New York Journal of Mathematics

New York J. Math. **27** (2021) 1375–1414.

Flow equivalence of topological Markov shifts and Ruelle algebras

Kengo Matsumoto

ABSTRACT. In this paper we study discrete flow equivalence of two-sided topological Markov shifts by using extended Ruelle algebra. We characterize flow equivalence of two-sided topological Markov shifts in terms of conjugacy of certain actions weighted by ceiling functions of two-dimensional torus on the stabilized extended Ruelle algebras for the Markov shifts.

CONTENTS

1. Introduction

Flow equivalence relation in two-sided topological Markov shifts is one of the most interesting and important equivalence relations in symbolic dynamics as seen in many papers [2], [3], [9], [22], etc. Let $(\bar{X}_A, \bar{\sigma}_A)$ be the two-sided topological Markov shift defined by an $N \times N$ irreducible matrix $A = [A(i,j)]_{i,j=1}^N$ with entries in $\{0,1\}$. The shift space $\bar{X}_A$ consists of bi-infinite sequences $(x_n)_{n\in\mathbb{Z}} \in \{1,\dots,N\}^{\mathbb{Z}}$ of $\{1,\dots,N\}$ such that $A(x_n, x_{n+1}) = 1$ for all $n \in \mathbb{Z}$. Take and fix a real number $\lambda_\circ$ such as $0 < \lambda_\circ < 1$. The space $\bar{X}_A$ is a

Received October 18, 2018.
2010 *Mathematics Subject Classification.* Primary 37A55, 46L35; Secondary, 37B10, 46L55.
Key words and phrases. Topological Markov shift, flow equivalence, topological conjugacy, Ruelle algerba, Cuntz–Krieger algebra.
The author would like to thank the referee for his careful reading, comments and suggestions on the first draft of the paper. This work was supported by JSPS KAKENHI Grant Number 15K04896, 19K03537.

ISSN 1076-9803/2021

compact metric space by the metric defined by for $x = (x_n)_{n\in\mathbb{Z}}, y = (y_n)_{n\in\mathbb{Z}}$ with $x \neq y$

$$d(x,y) = \begin{cases} 1 & \text{if } x_0 \neq y_0, \\ (\lambda_\circ)^m & \text{if } m = \mathrm{Max}\{n \mid x_k = y_k \text{ for all } k \text{ with } |k| < n\}. \end{cases}$$

The homeomorphism of the shift transformation $\bar{\sigma}_A$ on $\bar{X}_A$ is defined by

$$\bar{\sigma}_A((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}.$$

Two topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are said to be flow equivalent if they are realized as cross sections with their first return maps of a common flow space. Parry–Sullivan in [22] proved that $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent if and only if they are realized as discrete cross sections with their first return maps of a common topological Markov shift. Cuntz–Krieger have first found that there is an interesting relation between flow equivalence of topological Markov shifts and certain purely infinite simple C^*-algebras called Cuntz–Krieger algebras that they introduced in [7]. For an irreducible matrix A with entries in $\{0,1\}$, let $\mathcal{O}_A$ be the Cuntz–Krieger algebra and $\mathcal{D}_A$ its canonical maximal abelian C^*-subalgebra of $\mathcal{O}_A$. We denote by $\mathcal{K}$ and $\mathcal{C}$ the C^*-algebra of compact operators on the separable infinite dimensional Hilbert space $\ell^2(\mathbb{N})$ and its commutative C^*-subalgebra of diagonal operators on $\ell^2(\mathbb{N})$, respectively. Cuntz–Krieger proved that for irreducible non-permutation matrices A and B, if $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent, then there exists an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \longrightarrow \mathcal{O}_B \otimes \mathcal{K}$ of C^*-algebras such that $\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}$. Its converse implication holds by [21] (for more general matrices a similar assertion is shown in [5]). They also proved in [7] that the extension group $\mathrm{Ext}(\mathcal{O}_A)$, which is isomorphic to the K-group $K_0(\mathcal{O}_A)$ as groups, appears as the Bowen–Franks group $\mathrm{BF}(A)$ defined by Bowen–Franks in [2], that is an invariant of flow equivalence of $(\bar{X}_A, \bar{\sigma}_A)$ ([2]).

There is another kind of construction of C^*-algebras from two-sided topological Markov shifts by using groupoids and regarding the Markov shifts as Smale spaces ([1], [30], [33], etc.). The construction was initiated by D. Ruelle [30], [31] and I. Putnam [23], [24]. I. Putnam in [23] constructed several kinds of groupoids from each Smale space. Each of the groupoids yields a C^*-algebra. In this paper, we focus on asymptotic groupoids G_A^a among several groupoids studied in [12], [23], [24], [25], etc. and their semi-direct products defined below. The asymptotic étale groupoid G_A^a for $(\bar{X}_A, \bar{\sigma}_A)$ is defined by

$$G_A^a := \{(x,y) \in \bar{X}_A \times \bar{X}_A \mid \lim_{n\to\infty} d(\sigma_A^n(x), \sigma_A^n(y)) = \lim_{n\to-\infty} d(\sigma_A^n(x), \sigma_A^n(y)) = 0\}$$

with natural groupoid operations and topology (see [23]). It has been shown in [24] that the groupoid G_A^a is amenable and its C^*-algebra $C^*(G_A^a)$ is stably isomorphic to the tensor product $\mathcal{F}_{A^t} \otimes \mathcal{F}_A$ of the canonical AF-subalgebras $\mathcal{F}_{A^t}$ and $\mathcal{F}_A$ inside the Cuntz–Krieger algebras $\mathcal{O}_{A^t}$ and $\mathcal{O}_A$, respectively. The semi-direct product $G_A^a \rtimes \mathbb{Z}$ is defined by

$$G_A^a \rtimes \mathbb{Z} := \{(x, k-l, y) \in \bar{X}_A \times \mathbb{Z} \times \bar{X}_A \mid (\bar{\sigma}_A^k(x), \bar{\sigma}_A^l(x)) \in G_A^a\}$$

with natural groupoid operations and topology (see [23]). It is étale and amenable. The groupoid C^*-algebra $C^*(G_A^a \rtimes \mathbb{Z})$ is called the Ruelle algebra for the Markov shift $(\bar{X}_A, \bar{\sigma}_A)$ and written $\mathcal{R}_A$. Since the unit space $(G_A^a \rtimes \mathbb{Z})^\circ$ is $\{(x,0,x) \in G_A^a \rtimes \mathbb{Z} \mid x \in \bar{X}_A\}$ that is identified with the shift space $\bar{X}_A$, the algebra $\mathcal{R}_A$ has the commutative C^*-algebra $C(\bar{X}_A)$ of continuous functions on $\bar{X}_A$ as a maximal commutative C^*-subalgebra. It is the crossed product $C^*(G_A^a) \rtimes \mathbb{Z}$ of $C^*(G_A^a)$ induced by the automorphism of the shift $\bar{\sigma}_A$, and hence has the dual action written $\rho_t^A, t \in \mathbb{T}$. See [26] for the construction of C^*-algebras from groupoids.

Following [23], let us consider the groupoids G_A^s and G_A^u defined by stable equivalence relation and unstable equivalence relation on $(\bar{X}_A, \bar{\sigma}_A)$, respectively, which are defined by

$$G_A^s =\{(x,y) \in \bar{X}_A \times \bar{X}_A \mid \lim_{n\to\infty} d(\bar{\sigma}_A^n(x), \bar{\sigma}_A^n(y) = 0\},$$
$$G_A^u =\{(x,y) \in \bar{X}_A \times \bar{X}_A \mid \lim_{n\to-\infty} d(\bar{\sigma}_A^n(x), \bar{\sigma}_A^n(y) = 0\}.$$

In [19] and [20], the author introduced the groupoid $G_A^{s,u} \rtimes \mathbb{Z}^2$ defined by

$$G_A^{s,u} \rtimes \mathbb{Z}^2 := \{(x,p,q,y) \in \bar{X}_A \times \mathbb{Z} \times \mathbb{Z} \times \bar{X}_A \mid (\bar{\sigma}_A^p(x), y) \in G_A^s, (\bar{\sigma}_A^q(x)), y) \in G_A^u\}$$

which has a natural groupoid operations and topology making it étale and amenable. The groupoid C^*-algebra $C^*(G_A^{s,u} \rtimes \mathbb{Z}^2)$ is called the extended Ruelle algebra written $\widetilde{\mathcal{R}}_A$. Since the unit space $(G_A^{s,u} \rtimes \mathbb{Z}^2)^\circ$ is $\{(x,0,0,x) \in G_A^{s,u} \rtimes \mathbb{Z}^2 \mid x \in \bar{X}_A\}$ that is identified with the shift space $\bar{X}_A$, the algebra $\widetilde{\mathcal{R}}_A$ has $C(\bar{X}_A)$ as a maximal abelian C^*-subalgebra. As in [19] and [20], there exists a projection E_A in the tensor product $\mathcal{O}_{A^t} \otimes \mathcal{O}_A$ such that $E_A(\mathcal{O}_{A^t} \otimes \mathcal{O}_A)E_A$ is naturally isomorphic to the algebra $\widetilde{\mathcal{R}}_A$, so that the C^*-algebra $\widetilde{\mathcal{R}}_A$ is regarded as a version of the bilateral Cuntz–Krieger algebra. Let α_A denote the gauge action on the Cuntz–Krieger algebra $\mathcal{O}_A$. Under the identification between $E_A(\mathcal{O}_{A^t} \otimes \mathcal{O}_A)E_A$ and $\widetilde{\mathcal{R}}_A$, the tensor product $\alpha_r^{A^t} \otimes \alpha_s^A$ for $(r,s) \in \mathbb{T}^2$ yields an action of $\mathbb{T}^2$ written $\gamma_{(r,s)}^A, (r,s) \in \mathbb{T}^2$. In [20, Theorem 1.1], it was shown that the triplet $(\widetilde{\mathcal{R}}_A, C(\bar{X}_A), \gamma^A)$ is a complete invariant for the topological conjugacy class of $(\bar{X}_A, \bar{\sigma}_A)$. For a continuous function $f : \bar{X}_A \longrightarrow \mathbb{N}$, we may define an action $\gamma^{A,f}$ weighted by f on $\widetilde{\mathcal{R}}_A$. In this paper, we will characterize the flow equivalence class of $(\bar{X}_A, \bar{\sigma}_A)$ in terms of the stabilized version of $\widetilde{\mathcal{R}}_A$ with the weighted action $\gamma^{A,f}$. The continuous function $f : \bar{X}_A \longrightarrow \mathbb{N}$ exactly corresponds to a ceiling function of a discrete suspension. The main result of this paper is the following theorem.

Theorem 1.1 (Theorem 6.8)**.** *Let A, B be irreducible, non-permutation matrices with entries in $\{0,1\}$. The two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and*

$(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent if and only if there is an irreducible non-permutation matrix C with entries in $\{0,1\}$ and continuous functions $f_A, f_B : \bar{X}_C \longrightarrow \mathbb{N}$ with values in the positive integers such that $\widetilde{\mathcal{R}}_A \otimes \mathcal{K}$ and $\widetilde{\mathcal{R}}_B \otimes \mathcal{K}$ are isomorphic to $\widetilde{\mathcal{R}}_C \otimes \mathcal{K}$ via isomorphisms Φ_A and Φ_B satisfying

$$\Phi_A \circ (\gamma^A_{(r,s)} \otimes \mathrm{id}) = (\gamma^{C,f_A}_{(r,s)} \otimes \mathrm{id}) \circ \Phi_A,$$
$$\Phi_B \circ (\gamma^B_{(r,s)} \otimes \mathrm{id}) = (\gamma^{C,f_B}_{(r,s)} \otimes \mathrm{id}) \circ \Phi_B$$

for $(r,s) \in \mathbb{T}^2$.

The above statement exactly corresponds to the situation that the topological Markov shift $(\bar{X}_A, \bar{\sigma}_A)$ is realized as a discrete suspension of $(\bar{X}_C, \bar{\sigma}_C)$ by ceiling function f_A, and $(\bar{X}_B, \bar{\sigma}_B)$ is realized as a discrete suspension of $(\bar{X}_C, \bar{\sigma}_C)$ by ceiling function f_B. As a corollary we have the following.

Corollary 1.2 (Corollary 6.9)**.** *Let A, B be irreducible, non-permutation matrices with entries in $\{0,1\}$. Two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent if and only if there exist continuous functions $f_A : \bar{X}_A \longrightarrow \mathbb{N}$ and $f_B : \bar{X}_B \longrightarrow \mathbb{N}$ with values in the positive integers, and an isomorphism $\Phi : \widetilde{\mathcal{R}}_A \otimes \mathcal{K} \longrightarrow \widetilde{\mathcal{R}}_B \otimes \mathcal{K}$ of C^*-algebras such that*

$$\Phi \circ (\gamma^{A,f_A}_{(r,s)} \otimes \mathrm{id}) = (\gamma^{B,f_B}_{(r,s)} \otimes \mathrm{id}) \circ \Phi, \qquad (r,s) \in \mathbb{T}^2.$$

The organization of the paper is the following.

In Section 2, we will briefly recall basic notation and terminology on groupoid C^*-algebras, Cuntz-Krieger algebras, Ruelle algebras and flow equivalence of topological Markov shifts.

In Section 3, a bilateral version of the Krieger's dimension group for topological Markov shifts will be studied and called the dimension quadruplet that will be shown to be invariant for shift equivalence of the underlying matrices.

In Section 4, the dimension quadruplet is described by the K-group of the AF-algebra $C^*(G^a_A)$ of the groupoid G^a_A. As a result, a sufficient condition under which the two-sided Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent is given in terms of the stabilized action $\gamma^A \otimes \mathrm{id}$ of $\mathbb{T}^2$ on $\widetilde{\mathcal{R}}_A \otimes \mathcal{K}$ (Proposition 4.6).

In Section 5, the action $\gamma^{A,f}$ with potential function f on the algebra $\widetilde{\mathcal{R}}_A$ is introduced.

In Section 6, we characterize the flow equivalence of two-sided topological Markov shifts in terms of the actions with potential functions of two dimensional torus on the extended Ruelle algebras $\widetilde{\mathcal{R}}_A$.

In Section 7, we reformulate Theorem 1.1 and Corollary 1.2 by describing their statements including not only flow equivalence but also topological conjugacy of two-sided topological Markov shifts (Theorem 7.2 and Theorem 7.3).

Throughout the paper, we denote by $\mathbb{Z}_+$ and $\mathbb{N}$ the set of nonnegative integers and the set of positive integers, respectively.

2. Preliminaries

In this section, we briefly recall basic notation and terminology on the C^*-algebras of étale groupoids, Cuntz-Krieger algebras, Ruelle algebras and flow equivalence of topological Markov shifts. In what follows, a square matrix $A = [A(i,j)]_{i,j=1}^N$ is assumed to be an $N \times N$ irreducible, non-permutation matrix with entries in $\{0,1\}$.

2.1. C^*-algebras of étale groupoids. Let us construct C^*-algebras from étale groupoids. The general theory of the construction of groupoid C^*-algebras was initiated and studied by Renault [26] (see also [27], [28]). The construction will be used in the following sections. Let G be an étale groupoid with its unit space G° and range map, source map $r, s : G \longrightarrow G^\circ$ and $C_c(G)$ denote the $*$-algebra of continuous functions on G with compact support having its product and $*$-involution defined by

$$(f * g)(\gamma) = \sum_{\eta; r(\gamma)=r(\eta)} f(\eta)g(\eta^{-1}\gamma), \qquad f^*(\gamma) = \overline{f(\gamma^{-1})}$$

for $f, g \in C_c(G)$, $\gamma \in G$. We denote by $C_0(G^\circ)$ the commutative C^*-algebra of continuous functions on G° vanishing at infinity. The algebra $C_c(G)$ has a structure of right $C_0(G^\circ)$-module with $C_0(G^\circ)$-valued right inner product given by

$$(\xi g)(\gamma) = \xi(\gamma)g(s(\gamma)), \qquad <\xi, \zeta>(t) = \sum_{\eta; t=s(\eta)} \overline{\xi(\eta)}\zeta(\eta)$$

for $\xi, \zeta \in C_c(G)$, $g \in C_c(G^\circ)$, $\gamma \in G$, $t \in G^\circ$. The completion of $C_c(G)$ by the norm defined by the above inner product is denoted by $\ell^2(G)$, which is a Hilbert C^*-right module over $C_0(G^\circ)$. The algebra $C_c(G)$ is represented on $\ell^2(G)$ as bounded adjointable $C_0(G^\circ)$-right module maps by $\pi(f)\xi = f * \xi$ for $f \in C_c(G), \xi \in \ell^2(G)$. The closure of $\pi(C_c(G))$ by the operator norm on $\ell^2(G)$ is denoted by $C_r^*(G)$ and called the (reduced) groupoid C^*-algebra for the étale groupoid G. The completeion of $C_c(G)$ by the universal C^*-norm is called the (full) groupoid C^*-algebra for G. Now we treat the three kinds of groupoids $G_A^a, G_A^a \rtimes \mathbb{Z}, G_A^{s,u} \rtimes \mathbb{Z}^2$. They are all étale and amenable, so that the two groupoid C^*-algebras $C_r^*(G)$ and $C^*(G)$ are canonically isomorphic for such groupoids. We do not distinguish them, and write them as $C^*(G)$ for $G = G_A^a, G_A^a \rtimes \mathbb{Z}, G_A^{s,u} \rtimes \mathbb{Z}^2$.

2.2. Cuntz-Krieger algebras, Ruelle algebras and extended Ruelle algebras. The Cuntz–Krieger algebra $\mathcal{O}_A$ introduced by Cuntz–Krieger [7] is a universal unique C^*-algebra generated by partial isometries $S_1, \dots, S_N$ subject to the relations:

$$\sum_{j=1}^N S_j S_j^* = 1, \qquad S_i^* S_i = \sum_{j=1}^N A(i,j) S_j S_j^*, \qquad i = 1, \dots, N. \tag{2.1}$$

By the universality for the relations (2.1) of operators, the correspondence $S_i \longrightarrow \exp(2\pi\sqrt{-1}t)S_i$, $i = 1, \dots, N$ for each $t \in \mathbb{R}/\mathbb{Z} = \mathbb{T}$ yields an automorphism written α^A_t on the C^*-algebra $\mathcal{O}_A$. The automorphisms $\alpha^A_t, t \in \mathbb{T}$ define an action of $\mathbb{T}$ on $\mathcal{O}_A$ called the gauge action. It is well-known that the fixed point algebra $(\mathcal{O}_A)^{\alpha^A}$ of $\mathcal{O}_A$ under the gauge action is an AF-algebra written $\mathcal{F}_A$. Let us denote by $B_m(\bar{X}_A)$ the set of admissible words in $\bar{X}_A$ of length m and by $B_*(\bar{X}_A)$ the set of all admissible words of $\bar{X}_A$. For $\mu = (\mu_1, \dots, \mu_m) \in B_m(\bar{X}_A)$, we write $S_\mu = S_{\mu_1} \cdots S_{\mu_m}$. We denote by $\mathcal{D}_A$ the C^*-subalgebra of $\mathcal{F}_A$ generated by projections $S_\mu S_\mu^*, \mu \in B_*(\bar{X}_A)$.

As in [7] and [6] (cf. [18], [29]), the crossed product $\mathcal{O}_A \rtimes_{\alpha^A} \mathbb{T}$ is stably isomorphic to the AF-algebra $\mathcal{F}_A$. Hence the dual action $\hat{\alpha}^A$ on $\mathcal{O}_A \rtimes_{\alpha^A} \mathbb{T}$ induces an automorphism on $K_0(\mathcal{F}_A)$, that is written δ_A. The triplet $(K_0(\mathcal{F}_A), K_0^+(\mathcal{F}_A), \delta_A)$ appears as the (future) dimension triplet written $(\Delta_A, \Delta_A^+, \delta_A)$ for A defined by W. Krieger [16]. For the transposed matrix A^t of A, we similarly consider the Cuntz–Krieger algebra $\mathcal{O}_{A^t}$ and its AF-subalgebra $\mathcal{F}_{A^t}$. Let us denote by $T_1, \dots, T_N$ the generating partial isometries of $\mathcal{O}_{A^t}$ which satisfy the relations:

$$\sum_{i=1}^N T_i T_i^* = 1, \qquad T_j^* T_j = \sum_{i=1}^N A(i,j) T_i T_i^*, \quad j = 1, \dots, N. \tag{2.2}$$

For $\xi = (\xi_1, \dots, \xi_k) \in B_k(\bar{X}_A)$, we denote by $\bar{\xi}$ the transposed word $(\xi_k, \dots, \xi_1)$ which belongs to $B_k(\bar{X}_{A^t})$, and write $T_{\bar{\xi}} = T_{\xi_k} \cdots T_{\xi_1}$.

Define the projection $E_A \in \mathcal{F}_{A^t} \otimes \mathcal{F}_A$ by setting

$$E_A = \sum_{j=1}^N T_j^* T_j \otimes S_j S_j^*$$

which coincides with $\sum_{i=1}^N T_i T_i^* \otimes S_i^* S_i$ because of the equalities (2.1) and (2.2). Let $G_A^a, G_A^a \rtimes \mathbb{Z}, G_A^{s,u} \rtimes \mathbb{Z}^2$ denote the étale amenable groupoids stated in Section 1. For reference, we state [20, Proposition 2.1] as

Lemma 2.1.

(i) *The groupoid C^*-algebra $C^*(G_A^a)$ is canonically isomorphic to the C^*-subalgebra of $\mathcal{F}_{A^t} \otimes \mathcal{F}_A$ generated by elements $T_{\bar{\xi}} T_{\bar{\eta}}^* \otimes S_\mu S_\nu^*$ where $\mu = (\mu_1, \dots, \mu_m)$, $\nu = (\nu_1, \dots, \nu_n) \in B_*(\bar{X}_A)$, $\bar{\xi} = (\xi_k, \dots, \xi_1)$, $\bar{\eta} = (\eta_l, \dots, \eta_1) \in B_*(\bar{X}_{A^t})$ satisfying $A(\xi_k, \mu_1) = A(\eta_l, \nu_1) = 1$ and $k = l$, $m = n$. Hence $C^*(G_A^a)$ is canonically isomorphic to the C^*-algebra $E_A(\mathcal{F}_{A^t} \otimes \mathcal{F}_A)E_A$.*

(ii) *The Ruelle algebra $\mathcal{R}_A = C^*(G_A^a \rtimes \mathbb{Z})$ is canonically isomorphic to the C^*-subalgebra of $\mathcal{O}_{A^t} \otimes \mathcal{O}_A$ generated by elements $T_{\bar{\xi}} T_{\bar{\eta}}^* \otimes S_\mu S_\nu^*$ where $\mu = (\mu_1, \dots, \mu_m)$, $\nu = (\nu_1, \dots, \nu_n) \in B_*(\bar{X}_A)$, $\bar{\xi} = (\xi_k, \dots, \xi_1)$, $\bar{\eta} = (\eta_l, \dots, \eta_1) \in B_*(\bar{X}_{A^t})$ satisfying $A(\xi_k, \mu_1) = A(\eta_l, \nu_1) = 1$ and $m + k = n + l$.*

(iii) *The extended Ruelle algebra $\widetilde{\mathcal{R}}_A = C^*(G_A^{s,u} \rtimes \mathbb{Z}^2)$ is canonically isomorphic to the C^*-subalgebra of $\mathcal{O}_{A^t} \otimes \mathcal{O}_A$ generated by elements $T_{\bar{\xi}} T_{\bar{\eta}}^* \otimes S_\mu S_\nu^*$ where $\mu = (\mu_1, \dots, \mu_m)$, $\nu = (\nu_1, \dots, \nu_n) \in B_*(\bar{X}_A)$, $\bar{\xi} = (\xi_k, \dots, \xi_1)$, $\bar{\eta} =$*

$(\eta_l, \dots, \eta_1) \in B_*(\bar{X}_{A^t})$ satisfying $A(\xi_k, \mu_1) = A(\eta_l, \nu_1) = 1$. Hence $\widetilde{\mathcal{R}}_A$ is canonically isomorphic to the C^*-algebra $E_A(\mathcal{O}_{A^t} \otimes \mathcal{O}_A)E_A$.

Under the identification between $E_A(\mathcal{O}_{A^t} \otimes \mathcal{O}_A)E_A$ and $\widetilde{\mathcal{R}}_A$, the tensor product $\alpha_r^{A^t} \otimes \alpha_s^A$ of gauge actions on $\mathcal{O}_{A^t}$ and $\mathcal{O}_A$ yields an action of $\mathbb{T}^2$ on $\widetilde{\mathcal{R}}_A$ written $\gamma^A_{(r,s)}, (r,s) \in \mathbb{T}^2$, because $\gamma^A_{(r,s)}(E_A) = E_A$. We write

$$\delta_t^A := \alpha_t^{A^t} \otimes \alpha_t^A, \qquad \rho_t^A := \gamma^A_{(-\frac{t}{2}, \frac{t}{2})}, \qquad t \in \mathbb{T}.$$

Lemma 2.2.

(i) *The restriction of the action $\rho_t^A, t \in \mathbb{T}$ to the subalegera $\mathcal{R}_A$ is regarded as the dual action on $\mathcal{R}_A$ under a natural identification between $\mathcal{R}_A$ and the crossed product $C^*(G_A^a) \rtimes \mathbb{Z}$. Hence the fixed point algebra $(\mathcal{R}_A)^{\rho^A}$ is isomorphic to $C^*(G_A^a)$.*

(ii) *The fixed point algebra $(\widetilde{\mathcal{R}}_A)^{\delta^A}$ of $\widetilde{\mathcal{R}}_A$ under δ^A is isomorphic to $\mathcal{R}_A$, so that the fixed point algebra $(\widetilde{\mathcal{R}}_A)^{\gamma^A}$ of $\widetilde{\mathcal{R}}_A$ under γ^A is isomorphic to $C^*(G_A^a)$.*

2.3. Suspension and flow equivalence. We will briefly review discrete suspensions of topological Markov shifts. Let $f : \bar{X}_A \longrightarrow \mathbb{N}$ be a continuous function on the shift space $\bar{X}_A$ with values in the positive integers. Let $f(\bar{X}_A) = \{1, 2, \dots, L\}$. Put $X_j = \{x \in \bar{X}_A \mid f(x) = j\}, j = 1, \dots, L$. Define the suspension space $\bar{X}_{A,f} = \cup_{j=1}^L X_j \times \{0, 1, \dots, j-1\}$ with transformation $\bar{\sigma}_{A,f}$ on $\bar{X}_{A,f}$ by

$$\bar{\sigma}_{A,f}([x,k]) = \begin{cases} [x, k+1] & \text{if } 0 \le k \le j-2, \\ [\bar{\sigma}_A(x), 0] & \text{if } k = j-1 \end{cases}$$

for $[x,k] \in X_j \times \{0, 1, \dots, j-1\}$. The resulting topological dynamical system $(\bar{X}_{A,f}, \bar{\sigma}_{A,f})$ is called the discrete suspension of $(\bar{X}_A, \bar{\sigma}_A)$ by the ceiling function f, which is homeomorphic to a topological Markov shift. If, in particular, the function $f : \bar{X}_A \longrightarrow \mathbb{N}$ depends only on the 0th coordinate of $\bar{X}_A$, then f is written $f = \sum_{j=1}^N f_j \chi_{U_j(0)}$ for some integers $f_j \in \mathbb{N}$, where $\chi_{U_j(0)}$ is the characteristic function of the cylinder set

$$U_j(0) = \{(x_n)_{n\in\mathbb{Z}} \in \bar{X}_A \mid x_0 = j\}, \qquad j = 1, \dots, N.$$

Put $m_j = f_j - 1$ for $j = 1, \dots, N$. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the directed graph defined by the matrix A with the vertex set $\mathcal{V} = \{1, 2, \dots, N\}$. An edge of $\mathcal{G}$ is defined by a pair (i,j) of vertices $i, j = 1, \dots, N$ such that $A(i,j) = 1$, whose source is i and the terminal is j. The set of such pairs (i,j) is the edge set $\mathcal{E}$. Construct a new graph $\mathcal{G}_f = (\mathcal{V}_f, \mathcal{E}_f)$ with its transition matrix A_f from the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and the function f such that $\mathcal{V}_f = \cup_{j=1}^N \{j_0, j_1, j_2, \dots, j_{m_j}\}$ and if $A(j,k) = 1$, then

$$A_f(j_0, j_1) = A_f(j_1, j_2) = \cdots = A_f(j_{m_j - 1}, j_{m_j}) = A_f(j_{m_j}, k_0) = 1. \tag{2.3}$$

For other pairs $(j_i, j'_{i'}) \in \mathcal{V}_f \times \mathcal{V}_f$, we define $A_f(j_i, j'_{i'}) = 0$. Hence the size of the matrix A_f is $(f_1 + f_2 + \cdots + f_N) \times (f_1 + f_2 + \cdots + f_N)$. Then the discrete suspension $(\bar{X}_{A,f}, \bar{\sigma}_{A,f})$ is nothing but the topological Markov shift $(\bar{X}_{A_f}, \bar{\sigma}_{A_f})$ defined by the matrix A_f.

Two topological Markov shifts are said to be flow equivalent if they are realized as cross sections with their first return maps of a common one-dimensional flow space. Parry–Sullivan in [22] proved that $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent if and only if there exist another topological Markov shift $(\bar{X}_C, \bar{\sigma}_C)$ for some matrix C and continuous maps $f_A, f_B : \bar{X}_C \longrightarrow \mathbb{N}$ such that $(\bar{X}_A, \bar{\sigma}_A)$ is topologically conjugate to the discrete suspension $(\bar{X}_{C,f_A}, \bar{\sigma}_{C,f_A})$ and $(\bar{X}_B, \bar{\sigma}_B)$ is topologically conjugate to the discrete suspension $(\bar{X}_{C,f_B}, \bar{\sigma}_{C,f_B})$.

Cuntz and Krieger were the first to find interesting relations between flow equivalence of topological Markov shifts and Cuntz–Krieger algebras in [7]. Recall that $\mathcal{K}$ and $\mathcal{C}$ are the C^*-algebra of compact operators on the separable infinite dimensional Hilbert space $\ell^2(\mathbb{N})$ and its commutative C^*-subalgebra of diagonal operators on $\ell^2(\mathbb{N})$. Cuntz and Krieger proved that for irreducible non-permutation matrices A and B, if $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent, then there exists an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \longrightarrow \mathcal{O}_B \otimes \mathcal{K}$ of C^*-algebras such that $\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}$. Its converse implication holds by [21] (for more general matrices a similar assertion is shown in [5]).

In this paper, we will study flow equivalence of topological Markov shifts in terms of the extended Ruelle algebras with its action γ^A of $\mathbb{T}^2$.

3. Bilateral dimension groups

We keep an irreducible, non-permutation matrix $A = [A(i,j)]_{i,j=1}^N$ with entries in $\{0,1\}$. Following W. Krieger [16] (cf. [14], [15], [8], etc.), the dimension group (Δ_A, Δ_A^+) for the matrix A are defined as an ordered group by the inductive limits

$$\Delta_A = \mathbb{Z}^N \xrightarrow{A^t} \mathbb{Z}^N \xrightarrow{A^t} \cdots, \qquad \Delta_A^+ = \mathbb{Z}_+^N \xrightarrow{A^t} \mathbb{Z}_+^N \xrightarrow{A^t} \cdots.$$

The group Δ_A is identified with the equivalence classes of $\cup_{n=0}^{\infty}\{(v,n) \mid v \in \mathbb{Z}^N, n \in \mathbb{Z}_+\}$ by the equivalence relation generated by $(v,n) \sim (A^t v, n+1)$. The equivalence class of (v,n) is denoted by $[v,n]$. The dimension drop automorphism δ_A on (Δ_A, Δ_A^+) is defined by $\delta_A([v,n]) = [(v,n+1)]$ for $[v,n] \in \Delta_A$. The triplet $(\Delta_A, \Delta_A^+, \delta_A)$ is called the (future) dimension triplet for the topological Markov shift $(\bar{X}_A, \bar{\sigma}_A)$. We similarly have the (future) dimension triplet $(\Delta_{A^t}, \Delta_{A^t}^+, \delta_{A^t})$ for the topological Markov shift $(\bar{X}_{A^t}, \bar{\sigma}_{A^t})$ for the matrix A^t, which is called the (past) dimension triplet for $(\bar{X}_A, \bar{\sigma}_A)$. Hence we have two dimension triplets $(\Delta_A, \Delta_A^+, \delta_A)$ and $(\Delta_{A^t}, \Delta_{A^t}^+, \delta_{A^t})$ for the matrix A.

Let $e_i \in \mathbb{Z}^N$ be the vector of $\mathbb{Z}^N$ whose ith component is 1, other components are zeros. We will define a specific element $\tilde{u}_A$ in $\Delta_{A^t} \otimes \Delta_A$ by setting

$$\tilde{u}_A := \sum_{i,j=1}^N [e_j, 1] \otimes A(j,i)[e_i, 1] \in \Delta_{A^t} \otimes \Delta_A.$$

We then see that

$$\begin{aligned}\tilde{u}_A &= \sum_{j=1}^{N}\left([e_j,1]\otimes[\sum_{i=1}^{N}A(j,i)e_i,1]\right) = \sum_{j=1}^{N}[e_j,1]\otimes[A^t e_j,1]\\ &=(\mathrm{id}\otimes\delta_A^{-1})\sum_{j=1}^{N}([e_j,1]\otimes[e_j,1])\end{aligned}$$

and

$$\begin{aligned}\tilde{u}_A &= \sum_{i=1}^{N}\left([\sum_{j=1}^{N}A(j,i)e_j,1]\otimes[e_i,1]\right) = \sum_{i=1}^{N}[Ae_i,1]\otimes[e_i,1]\\ &=(\delta_{A^t}^{-1}\otimes\mathrm{id})\sum_{i=1}^{N}([e_i,1]\otimes[e_i,1]).\end{aligned}$$

Define an automorphism $\tilde{\delta}_A : \Delta_{A^t}\otimes\Delta_A \longrightarrow \Delta_{A^t}\otimes\Delta_A$ by $\tilde{\delta}_A = \delta_{A^t}^{-1}\otimes\delta_A$. It satisfies

$$\tilde{\delta}_A([u,n]\otimes[v,m]) = [Au,n]\otimes[v,m+1], \qquad [u,n]\otimes[v,m]\in\Delta_{A^t}\otimes\Delta_A.$$

We set the abelian group $\tilde{\Delta}_A = \Delta_{A^t}\otimes\Delta_A$ with its positive cone $\tilde{\Delta}_A^+ = \Delta_{A^t}^+\otimes\Delta_A^+$.

Definition 3.1. The quadruplet $(\tilde{\Delta}_A, \tilde{\Delta}_A^+, \tilde{\delta}_A, \tilde{u}_A)$ is called the *dimension quadruplet* for the two-sided topological Markov shift $(\bar{X}_A, \bar{\sigma}_A)$.

We note that a bilateral version of the dimension groups first appeared in Krieger's paper [14] (cf. [15], [16]).

Lemma 3.2. $\tilde{\delta}_A(\tilde{u}_A) = \tilde{u}_A$.

Proof. Since

$$\tilde{u}_A = (\mathrm{id}\otimes\delta_A^{-1})\sum_{j=1}^{N}([e_j,1]\otimes[e_j,1]) = (\delta_{A^t}^{-1}\otimes\mathrm{id})\sum_{i=1}^{N}([e_i,1]\otimes[e_i,1]),$$

and $\tilde{\delta}_A = \delta_{A^t}^{-1}\otimes\delta_A$, the assertion is immediate. □

We will next show that the dimension quadruplet $(\tilde{\Delta}_A, \tilde{\Delta}_A^+, \tilde{\delta}_A, \tilde{u}_A)$ is invariant under shift equivalence of the underlying matrices A. The notion of shift equivalence in square matrices with entries in nonnegative integers has been introduced by W. F. Williams [34]. Two matrices A and B are said to be shift equivalent if there exist rectangular matrices H, K with entries in nonnegative integers and a positive integer ℓ such that

$$A^\ell = HK, \quad B^\ell = KH, \quad AH = HB, \quad KA = BK. \tag{3.1}$$

W. Krieger has proved in [16] that two matrices A and B are shift equivalent if and only if their dimension triplet $(\Delta_A, \Delta_A^+, \delta_A)$ and $(\Delta_B, \Delta_B^+, \delta_B)$ are isomorphic. The following result has been already proved by C. G. Holton [10, Proposition

6.7] for primitive matrices by using Rohlin property of automorphisms on the AF-algebras $C^*(G_A^a)$. The proof given below does not use C^*-algebra theory, nor does it assume that the matrices are primitive.

Proposition 3.3 (C. G. Holton [10, Proposition 6.7])**.** *Suppose that A and B are shift equivalent. Then there exists an isomorphism $\Phi : \tilde{\Delta}_A \longrightarrow \tilde{\Delta}_B$ which yields an isomorphism between the dimension quadruplets $(\tilde{\Delta}_A, \tilde{\Delta}_A^+, \tilde{\delta}_A, \tilde{u}_A)$ and $(\tilde{\Delta}_B, \tilde{\Delta}_B^+, \tilde{\delta}_B, \tilde{u}_B)$.*

Proof. Let A and B be $N \times N$ matrix and $M \times M$ matrix, respectively. Assume that there exist rectangular matrices H, K with entries in nonnegative integers and a positive integer ℓ satisfying (3.1). Define

$$\begin{aligned} \Phi_+ : \Delta_A \longrightarrow \Delta_B \quad &\text{by} \quad \Phi_+([v,k]) = [H^t v, k], \\ \Phi_- : \Delta_{A^t} \longrightarrow \Delta_{B^t} \quad &\text{by} \quad \Phi_-([v,k]) = [Kv, k+\ell], \end{aligned}$$

so that

$$\begin{aligned} \Phi_+^{-1} : \Delta_B \longrightarrow \Delta_A \quad &\text{satisfies} \quad \Phi_+^{-1}([u,j]) = [K^t u, j+\ell], \\ \Phi_-^{-1} : \Delta_{B^t} \longrightarrow \Delta_{A^t} \quad &\text{satisfies} \quad \Phi_-^{-1}([u,j]) = [Hu, j]. \end{aligned}$$

As in [16], $\Phi_+ : \Delta_A \longrightarrow \Delta_B$ and $\Phi_- : \Delta_{A^t} \longrightarrow \Delta_{B^t}$ yield isomorphisms for each such that

$$\begin{aligned} \Phi_+(\Delta_A^+) = \Delta_B^+, \qquad & \Phi_+ \circ \delta_A = \delta_B \circ \Phi_+, \\ \Phi_-(\Delta_{A^t}^+) = \Delta_{B^t}^+, \qquad & \Phi_- \circ \delta_{A^t} = \delta_{B^t} \circ \Phi_-. \end{aligned}$$

Hence they induce isomorphisms

$$\begin{aligned} \Phi_+ &: (\Delta_A, \Delta_A^+, \delta_A) \longrightarrow (\Delta_B, \Delta_B^+, \delta_B), \\ \Phi_- &: (\Delta_{A^t}, \Delta_{A^t}^+, \delta_{A^t}) \longrightarrow (\Delta_{B^t}, \Delta_{B^t}^+, \delta_{B^t}). \end{aligned}$$

We define $\Phi = \Phi_- \otimes \Phi_+ : \tilde{\Delta}_A \longrightarrow \tilde{\Delta}_B$. Let $f_l \in \mathbb{Z}^M$ be the vector whose lth component is 1, and other components are zeros. It then follows that

$$
\begin{aligned}
\Phi(\tilde{u}_A) &= \sum_{i,j=1}^{N} \Phi_-([e_j, 1]) \otimes \Phi_+(A(j,i)[e_i, 1]) \\
&= \sum_{i,j=1}^{N} [Ke_j, 1+\ell] \otimes A(j,i)[H^t e_i, 1] \\
&= \sum_{j=1}^{N} [Ke_j, 1+\ell] \otimes [(AH)^t e_j, 1] \\
&= \sum_{j=1}^{N} [Ke_j, 1+\ell] \otimes [\sum_{l=1}^{M} (AH)(j,l) f_l, 1] \\
&= \sum_{l=1}^{M} \sum_{j=1}^{N} [\begin{bmatrix} K(1,j)(AH)(j,l) \\ K(2,j)(AH)(j,l) \\ \vdots \\ K(M,j)(AH)(j,l) \end{bmatrix}, 1+\ell] \otimes [f_l, 1] \\
&= \sum_{l=1}^{M} [(KAH) f_l, 1+\ell] \otimes [f_l, 1] \\
&= \sum_{l=1}^{M} [(BKH) f_l, 1+\ell] \otimes [f_l, 1] \\
&= \sum_{l=1}^{M} [B^{\ell+1} f_l, 1+\ell] \otimes [f_l, 1] \\
&= \sum_{l=1}^{M} [B f_l, 1] \otimes [f_l, 1] = \tilde{u}_B.
\end{aligned}
$$

□

R. F. Williams characterized topological conjugacy of two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ in terms of an equivalence relation of its underlying matrices, called strong shift equivalence ([34]). Two square matrices A and B with entries in nonnegative integers are said to be elementary equivalent if there exist rectangular matrices C, D with entries in nonnegative integers such that $A = CD, B = DC$. If two matrices are connected by a finite chain of elementary equivalences, they are said to be strong shift equivalent. Williams proved that two-sided topological Markov shift $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate if and only if the matrices A and B are strong shift equivalent ([34]). Since shift equivalence is weaker than strong shift equivalence, by virtue of the Williams' result, we have

Proposition 3.4. *The dimension quadruplet* $(\tilde{\Delta}_A, \tilde{\Delta}_A^+, \tilde{\delta}_A, \tilde{u}_A)$ *is invariant under topological conjugacy of the two-sided topological Markov shift* $(\bar{X}_A, \bar{\sigma}_A)$.

4. Dimension quadruplets and AF-algebras

In this section, we will study the dimension quadruplet $(\tilde{\Delta}_A, \tilde{\Delta}_A^+, \tilde{\delta}_A, \tilde{u}_A)$ by using K-theory for C^*-algebras. D. B. Killough and I. F. Putnam in [13] have deeply studied ring and module structure of the AF-algebras $C^*(G_A^s)$ as well as $C^*(G_A^a)$ from a different view point from ours below. Recall that $\mathcal{K}$ denotes the C^*-algebra of compact operators on the separable infinite dimensional Hilbert space $H = \ell^2(\mathbb{N})$.

Lemma 4.1. *Let* A *be an irreducible, non-permutation matrix with entries in* $\{0, 1\}$.

(i) *There exists a projection* p_0 *in the crossed product* $\widetilde{\mathcal{R}}_A \rtimes_{\gamma^A} \mathbb{T}^2$ *of* $\widetilde{\mathcal{R}}_A$ *by* γ^A *such that* $p_0(\widetilde{\mathcal{R}}_A \rtimes_{\gamma^A} \mathbb{T}^2)p_0$ *is isomorphic to* $C^*(G_A^a)$. *Hence* $\widetilde{\mathcal{R}}_A \rtimes_{\gamma^A} \mathbb{T}^2$ *is stably isomorphic to the AF-algebra* $C^*(G_A^a)$.

(ii) *The inclusion* $\iota_A : p_0(\widetilde{\mathcal{R}}_A \rtimes_{\gamma^A} \mathbb{T}^2)p_0 \hookrightarrow \widetilde{\mathcal{R}}_A \rtimes_{\gamma^A} \mathbb{T}^2$ *induces an isomorphism*

$$\iota_{A*} : K_0(C^*(G_A^a)) \longrightarrow K_0(\widetilde{\mathcal{R}}_A \rtimes_{\gamma^A} \mathbb{T}^2)$$

on K-theory where $C^*(G_A^a)$ *is identified with* $p_0(\widetilde{\mathcal{R}}_A \rtimes_{\gamma^A} \mathbb{T}^2)p_0$.

Proof. (i) The fixed point algebra $(\widetilde{\mathcal{R}}_A)^{\gamma^A}$ of $\widetilde{\mathcal{R}}_A$ under γ^A coincides with the fixed point algebra $(E_A(\mathcal{O}_{A^t} \otimes \mathcal{O}_A)E_A)^{\alpha^{A^t} \otimes \alpha^A}$ which is nothing but $E_A(\mathcal{F}_{A^t} \otimes \mathcal{F}_A)E_A$. Hence $(\widetilde{\mathcal{R}}_A)^{\gamma^A}$ is identified with $C^*(G_A^a)$. Let p_0 be the projection in $\in L^1(\mathbb{T}^2, \widetilde{\mathcal{R}}_A)$ defined by $p_0(r,s) = 1$ for all $(r,s) \in \mathbb{T}^2$. We know that p_0 is a full projection in $\widetilde{\mathcal{R}}_A \rtimes_{\gamma^A} \mathbb{T}^2$ and

$$p_0(\widetilde{\mathcal{R}}_A \rtimes_{\gamma^A} \mathbb{T}^2)p_0 = (\widetilde{\mathcal{R}}_A)^{\gamma^A} = C^*(G_A^a)$$

by [29] or a manner similar to [18]. This shows that the algebra $\widetilde{\mathcal{R}}_A \rtimes_{\gamma^A} \mathbb{T}^2$ is stably isomorphic to the AF-algebra $C^*(G_A^a)$ by [4].

(ii) By [4], there exists a partial isometry v_A in the multiplier algebra $M(\widetilde{\mathcal{R}}_A \rtimes_{\gamma^A} \mathbb{T}^2 \otimes \mathcal{K})$ of $\widetilde{\mathcal{R}}_A \rtimes_{\gamma^A} \mathbb{T}^2 \otimes \mathcal{K}$ such that $v_A^* v_A = p_0$, $v_A v_A^* = 1$. Put $\psi_A = \mathrm{Ad}(v_A) : p_0(\widetilde{\mathcal{R}}_A \rtimes_{\gamma^A} \mathbb{T}^2)p_0 \otimes \mathcal{K} \longrightarrow \widetilde{\mathcal{R}}_A \rtimes_{\gamma^A} \mathbb{T}^2 \otimes \mathcal{K}$, which is an isomorphism of C^*-algebras. We then have for a projection $p_0 f p_0 \otimes q \in \widetilde{\mathcal{R}}_A \rtimes_{\gamma^A} \mathbb{T}^2 \otimes \mathcal{K}$,

$$\begin{aligned}
(\iota_A \otimes \mathrm{id})_*([p_0 f p_0 \otimes q]) =& [p_0 f p_0 \otimes q] \\
=& [v_A^* v_A (p_0 f p_0 \otimes q) v_A^* v_A] \\
=& [v_A (p_0 f p_0 \otimes q) v_A^*] \\
=& \psi_{A*}([p_0 f p_0 \otimes q]).
\end{aligned}$$

Hence $\iota_{A*} = \psi_{A*} : K_0(C^*(G_A^a)) \longrightarrow K_0(\widetilde{\mathcal{R}}_A \rtimes_{\gamma^A} \mathbb{T}^2)$ is an isomorphism. $\square$

Let us denote by $\hat{\gamma}^A$ the dual action of the crossed product $\widetilde{\mathcal{R}}_A \rtimes_{\gamma^A} \mathbb{T}^2$. Under the identifications

$$C^*(G_A^a) = (\widetilde{\mathcal{R}}_A)^{\gamma^A} = p_0(\widetilde{\mathcal{R}}_A \rtimes_{\gamma^A} \mathbb{T}^2)p_0,$$

we define an action β of $\mathbb{Z}^2$ on $K_0(C^*(G_A^a))$ by

$$\beta_{(m,n)} := \iota_{A*}^{-1} \circ \hat{\gamma}^A_{(m,n)*} \circ \iota_{A*} : K_0(C^*(G_A^a)) \longrightarrow K_0(C^*(G_A^a)), \quad (m,n) \in \mathbb{Z}^2$$

such that the diagram

$$\begin{array}{ccc} K_0(\widetilde{\mathcal{R}}_A \rtimes_{\gamma^A} \mathbb{T}^2) & \xrightarrow{\hat{\gamma}^A_{(m,n)*}} & K_0(\widetilde{\mathcal{R}}_A \rtimes_{\gamma^A} \mathbb{T}^2) \\ {\scriptstyle \iota_{A*}}\big\uparrow & & \big\uparrow{\scriptstyle \iota_{A*}} \\ K_0(C^*(G_A^a)) & \xrightarrow{\beta_{(m,n)}} & K_0(C^*(G_A^a)) \end{array}$$

is commutative.

Let $U_A = \sum_{i=1}^N T_i^* \otimes S_i$ in $\mathcal{O}_{A^t} \otimes \mathcal{O}_A$. As in [19], U_A is a unitary in $\mathcal{R}_A$ and hence in $\widetilde{\mathcal{R}}_A$, so that $U_A U_A^* = U_A^* U_A = E_A$. We denote by $1_{C^*(G_A^a)}$ the unit of the C^*-algebra $C^*(G_A^a)$. By [6] and [7] (see also [8], [15], [16]), the ordered group Δ_A is naturally identified with the K-group $K_0(\mathcal{F}_A)$.

Lemma 4.2. *There exists an isomorphism* $\varphi_A : C^*(G_A^a) \otimes \mathcal{K} \longrightarrow \mathcal{F}_{A^t} \otimes \mathcal{F}_A \otimes \mathcal{K}$ *of* C^**-algebras such that the induced isomorphism*

$$\varphi_{A*} : K_0(C^*(G_A^a)) \longrightarrow K_0(\mathcal{F}_{A^t} \otimes \mathcal{F}_A) = K_0(\mathcal{F}_{A^t}) \otimes K_0(\mathcal{F}_A)$$

satisfies

$$\varphi_{A*}([1_{C^*(G_A^a)}]) = [E_A], \qquad \varphi_{A*} \circ \mathrm{Ad}(U_A)_* = \tilde{\delta}_A \circ \varphi_{A*},$$
$$\varphi_{A*} \circ \beta_{(m,n)} \circ \varphi_{A*}^{-1} = \delta_{A^t}^m \otimes \delta_A^n, \quad (m,n) \in \mathbb{Z}^2.$$

Hence the diagrams

$$\begin{array}{ccc} K_0(\widetilde{\mathcal{R}}_A \rtimes_{\gamma^A} \mathbb{T}^2) & \xrightarrow{\hat{\gamma}^A_{(m,n)*}} & K_0(\widetilde{\mathcal{R}}_A \rtimes_{\gamma^A} \mathbb{T}^2) \\ {\scriptstyle \iota_{A*}}\big\uparrow & & \big\uparrow{\scriptstyle \iota_{A*}} \\ K_0(C^*(G_A^a)) & \xrightarrow{\beta_{(m,n)}} & K_0(C^*(G_A^a)) \\ {\scriptstyle \varphi_{A*}}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_{A*}} \\ K_0(\mathcal{F}_{A^t}) \otimes K_0(\mathcal{F}_A) & \xrightarrow{\delta_{A^t}^m \otimes \delta_A^n} & K_0(\mathcal{F}_{A^t}) \otimes K_0(\mathcal{F}_A) \end{array}$$

are commutative.

Proof. Since the C^*-algebra $\mathcal{F}_{A^t} \otimes \mathcal{F}_A$ is simple, the projection E_A is full in $\mathcal{F}_{A^t} \otimes \mathcal{F}_A$. By using Brown's theorem [4], there exists an isometry u_A in the multiplier algebra $M((\mathcal{F}_{A^t} \otimes \mathcal{F}_A) \otimes \mathcal{K})$ of $(\mathcal{F}_{A^t} \otimes \mathcal{F}_A) \otimes \mathcal{K}$ such that $u_A^* u_A = 1, u_A u_A^* = E_A \otimes 1_H$. Define an isomorphism

$$\varphi_A = \mathrm{Ad}(u_A^*) : C^*(G_A^a) \otimes \mathcal{K}(= E_A(\mathcal{F}_{A^t} \otimes \mathcal{F}_A)E_A \otimes \mathcal{K}) \longrightarrow \mathcal{F}_{A^t} \otimes \mathcal{F}_A \otimes \mathcal{K}.$$

Let p_1 be a rank one projection in $\mathcal{K}$. We then have

$$\begin{aligned}
\varphi_{A*}([1_{C^*(G_A^a)}]) =&\varphi_{A*}([E_A \otimes p_1]) \\
=&[u_A^*(E_A \otimes p_1)u_A] \\
=&[((E_A \otimes p_1)u_A)((E_A \otimes p_1)u_A)^*] \\
=&[E_A \otimes p_1] = [E_A].
\end{aligned}$$

We will next see that $\varphi_{A*} \circ \mathrm{Ad}(U_A)_* = \tilde{\delta}_A \circ \varphi_{A*}$. We note that $K_0(C^*(G_A^a))$ is generated by the classes of projections of the form $T_{\bar{\xi}} T_{\bar{\xi}}^* \otimes S_\mu S_\mu^*$ where $\mu = (\mu_1, \dots, \mu_m) \in B_k(\bar{X}_A)$, $\bar{\xi} = (\xi_k, \dots, \xi_1) \in B_k(\bar{X}_{A^t})$ with $A(\xi_k, \mu_1) = 1$. We then have

$$\begin{aligned}
(\varphi_{A*} \circ \mathrm{Ad}(U_A)_*)([T_{\bar{\xi}} T_{\bar{\xi}}^* \otimes S_\mu S_\mu^*]) =&[\varphi_A(T_{\xi_{k-1}\cdots\xi_1} T_{\xi_{k-1}\cdots\xi_1}^* \otimes S_{\xi_k\mu} S_{\xi_k\mu}^*)] \\
=&[u_A^*(T_{\xi_{k-1}\cdots\xi_1} T_{\xi_{k-1}\cdots\xi_1}^* \otimes S_{\xi_k\mu} S_{\xi_k\mu}^*)u_A] \\
=&[T_{\xi_{k-1}\cdots\xi_1} T_{\xi_{k-1}\cdots\xi_1}^* \otimes S_{\xi_k\mu} S_{\xi_k\mu}^*].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\tilde{\delta}_A \circ \varphi_{A*}([T_{\bar{\xi}} T_{\bar{\xi}}^* \otimes S_\mu S_\mu^*]) =&(\delta_{A^t}^{-1} \otimes \delta_A)([u_A^*(T_{\bar{\xi}} T_{\bar{\xi}}^* \otimes S_\mu S_\mu^*)u_A]) \\
=&(\delta_{A^t}^{-1} \otimes \delta_A)([T_{\bar{\xi}} T_{\bar{\xi}}^* \otimes S_\mu S_\mu^*]) \\
=&\delta_{A^t}^{-1}([T_{\bar{\xi}} T_{\bar{\xi}}^*]) \otimes \delta_A([S_\mu S_\mu^*]).
\end{aligned}$$

As in [18, Lemma 4.5], we know

$$\delta_{A^t}^{-1}([T_{\bar{\xi}} T_{\bar{\xi}}^*]) = [T_{\xi_{k-1}\cdots\xi_1} T_{\xi_{k-1}\cdots\xi_1}^*]$$

and $\delta_A([S_\mu S_\mu^*]) = [S_{\xi_k\mu} S_{\xi_k\mu}^*]$. Hence we have

$$(\varphi_{A*} \circ \mathrm{Ad}(U_A)_*)([T_{\bar{\xi}} T_{\bar{\xi}}^* \otimes S_\mu S_\mu^*]) = \tilde{\delta}_A \circ \varphi_{A*}([T_{\bar{\xi}} T_{\bar{\xi}}^* \otimes S_\mu S_\mu^*]).$$

□

We note that the K-theoretic class $[E_A]$ of the projection E_A has appeared in studying of K-theoretic duality by J. Kaminker–I. F. Putnam [11].

Lemma 4.3. *Let $A = [A(i,j)]_{i,j=1}^N$ and $B = [B(i,j)]_{i,j=1}^M$ be irreducible, non-permutation matrices with entries in* $\{0,1\}$. *Suppose that there exists an isomorphism $\Phi : \widetilde{\mathcal{R}}_A \otimes \mathcal{K} \longrightarrow \widetilde{\mathcal{R}}_B \otimes \mathcal{K}$ of C^*-algebras such that*

$$\Phi \circ (\gamma_{(r,s)}^A \otimes \mathrm{id}) = (\gamma_{(r,s)}^B \otimes \mathrm{id}) \circ \Phi, \quad (r,s) \in \mathbb{T}^2. \tag{4.1}$$

(i) *Then Φ induces an isomorphism*

$$\Phi_{0*} : K_0(\mathcal{F}_{A^t}) \otimes K_0(\mathcal{F}_A) \longrightarrow K_0(\mathcal{F}_{B^t}) \otimes K_0(\mathcal{F}_B)$$

such that

$$\Phi_{0*}\circ(\delta^m_{A^t} \otimes \delta^n_A) = (\delta^m_{B^t} \otimes \delta^n_B)\circ\Phi_{0*}, \qquad (m,n) \in \mathbb{Z}^2.$$

(ii) *There exist an $N^2 \times M^2$-matrix H, an $M^2 \times N^2$-matrix K with entries in nonnegative integers and a natural number ℓ such that*

$$(A^t \otimes A)^\ell = HK, \qquad (B^t \otimes B)^\ell = KH, \tag{4.2}$$

$$(1 \otimes A)H = H(1 \otimes B), \qquad K(1 \otimes A) = (1 \otimes B)K, \tag{4.3}$$

$$(A^t \otimes 1)H = H(B^t \otimes 1), \qquad K(A^t \otimes 1) = (B^t \otimes 1)K. \tag{4.4}$$

Proof. (i) Since $\Phi : \widetilde{\mathcal{R}}_A \otimes \mathcal{K} \longrightarrow \widetilde{\mathcal{R}}_B \otimes \mathcal{K}$ is an isomorphism of C^*-algebras satisfying (4.1), it induces an isomorphism

$$\Phi_1 : (\widetilde{\mathcal{R}}_A \otimes \mathcal{K}) \rtimes_{\gamma^A\otimes\mathrm{id}} \mathbb{T}^2 \longrightarrow (\widetilde{\mathcal{R}}_B \otimes \mathcal{K}) \rtimes_{\gamma^B\otimes\mathrm{id}} \mathbb{T}^2$$

of C^*-algebras of the crossed products. Let $\hat{\gamma}^A, \hat{\gamma}^B$ be the dual actions on $\widetilde{\mathcal{R}}_A \rtimes_{\gamma^A} \mathbb{T}^2, \widetilde{\mathcal{R}}_B \rtimes_{\gamma^A} \mathbb{T}^2$, respectively. By identifying $(\widetilde{\mathcal{R}}_A \otimes \mathcal{K}) \rtimes_{\gamma^A\otimes\mathrm{id}} \mathbb{T}^2$ with $(\widetilde{\mathcal{R}}_A \rtimes_{\gamma^A} \mathbb{T}^2) \otimes \mathcal{K}$, and $(\widetilde{\mathcal{R}}_B \otimes \mathcal{K}) \rtimes_{\gamma^B\otimes\mathrm{id}} \mathbb{T}^2$ with $(\widetilde{\mathcal{R}}_B \rtimes_{\gamma^B} \mathbb{T}^2) \otimes \mathcal{K}$, we see that

$$\Phi_1\circ(\hat{\gamma}^A_{(m,n)} \otimes \mathrm{id}) = (\hat{\gamma}^B_{(m,n)} \otimes \mathrm{id})\circ\Phi_1, \quad (m,n) \in \mathbb{Z}^2.$$

Hence we have an isomorphism

$$\Phi_{1*} : K_0(\widetilde{\mathcal{R}}_A \rtimes_{\gamma^A} \mathbb{T}^2) \longrightarrow K_0(\widetilde{\mathcal{R}}_B \rtimes_{\gamma^B} \mathbb{T}^2)$$

such that

$$\Phi_{1*}\circ\hat{\gamma}^A_{(m,n)*} = \hat{\gamma}^B_{(m,n)*}\circ\Phi_{1*}, \qquad (m,n) \in \mathbb{Z}^2.$$

We then define $\Phi_{0*} : K_0(\mathcal{F}_{A^t}) \otimes K_0(\mathcal{F}_A) \longrightarrow K_0(\mathcal{F}_{B^t}) \otimes K_0(\mathcal{F}_B)$ by setting

$$\Phi_{0*} = \varphi_{B*}\circ\iota_{B*}^{-1}\circ\Phi_{1*}\circ\iota_{A*}\circ\varphi_{A*}^{-1},$$

where $\iota_{A*} : K_0(C^*(G^a_A)) = K_0(p_0(\widetilde{\mathcal{R}}_A \rtimes_{\gamma^A} \mathbb{T}^2)p_0) \longrightarrow K_0(\widetilde{\mathcal{R}}_A \rtimes_{\gamma^A} \mathbb{T}^2)$ is the isomorphism defined in Lemma 4.1 (ii). Hence the following diagram is commutative:

$$\begin{tikzcd}
K_0(\widetilde{\mathcal{R}}_A \rtimes_{\gamma^A} \mathbb{T}^2) \arrow[r, "\Phi_{1*}"] & K_0(\widetilde{\mathcal{R}}_B \rtimes_{\gamma^B} \mathbb{T}^2) \\
K_0(p_0(\widetilde{\mathcal{R}}_A \rtimes_{\gamma^A} \mathbb{T}^2)p_0) \arrow[u, "\iota_{A*}"] \arrow[d, equal] & K_0(p_0(\widetilde{\mathcal{R}}_B \rtimes_{\gamma^B} \mathbb{T}^2)p_0) \arrow[u, "\iota_{B*}"'] \arrow[d, equal] \\
K_0(C^*(G^a_A)) \arrow[d, "\varphi_{A*}"'] & K_0(C^*(G^a_A)) \arrow[d, "\varphi_{B*}"] \\
K_0(\mathcal{F}_{A^t}) \otimes K_0(\mathcal{F}_A) \arrow[r, "\Phi_{0*}"] & K_0(\mathcal{F}_{A^t}) \otimes K_0(\mathcal{F}_A).
\end{tikzcd}$$

We then have by Lemma 4.2

$$
\begin{aligned}
&\Phi_{0*}\circ(\delta^m_{A^t}\otimes\delta^n_A)\\
=&(\varphi_{B*}\circ\iota_{B*}^{-1}\circ\Phi_{1*}\circ\iota_{A*}\circ\varphi_{A*}^{-1})\circ(\varphi_{A*}\circ\iota_{A*}^{-1}\circ\hat{\gamma}^A_{(m,n)*}\circ\iota_{A*}\circ\varphi_{A*}^{-1})\\
=&\varphi_{B*}\circ\iota_{B*}^{-1}\circ\Phi_{1*}\circ\hat{\gamma}^A_{(m,n)*}\circ\iota_{A*}\circ\varphi_{A*}^{-1}\\
=&\varphi_{B*}\circ\iota_{B*}^{-1}\circ\hat{\gamma}^B_{(m,n)*}\circ\Phi_{1*}\circ\iota_{A*}\circ\varphi_{A*}^{-1}\\
=&(\varphi_{B*}\circ\iota_{B*}^{-1}\circ\hat{\gamma}^B_{(m,n)*}\circ\iota_{B*}\circ\varphi_{B*}^{-1})\circ(\varphi_{B*}\circ\iota_{B*}^{-1}\circ\Phi_{1*}\circ\iota_{A*}\circ\varphi_{A*}^{-1})\\
=&(\delta^m_{B^t}\otimes\delta^n_B)\circ\Phi_{0*}.
\end{aligned}
$$

(ii) By (i) the isomorphism $\Phi : \widetilde{\mathcal{R}}_A\otimes\mathcal{K}\longrightarrow\widetilde{\mathcal{R}}_B\otimes\mathcal{K}$ satisfying (4.1) induces an isomorphism

$$\Phi_{0*} : K_0(\mathcal{F}_{A^t})\otimes K_0(\mathcal{F}_A)\longrightarrow K_0(\mathcal{F}_{B^t})\otimes K_0(\mathcal{F}_B)$$

of ordered groups such that $\Phi_{0*}\circ(\delta^m_{A^t}\otimes\delta^n_A)=(\delta^m_{B^t}\otimes\delta^n_B)\circ\Phi_{0*}$. Now

$$
\begin{aligned}
K_0(\mathcal{F}_A)&=\lim\{\mathbb{Z}^N\xrightarrow{A^t}\mathbb{Z}^N\xrightarrow{A^t}\cdots\},\\
K_0(\mathcal{F}_{A^t})&=\lim\{\mathbb{Z}^N\xrightarrow{A}\mathbb{Z}^N\xrightarrow{A}\cdots\}
\end{aligned}
$$

and the dimension drop automorphisms $\delta_A : K_0(\mathcal{F}_A)\longrightarrow K_0(\mathcal{F}_A)$ and $\delta_{A^t} : K_0(\mathcal{F}_{A^t})\longrightarrow K_0(\mathcal{F}_{A^t})$ are defined by $\delta_A([x,n])=[x,n+1](=[Ax,n])$ for $[x,n]\in K_0(\mathcal{F}_A)$ and $\delta_{A^t}([y,n])=[y,n+1](=[A^ty,n])$ for $[y,n]\in K_0(\mathcal{F}_{A^t})$, respectively ([6], [7]). Since $K_0(\mathcal{F}_{A^t})\otimes K_0(\mathcal{F}_A)=K_0(\mathcal{F}_{A^t\otimes A})$ and $\delta_{A^t}\otimes\delta_A=\delta_{A^t\otimes A}$, we have an isomorphism of dimension triplets

$$(K_0(\mathcal{F}_{A^t\otimes A}),\delta_{A^t\otimes A})\cong(K_0(\mathcal{F}_{B^t\otimes B}),\delta_{B^t\otimes B})$$

with dimension drop automorphisms. Hence the two matrices $A^t\otimes A$ and $B^t\otimes B$ are shift equivalent by [16], which means that there exist an $N^2\times M^2$-matrix H, an $M^2\times N^2$-matrix K with entries in nonnegative integers and a natural number ℓ such that

$$
\begin{aligned}
(A^t\otimes A)^\ell&=HK, &\qquad (B^t\otimes B)^\ell&=KH,\\
(A^t\otimes A)H&=H(B^t\otimes B), &\qquad K(A^t\otimes A)&=(B^t\otimes B)K.
\end{aligned}
$$

Since

$$
\begin{aligned}
&K_0(\mathcal{F}_{A^t})\otimes K_0(\mathcal{F}_A)\\
=&\lim\{\mathbb{Z}^N\xrightarrow{A}\mathbb{Z}^N\xrightarrow{A}\cdots\}\otimes\lim\{\mathbb{Z}^N\xrightarrow{A^t}\mathbb{Z}^N\xrightarrow{A^t}\cdots\}\\
\cong&\lim\{\mathbb{Z}^N\otimes\mathbb{Z}^N\xrightarrow{A\otimes A^t}\mathbb{Z}^N\otimes\mathbb{Z}^N\xrightarrow{A\otimes A^t}\cdots\}
\end{aligned}
$$

and similarly

$$K_0(\mathcal{F}_{B^t})\otimes K_0(\mathcal{F}_{B^t})\cong\lim\{\mathbb{Z}^M\otimes\mathbb{Z}^M\xrightarrow{B\otimes B^t}\mathbb{Z}^M\otimes\mathbb{Z}^M\xrightarrow{B\otimes B^t}\cdots\},$$

Φ_{0*} induces an isomorphism

$$\lim\{\mathbb{Z}^N \otimes \mathbb{Z}^N \xrightarrow{A\otimes A^t} \mathbb{Z}^N \otimes \mathbb{Z}^N \xrightarrow{A\otimes A^t} \cdots\}$$
$$\longrightarrow \lim\{\mathbb{Z}^M \otimes \mathbb{Z}^M \xrightarrow{B\otimes B^t} \mathbb{Z}^M \otimes \mathbb{Z}^M \xrightarrow{B\otimes B^t} \cdots\},$$

which satisfies

$$\Phi_{0*}\circ(\mathrm{id}\otimes\delta_A) = (\mathrm{id}\otimes\delta_B)\circ\Phi_{0*}, \tag{4.5}$$
$$\Phi_{0*}\circ(\delta_{A^t}\otimes\mathrm{id}) = (\delta_{B^t}\otimes\mathrm{id}\circ\Phi_{0*}. \tag{4.6}$$

By the conditions (4.5) and (4.6), we may take the matrices H, K satisfying (4.2),(4.3) and (4.4). □

For a matrix A, let us denote by $\mathrm{Sp}^\times(A)$ and $\mathrm{Sp}^\times_{\mathrm{m}}(A)$ the set of nonzero eigenvalues of A and the list of nonzero repeated eigenvalues of A according to their multiplicity, respectively.

Lemma 4.4. *Let A, B be irreducible, non-permutation matrices with entries in $\{0,1\}$. Suppose that there exist an $N^2\times M^2$-matrix H, an $M^2\times N^2$-matrix K with entries in nonnegative integers and a natural number ℓ satisfying* (4.2),(4.3) *and* (4.4). *Then*

$$\mathrm{Sp}^\times_{\mathrm{m}}(A^t\otimes A) = \mathrm{Sp}^\times_{\mathrm{m}}(B^t\otimes B) \quad \textit{and} \quad \mathrm{Sp}^\times(A) = \mathrm{Sp}^\times(B). \tag{4.7}$$

Proof. We note that $\mathrm{Sp}^\times_{\mathrm{m}}(A) = \mathrm{Sp}^\times_{\mathrm{m}}(A^t)$. By the equalities (4.3) and (4.4), we see that

$$(A^t\otimes A)H = H(B^t\otimes B), \qquad K(A^t\otimes A) = (B^t\otimes B)K. \tag{4.8}$$

The equalities (4.2) together with (4.8) show us that the matrices $A^t\otimes A$ and $B^t\otimes B$ are shift equivalent, so that

$$\mathrm{Sp}^\times_{\mathrm{m}}(A^t\otimes A) = \mathrm{Sp}^\times_{\mathrm{m}}(B^t\otimes B) \tag{4.9}$$

by [17, Theorem 7.4.10]. For $\lambda\in\mathrm{Sp}^\times(A)$, one may take nonzero eigenvectors $u, v\in\mathbb{C}^N$ such that $Av = \lambda v$ and $A^t u = \lambda u$. By (4.3), we have

$$(1\otimes B)K(u\otimes v) = K(1\otimes A)(u\otimes v) = K(u\otimes\lambda v) = \lambda K(u\otimes v).$$

By (4.2), we have

$$HK(u\otimes v) = (A^t\otimes A)^\ell(u\otimes v) = \lambda^\ell(u\otimes v)$$

so that the vector $K(u\otimes v)$ is a nonzero eigenvector of the matrix $1\otimes B$ for the eigenvalue λ. Hence $\lambda\in\mathrm{Sp}^\times(1\otimes B)$. Since $\mathrm{Sp}^\times(1\otimes B) = \mathrm{Sp}^\times(B)$, we have $\lambda\in\mathrm{Sp}^\times(B)$, so that $\mathrm{Sp}^\times(A)\subset\mathrm{Sp}^\times(B)$. Similarly the inclusion relation $\mathrm{Sp}^\times(B)\subset\mathrm{Sp}^\times(A)$ holds and hence $\mathrm{Sp}^\times(A) = \mathrm{Sp}^\times(B)$. □

Lemma 4.5. *Suppose that two irreducible, non-permutation matrices A, B with entries in $\{0,1\}$ satisfy* (4.7). *Then we have*

$$\mathrm{Sp}^\times_{\mathrm{m}}(A) = \mathrm{Sp}^\times_{\mathrm{m}}(B) \quad \textit{and hence} \quad \det(1-A) = \det(1-B).$$

Proof. Since both A, B are irreducible, they have its periods as irreducible matrices, which we denote by p_A, p_B, respectively. Since $\mathrm{Sp}^\times(A) = \mathrm{Sp}^\times(B)$, their Perron-Frobenius eigenvalues coincide. We denote the common eigenvalue by λ_1 which is positive. There are exactly p_A eigenvalues λ of $\mathrm{Sp}^\times_{\mathrm{m}}(A)$ such that $|\lambda| = \lambda_1$, so that we have $p_A = p_B$ which we denote by p. Let ω be the p th root $e^{2\pi\sqrt{-1}\frac{1}{p}}$ of unity. By Perron-Frbobenius theorem for irreducible matrices, one may find distinct eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_L\} \subset \mathrm{Sp}^\times(A)(= \mathrm{Sp}^\times(B))$ such that

$$\lambda_1 > |\lambda_2| \ge |\lambda_3| \ge \cdots \ge |\lambda_L| \tag{4.10}$$

and the set $\{\omega^k \lambda_i \mid k = 0, 1, \dots, p-1,\ i = 1, \dots, L\}$ is the full list of $\mathrm{Sp}^\times(A)(= \mathrm{Sp}^\times(B))$ (cf. [32, Section 1.4]). For each $i = 1, \dots, L$, the p eigenvalues

$$\omega^k \lambda_i, \qquad k = 0, 1, \dots, p-1$$

have common multiplicities in $\mathrm{Sp}^\times_{\mathrm{m}}(A)$ and in $\mathrm{Sp}^\times_{\mathrm{m}}(B)$, respectively, which we denote by m_i^A and m_i^B, respectively. Hence we know that $m_1^A = m_1^B = 1$. We put $\lambda_i(k) = \omega^k \lambda_i$ for $k = 0, 1, \dots, p-1$, $i = 1, \dots, L$. Let m_0^A, m_0^B be the multiplicities of zero eigenvalues of A, B, respectively. Then the characteristic polynomials of the matrices $A^t \otimes A, B^t \otimes B$ are written such that

$$\varphi_{A^t\otimes A}(t) = t^{(m_0^A)^2} \prod_{i,j=1}^{L} \prod_{k,l=0}^{p-1} (t - \lambda_i(k)\lambda_j(l))^{m_i^A m_j^A},$$

$$\varphi_{B^t\otimes B}(t) = t^{(m_0^B)^2} \prod_{i,j=1}^{L} \prod_{k,l=0}^{p-1} (t - \lambda_i(k)\lambda_j(l))^{m_i^B m_j^B}.$$

By the assumption $\mathrm{Sp}^\times_{\mathrm{m}}(A^t \otimes A) = \mathrm{Sp}^\times_{\mathrm{m}}(B^t \otimes B)$, we have

$$\prod_{i,j=1}^{L} \prod_{k,l=0}^{p-1} (t - \lambda_i(k)\lambda_j(l))^{m_i^A m_j^A} = \prod_{i,j=1}^{L} \prod_{k,l=0}^{p-1} (t - \lambda_i(k)\lambda_j(l))^{m_i^B m_j^B}. \tag{4.11}$$

The above polynomial of the left (resp. right) hand side is denoted by $\phi_A(t)$ (resp. $\phi_B(t)$). Suppose that

$$\lambda_1\lambda_2 = \lambda_i(k)\lambda_j(l) \quad \text{for some } i, j = 1, \dots, L \text{ and } k, l = 0, 1, \dots, p-1.$$

We may assume $i \le j$. By the inequalities (4.10) with $|\lambda_i(k)| = |\lambda_i|$, $|\lambda_j(l)| = |\lambda_j|$, we have $i = 1$, so that $\lambda_i(k) = \omega^k \lambda_1$. Hence we have

$$\lambda_2 = \omega^k \lambda_j(l) = \omega^{k+l} \lambda_j$$

so that $j = 2$ and $k + l \equiv 0 \,(\mathrm{mod}\, p)$. We put $a = \lambda_1\lambda_2$. The power exponent of $(t - a)$ in the polynomial $\phi_A(t)$ is

$$(m_1^A m_2^A + m_2^A m_1^A) \times |\{(k, l) \in \{0, 1, \dots, p-1\}^2 \mid k + l \equiv 0 \,(\mathrm{mod}\, p)\}| = 2m_2^A p.$$

Similarly the power exponent of $(t - a)$ in the polynomial $\phi_B(t)$ is $2m_2^B p$. Hence we have $m_2^A = m_2^B$. Next assume that there exists $2 \le h \le L$ such that

$$m_n^A = m_n^B \quad \text{for all } n \le h. \tag{4.12}$$

Suppose that

$$\lambda_1\lambda_{h+1} = \lambda_i(k)\lambda_j(l) \quad \text{for some } i,j = 1,\dots,L \text{ and } \ k,l = 0,1,\dots,p-1.$$

We may assume $i \le j$. If $i = 1$, then $\lambda_i(k) = \omega^k\lambda_1$. Hence we have

$$\lambda_{h+1} = \omega^k\lambda_j(l) = \omega^{k+l}\lambda_j$$

so that $j = h+1$ and $k + l \equiv 0 \pmod p$. If $i \ne 1$, we have $j < h+1$ because of the inequalities (4.10). We put

$$\begin{aligned} p_1(1,h+1) = \{(i,j) \in \{2,\dots,L\}^2 \mid i<j,\ &\lambda_1\lambda_{h+1} = \lambda_i(k)\lambda_j(l) \\ &\text{for some } k,l = 0,1,\dots,p-1\}, \\ p_0(1,h+1) = \{i \in \{1,2,\dots,L\} \mid &\lambda_1\lambda_{h+1} = \lambda_i(k)^2 \\ &\text{for some } k = 0,1,\dots,p-1\}. \end{aligned}$$

Both sets $p_1(1,h+1)$ and $p_0(1,h+1)$ are possibly empty. We note that $\lambda_i(k)\lambda_j(l) = \omega^{k+l}\lambda_i\lambda_j$ and $|\{(k,l) \in \{1,\dots,p\}^2 \mid k+l \equiv 0 \pmod p)\}| = p$. Put $b = \lambda_1\lambda_{h+1}$. Hence the power exponent of $(t-b)$ in the polynomial $\phi_A(t)$ is

$$2m_1^A m_{h+1}^A p + 2(\sum_{(i,j)\in p_1(1,h+1)} m_i^A m_j^A)\cdot p + \epsilon_p \sum_{i\in p_0(1,h+1)} m_i^A$$

where $\epsilon_p = 2$ if p is even, and $\epsilon_p = 1$ if p is odd. Similarly the power exponent of $(t-b)$ in the polynomial $\phi_B(t)$ is

$$2m_1^B m_{h+1}^B p + 2(\sum_{(i,j)\in p_1(1,h+1)} m_i^B m_j^B)\cdot p + \epsilon_p \sum_{i\in p_0(1,h+1)} m_i^B.$$

Any pair $(i,j) \in p_1(1,h+1)$ satisfies $i < j < h+1$ and any element $i \in p_0(1,h+1)$ satisfies $i < h+1$. Hence the hypothesis (4.12) ensures that

$$m_{h+1}^A = m_{h+1}^B.$$

Therefore we obtain that $\mathrm{Sp}_{\mathrm{m}}^{\times}(A) = \mathrm{Sp}_{\mathrm{m}}^{\times}(B)$. Since

$$\det(1-A) = \prod_{i=1}^{L}\prod_{k=0}^{p-1}(1-\lambda_i(k)^{m_i^A}), \quad \det(1-B) = \prod_{i=1}^{L}\prod_{k=0}^{p-1}(1-\lambda_i(k)^{m_i^B}),$$

the equality $\det(1-A) = \det(1-B)$ follows from $\mathrm{Sp}_{\mathrm{m}}^{\times}(A) = \mathrm{Sp}_{\mathrm{m}}^{\times}(B)$. □

W. Parry and D. Sullivan in [22] proved that the determinant $\det(1-A)$ is invariant under flow equivalence of topological Markov shift $(\bar{X}_A, \bar{\sigma}_A)$. There is another crucial invariant of flow equivalence called the Bowen–Franks group written $\mathrm{BF}(A)$, which is defined by the abelian group $\mathbb{Z}^N/(1-A)\mathbb{Z}^N$ for the $N \times N$ matrix A with entries in $\{0,1\}$ ([2]). J, Franks in [9] proved that the pair $\det(1-A)$ and $\mathrm{BF}(A)$ is a complete set of invariants of flow equivalence. We note that the group $\mathrm{BF}(A)$ is isomorphic to the K_0-group $K_0(\mathcal{O}_A)$ of the Cuntz-Krieger algebra $\mathcal{O}_A$. We reach the following proposition.

Proposition 4.6. *Assume that A and B are irreducible, non-permutation matrices with entries in $\{0,1\}$. Suppose that there exists an isomorphism $\Phi : \widetilde{\mathcal{R}}_A \otimes \mathcal{K} \longrightarrow \widetilde{\mathcal{R}}_B \otimes \mathcal{K}$ such that*

$$\Phi\circ(\gamma^A_{(r,s)} \otimes \mathrm{id}) = (\gamma^B_{(r,s)} \otimes \mathrm{id})\circ\Phi, \quad (r,s) \in \mathbb{T}^2. \tag{4.13}$$

Then the two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent.

Proof. Suppose that there exists an isomorphism $\Phi : \widetilde{\mathcal{R}}_A \otimes \mathcal{K} \longrightarrow \widetilde{\mathcal{R}}_B \otimes \mathcal{K}$ satisfying (4.13). We then have $K_0(\widetilde{\mathcal{R}}_A) = K_0(\widetilde{\mathcal{R}}_B)$ so that $K_0(\mathcal{O}_{A^t} \otimes \mathcal{O}_A) \cong K_0(\mathcal{O}_{B^t} \otimes \mathcal{O}_B)$ and hence $K_0(\mathcal{O}_A) \cong K_0(\mathcal{O}_B)$ by Künneth formulas. This implies that $\mathrm{BF}(A)$ is isomorphic to $\mathrm{BF}(B)$. By Lemma 4.3, Lemma 4.4 and Lemma 4.5, we have $\det(1-A) = \det(1-B)$. Hence we conclude that $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent by Franks's theorem [9]. □

We will use Proposition 4.6 to prove Theorem 6.7 in Section 6.

5. Gauge actions with potentials

In this section, we will define gauge actions $\gamma^{A,f}$ with potential function $f : \bar{X}_A \longrightarrow \mathbb{Z}$ on the C^*-algebra $\widetilde{\mathcal{R}}_A$. For a continuous function $f \in C(\bar{X}_A, \mathbb{Z})$ on $\bar{X}_A$ and $n \in \mathbb{Z}$, we define a continuous function $f^n \in C(\bar{X}_A, \mathbb{Z})$ by setting

$$f^n(x) = \begin{cases} \sum_{i=0}^{n-1} f(\bar{\sigma}_A^i(x)) & \text{for } n \geq 1, \\ 0 & \text{for } n = 0, \\ -\sum_{i=n}^{-1} f(\bar{\sigma}_A^i(x)) & \text{for } n \leq -1. \end{cases}$$

It is easy to see that the identities

$$f^{n+m}(x) = f^n(x) + f^m(\bar{\sigma}_A^n(x)), \qquad n, m \in \mathbb{Z},\ x \in \bar{X}_A$$

hold. For $f \in C(\bar{X}_A, \mathbb{Z})$ and $(x,p,q,y) \in G_A^{s,u} \rtimes \mathbb{Z}^2$, define

$$\tilde{f}^+(x,p,q,y) = \lim_{n\to\infty}\{f^{n+p}(\bar{\sigma}_A(x)) - f^n(\bar{\sigma}_A(y))\},$$
$$\tilde{f}^-(x,p,q,y) = \lim_{n\to-\infty}\{f^{n+q}(x) - f^n(y)\}.$$

Lemma 5.1. *Both $\tilde{f}^+, \tilde{f}^- : G_A^{s,u} \rtimes \mathbb{Z}^2 \longrightarrow \mathbb{Z}$ are continuous groupoid homomorphisms from $G_A^{s,u} \rtimes \mathbb{Z}^2$ to $\mathbb{Z}$.*

Proof. Take an arbitrary point $(x,p,q,y) \in G_A^{s,u} \rtimes \mathbb{Z}^2$ so that

$$\lim_{n\to\infty} d(\bar{\sigma}_A^{n+p}(x), \bar{\sigma}_A^n(y)) = 0, \qquad \lim_{n\to-\infty} d(\bar{\sigma}_A^{n+q}(x), \bar{\sigma}_A^n(y)) = 0. \tag{5.1}$$

By the first equality above, we may find $N_1 \in \mathbb{N}$ locally such that

$$f(\bar{\sigma}_A^n(\bar{\sigma}_A^p(\bar{\sigma}_A(x)))) = f(\bar{\sigma}_A^n(\bar{\sigma}_A(y))) \quad \text{for all } n \geq N_1. \tag{5.2}$$

For $n \ge N_1$, we have

$$
\begin{aligned}
& f^{n+p}(\bar\sigma_A(x)) - f^n(\bar\sigma_A(y)) \\
=& f^p(\bar\sigma_A(x)) + f^n(\bar\sigma_A^p(\bar\sigma_A(x))) - f^n(\bar\sigma_A(y)) \\
=& f^p(\bar\sigma_A(x)) + f(\bar\sigma_A^p(\bar\sigma_A(x))) + f(\bar\sigma_A^{p+1}(\bar\sigma_A(x))) + \cdots + f(\bar\sigma_A^{p+N_1-1}(\bar\sigma_A(x))) \\
& - f(\bar\sigma_A(y)) - f(\bar\sigma_A^2(y)) - \cdots - f(\bar\sigma_A^{N_1-1}(\bar\sigma_A(y))) \\
=& f^p(\bar\sigma_A(x)) + f^{N_1}(\bar\sigma_A^p(\bar\sigma_A(x))) - f^{N_1}(\bar\sigma_A(y))
\end{aligned}
$$

so that

$$
\tilde f^+(x,p,q,y) = f^{p+N_1}(\bar\sigma_A(x)) - f^{N_1}(\bar\sigma_A(y)). \tag{5.3}
$$

By the second equality of (5.1), we may similarly find a negative integer $N_2 \in \mathbb{Z}$ such that

$$
\tilde f^-(x,p,q,y) = f^{q+N_2}(x) - f^{N_2}(y). \tag{5.4}
$$

Hence both the values $\tilde f^+(x,p,q,y), \tilde f^-(x,p,q,y)$ are defined.

For $(x,p,q,y),(x',p',q',y') \in G_A^{s,u} \rtimes \mathbb{Z}^2$ with $x' = y$, we have

$$
\begin{aligned}
& \tilde f^+((x,p,q,y)(x',p',q',y')) \\
=& \tilde f^+(x,p+p',q+q',y') \\
=& \lim_{n\to\infty}\{f^{n+p+p'}(\bar\sigma_A(x)) - f^n(\bar\sigma_A(y'))\} \\
=& \lim_{n\to\infty}\{f^{n+p}(\bar\sigma_A(x)) + f^{p'}(\bar\sigma_A^n(\bar\sigma_A^p(\bar\sigma_A(x)))) - f^n(\bar\sigma_A(y'))\}.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
& \tilde f^+(x,p,q,y) + \tilde f^+(x',p',q',y') \\
=& \lim_{n\to\infty}\{f^{n+p}(\bar\sigma_A(x)) - f^n(\bar\sigma_A(y))\} + \lim_{n\to\infty}\{f^{n+p'}(\bar\sigma_A(x')) - f^n(\bar\sigma_A(y'))\} \\
=& \lim_{n\to\infty}\{f^{n+p}(\bar\sigma_A(x)) + f^{n+p'}(\bar\sigma_A(y)) - f^n(\bar\sigma_A(y)) - f^n(\bar\sigma_A(y'))\} \\
=& \lim_{n\to\infty}\{f^{n+p}(\bar\sigma_A(x)) + f^{p'}(\bar\sigma_A^n(\bar\sigma_A(y))) - f^n(\bar\sigma_A(y'))\}.
\end{aligned}
$$

By (5.2), the equality

$$
\lim_{n\to\infty} f^{p'}(\bar\sigma_A^n(\bar\sigma_A^p(\bar\sigma_A(x)))) = \lim_{n\to\infty} f^{p'}(\bar\sigma_A^n(\bar\sigma_A(y)))
$$

holds, so that

$$
\tilde f^+((x,p,q,y)(x',p',q',y')) = \tilde f^+(x,p,q,y) + \tilde f^+(x',p',q',y')
$$

and similarly

$$
\tilde f^-((x,p,q,y)(x',p',q',y')) = \tilde f^-(x,p,q,y) + \tilde f^-(x',p',q',y').
$$

The identities

$$
\tilde f^+((x,p,q,y)^{-1}) = -\tilde f^+(x,p,q,y), \qquad \tilde f^-((x,p,q,y)^{-1}) = -\tilde f^-(x,p,q,y)
$$

are easily seen. As the continuity of $\tilde{f}^+, \tilde{f}^-$ follows from the formulas (5.3), (5.4) with the uniform continuity of f, we know that they are continuous groupoid homomorphisms from $G_A^{s,u} \rtimes \mathbb{Z}^2$ to $\mathbb{Z}$. $\square$

Define a continuous groupoid homomorphism $\tilde{f} : G_A^{s,u} \rtimes \mathbb{Z}^2 \longrightarrow \mathbb{Z}$ by $\tilde{f}(x,p,q,y) = \tilde{f}^+(x,p,q,y) - \tilde{f}^-(x,p,q,y)$. Recall that the C^*-algebra $\widetilde{\mathcal{R}}_A$ is represented on the Hilbert C^*-right module $\ell^2(G_A^{s,u} \rtimes \mathbb{Z}^2)$ over $C_0((G_A^{s,u} \rtimes \mathbb{Z}^2)^\circ)(= C(\bar{X}_A))$ as the reduced groupoid C^*-algebra. For $f \in C(\bar{X}_A, \mathbb{Z}), (r,s) \in \mathbb{T}^2$ and $\xi \in \ell^2(G_A^{s,u} \rtimes \mathbb{Z}^2)$, we set

$$[U_s(\tilde{f}^+)\xi](x,p,q,y) = \exp\{2\pi\sqrt{-1}\tilde{f}^+(x,p,q,y)s\}\xi(x,p,q,y),$$
$$[U_r(\tilde{f}^-)\xi](x,p,q,y) = \exp\{2\pi\sqrt{-1}\tilde{f}^-(x,p,q,y)r\}\xi(x,p,q,y),$$
$$U_{(r,s)}(\tilde{f}) = U_r(\tilde{f}^-)U_s(\tilde{f}^+).$$

Since $\tilde{f}^+, \tilde{f}^-$ and $\tilde{f}$ are groupoid homomorphisms from $G_A^{s,u} \rtimes \mathbb{Z}^2$ to $\mathbb{Z}$, the operators $U(\tilde{f}^+), U(\tilde{f}^-)$ yield unitary representations of $\mathbb{T}$ and $U(\tilde{f})$ does a unitary representation of $\mathbb{T}^2$.

Proposition 5.2. *For $f \in C(\bar{X}_A, \mathbb{Z})$, the correspondence*

$$a \in \widetilde{\mathcal{R}}_A \longrightarrow \mathrm{Ad}(U_{(r,s)}(\tilde{f}))(a) \left(= U_{(r,s)}(\tilde{f})aU_{(r,s)}(\tilde{f})^*\right) \in \widetilde{\mathcal{R}}_A$$

defines an automorphism on $\widetilde{\mathcal{R}}_A$ such that $(r,s) \in \mathbb{T}^2 \longrightarrow \mathrm{Ad}(U_{(r,s)}) \in \mathrm{Aut}(\widetilde{\mathcal{R}}_A)$ gives rise to an action of $\mathbb{T}^2$ on $\widetilde{\mathcal{R}}_A$ and its restriction to the subalgebra $C(\bar{X}_A)$ is the identity.

Proof. For $a \in C_c(G_A^{s,u} \rtimes \mathbb{Z}^2)$, $\xi \in \ell^2(G_A^{s,u} \rtimes \mathbb{Z}^2)$, $(x,p,q,y) \in G_A^{s,u} \rtimes \mathbb{Z}^2$, we have

$$\begin{aligned}&[\mathrm{Ad}(U_{(r,s)}(\tilde{f}))(a)\xi](x,p,q,y)\\ =&\exp\{2\pi\sqrt{-1}(\tilde{f}^+(x,p,q,y)s + \tilde{f}^-(x,p,q,y)r)\}[aU_{(r,s)}(\tilde{f})^*\xi](x,p,q,y).\end{aligned}$$

Now the equalities

$$\begin{aligned}&[aU_{(r,s)}(\tilde{f})^*\xi](x,p,q,y)\\ =&\sum_{\gamma;r(\gamma)=x} a(\gamma)[U_{(r,s)}(-\tilde{f})\xi](\gamma^{-1}\cdot(x,p,q,y))\\ =&\sum_{\gamma;r(\gamma)=x} a(\gamma)\cdot\exp\{2\pi\sqrt{-1}(\tilde{f}^+(\gamma)s + \tilde{f}^-(\gamma)r)\}\\ &\cdot\exp\{-2\pi\sqrt{-1}(\tilde{f}^+(x,p,q,y)s + \tilde{f}^-(x,p,q,y)r)\}\xi(\gamma^{-1}\cdot(x,p,q,y))\end{aligned}$$

hold, so that

$$\begin{aligned}&[\mathrm{Ad}(U_{(r,s)}(\tilde{f}))(a)\xi](x,p,q,y)\\ =&\sum_{\gamma;r(\gamma)=x} a(\gamma)\cdot\exp\{2\pi\sqrt{-1}(\tilde{f}^+(\gamma)s + \tilde{f}^-(\gamma)r)\}\xi(\gamma^{-1}\cdot(x,p,q,y)).\end{aligned}$$

Let us identify $U_{(r,s)}(\tilde{f})$ with the continuous function on $G_A^{s,u} \rtimes \mathbb{Z}^2$ defined by

$$U_{(r,s)}(\tilde{f})(\gamma) = \exp\{2\pi\sqrt{-1}(\tilde{f}^+(\gamma)s + \tilde{f}^-(\gamma)r)\}, \qquad \gamma \in G_A^{s,u} \rtimes \mathbb{Z}^2.$$

Hence we have

$$\begin{aligned}[\mathrm{Ad}(U_{(r,s)}(\tilde{f}))(a)\xi](x,p,q,y) &= \sum_{\gamma; r(\gamma)=x} (U_{(r,s)}(\tilde{f}) \cdot a)(\gamma)\xi(\gamma^{-1} \cdot (x,p,q,y)) \\ &= [(U_{(r,s)}(\tilde{f}) \cdot a)\xi](x,p,q,y)\end{aligned}$$

so that

$$\mathrm{Ad}(U_{(r,s)}(\tilde{f}))(a) = U_{(r,s)}(\tilde{f}) \cdot a \qquad \text{for } a \in C_c(G_A^{s,u} \rtimes \mathbb{Z}^2) \tag{5.5}$$

where $U_{(r,s)}(\tilde{f}) \cdot a$ is the pointwise product between the two functions $U_{(r,s)}(\tilde{f})$ and a. Thus $\mathrm{Ad}(U_{(r,s)}(\tilde{f}))(a)$ belongs to $C_c(G_A^{s,u} \rtimes \mathbb{Z}^2)$, so that $\mathrm{Ad}(U_{(r,s)}(\tilde{f}))$ yields an automorphism of the C^*-algebra $\widetilde{\mathcal{R}}_A$.

Especially for a continuous function $a \in C(\bar{X}_A)$ on $\bar{X}_A$, it is regarded as an element of $C_c(G_A^{s,u} \rtimes \mathbb{Z}^2)$ by

$$a(x,p,q,y) = \begin{cases} a(x) & \text{if } x = y,\ p = q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $x = y,\ p = q = 0$, we know that $\tilde{f}^+(x,p,q,y) = \tilde{f}^-(x,p,q,y) = 0$ so that

$$\mathrm{Ad}(U_{(r,s)}(\tilde{f}))(a) = U_{(r,s)}(\tilde{f})) \cdot a = a \qquad \text{for } a \in C_c(\bar{X}_A).$$

□

We denote by $\gamma_{(r,s)}^{A,f}$ the automorphism $\mathrm{Ad}(U_{(r,s)}(\tilde{f}))$ on $\widetilde{\mathcal{R}}_A$, which yields an action called gauge action with potential function f, or weighted gauge action. For the constant function $f \equiv 1$, the equalities $\tilde{f}^+(x,p,q,y) = p$ and $\tilde{f}^-(x,p,q,y) = q$ hold so that the action $\gamma_{(r,s)}^{A,f}$ for $f \equiv 1$ coincides with the previously defined action $\gamma_{(r,s)}^A$.

Let $U_j(0), j = 1,2,\dots,N$ be the cylinder sets on $\bar{X}_A$ such that

$$U_j(0) = \{(x_n)_{n\in\mathbb{Z}} \in \bar{X}_A \mid x_0 = j\}.$$

Let $\chi_{U_j(0)}$ be the characteristic function of the cylinder set $U_j(0)$ on $\bar{X}_A$.

Lemma 5.3. *Suppose that* $f = \sum_{j=1}^N f_j \chi_{U_j(0)}$ *for some integers* $f_j \in \mathbb{Z}$. *Then we have*

$$\gamma_{(r,s)}^{A,f} = \alpha_r^{A^t,f} \otimes \alpha_s^{A,f}, \qquad (r,s) \in \mathbb{T}^2 \quad \text{on } \widetilde{\mathcal{R}}_A = E_A(\mathcal{O}_{A^t} \otimes \mathcal{O}_A)E_A,$$

where $\alpha_r^{A^t,f} \in \mathrm{Aut}(\mathcal{O}_{A^t}), \alpha_s^{A,f} \in \mathrm{Aut}(\mathcal{O}_A)$ *are defined by*

$$\alpha_r^{A^t,f}(T_j) = \exp\left(2\pi\sqrt{-1}f_j r\right)T_j, \qquad j = 1,2,\dots,N,$$

$$\alpha_s^{A,f}(S_j) = \exp\left(2\pi\sqrt{-1}f_j s\right)S_j, \qquad j = 1,2,\dots,N.$$

Proof. For $m,n,k,l \in \mathbb{N}$ and

$$\mu = (\mu_1, \dots, \mu_m), \quad \nu = (\nu_1, \dots, \nu_n) \in B_*(\bar{X}_A),$$
$$\bar{\xi} = (\xi_k, \dots, \xi_1), \quad \bar{\eta} = (\eta_l, \dots, \eta_1) \in B_*(\bar{X}_{A^t})$$

satisfying $A(\xi_k, \mu_1) = A(\eta_l, \nu_1) = 1$, we write

$$\begin{aligned} U_{\xi\mu,\eta\nu} = \{(x, m-n, l-k, y) \in G_A^{s,u} \rtimes \mathbb{Z}^2 \mid \\ (\bar{\sigma}_A^m(x), \bar{\sigma}_A^n(y)) \in G_A^{s,0}, (\bar{\sigma}_A^k(x), \bar{\sigma}_A^l(y)) \in G_A^{u,0}, \\ x_{[1,m]} = \mu,\ y_{[1,n]} = \nu,\ x_{[-k+1,0]} = \xi,\ y_{[-l+1,0]} = \eta\} \end{aligned}$$

where

$$G_A^{s,0} = \{(x,y) \in \bar{X}_A \times \bar{X}_A \mid x_i = y_i \text{ for all } i \in \mathbb{Z}_+\},$$
$$G_A^{u,0} = \{(x,y) \in \bar{X}_A \times \bar{X}_A \mid x_{-i} = y_{-i} \text{ for all } i \in \mathbb{Z}_+\}.$$

As in [19, Section 9], the correspondence $\chi_{U_{\xi\mu,\eta\nu}} \longleftrightarrow T_{\bar{\xi}} T_{\bar{\eta}}^* \otimes S_\mu S_\nu^*$ gives rise to an isomorphism between the groupoid C^*-algebra $C^*(G_A^{s,u} \rtimes \mathbb{Z}^2)$ and the algebra $\widetilde{\mathcal{R}}_A$. For $(x,p,q,y) \in U_{\xi\mu,\eta\nu}$ with $p = m-n, q = l-k$, one may take $N_1 = n$ so that

$$\begin{aligned} & \tilde{f}^+(x,p,q,y) \\ =& f^p(\bar{\sigma}_A(x)) + f(\bar{\sigma}_A^p(\bar{\sigma}_A(x))) + f(\bar{\sigma}_A^{p+1}(\bar{\sigma}_A(x))) + \cdots + f(\bar{\sigma}_A^{p+N_1-1}(\bar{\sigma}_A(x))) \\ & - f(\bar{\sigma}_A(y)) - f(\bar{\sigma}_A^2(y)) - \cdots - f(\bar{\sigma}_A^{N_1}(y)) \\ =& f(\bar{\sigma}_A(x)) + f(\bar{\sigma}_A(\bar{\sigma}_A(x))) + \cdots + f(\bar{\sigma}_A^m(x)) \\ & - f(\bar{\sigma}_A(y)) - f(\bar{\sigma}_A^2(y)) - \cdots - f(\bar{\sigma}_A^n(y)) \\ =& (f_{\mu_1} + f_{\mu_2} + \cdots + f_{\mu_m}) - (f_{\nu_1} + f_{\nu_2} + \cdots + f_{\nu_n}) \end{aligned}$$

because $\bar{\sigma}_A(x)_{[0,m-1]} = \mu$, $\bar{\sigma}_A(y)_{[0,n-1]} = \nu$, and similarly

$$\begin{aligned} & \tilde{f}^-(x,p,q,y) \\ =& f(x) + f(\bar{\sigma}_A^{-1}(x)) + f(\bar{\sigma}_A^{-2}(x)) + \cdots + f(\bar{\sigma}_A^{-k+1}(x)) \\ & - f(y) - f(\bar{\sigma}_A^{-1}(y)) - f(\bar{\sigma}_A^{-2}(y)) - \cdots - f(\bar{\sigma}_A^{-l+1}(y)) \\ =& (f_{\xi_1} + f_{\xi_2} + \cdots + f_{\xi_k}) - (f_{\eta_1} + f_{\eta_2} + \cdots + f_{\eta_l}). \end{aligned}$$

It then follows that by (5.5)

$$\begin{aligned} & [\mathrm{Ad}(U_{(r,s)}(\tilde{f}))(\chi_{U_{\xi\mu,\eta\nu}})](x,p,q,y) \\ =& [U_{(r,s)}(\tilde{f})) \cdot \chi_{U_{\xi\mu,\eta\nu}})](x,p,q,y) \\ =& \exp\{2\pi\sqrt{-1}(\tilde{f}^+(x,p,q,y)s + \tilde{f}^-(x,p,q,y)r)\}\chi_{U_{\xi\mu,\eta\nu}}(x,p,q,y) \end{aligned}$$

so that

$$\begin{aligned}&\mathrm{Ad}(U_{(r,s)}(\tilde{f}))(\chi_{U_{\xi\mu,\eta\nu}})\\ =&\exp[2\pi\sqrt{-1}\{(f_{\mu_1}+f_{\mu_2}+\cdots+f_{\mu_m})-(f_{\nu_1}+f_{\nu_2}+\cdots+f_{\nu_n})\}s\\ &+\{(f_{\xi_1}+f_{\xi_2}+\cdots+f_{\xi_k})-(f_{\eta_1}+f_{\eta_2}+\cdots+f_{\eta_l})\}r]\cdot\chi_{U_{\xi\mu,\eta\nu}},\end{aligned}$$

proving that the equality

$$\gamma^{A,f}_{(r,s)}(\chi_{U_{\xi\mu,\eta\nu}})=\alpha^{A^t,f}_r(T_\xi T^*_\eta)\otimes\alpha^{A,f}_s(S_\mu S^*_\nu).$$

□

Proposition 5.4. *Let $\varphi:\bar{X}_A\longrightarrow\bar{X}_B$ be a topological conjugacy between two-sided topological Markov shifts $(\bar{X}_A,\bar{\sigma}_A)$ and $(\bar{X}_B,\bar{\sigma}_B)$. Suppose that $f\in C(\bar{X}_A,\mathbb{Z})$ and $g\in C(\bar{X}_B,\mathbb{Z})$ satisfy $f=g\circ\varphi$. Then there exists an isomorphism $\Phi:\widetilde{\mathcal{R}}_A\longrightarrow\widetilde{\mathcal{R}}_B$ of C^*-algebras such that*

$$\Phi(C(\bar{X}_A))=C(\bar{X}_B),\qquad \Phi\circ\gamma^{A,f}_{(r,s)}=\gamma^{B,g}_{(r,s)}\circ\Phi,\quad (r,s)\in\mathbb{T}^2.$$

Proof. The topological conjugacy $\varphi:\bar{X}_A\longrightarrow\bar{X}_B$ induces an isomorphism $\tilde{\varphi}:G^{s,u}_A\rtimes\mathbb{Z}^2\longrightarrow G^{s,u}_B\rtimes\mathbb{Z}^2$ of étale groupoids such that $\tilde{\varphi}(x,p,q,y)=(\varphi(x),p,q,\varphi(y))$ for $(x,p,q,y)\in G^{s,u}_A\rtimes\mathbb{Z}^2$. It gives rise to a unitary written $V_\varphi:\ell^2(G^{s,u}_B\rtimes\mathbb{Z}^2)\longrightarrow\ell^2(G^{s,u}_A\rtimes\mathbb{Z}^2)$ satisfying $V_\varphi(\xi)=\xi\circ\tilde{\varphi}$ for $\xi\in\ell^2(G^{s,u}_B\rtimes\mathbb{Z}^2)$. Assume that the C^*-algebras $\widetilde{\mathcal{R}}_A$ and $\widetilde{\mathcal{R}}_B$ are represented on $\ell^2(G^{s,u}_A\rtimes\mathbb{Z}^2)$ and $\ell^2(G^{s,u}_B\rtimes\mathbb{Z}^2)$ as reduced groupoid C^*-algebras. Since $V^*_\varphi aV_\varphi=a\circ\tilde{\varphi}^{-1}\in C_c(G^{s,u}_B\rtimes\mathbb{Z}^2)$ for $a\in C_c(G^{s,u}_A\rtimes\mathbb{Z}^2)$, we know that $\Phi(a)=V^*_\varphi aV_\varphi, a\in\widetilde{\mathcal{R}}_A$ gives rise to an isomorphism $\widetilde{\mathcal{R}}_A\longrightarrow\widetilde{\mathcal{R}}_B$ of C^*-algebras. Since $\varphi:\bar{X}_A\longrightarrow\bar{X}_B$ is a topological conjugacy, we know that

$$\tilde{g}^+(\varphi(x),p,q,\varphi(y))=\tilde{f}^+(x,p,q,y),\qquad \tilde{g}^-(\varphi(x),p,q,\varphi(y))=\tilde{f}^-(x,p,q,y)$$

for $(x,p,q,y)\in G^{s,u}_A\rtimes\mathbb{Z}^2$ so that for $\xi\in\ell^2(G^{s,u}_A\rtimes\mathbb{Z}^2),(x,p,q,y)\in G^{s,u}_A\rtimes\mathbb{Z}^2$, we have

$$\begin{aligned}&[V_\varphi U_{(r,s)}(\tilde{g})V^*_\varphi\xi](x,p,q,y)\\ =&[U_{(r,s)}(\tilde{g})V^*_\varphi\xi](\varphi(x),p,q,\varphi(y))\\ =&\exp\{2\pi\sqrt{-1}(\tilde{g}^+(\varphi(x),p,q,\varphi(y))s+\tilde{g}^-(\varphi(x),p,q,\varphi(y))r)\}\\ &[V^*_\varphi\xi](\varphi(x),p,q,\varphi(y))\\ =&\exp\{2\pi\sqrt{-1}(\tilde{f}^+(x,p,q,y)s+\tilde{f}^-(x,p,q,y)r)\}\xi(x,p,q,y)\\ =&[U_{(r,s)}(\tilde{f})\xi](x,p,q,y).\end{aligned}$$

Hence we have $V_\varphi U_{(r,s)}(\tilde{g})V^*_\varphi=U_{(r,s)}(\tilde{f})$, so that the equality $\Phi\circ\gamma^{A,f}_{(r,s)}=\gamma^{B,g}_{(r,s)}\circ\Phi$ holds. Since $a\circ\tilde{\varphi}^{-1}\in C_c((G^{s,u}_B\rtimes\mathbb{Z}^2)^\circ)$ for $a\in C_c((G^{s,u}_A\rtimes\mathbb{Z}^2)^\circ)$, the equality $\Phi(C(\bar{X}_A))=C(\bar{X}_B)$ is obvious (cf. [20, Theorem 1.1]). □

Corollary 5.5. *Let B be an $M \times M$ irreducible non-permutation matrix with entries in $\{0,1\}$. For any continuous function $g \in C(\bar{X}_B, \mathbb{Z})$ on $\bar{X}_B$, there exist an $N \times N$ irreducible non-permutation matrix A with entries in $\{0,1\}$ and a continuous function $f = \sum_{j=1}^N f_j \chi_{U_j(0)}$ for some integers $f_j \in \mathbb{Z}$ such that $(\bar{X}_A, \bar{\sigma}_A)$ is topologically conjugate to $(\bar{X}_B, \bar{\sigma}_B)$ and there exists an isomorphism $\Phi : \widetilde{\mathcal{R}}_A \longrightarrow \widetilde{\mathcal{R}}_B$ such that*

$$\Phi(C(\bar{X}_A)) = C(\bar{X}_B), \qquad \Phi \circ \gamma^{A,f}_{(r,s)} = \gamma^{B,g}_{(r,s)} \circ \Phi, \quad (r,s) \in \mathbb{T}^2.$$

Proof. There exists $K \in \mathbb{N}$ such that $g = \sum_{\mu \in B_K(\bar{X}_B)} g_\mu \chi_{U_\mu}$ for some $g_\mu \in \mathbb{Z}$ where U_μ is the cylinder set of $\bar{X}_B$ for a word $\mu \in B_K(\bar{X}_B)$. By taking K-higher block representation of $\bar{X}_B$ and its K higher block matrix of B as A (see [17, 1.4]), and shifting g, one may have a topological conjugacy $\varphi : \bar{X}_A \longrightarrow \bar{X}_B$ and a continuous function $f = \sum_{j=1}^N f_j \chi_{U_j(0)}$ for some integers $f_j \in \mathbb{Z}$ such that $f = g \circ \varphi$. Hence we get the desired assertion by Proposition 5.4. □

6. Flow equivalence

We fix an irreducible, non-permutation matrix A. Let $f : \bar{X}_A \longrightarrow \mathbb{N}$ be a continuous function on $\bar{X}_A$ such that $f = \sum_{j=1}^N f_j \chi_{U_j(0)}$ for some positive integers $f_j \in \mathbb{N}$. Put $m_j = f_j - 1$ for $j = 1, \dots, N$. Consider the new graph $\mathcal{G}_f = (\mathcal{V}_f, \mathcal{E}_f)$ with its transition matrix A_f from the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ for the matrix A defined in (2.3) in Section 2. The vertex set $\mathcal{V}_f$ is

$$\bigcup_{j=1}^N \{j_0, j_1, j_2, \dots, j_{m_j}\},$$

which is denoted by $\tilde{\Sigma}$, and if $A(j,k) = 1$, then

$$A_f(j_0, j_1) = A_f(j_1, j_2) = \cdots = A_f(j_{m_j - 1}, j_{m_j}) = A_f(j_{m_j}, k_0) = 1.$$

Let us denote by

$$\tilde{S}_{j_0}, \tilde{S}_{j_1}, \tilde{S}_{j_2}, \dots, \tilde{S}_{j_{m_j}} \quad \text{and} \quad \tilde{T}_{j_0}, \tilde{T}_{j_1}, \tilde{T}_{j_2}, \dots, \tilde{T}_{j_{m_j}}$$

the canonical generating partial isometries of $\mathcal{O}_{A_f}$ and $\mathcal{O}_{(A_f)^t}$ respectively which satisfy

$$\sum_{j=1}^N (\tilde{S}_{j_0} \tilde{S}^*_{j_0} + \tilde{S}_{j_1} \tilde{S}^*_{j_1} + \cdots + \tilde{S}_{j_{m_j}} \tilde{S}^*_{j_{m_j}}) = 1, \tag{6.1}$$

$$\tilde{S}^*_{j_n} \tilde{S}_{j_n} = \tilde{S}_{j_{n+1}} \tilde{S}^*_{j_{n+1}}, \qquad n = 0, 1, \dots, m_j - 1, \tag{6.2}$$

$$\tilde{S}^*_{j_{m_j}} \tilde{S}_{j_{m_j}} = \sum_{k=1}^N A(j,k) \tilde{S}_{k_0} \tilde{S}^*_{k_0}, \qquad j = 1, 2, \dots, N \tag{6.3}$$

and

$$\sum_{j=1}^{N}(\tilde{T}_{j_0}\tilde{T}^*_{j_0} + \tilde{T}_{j_1}\tilde{T}^*_{j_1} + \cdots + \tilde{T}_{j_{m_j}}\tilde{T}^*_{j_{m_j}}) = 1, \tag{6.4}$$

$$\tilde{T}^*_{j_{n+1}}\tilde{T}_{j_{n+1}} = \tilde{T}_{j_n}\tilde{T}^*_{j_n}, \qquad n = 0, 1, \dots, m_j - 1, \tag{6.5}$$

$$\tilde{T}^*_{j_0}\tilde{T}_{j_0} = \sum_{k=1}^{N} A^t(j,k)\tilde{T}_{k_{m_k}}\tilde{T}^*_{k_{m_k}}, \qquad j = 1, 2, \dots, N. \tag{6.6}$$

We set

$$S_j = \tilde{S}_{j_0}\tilde{S}_{j_1}\tilde{S}_{j_2}\cdots\tilde{S}_{j_{m_j}}, \quad T_j = \tilde{T}_{j_{m_j}}\tilde{T}_{j_{m_j-1}}\cdots\tilde{T}_{j_1}\tilde{T}_{j_0} \quad \text{for} \quad j = 1, \dots, N.$$

Define the projections

$$P_A = \sum_{j=1}^{N}\tilde{S}_{j_0}\tilde{S}^*_{j_0} \quad \text{and} \quad P_{A^t} = \sum_{j=1}^{N}\tilde{T}_{j_{m_j}}\tilde{T}^*_{j_{m_j}}.$$

We denote by $C^*(S_1, \dots, S_N)$ (resp. $C^*(T_1, \dots, T_N)$) the C^*-subalgebra of $\mathcal{O}_{A_f}$ (resp. $\mathcal{O}_{(A_f)^t}$) generated by $S_1, \dots, S_N$ (resp. $T_1, \dots, T_N$).

Lemma 6.1. *Keep the above notation. We have*

(i)

$$\sum_{j=1}^{N} S_j S_j^* = P_A, \qquad S_j^* S_j = \sum_{k=1}^{N} A(j,k) S_k S_k^* \quad \textit{for} \quad j = 1, \dots, N,$$

and the C^*-algebra $P_A\mathcal{O}_{A_f}P_A$ coincides with $C^*(S_1, \dots, S_N)$ that is isomorphic to $\mathcal{O}_A$.

(ii)

$$\sum_{j=1}^{N} T_j T_j^* = P_{A^t}, \qquad T_j^* T_j = \sum_{k=1}^{N} A^t(j,k) T_k T_k^* \quad \textit{for} \quad j = 1, \dots, N,$$

and the C^*-algebra $P_{A^t}\mathcal{O}_{(A_f)^t}P_{A^t}$ coincides with $C^*(T_1, \dots, T_N)$ that is isomorphic to $\mathcal{O}_{A^t}$.

Proof. We will prove (i). By (6.2), we have the following equalities

$$\begin{aligned}
S_j^* S_j =& (\tilde{S}_{j_0}\tilde{S}_{j_1}\tilde{S}_{j_2}\cdots\tilde{S}_{j_{m_j}})^*(\tilde{S}_{j_0}\tilde{S}_{j_1}\tilde{S}_{j_2}\cdots\tilde{S}_{j_{m_j}}) \\
=& \tilde{S}^*_{j_{m_j}}\tilde{S}^*_{j_{m_j-1}}\cdots\tilde{S}^*_{j_1}\tilde{S}^*_{j_0}\tilde{S}_{j_0}\tilde{S}_{j_1}\cdots\tilde{S}_{j_{m_j-1}}\tilde{S}_{j_{m_j}} \\
=& \tilde{S}^*_{j_{m_j}}\tilde{S}^*_{j_{m_j-1}}\cdots\tilde{S}^*_{j_1}\tilde{S}_{j_1}\tilde{S}^*_{j_1}\tilde{S}_{j_1}\cdots\tilde{S}_{j_{m_j-1}}\tilde{S}_{j_{m_j}} \\
=& \tilde{S}^*_{j_{m_j}}\tilde{S}^*_{j_{m_j-1}}\cdots\tilde{S}^*_{j_2}\tilde{S}^*_{j_1}\tilde{S}_{j_1}\tilde{S}_{j_2}\cdots\tilde{S}_{j_{m_j-1}}\tilde{S}_{j_{m_j}} \\
=& \tilde{S}^*_{j_{m_j}}\tilde{S}^*_{j_{m_j-1}}\cdots\tilde{S}^*_{j_2}\tilde{S}_{j_2}\cdots\tilde{S}_{j_{m_j-1}}\tilde{S}_{j_{m_j}}.
\end{aligned}$$

By continuing this procedure, the last term above goes to $\tilde{S}^*_{j_{m_j}}\tilde{S}_{j_{m_j}}$ so that $S_j^* S_j = \tilde{S}^*_{j_{m_j}}\tilde{S}_{j_{m_j}}$. We also have

$$
\begin{aligned}
S_j S_j^* =&(\tilde{S}_{j_0}\tilde{S}_{j_1}\tilde{S}_{j_2}\cdots\tilde{S}_{j_{m_j}})(\tilde{S}_{j_0}\tilde{S}_{j_1}\tilde{S}_{j_2}\cdots\tilde{S}_{j_{m_j}})^*\\
=&\tilde{S}_{j_0}\tilde{S}_{j_1}\cdots\tilde{S}_{j_{m_j-1}}\tilde{S}_{j_{m_j}}\tilde{S}^*_{j_{m_j}}\tilde{S}^*_{j_{m_j-1}}\cdots\tilde{S}^*_{j_1}\tilde{S}^*_{j_0}\\
=&\tilde{S}_{j_0}\tilde{S}_{j_1}\cdots\tilde{S}_{j_{m_j-1}}\tilde{S}^*_{j_{m_j-1}}\tilde{S}_{j_{m_j-1}}\tilde{S}^*_{j_{m_j-1}}\cdots\tilde{S}^*_{j_1}\tilde{S}^*_{j_0}\\
=&\tilde{S}_{j_0}\tilde{S}_{j_1}\cdots\tilde{S}_{j_{m_j-1}}\tilde{S}^*_{j_{m_j-1}}\cdots\tilde{S}^*_{j_1}\tilde{S}^*_{j_0}.
\end{aligned}
$$

By continuing this procedure, the last term above goes to $\tilde{S}_{j_0}\tilde{S}^*_{j_0}$. Hence we have

$$
S_j^* S_j = \tilde{S}^*_{j_{m_j}}\tilde{S}_{j_{m_j}}, \qquad S_j S_j^* = \tilde{S}_{j_0}\tilde{S}^*_{j_0}, \qquad j = 1, \dots, N \tag{6.7}
$$

so that

$$
\sum_{j=1}^N S_j S_j^* = P_A \quad \text{and} \quad S_j^* S_j = \sum_{k=1}^N A(j,k) S_k S_k^*, \qquad j = 1, \dots, N.
$$

Similarly we have

$$
T_j^* T_j = \tilde{T}^*_{j_0}\tilde{T}_{j_0}, \qquad T_j T_j^* = \tilde{T}_{j_{m_j}}\tilde{T}^*_{j_{m_j}}, \qquad j = 1, \dots, N \tag{6.8}
$$

so that

$$
\sum_{j=1}^N T_j T_j^* = P_{A^t} \quad \text{and} \quad T_j^* T_j = \sum_{k=1}^N A^t(j,k) T_k T_k^*, \qquad j = 1, \dots, N.
$$

As $S_j = P_A \tilde{S}_{j_0}\tilde{S}_{j_1}\tilde{S}_{j_2}\cdots\tilde{S}_{j_{m_j}} P_A$, one sees that $S_j \in P_A \mathcal{O}_{A_f} P_A$ for $j = 1, \dots, N$. Hence we have $C^*(S_1, \dots, S_N) \subset P_A \mathcal{O}_{A_f} P_A$. We will show the converse inclusion relation. We note that for $j_k, j'_{k'} \in \tilde{\Sigma}$, the equality $A_f(j_k, j'_{k'}) = 1$ holds if and only if either of the following two cases occurs

(1) $j' = j$ and $k' = k+1$

(2) $A(j, j') = 1$ and $k = m_j$, $k' = 0$.

For $\tilde{\mu} = (\tilde{\mu}_1, \dots, \tilde{\mu}_m)$, $\tilde{\nu} = (\tilde{\nu}_1, \dots, \tilde{\nu}_n) \in B_*(\bar{X}_{A_f})$, suppose that $P_A \tilde{S}_{\tilde{\mu}}\tilde{S}^*_{\tilde{\nu}} P_A \neq 0$. We first see that $\tilde{\mu}_1 = m(1)_0$ and $\tilde{\nu}_1 = n(1)_0$ for some $m(1), n(1) = 1, \dots, N$. By the conditions (1), (2), we know that

$$
\tilde{S}_{\tilde{\mu}} = S_{m(1)}\cdots S_{m(p)}\tilde{S}_{j_0}\cdots\tilde{S}_{j_k}, \quad \tilde{S}_{\tilde{\nu}} = S_{n(1)}\cdots S_{n(q)}\tilde{S}_{i_0}\cdots\tilde{S}_{i_l}
$$

for some $m(1), \dots, m(p), n(1), \dots, n(q), j, i \in \{1, \dots, N\}$ and $0 \le k \le m_j$, $0 \le l \le m_i$. Hence we may assume that $\tilde{\mu} = (j_0, j_1, \dots, j_k)$, $\tilde{\nu} = (i_0, i_1, \dots, i_l)$ with $0 \le k \le m_j$, $0 \le l \le m_i$, so that

$$
\tilde{S}_{\tilde{\mu}}\tilde{S}^*_{\tilde{\nu}} = \tilde{S}_{j_0}\tilde{S}_{j_1}\cdots\tilde{S}_{j_k}\tilde{S}^*_{i_l}\cdots\tilde{S}^*_{i_1}\tilde{S}^*_{i_0} \neq 0.
$$

Since

$$
\tilde{S}_{j_k} = \tilde{S}_{j_k}\tilde{S}^*_{j_k}\tilde{S}_{j_k} = \tilde{S}_{j_k}\tilde{S}_{j_{k+1}}\tilde{S}^*_{j_{k+1}}, \qquad \tilde{S}^*_{i_l} = \tilde{S}^*_{i_l}\tilde{S}_{i_l}\tilde{S}^*_{i_l} = \tilde{S}_{i_{l+1}}\tilde{S}^*_{i_{l+1}}\tilde{S}^*_{i_l},
$$

we have

$$\tilde{S}_{\tilde{\mu}}\tilde{S}_{\tilde{\nu}}^* = \tilde{S}_{j_0}\tilde{S}_{j_1}\cdots\tilde{S}_{j_{k-1}}\tilde{S}_{j_k}\tilde{S}_{j_{k+1}}\tilde{S}_{j_{k+1}}^*\tilde{S}_{i_{l+1}}\tilde{S}_{i_{l+1}}^*\tilde{S}_{i_l}^*\tilde{S}_{i_{l-1}}^*\cdots\tilde{S}_{i_1}^*\tilde{S}_{i_0}^*.$$

The condition $\tilde{S}_{\tilde{\mu}}\tilde{S}_{\tilde{\nu}}^* \neq 0$ leads $j_{k+1} = i_{l+1}$, so that we have $j_k = i_l, \cdots, j_1 = i_1,\ j_0 = i_0$. Hence

$$\begin{aligned}&\tilde{S}_{j_0}\tilde{S}_{j_1}\cdots\tilde{S}_{j_{k-1}}\tilde{S}_{j_k}\tilde{S}_{j_{k+1}}\tilde{S}_{j_{k+1}}^*\tilde{S}_{i_{l+1}}\tilde{S}_{i_{l+1}}^*\tilde{S}_{i_l}^*\tilde{S}_{i_{l-1}}^*\cdots\tilde{S}_{i_1}^*\tilde{S}_{i_0}^*\\ =&\tilde{S}_{j_0}\tilde{S}_{j_1}\cdots\tilde{S}_{j_{k-1}}\tilde{S}_{j_k}\tilde{S}_{j_{k+1}}\tilde{S}_{j_{k+1}}^*\tilde{S}_{j_{k+1}}\tilde{S}_{j_{k+1}}^*\tilde{S}_{j_k}^*\tilde{S}_{j_{k-1}}^*\cdots\tilde{S}_{j_1}^*\tilde{S}_{j_0}^*.\end{aligned}$$

As $\tilde{S}_{j_{k+1}}^*\tilde{S}_{j_{k+1}} = \tilde{S}_{j_{k+2}}\tilde{S}_{j_{k+2}}^*$, by continuing this procedure we know that

$$\tilde{S}_{\tilde{\mu}}\tilde{S}_{\tilde{\nu}}^* = \tilde{S}_{j_0}\tilde{S}_{j_1}\cdots\tilde{S}_{j_{k-1}}\tilde{S}_{j_k}\cdots\tilde{S}_{j_{m_j}}\tilde{S}_{j_{m_j}}^*\cdots\tilde{S}_{j_k}^*\tilde{S}_{j_{k-1}}^*\cdots\tilde{S}_{j_1}^*\tilde{S}_{j_0}^* = S_jS_j^*.$$

This shows that the element $P_A\tilde{S}_{\tilde{\mu}}\tilde{S}_{\tilde{\nu}}^*P_A$ belongs to $C^*(S_1,\dots,S_N)$, so that

$$P_A\mathcal{O}_{A_f}P_A = C^*(S_1,\dots,S_N)$$

and similarly

$$P_{A^t}\mathcal{O}_{(A_f)^t}P_{A^t} = C^*(T_1,\dots,T_N).$$

□

Recall that the vertex set $\mathcal{V}_f$ defined by $\cup_{j=1}^N\{j_0,j_1,\dots,j_{m_j}\}$ of the graph $\mathcal{G}_f$ is denoted by $\tilde{\Sigma}$. We set

$$\tilde{U}_{j_k} = \tilde{T}_{j_k}^*\otimes\tilde{S}_{j_k}, \qquad j_k\in\tilde{\Sigma}.$$

The partial isometries $\tilde{U}_{j_k}, j_k \in \tilde{\Sigma}$ belong to the Ruelle algebra $\mathcal{R}_{A_f}$ for the matrix A_f. The following lemma is direct from the identities (6.2), (6.5).

Lemma 6.2. *For $j = 1,\dots,N,\ k = 0,1,\dots,m_j$, we have*

$$\begin{aligned}(\tilde{U}_{j_k}\tilde{U}_{j_{k+1}}\cdots\tilde{U}_{j_{m_j}})(\tilde{U}_{j_k}\tilde{U}_{j_{k+1}}\cdots\tilde{U}_{j_{m_j}})^* =&\tilde{U}_{j_k}\tilde{U}_{j_k}^* = \tilde{T}_{j_k}^*\tilde{T}_{j_k}\otimes\tilde{S}_{j_k}\tilde{S}_{j_k}^*,\\ (\tilde{U}_{j_k}\tilde{U}_{j_{k+1}}\cdots\tilde{U}_{j_{m_j}})^*(\tilde{U}_{j_k}\tilde{U}_{j_{k+1}}\cdots\tilde{U}_{j_{m_j}}) =&\tilde{U}_{j_{m_j}}^*\tilde{U}_{j_{m_j}} = T_jT_j^*\otimes S_j^*S_j.\end{aligned}$$

Hence we have

$$\begin{aligned}E_{A_f} &= \sum_{j=1}^N\sum_{k=0}^{m_j}\tilde{T}_{j_k}\tilde{T}_{j_k}^*\otimes\tilde{S}_{j_k}^*\tilde{S}_{j_k} = \sum_{j=1}^N\sum_{k=0}^{m_j}\tilde{U}_{j_k}\tilde{U}_{j_k}^* \quad \text{in} \quad \mathcal{R}_{A_f},\\ E_A &= \sum_{j=1}^N T_jT_j^*\otimes S_j^*S_j = \sum_{j=1}^N\tilde{U}_{j_{m_j}}^*\tilde{U}_{j_{m_j}} \quad \text{in} \quad \mathcal{R}_A.\end{aligned}$$

Let H be the separable infinite dimensional Hilbert space $\ell^2(\mathbb{N})$. Take isometries $s_{j_k}, j_k\in\tilde{\Sigma}$ on H such that

$$\sum_{k=0}^{m_j}s_{j_k}s_{j_k}^* = 1_H, \qquad j = 1,\dots,N. \tag{6.9}$$

We define a partial isometry $\tilde{V}_f$ in the tensor product C^*-algebra $\mathcal{R}_{A_f} \otimes B(H)$ by setting

$$\tilde{V}_f = \sum_{j=1}^{N} \sum_{k=0}^{m_j} \tilde{U}_{j_k} \tilde{U}_{j_{k+1}} \cdots \tilde{U}_{j_{m_j}} \otimes s_{j_k}^*. \tag{6.10}$$

Lemma 6.3. *$\tilde{V}_f \tilde{V}_f^* = E_{A_f} \otimes 1_H$ and $\tilde{V}_f^* \tilde{V}_f = E_A \otimes 1_H$.*

Proof. By Lemma 6.2 we have the following equalities.

$$\begin{aligned}
\tilde{V}_f^* \tilde{V}_f &= \sum_{j=1}^{N} \sum_{k=0}^{m_j} (\tilde{U}_{j_k} \tilde{U}_{j_{k+1}} \cdots \tilde{U}_{j_{m_j}})^* (\tilde{U}_{j_k} \tilde{U}_{j_{k+1}} \cdots \tilde{U}_{j_{m_j}}) \otimes s_{j_k} s_{j_k}^* \\
&= \sum_{j=1}^{N} (T_j T_j^* \otimes S_j^* S_j) \otimes (\sum_{k=0}^{m_j} s_{j_k} s_{j_k}^*) \\
&= \sum_{j=1}^{N} T_j T_j^* \otimes S_j^* S_j \otimes 1_H = E_A \otimes 1_H
\end{aligned}$$

and

$$\begin{aligned}
\tilde{V}_f \tilde{V}_f^* &= \sum_{j=1}^{N} \sum_{k=0}^{m_j} (\tilde{U}_{j_k} \tilde{U}_{j_{k+1}} \cdots \tilde{U}_{j_{m_j}})(\tilde{U}_{j_k} \tilde{U}_{j_{k+1}} \cdots \tilde{U}_{j_{m_j}})^* \otimes s_{j_k}^* s_{j_k} \\
&= \sum_{j=1}^{N} \sum_{k=0}^{m_j} \tilde{U}_{j_k} \tilde{U}_{j_k}^* \otimes 1_H = E_{A_f} \otimes 1_H.
\end{aligned}$$

□

Recall that $\mathcal{K}$ denotes the C^*-algebra $\mathcal{K}(H)$ of compact operators on the Hilbert space H. As $\tilde{U}_{j_k}$ belongs to $\mathcal{R}_{A_f}$ and $\mathcal{R}_{A_f} \otimes B(H)$ is contained in the multiplier algebra $M(\mathcal{R}_{A_f} \otimes \mathcal{K})$ of $\mathcal{R}_{A_f} \otimes \mathcal{K}$, the partial isometry $\tilde{V}_f$ belongs to $M(\mathcal{R}_{A_f} \otimes \mathcal{K})$.

Lemma 6.4. $E_{A_f}(P_{A^t} \otimes P_A) = (P_{A^t} \otimes P_A) E_{A_f} = E_A$.

Proof. We have

$$\begin{aligned}
E_{A_f}(P_{A^t} \otimes P_A) &= \sum_{j=1}^{N} (\sum_{k=0}^{m_j} \tilde{T}_{j_k} \tilde{T}_{j_k}^* \otimes \tilde{S}_{j_k}^* \tilde{S}_{j_k}) \cdot (\sum_{j=1}^{N} \tilde{T}_{j_{m_j}} \tilde{T}_{j_{m_j}}^* \otimes \sum_{i=1}^{N} \tilde{S}_{i_0} \tilde{S}_{i_0}^*) \\
&= \sum_{j=1}^{N} \{\tilde{T}_{j_{m_j}} \tilde{T}_{j_{m_j}}^* \otimes (\sum_{i=1}^{N} \tilde{S}_{j_{m_j}}^* \tilde{S}_{j_{m_j}} \cdot \tilde{S}_{i_0} \tilde{S}_{i_0}^*)\}.
\end{aligned}$$

By the identities (6.7) and (6.8), we know

$$\tilde{S}_{j_{m_j}}^* \tilde{S}_{j_{m_j}} = S_j^* S_j, \quad \tilde{S}_{i_0} \tilde{S}_{i_0}^* = S_i S_i^*, \quad \tilde{T}_{j_{m_j}} \tilde{T}_{j_{m_j}}^* = T_j T_j^*,$$

so that we have

$$\begin{aligned} & E_{A_f}(P_{A^t} \otimes P_A) \\ =& \sum_{j=1}^{N} \{T_j T_j^* \otimes S_j^* S_j (\sum_{i=1}^{N} A(j,i) S_i S_i^*)\} = \sum_{j=1}^{N} (T_j T_j^* \otimes S_j^* S_j) = E_A, \end{aligned}$$

and hence $(P_{A^t} \otimes P_A) E_{A_f} = E_A$. □

Theorem 6.5. *Let A be an irreducible, non-permutation matrix with entries in $\{0,1\}$. For a continuous function $f : \bar{X}_A \longrightarrow \mathbb{N}$, there exists an isomorphism $\Phi_f : \widetilde{\mathcal{R}}_{A_f} \otimes \mathcal{K} \longrightarrow \widetilde{\mathcal{R}}_A \otimes \mathcal{K}$ of C^*-algebras such that*

$$\Phi_f \circ (\gamma_{(r,s)}^{A_f} \otimes \mathrm{id}) = (\gamma_{(r,s)}^{A,f} \otimes \mathrm{id}) \circ \Phi_f, \quad (r,s) \in \mathbb{T}^2.$$

Proof. By Corollary 5.5, we may assume that f is of the form $\sum_{j=1}^{N} f_j \chi_{U_j(0)}$. Define $\Phi_f : \widetilde{\mathcal{R}}_{A_f} \otimes \mathcal{K} \longrightarrow \widetilde{\mathcal{R}}_A \otimes \mathcal{K}$ by setting $\Phi_f(x \otimes K) = \tilde{V}_f^*(x \otimes K)\tilde{V}_f$ for $x \otimes K \in \widetilde{\mathcal{R}}_{A_f} \otimes \mathcal{K}$. We note that

$$\begin{gathered} \widetilde{\mathcal{R}}_{A_f} = E_{A_f}(\mathcal{O}_{(A_f)^t} \otimes \mathcal{O}_{A_f}) E_{A_f}, \qquad \widetilde{\mathcal{R}}_A = E_A(\mathcal{O}_{A^t} \otimes \mathcal{O}_A) E_A, \\ \tilde{V}_f \tilde{V}_f^* = E_{A_f} \otimes 1_H, \qquad \tilde{V}_f^* \tilde{V}_f = E_A \otimes 1_H, \\ (P_{A^t} \otimes P_A)(\mathcal{O}_{(A_f)^t} \otimes \mathcal{O}_{A_f})(P_{A^t} \otimes P_A) = \mathcal{O}_{A^t} \otimes \mathcal{O}_A. \end{gathered}$$

For $x \otimes K \in \widetilde{\mathcal{R}}_{A_f} \otimes \mathcal{K}$, we have

$$\begin{aligned} & \tilde{V}_f^*(x \otimes K)\tilde{V}_f \\ =& (E_A \otimes 1_H)\tilde{V}_f^*(x \otimes K)\tilde{V}_f(E_A \otimes 1_H) \\ =& (E_A \otimes 1_H)((P_{A^t} \otimes P_A) \otimes 1_H)\tilde{V}_f^*(x \otimes K)\tilde{V}_f((P_{A^t} \otimes P_A) \otimes 1_H)(E_A \otimes 1_H). \end{aligned}$$

As $\tilde{V}_f \in M(\widetilde{\mathcal{R}}_{A_f} \otimes \mathcal{K})$, the element $\tilde{V}_f^*(x \otimes K)\tilde{V}_f$ belongs to $\widetilde{\mathcal{R}}_{A_f} \otimes \mathcal{K}$ which is $(E_{A_f} \otimes 1_H)((\mathcal{O}_{(A_f)^t} \otimes \mathcal{O}_{A_f}) \otimes \mathcal{K})(E_{A_f} \otimes 1_H)$. By Lemma 6.4, we have

$$(E_A \otimes 1_H)((P_{A^t} \otimes P_A) \otimes 1_H)(E_{A_f} \otimes 1_H) = (E_A \otimes 1_H)((P_{A^t} \otimes P_A) \otimes 1_H)$$

so that $\tilde{V}_f^*(x \otimes K)\tilde{V}_f$ belongs to the algebra

$$(E_A \otimes 1_H)((P_{A^t} \otimes P_A) \otimes 1_H)((\mathcal{O}_{(A_f)^t} \otimes \mathcal{O}_{A_f}) \otimes \mathcal{K})((P_{A^t} \otimes P_A) \otimes 1_H)(E_A \otimes 1_H)$$

which is

$$E_A(\mathcal{O}_{A^t} \otimes \mathcal{O}_A)E_A \otimes \mathcal{K} = \widetilde{\mathcal{R}}_A \otimes \mathcal{K}.$$

Hence $\tilde{V}_f^*(x \otimes K)\tilde{V}_f$ belongs to the algebra $\widetilde{\mathcal{R}}_A \otimes \mathcal{K}$. This shows that the inclusion relation

$$\tilde{V}_f^*(\widetilde{\mathcal{R}}_{A_f} \otimes K)\tilde{V}_f \subset \widetilde{\mathcal{R}}_A \otimes \mathcal{K} \tag{6.11}$$

holds. Conversely, for $y \otimes K \in \widetilde{\mathcal{R}}_A \otimes \mathcal{K}$, we have

$$\tilde{V}_f(y \otimes K)\tilde{V}_f^* = (E_{A_f} \otimes 1_H)\tilde{V}_f(y \otimes K)\tilde{V}_f^*(E_{A_f} \otimes 1_H).$$

As $\tilde{V}_f \in M(\widetilde{\mathcal{R}}_{A_f} \otimes \mathcal{K})$ and $y \otimes K \in \widetilde{\mathcal{R}}_A \otimes \mathcal{K} \subset \widetilde{\mathcal{R}}_{A_f} \otimes \mathcal{K}$, the element $\tilde{V}_f(y \otimes K)\tilde{V}_f^*$ belongs to $\widetilde{\mathcal{R}}_{A_f} \otimes \mathcal{K}$ which is $(E_{A_f} \otimes 1_H)((\mathcal{O}_{(A_f)^t} \otimes \mathcal{O}_{A_f}) \otimes \mathcal{K})(E_{A_f} \otimes 1_H)$. Hence $\tilde{V}_f(x \otimes K)\tilde{V}_f^*$ belongs to the algebra

$$(E_{A_f} \otimes 1_H)(E_{A_f} \otimes 1_H)((\mathcal{O}_{(A_f)^t} \otimes \mathcal{O}_{A_f}) \otimes \mathcal{K})(E_{A_f} \otimes 1_H)$$

which is

$$E_{A_f}(\mathcal{O}_{(A_f)^t} \otimes \mathcal{O}_{A_f})E_{A_f} \otimes \mathcal{K} = \widetilde{\mathcal{R}}_{A_f} \otimes \mathcal{K}.$$

Hence $\tilde{V}_f(y \otimes K)\tilde{V}_f^*$ belongs to the algebra $\widetilde{\mathcal{R}}_{A_f} \otimes \mathcal{K}$. This shows that the inclusion relation

$$\tilde{V}_f(\widetilde{\mathcal{R}}_A \otimes K)\tilde{V}_f^* \subset \widetilde{\mathcal{R}}_{A_f} \otimes \mathcal{K} \tag{6.12}$$

holds. Since $\tilde{V}_f^*\tilde{V}_f = E_A \otimes 1_H$ and $E_A\widetilde{\mathcal{R}}_A E_A = \widetilde{\mathcal{R}}_A$, the inclusion relation (6.12) implies

$$\widetilde{\mathcal{R}}_A \otimes K \subset \tilde{V}_f^*(\widetilde{\mathcal{R}}_{A_f} \otimes \mathcal{K})\tilde{V}_f. \tag{6.13}$$

By (6.11) and (6.13), we have $\tilde{V}_f^*(\widetilde{\mathcal{R}}_{A_f} \otimes \mathcal{K})\tilde{V}_f = \widetilde{\mathcal{R}}_A \otimes \mathcal{K}$. Therefore we have an isomorphism $\Phi_f = \mathrm{Ad}(\tilde{V}_f^*) : \widetilde{\mathcal{R}}_{A_f} \otimes \mathcal{K} \longrightarrow \widetilde{\mathcal{R}}_A \otimes \mathcal{K}$.

Since $\gamma_{(r,s)}^{A_f}(\tilde{U}_{j_k}) = \tilde{U}_{j_k}$ for $j_k \in \tilde{\Sigma}$ by Lemma 5.3, we know the equality $(\gamma_{(r,s)}^{A_f} \otimes \mathrm{id})(\tilde{V}_f) = \tilde{V}_f$. For $x \otimes K \in \widetilde{\mathcal{R}}_{A_f} \otimes \mathcal{K}$, we have

$$\begin{aligned}(\Phi_f \circ(\gamma_{(r,s)}^{A_f} \otimes \mathrm{id}))(x \otimes K) =&\tilde{V}_f^*(\gamma_{(r,s)}^{A_f}(x) \otimes K)\tilde{V}_f \\ =&(\gamma_{(r,s)}^{A_f} \otimes \mathrm{id})(\tilde{V}_f^*(x \otimes K)\tilde{V}_f) \\ =&(\gamma_{(r,s)}^{A_f} \otimes \mathrm{id})(\Phi(x \otimes K)).\end{aligned}$$

Now

$$\begin{aligned}&\gamma_{(r,s)}^{A_f}(T_j^* \otimes S_k) \\ =&\gamma_{(r,s)}^{A_f}((\tilde{T}_{j_{m_j}} \cdots \tilde{T}_{j_1}\tilde{T}_{j_0})^* \otimes \tilde{S}_{k_0}\tilde{S}_{k_1} \cdots \tilde{S}_{k_{m_k}}) \\ =&\exp(2\pi\sqrt{-1}(f_k s - f_j r))((\tilde{T}_{j_{m_j}} \cdots \tilde{T}_{j_1}\tilde{T}_{j_0})^* \otimes (\tilde{S}_{k_0}\tilde{S}_{k_1} \cdots \tilde{S}_{k_{m_k}})) \\ =&\alpha_r^{A^t,f}(T_j^*) \otimes \alpha_s^{A,f}(S_k).\end{aligned}$$

Hence the restriction of $\gamma_{(r,s)}^{A_f} \otimes \mathrm{id}$ to the subalgebra $\widetilde{\mathcal{R}}_A \otimes \mathcal{K}$ coincides with $\gamma_{(r,s)}^{A,f} \otimes \mathrm{id}$ so that we conclude that

$$\Phi_f \circ(\gamma_{(r,s)}^{A_f} \otimes \mathrm{id}) = (\gamma_{(r,s)}^{A,f} \otimes \mathrm{id})\circ\Phi_f.$$

□

Remark 6.6. We note that it is not difficult to see that the above isomorphism $\Phi_f : \widetilde{\mathcal{R}}_{A_f} \otimes \mathcal{K} \longrightarrow \widetilde{\mathcal{R}}_A \otimes \mathcal{K}$ satisfies $\Phi_f(C(\bar{X}_{A_f}) \otimes \mathcal{C}) = C(\bar{X}_A) \otimes \mathcal{C}$, where $\mathcal{C}$ is the commutative C^*-algebra of diagonal operators on the Hilbert soace $H = \ell^2(\mathbb{N})$.

By using Proposition 4.6, we have the converse implication of Theorem 6.5 in the following way.

Theorem 6.7. *Let A, B be irreducible, non-permutation matrices with entries in $\{0,1\}$. Suppose that there exist a continuous function $f : \bar{X}_A \longrightarrow \mathbb{N}$ and an isomorphism $\Phi : \widetilde{\mathcal{R}}_B \otimes \mathcal{K} \longrightarrow \widetilde{\mathcal{R}}_A \otimes \mathcal{K}$ such that*

$$\Phi\circ(\gamma^B_{(r,s)} \otimes \mathrm{id}) = (\gamma^{A,f}_{(r,s)} \otimes \mathrm{id})\circ\Phi, \qquad (r,s) \in \mathbb{T}^2.$$

Then the two-sided topological Markov shifts $(\bar{X}_B, \bar{\sigma}_B)$ and $(\bar{X}_A, \bar{\sigma}_A)$ are flow equivalent.

Proof. By Theorem 6.5, there exists an isomorphism $\Phi_f : \widetilde{\mathcal{R}}_{A_f} \otimes \mathcal{K} \longrightarrow \widetilde{\mathcal{R}}_A \otimes \mathcal{K}$ of C^*-algebras such that

$$\Phi_f\circ(\gamma^{A_f}_{(r,s)} \otimes \mathrm{id}) = (\gamma^{A,f}_{(r,s)} \otimes \mathrm{id})\circ\Phi_f, \quad (r,s) \in \mathbb{T}^2.$$

We define the isomorphism $\Phi_\circ = \Phi_f^{-1}\circ\Phi : \widetilde{\mathcal{R}}_B \otimes \mathcal{K} \longrightarrow \widetilde{\mathcal{R}}_{A_f} \otimes \mathcal{K}$ which satisfies

$$\Phi_\circ\circ(\gamma^B_{(r,s)} \otimes \mathrm{id}) = (\gamma^{A_f}_{(r,s)} \otimes \mathrm{id})\circ\Phi_\circ, \quad (r,s) \in \mathbb{T}^2.$$

By Proposition 4.6, we know that $(\bar{X}_B, \bar{\sigma}_B)$ and $(\bar{X}_{A_f}, \bar{\sigma}_{A_f})$ are flow equivalent. Since $(\bar{X}_{A_f}, \bar{\sigma}_{A_f})$ is a discrete suspension of $(\bar{X}_A, \bar{\sigma}_A)$, they are flow equivalent, so $(\bar{X}_B, \bar{\sigma}_B)$ and $(\bar{X}_A, \bar{\sigma}_A)$ are flow equivalent. □

Therefore we have a characterization of flow equivalence in terms of the C^*-algebras $\widetilde{\mathcal{R}}_A$ with their gauge actions with potentials.

Theorem 6.8. *Let A, B be irreducible, non-permutation matrices with entries in $\{0,1\}$. Two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent if and only if there exist an irreducible non-permutation matrix C with entries in $\{0,1\}$ and continuous functions $f_A, f_B : \bar{X}_C \longrightarrow \mathbb{N}$ such that there exist isomorphisms $\Phi_A : \widetilde{\mathcal{R}}_A \otimes \mathcal{K} \longrightarrow \widetilde{\mathcal{R}}_C \otimes \mathcal{K}$ and $\Phi_B : \widetilde{\mathcal{R}}_B \otimes \mathcal{K} \longrightarrow \widetilde{\mathcal{R}}_C \otimes \mathcal{K}$ satisfying*

$$\Phi_A\circ(\gamma^A_{(r,s)} \otimes \mathrm{id}) = (\gamma^{C,f_A}_{(r,s)} \otimes \mathrm{id})\circ\Phi_A \tag{6.14}$$

and

$$\Phi_B\circ(\gamma^B_{(r,s)} \otimes \mathrm{id}) = (\gamma^{C,f_B}_{(r,s)} \otimes \mathrm{id})\circ\Phi_B. \tag{6.15}$$

Proof. Suppose the two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent. By Parry–Sullivan [22], there exist an irreducible non-permutation matrix C with entries in $\{0,1\}$ and continuous functions $f_A, f_B : \bar{X}_C \longrightarrow \mathbb{N}$ such that $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_{C_{f_A}}, \bar{\sigma}_{C_{f_A}})$ are topologically conjugate, and $(\bar{X}_B, \bar{\sigma}_B)$ and $(\bar{X}_{C_{f_B}}, \bar{\sigma}_{C_{f_B}})$ are topologically conjugate. By [20, Theorem 1.1] and Theorem 6.5, we have isomorphisms $\Phi_A : \widetilde{\mathcal{R}}_A \otimes \mathcal{K} \longrightarrow \widetilde{\mathcal{R}}_C \otimes \mathcal{K}$ and $\Phi_B : \widetilde{\mathcal{R}}_B \otimes \mathcal{K} \longrightarrow \widetilde{\mathcal{R}}_C \otimes \mathcal{K}$ of C^*-algebras satisfying (6.14) and (6.15), respectively.

The converse implication immediately follows from Theorem 6.7. □

As a corollary we have

Corollary 6.9. *Let A, B be irreducible, non-permutation matrices with entries in $\{0,1\}$. Two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent if and only if there exist continuous functions $f_A : \bar{X}_A \longrightarrow \mathbb{N}$ and $f_B : \bar{X}_B \longrightarrow \mathbb{N}$ such that there exists an isomorphism $\Phi : \widetilde{\mathcal{R}}_A \otimes \mathcal{K} \longrightarrow \widetilde{\mathcal{R}}_B \otimes \mathcal{K}$ of C^*-algebras satisfying*

$$\Phi \circ (\gamma^{A,f_A}_{(r,s)} \otimes \mathrm{id}) = (\gamma^{B,f_B}_{(r,s)} \otimes \mathrm{id}) \circ \Phi, \qquad (r,s) \in \mathbb{T}^2.$$

Proof. By Parry–Sullivan [22], $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent if and only if there exist continuous functions $f_A : \bar{X}_A \longrightarrow \mathbb{N}$ and $f_B : \bar{X}_B \longrightarrow \mathbb{N}$ such that $(\bar{X}_{A_{f_A}}, \bar{\sigma}_{A_{f_A}})$ and $(\bar{X}_{B_{f_B}}, \bar{\sigma}_{B_{f_B}})$ are topologically conjugate. The assertion follows from [20, Theorem 1.1] and Theorem 6.5, Theorem 6.7. □

7. Flow equivalence and topological conjugacy

In [20, Theorem 1.1], it was proved that two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate if and only if there exists an isomorphism $\Phi : \widetilde{\mathcal{R}}_A \longrightarrow \widetilde{\mathcal{R}}_B$ of C^*-algebras such that $\Phi(C(\bar{X}_A)) = C(\bar{X}_B)$ and $\Phi \circ \gamma^A_{(r,s)} = \gamma^B_{(r,s)} \circ \Phi, (r,s) \in \mathbb{T}^2$. Although topological conjugacy is a special case of flow equivalence, Theorem 6.8 and Corollary 6.9 do not refer to the case of topological conjugacy. In this final section, we reformulate both Theorem 6.8 and Corollary 6.9 to give characterizations of flow equivalence in terms of C^*-algebras that simultaneously include characterzations of topological conjugacy as a special case. They appear as Theorem 7.2 and Theorem 7.3.

We keep the assumption that A is an irreducible, non-permutation matrix with entries in $\{0,1\}$, and $f : \bar{X}_A \longrightarrow \mathbb{N}$ is a continuous function such that $f = \sum_{j=1}^N f_j \chi_{U_j(0)}$ for some positive integers f_j. Let $e_n, n \in \mathbb{N}$ be a sequence of vectors of complete orthonormal basis of the separable infinite dimensional Hilbert space $H = \ell^2(\mathbb{N})$. We fix $j = 1, \dots, N$. For $k = 0, 1, \dots, m_j$, where $m_j = f_j - 1$, we set

$$\mathbb{N}_k = \{n \in \mathbb{N} \mid n \equiv k \,(\mathrm{mod}\ f_j)\}$$

so that we have a disjoint union $\mathbb{N} = \mathbb{N}_0 \cup \mathbb{N}_1 \cup \cdots \cup \mathbb{N}_{m_j}$. We write

$$\mathbb{N}_k = \{1_k, 2_k, 3_k, \dots\} \quad \text{where } 1_k < 2_k < 3_k < \dots .$$

Define an isometry s_{j_k} on H by setting

$$s_{j_k} e_n = e_{n_k}, \qquad n \in \mathbb{N},\ k = 0, 1, \dots, m_j.$$

The family $\{s_{j_k}\}_{k=0}^{m_j}$ satisfy (6.9). We may construct the operator $\tilde{V}_f$ from them by the formula (6.10). As in the proof of Theorem 6.5, define the isomorphism $\Phi_f : \widetilde{\mathcal{R}}_{A_f} \otimes \mathcal{K} \longrightarrow \widetilde{\mathcal{R}}_A \otimes \mathcal{K}$ of C^*-algebras by setting $\Phi_f(x \otimes K) = \tilde{V}_f^*(x \otimes k)\tilde{V}_f$

for $x \otimes K \in \widetilde{\mathcal{R}}_{A_f} \otimes \mathcal{K}$. Let p_n be the projection on H of rank one onto the vector e_n. As $E_{A_f} \otimes 1_H = \tilde{V}_f \tilde{V}_f^*$, we have

$$\begin{aligned}\tilde{V}_f^*(E_{A_f} \otimes p_1)\tilde{V}_f &= \sum_{j=1}^{N}\sum_{k=0}^{m_j}(\tilde{U}_{j_k}\tilde{U}_{j_{k+1}}\cdots\tilde{U}_{j_{m_j}})^*(\tilde{U}_{j_k}\tilde{U}_{j_{k+1}}\cdots\tilde{U}_{j_{m_j}}) \otimes s_{j_k} p_1 s_{j_k}^* \\ &= \sum_{j=1}^{N}\{(T_jT_j^* \otimes S_j^*S_j) \otimes (\sum_{k=0}^{m_j} s_{j_k} p_1 s_{j_k}^*)\}.\end{aligned}$$

Since $s_{j_k} p_1 s_{j_k}^* = p_{1_k}$, we have

$$\tilde{V}_f^*(E_{A_f} \otimes p_1)\tilde{V}_f = \sum_{j=1}^{N}\{(T_jT_j^* \otimes S_j^*S_j) \otimes (\sum_{k=0}^{m_j} p_{1_k})\} \in C(\bar{X}_A) \otimes \mathcal{K}.$$

Hence we obtain the following formula in the K_0-group $K_0(C(\bar{X}_A))$:

$$[\tilde{V}_f^*(E_{A_f} \otimes p_1)\tilde{V}_f] = \sum_{j=1}^{N} f_j [T_jT_j^* \otimes S_j^*S_j] \quad \text{in } K_0(C(\bar{X}_A)).$$

Let us denote by $C(\bar{X}_A, \mathbb{Z})$ the abelian group of $\mathbb{Z}$-valued continuous functions on $\bar{X}_A$. Under the natural identification between $K_0(C(\bar{X}_A))$ and $C(\bar{X}_A, \mathbb{Z})$, one may regard the class $\sum_{j=1}^{N} f_j [T_jT_j^* \otimes S_j^*S_j]$ in $K_0(C(\bar{X}_A))$ as the function f in $C(\bar{X}_A, \mathbb{Z})$. We write it as $f([E_A])$ in $K_0(C(\bar{X}_A))$. For a general continuous function $f \in C(\bar{X}_A, \mathbb{Z})$, we write it as $f([E_A])$ as an element of $K_0(C(\bar{X}_A))$ under the identification between $K_0(C(\bar{X}_A))$ and $C(\bar{X}_A, \mathbb{Z})$. Recall that $\mathcal{C}$ denotes the commutative C^*-algebra on $\ell^2(\mathbb{N})$ consisting of diagonal operators with respect to the basis $\{e_n\}_{n\in\mathbb{N}}$. The construction of the operator $\tilde{V}_f$ tells us that the equality

$$\tilde{V}_f^*(C(\bar{X}_{A_f}) \otimes \mathcal{C})\tilde{V}_f = C(\bar{X}_A) \otimes \mathcal{C}$$

holds. Therefore we obtain the following proposition that is a reformulation of Theorem 6.5.

Proposition 7.1. *Let A be an irreducible, non-permutation matrix with entries in $\{0,1\}$. Then there exists an isomorphism $\Phi_f : \widetilde{\mathcal{R}}_{A_f} \otimes \mathcal{K} \longrightarrow \widetilde{\mathcal{R}}_A \otimes \mathcal{K}$ of C^*-algebras such that*

$$\Phi_f(C(\bar{X}_{A_f}) \otimes \mathcal{C}) = C(\bar{X}_A) \otimes \mathcal{C}, \tag{7.1}$$

$$\Phi_f \circ (\gamma_{(r,s)}^{A_f} \otimes \mathrm{id}) = (\gamma_{(r,s)}^{A,f} \otimes \mathrm{id}) \circ \Phi_f, \quad (r,s) \in \mathbb{T}^2, \tag{7.2}$$

$$\Phi_{f*}([E_{A_f}]) = f([E_A]) \quad \textit{in } K_0(C(\bar{X}_A)). \tag{7.3}$$

By virtue of Proposition 7.1, we may finally state two theorems. The first one is a reformulation of Theorem 6.8 in the following way.

Theorem 7.2. *Let A, B be irreducible, non-permutation matrices with entries in $\{0,1\}$. Two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent if and only if there exist an irreducible non-permutation matrix C with entries in $\{0,1\}$ and continuous functions $f_A, f_B : \bar{X}_C \longrightarrow \mathbb{N}$ such that $\widetilde{\mathcal{R}}_A \otimes \mathcal{K}$ and $\widetilde{\mathcal{R}}_B \otimes \mathcal{K}$ are isomorphic to $\widetilde{\mathcal{R}}_C \otimes \mathcal{K}$ via isomorphisms Φ_A and Φ_B satisfying*

$$\begin{cases} \Phi_A \circ (\gamma^A_{(r,s)} \otimes \mathrm{id}) & = (\gamma^{C,f_A}_{(r,s)} \otimes \mathrm{id}) \circ \Phi_A, \\ \Phi_A(C(\bar{X}_A) \otimes \mathcal{C}) & = C(\bar{X}_C) \otimes \mathcal{C}, \\ \qquad \Phi_{A*}([E_A]) & = f_A([E_C]) \quad \text{in } K_0(C(\bar{X}_C)), \end{cases} \tag{7.4}$$

$$\begin{cases} \Phi_B \circ (\gamma^B_{(r,s)} \otimes \mathrm{id}) & = (\gamma^{C,f_B}_{(r,s)} \otimes \mathrm{id}) \circ \Phi_B, \\ \Phi_B(C(\bar{X}_B) \otimes \mathcal{C}) & = C(\bar{X}_C) \otimes \mathcal{C}, \\ \qquad \Phi_{B*}([E_B]) & = f_B([E_C]) \quad \text{in } K_0(C(\bar{X}_C)). \end{cases} \tag{7.5}$$

In particular, $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate if and only if the equalities (7.4) *and* (7.5) *hold for $f_A \equiv f_B \equiv 1$.*

Proof. Assume the two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent. By Parry–Sullivan [22], there exist an irreducible non-permutation matrix C with entries in $\{0,1\}$ and continuous functions $f_A, f_B : \bar{X}_C \longrightarrow \mathbb{N}$ such that $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_{C_{f_A}}, \bar{\sigma}_{C_{f_A}})$ are topologically conjugate, and $(\bar{X}_B, \bar{\sigma}_B)$ and $(\bar{X}_{C_{f_B}}, \bar{\sigma}_{C_{f_B}})$ are topologically conjugate. By [20, Theorem 1.1] there exist isomorphisms $\mathit{\Phi}_A : \widetilde{\mathcal{R}}_A \longrightarrow \widetilde{\mathcal{R}}_{C_{f_A}}$ and $\mathit{\Phi}_B : \widetilde{\mathcal{R}}_B \longrightarrow \widetilde{\mathcal{R}}_{C_{f_B}}$ of C^*-algebras such that $\mathit{\Phi}_A \circ \gamma^A_{(r,s)} = \gamma^{C_{f_A}}_{(r,s)} \circ \mathit{\Phi}_A$, $\mathit{\Phi}_B \circ \gamma^B_{(r,s)} = \gamma^{C_{f_B}}_{(r,s)} \circ \mathit{\Phi}_B$ and $\mathit{\Phi}_A(C(\bar{X}_A)) = C(\bar{X}_{C_{f_A}})$, $\mathit{\Phi}_B(C(\bar{X}_B)) = C(\bar{X}_{C_{f_B}})$. By Proposition 7.1, there exist isomorphisms $\Phi_{f_A} : \widetilde{\mathcal{R}}_{C_{f_A}} \otimes \mathcal{K} \longrightarrow \widetilde{\mathcal{R}}_C \otimes \mathcal{K}$ and $\Phi_{f_B} : \widetilde{\mathcal{R}}_{C_{f_B}} \otimes \mathcal{K} \longrightarrow \widetilde{\mathcal{R}}_C \otimes \mathcal{K}$ satsifying the equalities sated as (7.1), (7.2) and (7.3). By putting

$$\Phi_A := \Phi_{f_A} \circ (\mathit{\Phi}_A \otimes \mathrm{id}) : \widetilde{\mathcal{R}}_A \otimes \mathcal{K} \longrightarrow \widetilde{\mathcal{R}}_C \otimes \mathcal{K},$$
$$\Phi_B := \Phi_{f_B} \circ (\mathit{\Phi}_B \otimes \mathrm{id}) : \widetilde{\mathcal{R}}_B \otimes \mathcal{K} \longrightarrow \widetilde{\mathcal{R}}_C \otimes \mathcal{K},$$

we know the isomorphisms Φ_A and Φ_B satisfy the desired properties. The converse implication immediately follows from Theorem 6.8.

It remains to show that if the equalities (7.4) and (7.5) hold for $f_A \equiv f_B \equiv 1$, then $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate. Let $e_n, n \in \mathbb{N}$ be a complete orthonormal system of the Hilbert space $\ell^2(\mathbb{N})$, and p_1 be the projection on $\ell^2(\mathbb{N})$ of rank one onto the vector e_1. By (7.4) for $f_A \equiv 1$, we have $[\Phi_A(E_A \otimes p_1)] = [E_C \otimes p_1]$ in $K_0(C(\bar{X}_C))$. Hence we may take a prtial isometry $V_A \in C(\bar{X}_C) \otimes \mathcal{K}$ such that $V_A^* V_A = E_C \otimes p_1, V_A^* V_A = \Phi_A(E_A \otimes p_1)$. Put $\Psi_A = \mathrm{Ad}(V_A) \circ \Phi_A$. Since we see

$$\Psi_A(E_A \otimes p_1) = V_A \Phi_A(E_A \otimes p_1) V_A^* = V_A V_A^* = E_C \otimes p_1,$$

we have an isomorphism $\Psi_A : \widetilde{\mathcal{R}}_A \otimes \mathbb{C}p_1 \longrightarrow \widetilde{\mathcal{R}}_C \otimes \mathbb{C}p_1$. For any $a \in C(\bar{X}_A)$, we have

$$\begin{aligned}\Psi_A(a \otimes p_1) &= V_A\Phi_A((E_A \otimes p_1)(a \otimes p_1)(E_A \otimes p_1))V_A^* \\ &= V_AV_A^*V_A\Phi_A(a \otimes p_1)V_A^*V_AV_A^* \\ &= (E_C \otimes p_1)V_A\Phi_A(a \otimes p_1)V_A^*(E_C \otimes p_1),\end{aligned}$$

so that we know that $\Psi_A(a \otimes p_1) \in C(\bar{X}_C) \otimes p_1$. We thus have $\Psi_A(C(\bar{X}_A) \otimes \mathbb{C}p_1) = C(\bar{X}_C)\otimes\mathbb{C}p_1$. Since V_A belongs to $C(\bar{X}_C)\otimes\mathcal{K}$, one knows that $(\gamma^{C,f_A}_{(r,s)} \otimes \mathrm{id})(V_A) = V_A$ so that

$$\Psi_A\circ(\gamma^{A}_{(r,s)} \otimes \mathrm{id}) = (\gamma^{C,f_A}_{(r,s)} \otimes \mathrm{id})\circ\Psi_A$$

for $f_A \equiv 1$. By restricting Ψ_A to $\widetilde{\mathcal{R}}_A \otimes \mathbb{C}p_1$, we have an isomorphism $\mathit{\Phi}_A : \widetilde{\mathcal{R}}_A \longrightarrow \widetilde{\mathcal{R}}_C$ such that $\mathit{\Phi}_A\circ\gamma^A_{(r,s)} = \gamma^C_{(r,s)}\circ\mathit{\Phi}_A$ and $\mathit{\Phi}_A(C(\bar{X}_A)) = C(\bar{X}_C)$. This shows that $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_C, \bar{\sigma}_C)$ are topologically conjugate by [20]. We similarly know that $(\bar{X}_B, \bar{\sigma}_B)$ and $(\bar{X}_C, \bar{\sigma}_C)$ are topologically conjugate, so that $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate.

Converse implication under the condition that $f_A \equiv 1$ and $f_B \equiv 1$ is obvious by [20, Theorem 1.1]. □

The second one is a reformulation of Corollary 6.9 in the following way.

Theorem 7.3. *Let A, B be irreducible, non-permutation matrices with entries in $\{0,1\}$. Two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent if and only if there exist continuous functions $f_A : \bar{X}_A \longrightarrow \mathbb{N}$ and $f_B : \bar{X}_B \longrightarrow \mathbb{N}$ such that there exists an isomorphism $\Phi : \widetilde{\mathcal{R}}_A \otimes \mathcal{K} \longrightarrow \widetilde{\mathcal{R}}_B \otimes \mathcal{K}$ of C^*-algebras satisfying*

$$\Phi\circ(\gamma^{A,f_A}_{(r,s)} \otimes \mathrm{id}) = (\gamma^{B,f_B}_{(r,s)} \otimes \mathrm{id})\circ\Phi, \qquad (r,s) \in \mathbb{T}^2, \tag{7.6}$$

$$\Phi(C(\bar{X}_A) \otimes \mathcal{C}) = C(\bar{X}_B) \otimes \mathcal{C}, \tag{7.7}$$

$$\Phi_*(f_A([E_A])) = f_B([E_B]) \quad \text{in } K_0(C(\bar{X}_B)). \tag{7.8}$$

In particular, $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate if and only if the equalities (7.6), (7.7) *and* (7.8) *hold for $f_A \equiv f_B \equiv 1$.*

Proof. It suffices to show the only if part. Assume that $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent. By Parry–Sullivan [22], there exist continuous functions $f_A : \bar{X}_A \longrightarrow \mathbb{N}$ and $f_B : \bar{X}_B \longrightarrow \mathbb{N}$ such that $(\bar{X}_{A_{f_A}}, \bar{\sigma}_{A_{f_A}})$ and $(\bar{X}_{B_{f_B}}, \bar{\sigma}_{B_{f_B}})$ are topologically conjugate. By [20], there exists an isomorphism $\mathit{\Phi}_\circ : \widetilde{\mathcal{R}}_{A_{f_A}} \longrightarrow \widetilde{\mathcal{R}}_{B_{f_B}}$ of C^*-algebras such that $\mathit{\Phi}_\circ(C(\bar{X}_{A_{f_A}})) = C(\bar{X}_{B_{f_B}})$ and

$$\mathit{\Phi}_\circ\circ\gamma^{A_{f_A}}_{(r,s)} = \gamma^{B_{f_B}}_{(r,s)}\circ\mathit{\Phi}_\circ, \qquad (r,s) \in \mathbb{T}^2.$$

By Theorem 6.5 and Remark 6.6 with Proposition 7.1, we have an isomorphism $\Phi_{f_A} : \widetilde{\mathcal{R}}_{A_{f_A}} \otimes \mathcal{K} \longrightarrow \widetilde{\mathcal{R}}_A \otimes \mathcal{K}$ of C^*-algebras satisfying

$$\Phi_{f_A} \circ (\gamma^{A_{f_A}}_{(r,s)} \otimes \mathrm{id}) = (\gamma^{A,f_A}_{(r,s)} \otimes \mathrm{id}) \circ \Phi_{f_A}, \qquad (r,s) \in \mathbb{T}^2,$$
$$\Phi_{f_A}(C(\bar{X}_{A_{f_A}}) \otimes \mathcal{C}) = C(\bar{X}_A) \otimes \mathcal{C},$$
$$\Phi_{f_A *}([E_{A_{f_A}}]) = f_A([E_A]) \quad \text{in } K_0(C(\bar{X}_A)).$$

Similarly we have an isomorphism $\Phi_{f_B} : \widetilde{\mathcal{R}}_{B_{f_B}} \otimes \mathcal{K} \longrightarrow \widetilde{\mathcal{R}}_B \otimes \mathcal{K}$ of C^*-algebras satisfying

$$\Phi_{f_B} \circ (\gamma^{B_{f_B}}_{(r,s)} \otimes \mathrm{id}) = (\gamma^{B,f_B}_{(r,s)} \otimes \mathrm{id}) \circ \Phi_{f_B}, \qquad (r,s) \in \mathbb{T}^2,$$
$$\Phi_{f_B}(C(\bar{X}_{B_{f_B}}) \otimes \mathcal{C}) = C(\bar{X}_B) \otimes \mathcal{C},$$
$$\Phi_{f_B *}([E_{B_{f_B}}]) = f_B([E_B]) \quad \text{in } K_0(C(\bar{X}_B)).$$

Put $\Phi := \Phi_{f_B} \circ (\mathit{\Phi}_\circ \otimes \mathrm{id}) \circ \Phi_{f_A}^{-1} : \widetilde{\mathcal{R}}_A \otimes \mathcal{K} \longrightarrow \widetilde{\mathcal{R}}_B \otimes \mathcal{K}$. We then see that the equality $\Phi(C(\bar{X}_A) \otimes \mathcal{C}) = C(\bar{X}_B) \otimes \mathcal{C}$ holds. Since $E_{A_{f_A}}, E_{B_{f_B}}$ are the units of $\widetilde{\mathcal{R}}_{B_{f_A}}, \widetilde{\mathcal{R}}_{B_{f_B}}$, respectively, we know that $(\mathit{\Phi}_\circ \otimes \mathrm{id})(E_{A_{f_A}} \otimes p_1) = E_{B_{f_B}} \otimes p_1$, where p_1 is the projection on $\ell^2(\mathbb{N})$ of rank one as in the proof of the preceding theorem. We thus have

$$\Phi_*(f_A[E_A]) = \Phi_{f_B *}((\mathit{\Phi}_{\circ *} \otimes \mathrm{id})([E_{f_A}} \otimes p_1])) = \Phi_{f_B *}([E_{f_B} \otimes p_1]) = f_B([E_B]).$$

Suppose next that $f_A \equiv 1$ and $f_B \equiv 1$. The condition (7.8) goes to $\Phi_*([E_A]) = [E_B]$ in $K_0(C(\bar{X}_B))$. Hence we may take a partial isometry $V \in C(\bar{X}_B) \otimes \mathcal{K}$ such that $\Phi(E_A \otimes p_1) = V^*V$ and $E_B \otimes p_1 = VV^*$. By a manner similar to the proof of Theorem 7.2, we obtain an isomorphism $\mathit{\Phi} : \widetilde{\mathcal{R}}_A \longrightarrow \widetilde{\mathcal{R}}_B$ such that $\mathit{\Phi}(C(\bar{X}_A)) = C(\bar{X}_B)$ and $\mathit{\Phi} \circ \gamma^{A,f_A}_{(r,s)} = \gamma^{B,f_B}_{(r,s)} \circ \mathit{\Phi}$. Now $f_A \equiv 1$, $f_B \equiv 1$, we conclude that $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate by [20, Theorem 1.1].

The converse implication under the condition $f_A \equiv 1$, $f_B \equiv 1$ is obvious. $\square$

References

[1] BOWEN, RUFUS. Equilibrium states and the ergodic theory of Anosov diffeomorphisms. Lecture Notes in Mathematics, 470. *Springer-Verlag, Berlin-New York*, 1975. i+108 pp. MR442989, Zbl 0308.28010. 1376

[2] BOWEN, RUFUS; FRANKS, JOHN. Homology for zero-dimensional nonwandering sets. *Ann. of Math.* (2) **106** (1977), no. 1, 73–92. MR0458492, Zbl 0375.58018, doi: 10.2307/1971159. 1375, 1376, 1393

[3] BOYLE, MIKE; HANDELMAN, DAVID. Orbit equivalence, flow equivalence and ordered cohomology. *Israel J. Math.* **95** (1996), 169–210. MR1418293, Zbl 0871.58071, doi: 10.1007/BF02761039. 1375

[4] BROWN, LAWRENCE G. Stable isomorphism of hereditary subalgebras of C^*-algebras. *Pacific J. Math.* **71** (1977), no. 2, 335–348. MR0454645, Zbl 0362.46042, doi: 10.2140/pjm.1977.71.335. 1386, 1387

[5] CARLSEN, TOKE MEIER; EILERS, SØREN; ORTEGA, EDUARD; RESTORFF, GUNNAR. Flow equivalence and orbit equivalence for shifts of finite type and isomorphism of their

groupoids. *J. Math. Anal. Appl.* **469** (2019), no. 2, 1088–1110. MR3860463, Zbl 1416.37020, arXiv:1610.09945, doi: 10.1016/j.jmaa.2018.09.056. 1376, 1382

[6] Cuntz, Joachim. A class of C^*-algebras and topological Markov chains. II. Reducible chains and the Ext-functor for C^*-algebras. *Invent. Math.* **63** (1981), no. 1, 25–40. MR0608527, Zbl 0461.46047, doi: 10.1007/BF01389192. 1380, 1387, 1390

[7] Cuntz, Joachim; Krieger, Wolfgang. A class of C^*-algebras and topological Markov chains. *Invent. Math.* **56** (1980), no. 3, 251–268. MR0561974, Zbl 0434.46045, doi: 10.1007/BF01390048. 1376, 1379, 1380, 1382, 1387, 1390

[8] Effros, Edward G. Dimensions and C^*-algebras. CBMS Regional Conference Series in Mathematics, 46. *Conference Board of the Mathematical Sciences, Washington, D.C.*, 1981. v+74 pp. ISBN: 0-8218-1697-7. MR0623762, Zbl 0475.46050. 1382, 1387

[9] Franks, John. Flow equivalence of subshifts of finite type. *Ergodic Theory Dynam. Systems* **4** (1984), no. 1, 53–66. MR0758893, Zbl 0555.54026, doi: 10.1017/S0143385700002261. 1375, 1393, 1394

[10] Holton, Charles G. The Rohlin property for shifts of finite type. *J. Funct. Anal.* **229** (2005), no. 2, 277–299. MR2182590, Zbl 1100.46034, arXiv:math/0411394, doi: 10.1016/j.jfa.2005.05.002. 1384

[11] Kaminker, Jerome; Putnam, Ian. K-theoretic duality of shifts of finite type. *Comm. Math. Phys.* **187** (1997), no. 3, 509–522. MR1468312, Zbl 0887.19002, arXiv:funct-an/9612010, doi: 10.1007/s002200050147. 1388

[12] Kaminker, Jerome; Putnam, Ian; Spielberg, Jack. Operator algebras and hyperbolic dynamics. *Operator algebras and quantum field theory* (Rome, 1996), 525–532. *Int. Press, Cambridge, MA*, 1997. MR1491139, Zbl 0906.46052. 1376

[13] Killough, D. Brady; Putnam, Ian F. Ring and module structures on dimension groups associated with a shift of finite type. *Ergodic Theory Dynam. Systems* **32** (2012), no. 4, 1370–1399. MR2955318, Zbl 1278.37018, arXiv:1104.1154, doi: 10.1017/S0143385712000272. 1386

[14] Krieger, Wolfgang. On topological Markov chains. *Dynamical systems, Vol. II Warsaw*, 193–196. Astérisque, 50. *Soc. Math. France, Paris*, 1977. MR0500874, Zbl 0393.60070. 1382, 1383

[15] Krieger, Wolfgang. On a dimension for a class of homeomorphism groups. *Math. Ann.* **252** (1979/80), no. 2, 87–95. MR0593623, Zbl 0472.54028, doi: 10.1007/BF01420115. 1382, 1383, 1387

[16] Krieger, Wolfgang. On dimension functions and topological Markov chains. *Invent. Math.* **56** (1980), no. 3, 239–250. MR0561973, Zbl 0431.54024, doi: 10.1007/BF01390047. 1380, 1382, 1383, 1384, 1387, 1390

[17] Lind, Douglas; Marcus, Brian. An introduction to symbolic dynamics and coding. *Cambridge University Press, Cambridge*, 1995. xvi+495 pp. ISBN: 0-521-55124-2; 0-521-55900-6. MR1369092, Zbl 1106.37301, doi: 10.1017/CBO9780511626302. 1391, 1400

[18] Matsumoto, Kengo. K-theory for C^*-algebras associated with subshifts. *Math. Scand.* **82** (1998), no. 2, 237–255. MR1646513, Zbl 0921.46078, doi: 10.7146/math.scand.a-13835. 1380, 1386, 1388

[19] Matsumoto, Kengo. Asymptotic continuous orbit equivalence of Smale spaces and Ruelle algebras. *Canad. J. Math.* **71** (2019), no. 5, 1243–1296. MR4010427, Zbl 1428.37007, arXiv:1703.07011, doi: 10.4153/CJM-2018-012-x. 1377, 1387, 1398

[20] Matsumoto, Kengo. Topological conjugacy of topological Markov shifts and Ruelle algebras. *J. Operator Theory* **82** (2019), no. 2, 253–284. MR4015953, Zbl 1438.46077, arXiv:1706.07155. 1377, 1380, 1399, 1407, 1408, 1410, 1411, 1412

[21] Matsumoto, Kengo; Matui, Hiroki. Continuous orbit equivalence of topological Markov shifts and Cuntz–Krieger algebras. *Kyoto J. Math.* **54** (2014), no. 4, 863–877. MR3276420, Zbl 1320.46055, arXiv:1403.0719, doi: 10.1215/21562261-2801849. 1376, 1382

[22] PARRY, BILL; SULLIVAN, DENNIS. A topological invariant of flows on 1-dimensional spaces. *Topology* **14** (1975), no. 4, 297–299. MR0405385, Zbl 0314.54045, doi: 10.1016/0040-9383(75)90012-9. 1375, 1376, 1382, 1393, 1407, 1408, 1410, 1411

[23] PUTNAM, IAN F. C^*-algebras from Smale spaces. *Canad. J. Math.* **48** (1996), no. 1, 175–195. MR1382481, Zbl 0855.46034, doi: 10.4153/CJM-1996-008-2. 1376, 1377

[24] PUTNAM, IAN F. Hyperbolic systems and generalized Cuntz–Krieger algebras. Lecture Notes, Summer School in Operator Algebras, Odense. August 1996. 1376

[25] PUTNAM, IAN F.; SPIELBERG, JACK. The structure of C^*-algebras associated with hyperbolic dynamical systems. *J. Funct. Anal.* **163** (1999), no. 2, 279–299. MR1680475, Zbl 0932.46059, doi: 10.1006/jfan.1998.3379. 1376

[26] RENAULT, JEAN. A groupoid approach to C^*-algebras. Lecture Notes in Mathematics, 793. *Springer, Berlin*, 1980. ii+160 pp. ISBN: 3-540-09977-8. MR0584266, Zbl 0433.46049. 1377, 1379

[27] RENAULT, JEAN. Cartan subalgebras in C^*-algebras. *Irish Math. Soc. Bull.* **61** (2008), 29–63. MR2460017, Zbl 1175.46050, arXiv:0803.2284. 1379

[28] RENAULT, JEAN. Examples of masas in C^*-algebras. *Operator structures and dynamical systems*, 259–265, Contemp. Math., 503. *Amer. Math. Soc., Providence, RI*, 2009. MR2590626, Zbl 1194.46105, arXiv:0903.3868. 1379

[29] ROSENBERG, JONATHAN. Appendix to: "Crossed products of UHF algebras by product type actions" by O. Bratteli. *Duke Math. J.* **46** (1979), no. 1, 25–26. MR0523599, Zbl 0395.46049, doi: 10.1215/S0012-7094-79-04602-7. 1380, 1386

[30] RUELLE, DAVID. Thermodynamic formalism. The mathematical structures of classical equilibrium statistical mechanics. Encyclopedia of Mathematics and its Applications, 5. *Addison-Wesley Publishing Co., Reading, Mass.*, 1978. xix+183 pp. ISBN: 0-201-13504-3. MR0511655, Zbl 0401.28016. 1376

[31] RUELLE, DAVID. Non-commutative algebras for hyperbolic diffeomorphisms. *Invent. Math.* **93** (1988), no. 1, 1–13. MR0943921, Zbl 0657.58026, doi: 10.1007/BF01393685. 1376

[32] SENETA, EUGENE. Non-negative matrices and Markov chains. Second Edition. Springer Series in Statistics. *Springer-Verlag, New York*, 1981. xiii+279 pp. ISBN: 0-387-90598-7. MR0719544, Zbl 0471.60001. 1392

[33] SMALE, STEVE. Differentiable dynamical systems. *Bull. Amer. Math. Soc.* **73** (1967), 747–817. MR0228014, Zbl 0202.55202, doi: 10.1090/S0002-9904-1967-11798-1. 1376

[34] WILLIAMS, ROBERT F. Classification of subshifts of finite type. *Ann. Math.* (2) **98** (1973), 120–153. erratum, *Ann. Math.* (2) **99** (1974), 380–381. MR0331436, Zbl 0282.58008, doi: 10.2307/1970908. 1383, 1385

(Kengo Matsumoto) DEPARTMENT OF MATHEMATICS, JOETSU UNIVERSITY OF EDUCATION, JOETSU, 943-8512, JAPAN
kengo@juen.ac.jp

This paper is available via http://nyjm.albany.edu/j/2021/27-53.html.