, $q$ is prime, a contradiction.

(2) is immediate from (1).

(3) Suppose that $p_{i,\alpha} + q_{j,\beta} = p_{k,\gamma} + q_{l,\delta}$. Then by [1, Corollary 6.21], either $q_{j,\beta} \in D_j$ or $q_{l,\delta} \in D_l$. In either case $q_{j,\beta} = q_{l,\delta}$, since both of them are prime and in $\mathbb{H}$, so by (2), $(j, \beta) = (l, \delta)$. We thus have that $p_{i,\alpha} + q_{j,\beta} = p_{k,\gamma} + q_{j,\beta}$. But then $p_{i,\alpha} = p_{k,\gamma}$, since $q_{j,\beta}$ is right cancelable, and so $(i, \alpha) = (k, \gamma)$.

We have constructed $Q$ as a subsemigroup of $\mathbb{N}^*$. We now describe it without mentioning ultrafilters.

Given a semilattice $I$ and a cardinal $\kappa$, let $S = S(I, \kappa)$ denote the semigroup whose underlying set is $I \times \kappa \times I \times \kappa$ and the operation is defined by

$$(i, \alpha, j, \beta) + (k, \gamma, l, \delta) = (i \land j \land k, \alpha, l, \delta).$$

The semigroup $S$ decomposes into the semilattice $I$ of the subsemigroups

$$S_t = \{(i, \alpha, j, \beta) : i \land j = t\},$$

where $t \in I$ (that is, $S_I + S_J \subseteq S_{(i,j)}$). For every $(i, \alpha, j, \beta) \in S_t$, if $i = t$, then

$$(t, \alpha, j, \beta) + (t, \alpha, j, \beta) = (t, \alpha, j, \beta),$$

so $(t, \alpha, j, \beta)$ is an idempotent, and if $i \neq t$, then

$$(i, \alpha, j, \beta) + (i, \alpha, j, \beta) = (t, \alpha, j, \beta)$$

$$= (i, \alpha, j, \beta) + (t, \alpha, j, \beta) = (t, \alpha, j, \beta) + (i, \alpha, j, \beta),$$

so $\{(i, \alpha, j, \beta), (t, \alpha, j, \beta)\}$ is a null semigroup.

If $I$ is a decreasing chain $1 > \ldots > m$, we write $S(m, \kappa)$ instead of $S(I, \kappa)$. For each $t \in \{1, \ldots, m\}$, the component $S_t$ of $S = S(m, \kappa)$ is the union of $\kappa \times (\{1, \ldots, t\} \times \kappa)$ rectangular band

$$B_t = \{(t, \alpha, j, \beta) : j \in \{1, \ldots, t\} \text{ and } \alpha, \beta < \kappa\}.$$
which is the smallest ideal of \( S_t \), and the subsemigroup
\[
S_{t,t} = \{(i, \alpha, t, \beta) : i \in \{1, \ldots, t\} \text{ and } \alpha, \beta < \kappa\}.
\]
The intersection of \( B_t \) and \( S_{t,t} \) is \( \kappa \times \kappa \) rectangular band
\[
B_{t,t} = \{(t, \alpha, t, \beta) : \alpha, \beta < \kappa\},
\]
which is the smallest ideal of \( S_{t,t} \), and \( S_{t,t} \) is a disjoint union of \( t \)-element null subsemigroups \( \{(i, \alpha, t, \beta) : i \in \{1, \ldots, t\}\} \), where \( \alpha, \beta < \kappa \), so \( S_{t,t} \) is isomorphic to the direct product of \( t \)-element null semigroup and \( B_{t,t} \).

Define \( \varepsilon : (m, 2^t) \to Q \) by
\[
\varepsilon(i, \alpha, j, \beta) = p_{i,\alpha} + q_{j,\beta}.
\]
Then \( \varepsilon \) is an isomorphism. Furthermore,
\[
\varepsilon(m, \alpha, j, \beta) = p_{m,\alpha} + q_{j,\beta} \in K(\beta N)
\]
because \( p_{m,\alpha} \in K(\beta N) \), and
\[
\varepsilon(i, \alpha, m, \beta) = p_{i,\alpha} + q_{m,\beta} \in \overline{K(\beta N)}
\]
because \( q_{m,\beta} \in \overline{D_{m,\beta}} \subseteq \overline{K(\beta N)} \) and \( \overline{K(\beta N)} \) is an ideal of \( \beta N \) [1, Theorem 4.44].

Thus, we have established the following result.

**Theorem 8.** Let \( m \in \mathbb{N} \) and \( S = (m, 2^t) \). Then there is an isomorphic embedding \( \varepsilon : S \to \mathbb{N}^n \). Furthermore, \( \varepsilon \) can be chosen so that \( \varepsilon(S_m) \subseteq \overline{K(\beta N)} \) and \( \varepsilon(K(S_m)) \subseteq K(\beta N) \).

Since \( S_{m,m} \) is isomorphic to the direct product of the \( m \)-element null semigroup and the \( 2^t \times 2^t \) rectangular band, Theorem 1 is a partial case of Theorem 8.

**References**


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This paper is available via http://nyjm.albany.edu/j/2022/28-24.html.