

# Torsion groups of elliptic curves over the $\mathbb{Z}_p$ -extensions of $\mathbb{Q}$

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ABSTRACT. We determine, for an elliptic curve  $E/\mathbb{Q}$  and for a prime  $p$ , all the possible torsion groups  $E(\mathbb{Q}_{\infty,p})_{\text{tors}}$ , where  $\mathbb{Q}_{\infty,p}$  is the  $\mathbb{Z}_p$ -extension of  $\mathbb{Q}$ .

## CONTENTS

1. Introduction	99
2. Notation and auxiliary results	101
3. Proof of Theorem 1.1	105
4. Proof of Theorem 1.2	107
5. Proof of Theorem 1.3	110
6. Examples of torsion growth	113
Acknowledgments	121
References	121

## 1. Introduction

For a prime number  $p$ , denote by  $\mathbb{Q}_{\infty,p}$  the unique  $\mathbb{Z}_p$ -extension of  $\mathbb{Q}$ , and for a positive integer  $n$ , denote by  $\mathbb{Q}_{n,p}$  the  $n^{\text{th}}$  layer of  $\mathbb{Q}_{\infty,p}$ , i.e. the unique subfield of  $\mathbb{Q}_{\infty,p}$  such that  $\text{Gal}(\mathbb{Q}_{n,p}/\mathbb{Q}) \simeq \mathbb{Z}/p^n\mathbb{Z}$ . Recall that the  $\mathbb{Z}_p$ -extension of  $\mathbb{Q}$  is the unique Galois extension  $\mathbb{Q}_{\infty,p}$  of  $\mathbb{Q}$  such that

$$\text{Gal}(\mathbb{Q}_{\infty,p}/\mathbb{Q}) \simeq \mathbb{Z}_p,$$

where  $\mathbb{Z}_p$  is the additive group of the  $p$ -adic integers and is constructed as follows. Let  $\zeta_k$  denote a primitive  $k$ -th root of unity and let  $\mathbb{Q}(\zeta_{p^\infty})$  be the

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field obtained by adjoining  $\zeta_{p^n}$  for all positive integers  $n$  to  $\mathbb{Q}$ . Let

$$G = \text{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}) = \varprojlim_n \text{Gal}(\mathbb{Q}(\zeta_{p^{n+1}})/\mathbb{Q}) \xrightarrow{\sim} \varprojlim_n (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times = \mathbb{Z}_p^\times.$$

Here we know that  $G = \Delta \times \Gamma$ , where  $\Gamma \simeq \mathbb{Z}_p$  and  $\Delta \simeq \mathbb{Z}/(p-1)\mathbb{Z}$  for  $p \geq 3$  and  $\Delta \simeq \mathbb{Z}/2\mathbb{Z}$  (generated by complex conjugation) for  $p = 2$ , so we define

$$\mathbb{Q}_{\infty,p} := \mathbb{Q}(\zeta_{p^\infty})^\Delta.$$

We also see that every layer is uniquely determined by

$$\mathbb{Q}_{n,p} = \mathbb{Q}(\zeta_{p^{n+1}})^\Delta,$$

so for  $p \geq 3$  it is the unique subfield of  $\mathbb{Q}(\zeta_{p^{n+1}})$  of degree  $p^n$  over  $\mathbb{Q}$ . More details and proofs of these facts about  $\mathbb{Z}_p$ -extensions and Iwasawa theory can be found in [26, Chapter 13].

Iwasawa theory for elliptic curves (see [9]) studies elliptic curves in  $\mathbb{Z}_p$ -extensions, in particular the growth of the rank and  $n$ -Selmer groups in the layers of the  $\mathbb{Z}_p$ -extensions.

In this paper we completely solve the problem of determining how the torsion of an elliptic curve defined over  $\mathbb{Q}$  grows in the  $\mathbb{Z}_p$ -extensions of  $\mathbb{Q}$ . As such, our results can be considered complementary to Greenberg's results [9] about the rank growth in  $\mathbb{Z}_p$ -extensions. We feel that they are interesting in their own right and the results might also find applications in other problems in Iwasawa theory for elliptic curves and in general. For example, to show that elliptic curves over  $\mathbb{Q}_{\infty,p}$  are modular for all  $p$ , Thorne [25] needed to show that  $E(\mathbb{Q}_{\infty,p})_{\text{tors}} = E(\mathbb{Q})_{\text{tors}}$  for two particular elliptic curves.

Our results are the following.

**Theorem 1.1.** *Let  $p \geq 5$  be a prime number, and  $E/\mathbb{Q}$  an elliptic curve. Then*

$$E(\mathbb{Q}_{\infty,p})_{\text{tors}} = E(\mathbb{Q})_{\text{tors}}.$$

**Theorem 1.2.** *Let  $E/\mathbb{Q}$  be an elliptic curve.  $E(\mathbb{Q}_{\infty,2})_{\text{tors}}$  is one of the following groups:*

$$\begin{aligned} \mathbb{Z}/N\mathbb{Z}, & \quad 1 \leq N \leq 10, \text{ or } N = 12, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N\mathbb{Z}, & \quad 1 \leq N \leq 4, \end{aligned}$$

*and for each group  $G$  from the list above there exists an  $E/\mathbb{Q}$  such that  $E(\mathbb{Q}_{\infty,2})_{\text{tors}} \simeq G$ .*

**Theorem 1.3.** *Let  $E/\mathbb{Q}$  be an elliptic curve.  $E(\mathbb{Q}_{\infty,3})_{\text{tors}}$  is one of the following groups:*

$$\begin{aligned} \mathbb{Z}/N\mathbb{Z}, & \quad 1 \leq N \leq 10, \text{ or } N = 12, 21 \text{ or } 27, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N\mathbb{Z}, & \quad 1 \leq N \leq 4. \end{aligned}$$

*and for each group  $G$  from the list above there exists an  $E/\mathbb{Q}$  such that  $E(\mathbb{Q}_{\infty,3})_{\text{tors}} \simeq G$ .*

*Remark.* By Mazur's theorem [20] we see that

$$\{E(\mathbb{Q}_{\infty,2})_{\text{tors}} : E/\mathbb{Q} \text{ elliptic curve}\} = \{E(\mathbb{Q})_{\text{tors}} : E/\mathbb{Q} \text{ elliptic curve}\},$$

and

$$\begin{aligned} \{E(\mathbb{Q}_{\infty,3})_{\text{tors}} : E/\mathbb{Q} \text{ elliptic curve}\} \\ = \{E(\mathbb{Q})_{\text{tors}} : E/\mathbb{Q} \text{ elliptic curve}\} \cup \{\mathbb{Z}/21\mathbb{Z}, \mathbb{Z}/27\mathbb{Z}\}. \end{aligned}$$

However, given a specific  $E/\mathbb{Q}$ , we do not necessarily have  $E(\mathbb{Q}_{\infty,p})_{\text{tors}} = E(\mathbb{Q})_{\text{tors}}$ . Indeed there are many elliptic curves for which torsion grows from  $\mathbb{Q}$  to  $\mathbb{Q}_{\infty,p}$ , and we investigate this question further in Section 6. Specifically, for each prime  $p$  we find for which groups  $G$  there exists infinitely many  $j$ -invariants  $j$  such that there exists an elliptic curve  $E/\mathbb{Q}$  with  $j(E) = j$  and such that  $E(\mathbb{Q})_{\text{tors}} \subsetneq E(\mathbb{Q}_{\infty,p})_{\text{tors}} \simeq G$ .

## 2. Notation and auxiliary results

In this paper we deal with elliptic curves defined over  $\mathbb{Q}$ , so unless noted otherwise, all elliptic curves will be assumed to be defined over  $\mathbb{Q}$ .

We will use the following notation throughout the paper:

- For a positive integer  $n$ ,  $\rho_{E,n}$  is the mod  $n$  Galois representation attached to elliptic curve  $E$ ; we will write just  $\rho_n$  when it is obvious what  $E$  is.
- For a number field  $K$ , we denote  $G_K := \text{Gal}(\overline{K}/K)$ .
- By  $G_{E,K}(n)$  (or just  $G_E(n)$ ) we will denote the image (after a choice of basis of  $E[n]$ ) of  $\rho_{E,n}(G_K)$  in  $\text{GL}_2(\mathbb{Z}/n\mathbb{Z})$  i.e.

$$G_{E,K}(n) = \{\rho_{E,n}(\sigma) : \sigma \in \text{Gal}(\overline{K}/K)\}.$$

- For a prime number  $\ell$ ,  $\rho_{E,\ell^\infty}$  is the  $\ell$ -adic Galois representation and  $T_\ell(E)$  is  $\ell$ -adic Tate module attached to  $E$ .
- We say that an elliptic curve  $E$  has or admits an  $n$ -isogeny over  $K$  if there exists an isogeny  $f : E \rightarrow E'$  for some elliptic curve  $E'$  of degree  $n$  with cyclic kernel and such that  $E, E'$  and  $f$  are all defined over  $K$ , or equivalently if  $G_K$  acts on  $\ker f$ .

To make this paper as self-contained as reasonably possible, we now list the most important known results that we will use.

**Proposition 2.1.** [24, Ch. III, Cor. 8.1.1] *Let  $E/L$  be an elliptic curve with  $L \subseteq \overline{\mathbb{Q}}$ . For each integer  $n \geq 1$ , if  $E[n] \subseteq E(L)$  then the  $n^{\text{th}}$  cyclotomic field  $\mathbb{Q}(\zeta_n)$  is a subfield of  $L$ .*

An immediate consequence of this proposition is

**Corollary 2.2.** *Let  $p$  and  $q$  be prime numbers and let  $q \neq 2$ . Then*

$$E(\mathbb{Q}_{\infty,p})[q] \simeq \{O\} \quad \text{or} \quad \mathbb{Z}/q\mathbb{Z}.$$

*Remark.* We have that  $E[q^n] \not\subseteq E(\mathbb{Q}_{\infty,p})$ , for each positive integer  $n$ .

**Proof.** Since  $-1$  and  $1$  are the only roots of unity contained in  $\mathbb{Q}_{\infty,p}$ , by Proposition 2.1  $E[q]$  cannot be contained in  $E(\mathbb{Q}_{\infty,p})$ .  $\square$

**Lemma 2.3.** [4, Lemma 4.6] *Let  $E$  be an elliptic curve over a number field  $K$ , let  $F$  be a Galois extension of  $\mathbb{Q}$ , let  $p$  be a prime, and let  $k$  be the largest integer for which  $E[p^k] \subseteq E(F)$ . If  $E(F)_{\text{tors}}$  contains a subgroup isomorphic to  $\mathbb{Z}/p^k\mathbb{Z} \oplus \mathbb{Z}/p^j\mathbb{Z}$  with  $j \geq k$ , then  $E$  admits a  $K$ -rational  $p^{j-k}$ -isogeny.*

Note that Lemma 2.3 as stated in [4, Lemma 4.6] requires additional assumptions, such as the ground field being  $\mathbb{Q}$  and  $F$  having finitely many roots of unity, but these are not necessary in the proof, so Lemma 2.3 is correct as stated above, without any additional assumptions.

**Theorem 2.4.** [20, 14, 15, 16, 17] *Let  $E/\mathbb{Q}$  be an elliptic curve with a rational  $n$ -isogeny. Then*

$$n \leq 19 \text{ or } n \in \{21, 25, 27, 37, 43, 67, 163\}.$$

**Corollary 2.5.** *Let  $p$  be an odd prime number,  $E/\mathbb{Q}$  elliptic curve and  $P \in E(\mathbb{Q}_{\infty,p})_{\text{tors}}$  a point of order  $q^n$  for some prime  $q$  and positive integer  $n$ , then*

$$q^n \in \{2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19, 25, 27, 32, 37, 43, 67, 163\}.$$

**Proof.** For  $q \geq 3$  we have  $E[q] \not\subseteq E(\mathbb{Q}_{\infty,p})$  by Corollary 2.2, so by Lemma 2.3 we conclude that  $E$  admits a rational  $q^n$ -isogeny.

For  $q = 2$ , having in mind that the only roots of unity in  $E(\mathbb{Q}_{\infty,p})_{\text{tors}}$  are  $\pm 1$ , by Lemma 2.3 we conclude that  $E$  admits a rational  $2^{n-1}$ -isogeny.

The result now follows from Theorem 2.4.  $\square$

**Theorem 2.6.** [8, Theorem 5.8] *Let  $E/\mathbb{Q}$  be an elliptic curve,  $p$  a prime and  $P$  a point of order  $p$  on  $E$ . Then all of the cases in the table below occur for  $p \leq 13$  or  $p = 37$ , and they are the only ones possible.*

$p$	$[\mathbb{Q}(P) : \mathbb{Q}]$
2	1, 2, 3
3	1, 2, 3, 4, 6, 8
5	1, 2, 4, 5, 8, 10, 16, 20, 24
7	1, 2, 3, 6, 7, 9, 12, 14, 18, 21, 24, 36, 42, 48
11	5, 10, 20, 40, 55, 80, 100, 110, 120
13	3, 4, 6, 12, 24, 39, 48, 52, 72, 78, 96, 144, 156, 168
37	12, 36, 72, 444, 1296, 1332, 1368

For all other  $p$ , for  $[\mathbb{Q}(P) : \mathbb{Q}]$  the following cases do occur:

- (1)  $p^2 - 1$ , for all  $p$ ,
- (2) 8, 16, 32, 136, 256, 272, 288, for  $p = 17$ ,
- (3)  $\frac{p-1}{2}$ ,  $p-1$ ,  $\frac{p(p-1)}{2}$ ,  $p(p-1)$ , if  $p \in \{19, 43, 67, 163\}$ ,

- (4)  $2(p-1), (p-1)^2,$  if  $p \equiv 1 \pmod{3}$  or  $\left(\frac{-D}{p}\right) = 1,$   
for some  $D \in \{1, 2, 7, 11, 19, 43, 67, 163\},$
- (5)  $\frac{(p-1)^2}{3}, \frac{2(p-1)^2}{3},$  if  $p \equiv 4, 7 \pmod{9},$
- (6)  $\frac{p^2-1}{3}, \frac{2(p^2-1)}{3},$  if  $p \equiv 2, 5 \pmod{9},$

Apart from the cases above that have been proven to appear, the only other options that might be possible are:

$$\frac{p^2-1}{3}, \frac{2(p^2-1)}{3}, \text{ for } p \equiv 8 \pmod{9}.$$

**Theorem 2.7.** [8, Theorem 7.2.] *Let  $E/\mathbb{Q}$  be an elliptic curve and  $p$  be the smallest prime divisor of a positive integer  $d$  and let  $K/\mathbb{Q}$  be a number field of degree  $d$ .*

- If  $p \geq 11$ , then

$$E(K)_{tors} = E(\mathbb{Q})_{tors}.$$

- If  $p = 7$ , then

$$E(K)[q^\infty] = E(\mathbb{Q})[q^\infty]$$

for all primes  $q \neq 7$ .

- If  $p = 5$ , then

$$E(K)[q^\infty] = E(\mathbb{Q})[q^\infty]$$

for all primes  $q \neq 5, 7, 11$ .

- If  $p = 3$ , then

$$E(K)[q^\infty] = E(\mathbb{Q})[q^\infty]$$

for all primes  $q \neq 2, 3, 5, 7, 11, 13, 19, 43, 67, 163$ .

We now prove a lemma that we will find useful.

**Lemma 2.8.** *Let  $p$  and  $q$  be prime numbers such that  $q-1 \nmid p$  and  $p \nmid q-1$ . Let  $K/\mathbb{Q}$  be a cyclic extension of degree  $p$ , and  $P \in E$  a point of order  $q$ . If  $P \in E(K)$ , then  $P \in E(\mathbb{Q})$ .*

**Proof.** If we assume that  $\mathbb{Q}(\zeta_q) \subseteq K$ , it follows that  $q-1 = [\mathbb{Q}(\zeta_q) : \mathbb{Q}] \mid [K : \mathbb{Q}] = p$ , and that is impossible by the assumption that  $q-1 \nmid p$ . Therefore, by Corollary 2.2 we conclude that  $E(K)[q] \simeq \mathbb{Z}/q\mathbb{Z}$ .

Let us assume that there is  $\sigma \in \text{Gal}(K/\mathbb{Q})$  such that  $P^\sigma \neq P$  (i.e. that  $P \notin E(\mathbb{Q})$ ). That means that there is some  $a \in \{2, 3, \dots, q-1\}$  such that  $P^\sigma = aP$ . Furthermore, we know that  $\sigma^p = 1$ , so

$$P = P^{\sigma^p} = a^p P,$$

which means that  $a^p \equiv 1 \pmod{q}$ , but there exists such an  $a \in \{2, 3, \dots, q-1\}$  if and only if  $p \mid q-1$  or  $q-1 \mid p$ , which is a contradiction.  $\square$

The following lemma will tell us how far up the tower we have to go to find a point of order  $n$ , if such a point exists.

**Lemma 2.9.** *Let  $E/\mathbb{Q}$  be an elliptic curve and  $P \in E$  a point of order  $n$  such that  $\mathbb{Q}(P)/\mathbb{Q}$  is Galois and let  $E(\mathbb{Q}(P))[n] \simeq \mathbb{Z}/n\mathbb{Z}$ . Then  $\text{Gal}(\mathbb{Q}(P)/\mathbb{Q})$  is isomorphic to a subgroup of  $(\mathbb{Z}/n\mathbb{Z})^\times$ .*

**Proof.** We see that  $G_{\mathbb{Q}}$  acts on  $P$  through  $G := \text{Gal}(\mathbb{Q}(P)/\mathbb{Q})$  so that for any  $\sigma \in G_{\mathbb{Q}}$  we have  $P^\sigma = aP$ , for some  $a \in (\mathbb{Z}/n\mathbb{Z})^\times$ , since  $|P^\sigma| = |P|$ . Since  $G$  acts faithfully on  $\langle P \rangle$ , this implies that  $G$  is isomorphic to a subgroup of  $(\mathbb{Z}/n\mathbb{Z})^\times$ .  $\square$

We immediately obtain the following corollary.

**Corollary 2.10.** *Let  $P \in E$  be a point of odd order  $n$  such that  $\mathbb{Q}(P) \subseteq \mathbb{Q}_{\infty,p}$ . Then  $\mathbb{Q}(P) \subseteq \mathbb{Q}_{m,p}$ , where  $m = v_p(\phi(n))$ .*

**Proof.** Since  $E(\mathbb{Q}(P))[n] \simeq \mathbb{Z}/n\mathbb{Z}$  by Corollary 2.2, the result follows from Lemma 2.9.  $\square$

**Proposition 2.11.** *Let  $E/F$  be an elliptic curve over a number field  $F$ ,  $n$  a positive integer,  $p$  a prime, and  $P \in E$  a point of order  $p^{n+1}$  such that  $F(P)/F(pP)$  is Galois and  $E(F(P))[p] \simeq \mathbb{Z}/p\mathbb{Z}$ . Then  $F(P) = F(pP)$  or  $[F(P) : F(pP)] = p$ .*

**Proof.** Let  $Q := pP$ , and consider the equation

$$pX = Q. \tag{1}$$

The solutions of (1) are of the form  $P + T$  for some  $T \in E[p]$  and  $P + T$  is defined over  $F(P)$  if and only if  $T$  is defined over  $F(P)$ . We see that the solutions of (1) defined over  $F(P)$  are in bijection with  $E(F(P))[p]$ , so by our assumptions, there are  $p$  of them. Let  $S := \{(1 + ap^n)P \mid a = 0, \dots, p-1\}$ . All elements of  $S$  are solutions to (1) and all are defined over  $F(P)$ , as they are multiples of  $P$ . By our assumption, these are the only solutions of (1) defined over  $F(P)$  and hence  $S$  contains all the solutions of (1) defined over  $F(P)$ . It follows that  $G := \text{Gal}(F(P)/F(Q))$  acts on  $S$ .

The degree  $[F(P) : F(Q)]$  is the same as the length of the orbit of  $P$  under the action of  $G$  on  $S$ . For any  $X_1, X_2 \in S$ , if  $X_1$  is defined over a number field  $K$ , then so is  $X_2$ , as  $X_2$  is a multiple of  $X_1$ . We conclude that  $F(X_1) = F(X_2)$  and hence  $[F(X_1) : F(Q)] = [F(X_2) : F(Q)]$  for all  $X_1, X_2 \in S$ , so the set  $S$  decomposes into orbits of equal length under the action of  $G$ . If all orbits are of length  $n$ , and there are  $x$  orbits, it follows that  $n \cdot x = p$ . Hence  $n$  will have to be either 1 or  $p$ , proving the proposition.  $\square$

*Remark.* Proposition 2.11 is a version of [8, Proposition 4.6.] with stronger assumptions.

**Corollary 2.12.** *Let  $p$  and  $q$  be primes such that  $q \neq 2$  and  $p \neq q$ . Let  $E/\mathbb{Q}$  be an elliptic curve,  $F$  some number field contained in  $\mathbb{Q}_{\infty,p}$  such that  $E(F)$  contains a point of order  $q^n$  but no points of order  $q^{n+1}$ . Then  $E(\mathbb{Q}_{\infty,p})$  contains no points of order  $q^{n+1}$ .*

**Proof.** Suppose the opposite, and let  $P \in E(\mathbb{Q}_{\infty,p})$  be a point of order  $q^{n+1}$ . By Corollary 2.2 we have that  $E(F(P))[q^{n+1}] \simeq \mathbb{Z}/q^{n+1}\mathbb{Z}$  so it follows that  $qP \in E(F)$  (as all of the  $q^n$ -torsion is defined already over  $F$ ). Now from Proposition 2.11 it follows that  $[F(P) : F(qP)] = [F(P) : F] = q$ , which is a contradiction, as a number field of degree divisible by  $q$  cannot be a subfield of  $\mathbb{Q}_{\infty,p}$ .  $\square$

### 3. Proof of Theorem 1.1

For primes  $p \geq 11$ , by Theorem 2.7 we know that  $E(\mathbb{Q}_{n,p})_{\text{tors}} = E(\mathbb{Q})_{\text{tors}}$ , for each positive integer  $n$ . It follows that  $E(\mathbb{Q}_{\infty,p})_{\text{tors}} = E(\mathbb{Q})_{\text{tors}}$ . It remains to prove this fact for the cases  $p = 7$  and  $p = 5$ .

**Theorem 3.1.**  $E(\mathbb{Q}_{\infty,7})_{\text{tors}} = E(\mathbb{Q})_{\text{tors}}$ .

**Proof.** From Theorem 2.7, we immediately conclude that  $E(\mathbb{Q}_{\infty,7})[q^\infty] = E(\mathbb{Q})[q^\infty]$  for all primes  $q \neq 7$ . It remains to prove that  $E(\mathbb{Q}_{\infty,7})[7^\infty] = E(\mathbb{Q})[7^\infty]$ .

By Corollary 2.5 we conclude that there is no 49-torsion in  $E(\mathbb{Q}_{\infty,7})$ , so it remains to prove that  $E(\mathbb{Q}_{\infty,7})[7] = E(\mathbb{Q})[7]$ .

Let  $P \in E(\mathbb{Q}_{\infty,7})$  be a point of order 7. By Theorem 2.6,  $P$  is defined over some field of degree at most  $7^2 - 1$ . Therefore,  $P \in E(\mathbb{Q}_{1,7})$ . From Lemma 2.8 it now follows that  $P \in E(\mathbb{Q})$  and we are done.  $\square$

**Lemma 3.2.**  $E(\mathbb{Q}_{\infty,5})[11^\infty] = \{O\}$ .

**Proof.** Again by Corollary 2.5 we conclude that there is no 121-torsion in  $E(\mathbb{Q}_{\infty,5})$ . It remains to prove that  $E(\mathbb{Q}_{\infty,5})[11] = \{O\}$ .

Let  $P \in E(\mathbb{Q}_{\infty,5})$  be a point of order 11. From Theorem 2.6 we conclude that  $P \in E(\mathbb{Q}_{1,5})$ . The modular curve  $X_1(11)$  is the elliptic curve

$$y^2 + y = x^3 - x^2.$$

We can easily compute (using Magma [1]) that  $X_1(11)$  has rank 0 and torsion  $\mathbb{Z}/5\mathbb{Z}$  over  $\mathbb{Q}_{1,5}$ , and all the torsion points are cusps, so there are no elliptic curves with 11-torsion over  $\mathbb{Q}_{1,5}$ .  $\square$

Before proving  $E(\mathbb{Q}_{\infty,5})[5^\infty] = E(\mathbb{Q})[5^\infty]$  we will need some technical results.

**Theorem 3.3.** [10, Theorem 2] *Let  $E/\mathbb{Q}$  be an elliptic curve with a rational 5-isogeny. The index  $[\text{Aut}_{\mathbb{Z}_5}(T_5(E)) : \text{im}(\rho_{E,5^\infty})]$  isn't divisible by 25.*

**Lemma 3.4.** *Let  $n$  be a positive integer and  $\zeta$  an  $n^{\text{th}}$  root of unity. Then for every  $\sigma \in \text{Gal}(\mathbb{Q}(E[n])/\mathbb{Q})$  we have*

$$\sigma(\zeta) = \zeta^{\det \rho_n(\sigma)}.$$

**Proof.** Let  $\{P, Q\}$  be a basis for  $E[n]$  and  $e_n(P, Q) = \zeta_n$ , where  $\zeta_n$  is a  $n^{\text{th}}$  primitive root of unity. For any  $n^{\text{th}}$  root of unity  $\zeta$ , there exists an  $m \in \mathbb{Z}$  such that  $\zeta = \zeta_n^m$ . So, it suffices to show that  $\sigma(\zeta_n) = \zeta_n^{\det \rho_n(\sigma)}$ .

Then there are some  $a, b, c, d \in (\mathbb{Z}/n\mathbb{Z})^\times$  such that

$$P^\sigma = aP + bQ \quad \text{and} \quad Q^\sigma = cP + dQ.$$

Using properties of the Weil pairing [24, Ch. III, §8.] we calculate:

$$\begin{aligned} \sigma(\zeta_n) &= \sigma(e_n(P, Q)) = e_n(P^\sigma, Q^\sigma) = e_n(aP + bQ, cP + dQ) \\ &= e_n(P, P)^{ac} e_n(P, Q)^{ad} e_n(Q, P)^{bc} e_n(Q, Q)^{bd} \\ &= 1 \cdot \zeta_n^{ad} \cdot \zeta_n^{-bc} \cdot 1^{bd} = \zeta_n^{\det \rho_n(\sigma)}. \end{aligned} \quad \square$$

**Proposition 3.5.** *Let  $n$  be a positive integer and  $\zeta_n$  an  $n^{\text{th}}$  primitive root of unity. Let  $K$  be a number field and  $E/K$  an elliptic curve. Then*

$$\det \rho_{E,n}(G_K) \simeq \text{Gal}(\mathbb{Q}(\zeta_n)/K \cap \mathbb{Q}(\zeta_n)).$$

**Proof.** Let

$$f: G_K \rightarrow \text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \quad \sigma \mapsto g \circ \det \circ \rho_{E,n}(\sigma),$$

where  $g$  is the canonical isomorphism mapping  $(\mathbb{Z}/n\mathbb{Z})^\times \rightarrow \text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$  by sending  $a$  to  $\sigma_a$ , where  $\sigma_a(\zeta_n) = \zeta_n^a$ .

As  $\det \rho_{E,n}(G_K) \leq (\mathbb{Z}/n\mathbb{Z})^\times$ , it follows that  $f(G_K) \leq \text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$  and hence, by Galois theory,  $f(G_K) = \text{Gal}(\mathbb{Q}(\zeta_n)/K')$  for some subfield  $K'$  of  $\mathbb{Q}(\zeta_n)$ . From Proposition 3.4 it follows that  $f$  is the restriction map sending  $\sigma \in G_K$  to  $\sigma|_{\mathbb{Q}(\zeta_n)} = \sigma_{\det \rho_n(\sigma)}$ .

It follows that  $f(G_K)$  comprises exactly those  $\sigma_a$  that leave  $K \cap \mathbb{Q}(\zeta_n)$  fixed, proving  $K' = K \cap \mathbb{Q}(\zeta_n)$  and hence the proposition.  $\square$

**Lemma 3.6.**  $E(\mathbb{Q}_{\infty,5})[5^\infty] = E(\mathbb{Q})[5^\infty]$ .

**Proof.** There is no 125-torsion in  $E(\mathbb{Q}_{\infty,5})$  by Corollary 2.5, so it remains to prove that  $E(\mathbb{Q}_{\infty,5})[25] = E(\mathbb{Q})[25]$ .

If  $P \in E(\mathbb{Q}_{\infty,5})$  is a point of order 5, then  $P$  is defined over  $\mathbb{Q}_{1,5}$  by Theorem 2.6, but then by Lemma 2.8 it follows that  $P \in E(\mathbb{Q})$ . Let us assume that there is a point  $P \in E(\mathbb{Q}_{\infty,5})_{\text{tors}}$  of order 25; obviously  $P \notin E(\mathbb{Q})$ . By Corollary 2.10 the order 5 point  $5P$  must be defined over  $\mathbb{Q}$ . Since  $P \in E(\mathbb{Q}_{1,5})$  and the extension  $\mathbb{Q}_{1,5}/\mathbb{Q}$  is cyclic, it follows that for every  $\sigma \in G_{\mathbb{Q}}$  there exist some  $a \in (\mathbb{Z}/25\mathbb{Z})^\times$  such that  $P^\sigma = aP$ . Since  $5P \in E(\mathbb{Q})$ , we have  $(5P)^\sigma = 5P$ , so  $a \equiv 1 \pmod{5}$ . Hence,  $G_{\mathbb{Q}}(25)$  is of the form

$$\left\{ \begin{pmatrix} a & * \\ 0 & * \end{pmatrix} : a \in 1 + 5\mathbb{Z}/25\mathbb{Z} \right\}.$$

Furthermore,  $\det G_{\mathbb{Q}_{1,5},E}(25)$  is by Proposition 3.5 the unique subgroup of  $(\mathbb{Z}/25\mathbb{Z})^\times$  of order 4:  $\{7, -1, -7, 1\}$ . The group  $G_{\mathbb{Q}_{1,5}}$  fixes the point  $P$  so we conclude that  $G_{\mathbb{Q}_{1,5},E}(25)$  is of the form

$$\left\{ \begin{pmatrix} 1 & * \\ 0 & b \end{pmatrix} : b \in \{7, -1, -7, 1\} \right\}.$$

Since  $G_{\mathbb{Q}_{1,5},E}(25)$  is a subgroup of  $G_{\mathbb{Q}}(25)$  of index 5, and  $P \notin E(\mathbb{Q})$ , so it follows that

$$G_{\mathbb{Q}}(25) = \left\{ \begin{pmatrix} a & * \\ 0 & b \end{pmatrix} : a \in 1 + 5\mathbb{Z}/25\mathbb{Z}, b \in \{7, -1, -7, 1\} \right\}.$$

Finally, we calculate that  $600 \mid [\mathrm{GL}_2(\mathbb{Z}/25\mathbb{Z}) : G_E(25)] \mid [\mathrm{Aut}_{\mathbb{Z}_5}(T_5(E)) : \mathrm{im}(\rho_{E,5^\infty})]$ , a contradiction with Theorem 3.3.  $\square$

**Theorem 3.7.**  $E(\mathbb{Q}_{\infty,5})_{\mathrm{tors}} = E(\mathbb{Q})_{\mathrm{tors}}$ .

**Proof.** From Theorem 2.7, we immediately conclude that  $E(\mathbb{Q}_{\infty,5})[q^\infty] = E(\mathbb{Q})[q^\infty]$  for all primes  $q \neq 5, 7, 11$ .

There is no 49-torsion in  $E(\mathbb{Q}_{\infty,5})$  by Corollary 2.5, so it remains to prove that  $E(\mathbb{Q}_{\infty,5})[7] = E(\mathbb{Q})[7]$ . Let  $P \in E(\mathbb{Q}_{\infty,5})$  be a point of degree 7 such that  $P \notin E(\mathbb{Q})$ . By Theorem 2.6 we conclude that  $\mathrm{gcd}(5, [\mathbb{Q}(P) : \mathbb{Q}]) = 1$ , which is a contradiction.

The cases  $q = 11$  and  $q = 5$  follow from Lemmas 3.2 and 3.6.  $\square$

#### 4. Proof of Theorem 1.2

By Theorem 2.6 and the following easy observations:

- $p^2 - 1 = (p - 1)(p + 1)$  isn't a power of 2 for primes  $p > 3$ . This follows from the fact that  $\mathrm{gcd}(p - 1, p + 1) = 2$ ,
- $p - 1$  isn't a power of 2 for  $p \in \{19, 43, 67, 163\}$ ,
- for a prime  $p$ ,  $p - 1$  is a power of 2 if and only if  $p$  is of the form  $2^{2^k} + 1$ ,
- $3 \mid \frac{(p - 1)^2}{3}$  for  $p \equiv 4, 7 \pmod{9}$ ,
- $\frac{p^2 - 1}{3}$  isn't a power of 2 for  $p \equiv 2, 5 \pmod{9}$  and  $p > 5$ . This follows from the fact that  $p$  is of the form  $3k - 1$  for some integer  $k \geq 4$ , and that means that  $\frac{p^2 - 1}{3} = (3k - 2) \cdot k$ , but  $\mathrm{gcd}(3k - 2, k) \leq 2$ , and  $3k - 2$  and  $k$  cannot both be powers of 2 simultaneously for  $k \geq 4$ ,
- $3 \mid \frac{p^2 - 1}{3}$  for  $p \equiv 8 \pmod{9}$ ,

we can conclude that  $E(\mathbb{Q}_{\infty,2})[q] = E(\mathbb{Q})[q]$  for all primes  $q \neq 2, 3, 5, 7, 13, 17$ .

**Lemma 4.1.**  $E(\mathbb{Q}_{\infty,2})$  does not contain a point of order 13.

**Proof.** Suppose that there exists such a curve; then the point  $P$  of order 13 is defined over the quartic field  $\mathbb{Q}_{2,2} = \mathbb{Q}(\sqrt{2 + \sqrt{2}})$  by Theorem 2.6. Let  $\delta = 2 + \sqrt{2}$ . Then  $E^\delta$  becomes isomorphic to  $E$  over  $\mathbb{Q}_{2,2}$  and

$$E(\mathbb{Q}_{2,2})[13] \simeq E(\mathbb{Q}_{1,2})[13] \times E^\delta(\mathbb{Q}_{1,2})[13],$$

so either  $E$  or  $E^\delta$  would have 13-torsion over  $\mathbb{Q}_{1,2} = \mathbb{Q}(\sqrt{2})$ , which is not possible by [13, Theorem 3].  $\square$

**Lemma 4.2.**  $E(\mathbb{Q}_{\infty,2})$  does not contain a point of order 17.

**Proof.** By Corollary 2.10 and Theorem 2.6, a point  $P$  of order 17 can be defined over a number field of degree 8 or 16. An elliptic curve  $E/\mathbb{Q}$  with such a point has a 17-isogeny over  $\mathbb{Q}$  (see [8, Table 2]), so  $j(E) = -(17^2 \cdot 101^3)/2$  or  $j(E) = -(17 \cdot 373^3)/2^{17}$ . We factor the 17th division polynomials of an elliptic curve with each of these invariants (the choice of the exact quadratic twist we choose with each  $j$ -invariant will be irrelevant) over  $\mathbb{Q}_{3,2}$ , and obtain that in one case the smallest degree of an irreducible factor is 4, while in the other case the smallest degree of an irreducible factor is 8.  $\square$

So we have  $E(\mathbb{Q}_{\infty,2})[q^\infty] = E(\mathbb{Q})[q^\infty]$  for all primes  $q \neq 2, 3, 5, 7$ .

**Lemma 4.3.** If  $E(\mathbb{Q}_{\infty,2})_{\text{tors}}$  has a point of order 7, then  $E(\mathbb{Q}_{\infty,2})_{\text{tors}} \simeq \mathbb{Z}/7\mathbb{Z}$

**Proof.** If  $E(\mathbb{Q}_{\infty,2})_{\text{tors}}$  has a point of order 7, then by Corollary 2.10, then  $P$  is defined over  $\mathbb{Q}_{1,2}$ . Let  $E^2$  be the quadratic twist of  $E$  by 2, which becomes isomorphic to  $E$  over  $\mathbb{Q}_{1,2}$ . Since  $E(\mathbb{Q}_{1,2})[7] \simeq E(\mathbb{Q})[7] \times E^2(\mathbb{Q})[7]$ , we conclude that either  $E(\mathbb{Q})$  or  $E^2(\mathbb{Q})$  has a point of order 7.

If  $E(\mathbb{Q}_{\infty,2})_{\text{tors}}$  contained a point of order 2, then both  $E(\mathbb{Q})$  and  $E^{(2)}(\mathbb{Q})$  would also have to contain a point of order 2, which would mean that one of these groups has a point of order 14, which is by Mazur's theorem impossible.

If  $E(\mathbb{Q}_{\infty,2})_{\text{tors}}$  contained a point of order 3, then by Corollary 2.10 it would have to be defined over  $\mathbb{Q}_{1,2}$ , which would mean that  $E(\mathbb{Q}_{1,2})$  has a point of order 21, which is impossible by [12, 18].

From the first paragraph of this proof we conclude that since either  $E$  or  $E^2$  has a point of order 7 over  $\mathbb{Q}$ , we conclude that  $E$  certainly has a 7-isogeny over  $\mathbb{Q}$ . So, if  $E(\mathbb{Q}_{\infty,2})_{\text{tors}}$  contained a point of order 5, it would have a 5-isogeny over  $\mathbb{Q}$  by Lemma 2.3, and hence also a 35-isogeny over  $\mathbb{Q}$ , contradicting Theorem 2.4.  $\square$

**Lemma 4.4.** If  $E(\mathbb{Q}_{\infty,2})_{\text{tors}}$  has a point of order 5, then  $E(\mathbb{Q}_{\infty,2})_{\text{tors}} \simeq \mathbb{Z}/5\mathbb{Z}$  or  $\mathbb{Z}/10\mathbb{Z}$

**Proof.** Suppose  $E(\mathbb{Q}_{\infty,2})_{\text{tors}}$  has a point of order 5. By Corollary 2.10, the point of order  $P$  on  $E$  has to be defined over  $\mathbb{Q}_{2,2}$ .

If  $E(\mathbb{Q}_{\infty,2})_{\text{tors}}$  had a point of order 15, then both a point of order 3 and of order 5 have to be defined over  $\mathbb{Q}_{2,2}$ , so the 15-torsion point must already be defined over  $\mathbb{Q}_{2,2}$ . But  $X_1(15)(\mathbb{Q}) = X_1(15)(\mathbb{Q}_{2,2})$ , so there are no elliptic curves with a point of order 15 over  $\mathbb{Q}_{2,2}$ .

Suppose now  $E(\mathbb{Q}_{\infty,2})_{\text{tors}}$  has a point of order 10. Then  $E(\mathbb{Q})$  has a point of order 2. Suppose  $E(\mathbb{Q}_{\infty,2})_{\text{tors}} \supseteq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z}$ ; then it would follow that  $E[2]$  is defined over  $\mathbb{Q}_{1,2}$  (as it has to be defined over some quadratic field if there is a point of order 2 over  $\mathbb{Q}$ ). Since  $E(\mathbb{Q}_{2,2})[5] \simeq \mathbb{Z}/5\mathbb{Z}$  and  $E(\mathbb{Q}_{2,2})[5] \simeq E(\mathbb{Q}_{1,2})[5] \oplus E^\delta(\mathbb{Q}_{1,2})[5]$ , for some quadratic twist (over  $\mathbb{Q}_{1,2}$ )  $E^\delta$  of  $E$ . So, since quadratic twisting does not change the 2-torsion, either  $E^\delta(\mathbb{Q}_{1,2}) \supseteq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z}$  or  $E(\mathbb{Q}_{1,2}) \supseteq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z}$ . But we

compute  $X_1(2, 10)(\mathbb{Q}(\sqrt{2})) = X_1(2, 10)(\mathbb{Q})$ , so there are no elliptic curves over  $\mathbb{Q}_{1,2} = \mathbb{Q}(\sqrt{2})$  with  $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z}$  torsion.

Finally  $E(\mathbb{Q}_{\infty,2})[5n]$  cannot be isomorphic to  $\mathbb{Z}/5n\mathbb{Z}$  for  $n = 4, 5, 7$  as then  $E$  would have a rational  $5n$ -isogeny which is impossible by Lemma 2.3 and Theorem 2.4.  $\square$

**Lemma 4.5.** *If  $E(\mathbb{Q}_{\infty,2})_{\text{tors}}$  has a point of order 9, then  $E(\mathbb{Q}_{\infty,2})_{\text{tors}} \simeq \mathbb{Z}/9\mathbb{Z}$ .*

**Proof.** Suppose  $E(\mathbb{Q}_{\infty,2})_{\text{tors}}$  contained a point  $P$  of order 27. The point  $3P$  of order 9 would have to be defined over  $\mathbb{Q}_{1,2}$  by Corollary 2.10. There cannot be any 27-torsion over  $\mathbb{Q}_{1,2}$  [12, 18]. But on the other hand, the point of order 9 cannot become divisible by 3 in a Galois extension of  $\mathbb{Q}_{1,2}$  of degree  $2^n$  by Corollary 2.12.

Suppose  $E(\mathbb{Q}_{\infty,2})_{\text{tors}}$  contained a point of order 18. Then  $E(\mathbb{Q}_{\infty,2})_{\text{tors}}$  would have to contain a 2-torsion point, and a point  $P$  of order 9 would have to be defined over  $\mathbb{Q}_{1,2}$  by Corollary 2.10. So  $E(\mathbb{Q}_{1,2})$  would contain a point of order 18. But there are no elliptic curves defined over  $\mathbb{Q}$  with a point of order 18 over a quadratic field by [22, Theorem 2].  $\square$

**Lemma 4.6.** *If  $E(\mathbb{Q}_{\infty,2})_{\text{tors}}$  has a point of order 12, then  $E(\mathbb{Q}_{\infty,2})_{\text{tors}} \simeq \mathbb{Z}/12\mathbb{Z}$ .*

**Proof.** By the previous lemmas,  $E(\mathbb{Q}_{\infty,2})_{\text{tors}}$  cannot contain a points of order 5, 7 or 9, so it remains to show that  $E(\mathbb{Q}_{\infty,2})_{\text{tors}}$  does not have full 2-torsion and that it doesn't have a point of order 24. The fact that there are no points of order 24 follows from the fact that there are no points of order 24 over  $\mathbb{Q}^{ab}$  of which  $\mathbb{Q}_{\infty,2}$  is a subfield by [2, Theorem 1.2].

Suppose now that  $E(\mathbb{Q}_{\infty,2})_{\text{tors}} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}$ , and let  $E(\mathbb{Q}_{\infty,2})_{\text{tors}} = \langle P, Q \rangle$ , where  $P$  is of order 12 and  $Q$  is of order 2. We have  $2E(\mathbb{Q}_{\infty,2})_{\text{tors}} = \langle 2P \rangle \simeq \mathbb{Z}/6\mathbb{Z}$  is a  $G_{\mathbb{Q}}$ -invariant subgroup, and hence  $(6P)^\sigma = 6P$  for all  $\sigma \in G_{\mathbb{Q}}$ , i.e.  $6P$  is defined over  $\mathbb{Q}$ . By [8, Proposition 4.8],  $3P$  has to be defined over an extension of  $\mathbb{Q}$  which has Galois group  $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$  or  $D_4$ . So we conclude that it has to be  $\mathbb{Z}/2\mathbb{Z}$  (since by assumption  $\mathbb{Q}(3P)$  is defined over  $\mathbb{Q}_{\infty,2}$ ). So the point  $3P$  of order 4 is defined over  $\mathbb{Q}_{1,2}$ .

The point  $4P$  is also defined over  $\mathbb{Q}_{1,2}$  by Corollary 2.10. Since  $6P$  is defined over  $\mathbb{Q}$ , then  $Q$  has to be defined over  $\mathbb{Q}_{1,2}$ .

So since  $4P, 3P$  and  $Q$  are all defined over  $\mathbb{Q}_{1,2}$ , we conclude that

$$E(\mathbb{Q}_{1,2})_{\text{tors}} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z},$$

but this is impossible by [13, Theorem 10.].  $\square$

Finally, we have the following result that controls the 2-power torsion.

**Theorem 4.7.** [7, Theorem 1] *For an elliptic curve  $E/\mathbb{Q}$ ,  $E(\mathbb{Q}_{\infty,2})[2^\infty] \subseteq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$ .*

**Proof of Theorem 1.2.** The results above combined prove that all the possible torsion groups  $E(\mathbb{Q}_{\infty,2})_{\text{tors}}$  are contained in the list given in Theorem

1.2. Theorem 6.1 proves that all the groups listed, apart from maybe  $\{O\}$  and  $\mathbb{Z}/2\mathbb{Z}$ , do appear as  $E(\mathbb{Q}_{\infty,2})_{\text{tors}}$ , so it remains to show that these two groups appear to complete the proof of Theorem 1.2.

To prove the existence of  $E/\mathbb{Q}$  such that  $E(\mathbb{Q}_{\infty,2})_{\text{tors}} \simeq \{O\}$ , take  $E$  to be an elliptic curve whose mod  $q$  representation is surjective for all primes  $q$  (almost all elliptic curves satisfy this [6]). Then if  $E$  has a point of order  $q$  over a number field  $K$ , then  $[K : \mathbb{Q}]$  is divisible by  $q^2 - 1$  [19, Theorem 5.1.]. But  $q^2 - 1$  cannot be a power of 2, so it follows that  $E(\mathbb{Q}_{\infty,2})_{\text{tors}} \simeq \{O\}$ .

Finally, to obtain  $E/\mathbb{Q}$  such that  $E(\mathbb{Q}_{\infty,2})_{\text{tors}} \simeq \mathbb{Z}/2\mathbb{Z}$ , take an elliptic curve with  $E(\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z}$  such that  $E$  has no 4-isogenies over  $\mathbb{Q}$ , that its discriminant  $\Delta(E)$  is not twice a rational square and such that its mod  $q$  representations are surjective for all primes  $q \neq 2$ . One would expect that almost all elliptic curves with  $\mathbb{Z}/2\mathbb{Z}$  torsion over  $\mathbb{Q}$  satisfy this, but as an explicit example one can take the curve with Cremona reference 69a2. The fact that the discriminant is not twice a rational square implies that  $\mathbb{Q}(E[2]) \cap \mathbb{Q}_{\infty,2} = \mathbb{Q}$  and hence  $E(\mathbb{Q}_{\infty,2})[2] \simeq \mathbb{Z}/2\mathbb{Z}$  and the non-existence of a rational 4-isogeny then proves, by Lemma 2.3, that there are no points of order 4 in  $E(\mathbb{Q}_{\infty,2})$ . Finally, as before, the fact that the mod  $q$  representations of  $E$  are surjective for all primes  $q \neq 2$  proves that  $E(\mathbb{Q}_{\infty,2})[q] = \{O\}$  for all primes  $q \neq 2$ .  $\square$

## 5. Proof of Theorem 1.3

Combining Theorem 2.6, Corollary 2.5 and the following facts:

- $p^2 - 1$  isn't a power of 3 for any odd prime  $p$  (it's divisible by 2),
- $p - 1$  isn't a power of 3 for any odd prime  $p$  (it's divisible by 2),

we conclude that  $E(\mathbb{Q}_{\infty,3})[q] = E(\mathbb{Q})[q]$  for all primes  $q \neq 2, 3, 7, 13, 19, 163$ .

**Lemma 5.1.**  $E(\mathbb{Q}_{\infty,3})[19] = \{O\}$ .

**Proof.** From Corollary 2.10 we deduce that a point of order 19 on  $E/\mathbb{Q}$  must be defined over  $\mathbb{Q}_{2,3}$ . By Lemma 2.3 it follows that  $E$  must have a rational 19-isogeny. There is only one family of quadratic twists (with  $j$ -invariant  $-2^{15} \cdot 3^3$ ), with complex multiplication by  $\mathbb{Z}[(1 + \sqrt{-19})/2]$ . We check that the 19<sup>th</sup> division polynomials of these elliptic curves with 19-isogeny don't have a root over the field  $\mathbb{Q}_{2,3}$ . It is enough to check this for one curve with  $j$ -invariant  $-2^{15} \cdot 3^3$ , as if the 19<sup>th</sup> division polynomial of this one curve with this  $j$ -invariant doesn't have a root over  $\mathbb{Q}_{2,3}$ , then neither does any quadratic twist of  $E$ . So  $E(\mathbb{Q}_{\infty,3})[19] = \{O\}$ .  $\square$

**Lemma 5.2.**  $E(\mathbb{Q}_{\infty,3})[13] = \{O\}$

**Proof.** From Corollary 2.10, a point of order 13 can be defined only over  $\mathbb{Q}_{1,3} = \mathbb{Q}(\zeta_9)^+$  (the maximal real subfield of  $\mathbb{Q}(\zeta_9)$ ). The modular curve  $X_1(13)$  is a curve of genus 2 with the following model (as we can see in [13]):

$$y^2 = x^6 - 2x^5 + x^4 - 2x^3 + 6x^2 - 4x + 1.$$

The rank of the Jacobian of this curve over  $\mathbb{Q}(\zeta_9)^+$  is 0, and the torsion is  $\mathbb{Z}/19\mathbb{Z}$ , and we easily check that  $X_1(13)(\mathbb{Q}_{1,3}) = X_1(13)(\mathbb{Q})$ , so there are no elliptic curves with 13-torsion over  $\mathbb{Q}_{1,3}$ . So  $E(\mathbb{Q}_{\infty,3})[13] = \{O\}$ .  $\square$

**Lemma 5.3.**  $E(\mathbb{Q}_{\infty,3})[5^\infty] = E(\mathbb{Q})[5^\infty]$ .

**Proof.** From Theorem 2.6, we see that if  $E(\mathbb{Q})[5] = \{O\}$ , then  $E(\mathbb{Q}_{\infty,3})[5] = \{O\}$ . If  $E(\mathbb{Q})[5] \neq \{O\}$ , then by Corollary 2.12, we see that  $E(\mathbb{Q})[5^\infty] = E(\mathbb{Q}_{\infty,3})[5^\infty]$ .  $\square$

**Lemma 5.4.** *If  $E(\mathbb{Q}_{\infty,3})_{tors}$  has a point of order 7, then  $E(\mathbb{Q}_{\infty,3})_{tors} \simeq \mathbb{Z}/7\mathbb{Z}$  or  $\mathbb{Z}/21\mathbb{Z}$*

**Proof.** By Corollary 2.12, the 7-power torsion can grow only if  $E(\mathbb{Q})[7] = \{O\}$ . We now determine, for an  $E/\mathbb{Q}$  such that

$$E(\mathbb{Q})[7] = \{O\}, \quad E(\mathbb{Q}_{\infty,3})[7] \neq \{O\},$$

what are the possible torsion groups of  $E(\mathbb{Q}_{\infty,3})$ . By Corollary 2.10 we conclude that a point of order 7 appears over  $\mathbb{Q}_{1,3}$ .

We first note that  $E(\mathbb{Q}_{\infty,3})[7^\infty] \simeq \mathbb{Z}/7\mathbb{Z}$ , as there cannot be any 49-torsion by Proposition 2.5. Also  $E(\mathbb{Q}_{\infty,3})$  obviously cannot contain a subgroup isomorphic to  $\mathbb{Z}/35\mathbb{Z}$  due to Theorem 2.4.

Suppose  $E(\mathbb{Q}_{\infty,3})$  contains  $\mathbb{Z}/14\mathbb{Z}$ . Then  $E(\mathbb{Q})$  has a point of order 2, and so  $E(\mathbb{Q}_{1,3})$  contains a subgroup isomorphic to  $\mathbb{Z}/14\mathbb{Z}$ . But we compute that  $X_1(14)(\mathbb{Q}_{1,3}) = X_1(14)(\mathbb{Q})$ , which shows that this is impossible.

Suppose  $E(\mathbb{Q}_{\infty,3})$  contains  $\mathbb{Z}/21\mathbb{Z}$ . First note that then  $E(\mathbb{Q}_{\infty,3})_{tors} \simeq \mathbb{Z}/21\mathbb{Z}$ , as a larger torsion group would contradict Theorem 2.4. By Corollary 2.10, we see that the 21-torsion point has to be defined over  $\mathbb{Q}_{1,3}$ . By [22, Theorem 1], there is a unique such curve  $E = 162b1$  satisfying this property.  $\square$

**Lemma 5.5.** *If  $E(\mathbb{Q})[2^\infty] \neq E(\mathbb{Q}_{\infty,3})[2^\infty]$ , then  $E(\mathbb{Q}_{\infty,3})[2^\infty] \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ .*

**Proof.** By [21, Lemma 1], if  $E(\mathbb{Q})[2] \neq \{O\}$ , then  $E(\mathbb{Q})[2^\infty] = E(\mathbb{Q}_{\infty,3})[2^\infty]$ . The 2-power torsion can grow only if  $E(\mathbb{Q})[2] = \{O\}$  and all the growth occurs over the field obtained by adjoining a 2-torsion point. By [22, Proposition 9], if  $E(\mathbb{Q}_{1,3})[2] \neq \{O\}$  then  $E(\mathbb{Q}_{1,3})[2^\infty] \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$  and so  $E(\mathbb{Q}_{\infty,3})[2^\infty] \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ .  $\square$

**Lemma 5.6.** *There are no points of order 18 in  $E(\mathbb{Q}_{\infty,3})$ .*

**Proof.** Suppose that  $E/\mathbb{Q}$  has a point  $P$  of order 18 over  $\mathbb{Q}_{\infty,3}$ . By Corollary 2.10, we see that the point  $2P$  of order 9 is defined over  $\mathbb{Q}_{1,3} = \mathbb{Q}(\zeta_9)^+$ . By Lemma 5.5 and the arguments in its proof, we have  $E(\mathbb{Q}_{\infty,3})[2^\infty] = E(\mathbb{Q}_{1,3})[2^\infty]$ . So it follows that  $9P$  and hence also  $9P + 5 \cdot (2P) = P$  is defined over  $\mathbb{Q}_{1,3} = \mathbb{Q}(\zeta_9)^+$ .

We will prove that  $X_1(18)(\mathbb{Q}(\zeta_9)^+)$  consists of only cusps. We compute that the rank of  $J_1(18)$  over  $\mathbb{Q}(\zeta_9)^+$  is 0. By considering reduction modulo

small primes of good reduction, we obtain that the torsion of  $J_1(18)(\mathbb{Q}(\zeta_9)^+)$  is a subgroup of  $\mathbb{Z}/21\mathbb{Z}^2$ . We find 12 points in  $X_1(18)(\mathbb{Q}(\zeta_9)^+)$ , all of which are cusps, and the differences of pairs of these cusps generate a group isomorphic to  $\mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/21\mathbb{Z}$ . To prove that  $J_1(18)(\mathbb{Q}(\zeta_9)^+) \simeq \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/21\mathbb{Z}$ , we use the following argument. One finds that

$$J_1(18)(\mathbb{Q})_{\text{tors}} \simeq \mathbb{Z}/21\mathbb{Z},$$

$$J_1(18)(\mathbb{Q}(\zeta_3))_{\text{tors}} \simeq \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/21\mathbb{Z},$$

and by considering reduction modulo small primes, that

$$J_1(18)(\mathbb{Q}(\zeta_3)) \leq \mathbb{Z}/21\mathbb{Z} \oplus \mathbb{Z}/21\mathbb{Z}.$$

Since  $\mathbb{Q}(\zeta_9) = \mathbb{Q}(\zeta_3)\mathbb{Q}(\zeta_9)^+$  and  $\mathbb{Q}(\zeta_3) \cap \mathbb{Q}(\zeta_9)^+ = \mathbb{Q}$ , we see that the field of definition of the non-rational elements of  $J_1(18)(\mathbb{Q}(\zeta_9)^+)[3]$  is  $\mathbb{Q}(\zeta_3)$ . Hence  $J_1(18)(\mathbb{Q}(\zeta_9)^+)[3] \simeq \mathbb{Z}/3\mathbb{Z}$  and after checking that none of the points in  $J_1(18)(\mathbb{Q}(\zeta_9)^+)$  come from points in  $X_1(18)(\mathbb{Q}(\zeta_9)^+)$  apart from the 12 known ones, we are done.  $\square$

**Lemma 5.7.** *There are no points of order 163 in  $E(\mathbb{Q}_{\infty,3})$ .*

**Proof.** Suppose  $E(\mathbb{Q}_{\infty,3})$  has a point  $P$  of order 163. Then by Lemma 2.3 we conclude that  $E$  has a 163-isogeny over  $\mathbb{Q}$  and hence  $E$  has  $j$ -invariant  $j_0 = -2^{18}3^35^323^329^3$ . By Corollary 2.10 it follows that  $\mathbb{Q}(P) \subseteq \mathbb{Q}_{3,3}$ . We factor the 163-division polynomial  $\psi_{163}$  of  $E$  (we can choose any elliptic curve with  $j$ -invariant  $j$ ) over  $\mathbb{F}_{19}$  and obtain that the smallest factor is of degree 81. Since the rational prime 19 splits completely in  $\mathbb{Q}_{1,3}$ , this implies that the smallest factor of  $\psi_{163}$  over  $\mathbb{Q}_{1,3}$  is at least 81. But this means that  $[\mathbb{Q}(P) : \mathbb{Q}_{1,3}] \geq 81$ , so  $[\mathbb{Q}(P) : \mathbb{Q}] \geq 243$ , which is a contradiction with  $\mathbb{Q}(P)$  being a subfield of  $\mathbb{Q}_{3,3}$ .  $\square$

**Proof of Theorem 1.3.** It remains to determine when the 3-power torsion grows.

By Lemma 2.8, if  $E(\mathbb{Q}_{\infty,3})[3] \neq \{O\}$ , then  $E(\mathbb{Q})[3] \neq \{O\}$ .

Suppose  $E(\mathbb{Q}_{\infty,3})$  has a point  $P$  of order 27; then  $E$  has a rational 27-isogeny over  $\mathbb{Q}$ , so  $j(E) = -2^{15} \cdot 3 \cdot 53$ . By Corollary 2.10,  $P \in E(\mathbb{Q}_{2,3})$ . Let  $E = 27a2$ ; we have  $j(E) = -2^{15} \cdot 3 \cdot 53$ . We factor  $\psi := \psi_{27}/\psi_9$ , where  $\psi_n$  denotes the  $n$ -division polynomial of  $E$  - the polynomial which is the product of all the  $x$ -coordinates of the points of order  $n$  of  $E$  and obtain that this polynomial has roots over  $\mathbb{Q}_{2,3}$ . This implies that there exists a single quadratic twist (over  $\mathbb{Q}_{2,3}$ )  $E^\delta$  of  $E$ , for some  $\delta \in L^*/(L^*)^2$  such that  $E^\delta(\mathbb{Q}_{2,3})$  has a point of order 27. It remains to check whether  $E^\delta$  is defined over  $\mathbb{Q}$ , or equivalently, whether  $\delta \cdot u^2 = d$  for some  $u \in L^*$  and some  $d \in \mathbb{Q}^*/(\mathbb{Q}^*)^2$ . We compute a  $\delta$  and obtain that  $N_{\mathbb{Q}_{2,3}/\mathbb{Q}}(\delta) = 3^{49}$ , so the only twists that we can consider are  $d = 3$  and  $-3$ . We obtain that  $E^{-3}$ , which is  $27a4$  has a point of order 27 over  $\mathbb{Q}_{2,3}$ . Note that from our argumentation it follows that  $27a4$  is the only elliptic curve with a point of order 27 over  $\mathbb{Q}_{\infty,3}$ . For

$E = 27a4$ , we have that  $E(\mathbb{Q}_{\infty,3})_{\text{tors}} \simeq \mathbb{Z}/27\mathbb{Z}$ , as any larger torsion would violate Theorem 2.4.

If  $E(\mathbb{Q})[3^\infty] \simeq \mathbb{Z}/9\mathbb{Z}$ , then we claim that  $E(\mathbb{Q}_{\infty,3})_{\text{tors}} \simeq \mathbb{Z}/9\mathbb{Z}$ . Indeed there cannot be any  $q$ -torsion for any  $q \neq 2, 3$  as this would force the existence of a  $9q$ -isogeny over  $\mathbb{Q}$ . It is impossible that  $E(\mathbb{Q}_{\infty,3})$  gains any 2-torsion, as this would imply that there is  $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/18\mathbb{Z}$  torsion over  $\mathbb{Q}_{1,3}$ , which cannot occur by Lemma 5.6.

Finally from what we have already proved, when  $E(\mathbb{Q}_{\infty,3})[3^\infty] \simeq \mathbb{Z}/3\mathbb{Z}$  then  $E(\mathbb{Q}_{\infty,3})_{\text{tors}}$  has to be either  $\mathbb{Z}/3\mathbb{Z}$ ,  $\mathbb{Z}/6\mathbb{Z}$ ,  $\mathbb{Z}/12\mathbb{Z}$ ,  $\mathbb{Z}/21\mathbb{Z}$  or  $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$ .

These results combined prove that all the possible groups  $E(\mathbb{Q}_{\infty,3})_{\text{tors}}$  are contained in the list given in Theorem 1.3. The results of Section 6 show that each group on the list appears, apart from maybe  $\{O\}$  and  $\mathbb{Z}/3\mathbb{Z}$ . Hence, to complete the proof of Theorem 1.3, we need to show that there exist elliptic curves  $E/\mathbb{Q}$  such that  $E(\mathbb{Q}_{\infty,3})_{\text{tors}} \simeq \{O\}$  and  $E(\mathbb{Q}_{\infty,3})_{\text{tors}} \simeq \mathbb{Z}/3\mathbb{Z}$ .

For the trivial group  $\{O\}$  one can, as in the proof of Theorem 1.2, take an elliptic curve with surjective mod  $q$  Galois representation for all primes  $q$ . Such an elliptic curve  $E$  has no  $q$ -torsion over  $\mathbb{Q}_{\infty,3}$  for  $q > 2$  by the same arguments as in the proof of Theorem 1.2 and has no 2-torsion over  $\mathbb{Q}_{\infty,3}$ , as the cubic field generated by a 2-torsion point of  $E$  is not Galois over  $\mathbb{Q}$ , and is hence not contained in  $\mathbb{Q}_{\infty,3}$ .

To obtain an  $E/\mathbb{Q}$  such that  $E(\mathbb{Q}_{\infty,3})_{\text{tors}} \simeq \mathbb{Z}/3\mathbb{Z}$ , take an elliptic curve with  $E(\mathbb{Q})_{\text{tors}} \simeq \mathbb{Z}/3\mathbb{Z}$ , no 9-isogenies over  $\mathbb{Q}$ , and surjective mod  $q$  Galois representations for all  $q \neq 3$ , such as the one with Cremona reference 106c1. The same arguments as above and as in the proof Theorem 1.2 show that  $E(\mathbb{Q}_{\infty,3})_{\text{tors}} \simeq \mathbb{Z}/3\mathbb{Z}$ .  $\square$

## 6. Examples of torsion growth

In this last section, we address the following question. Fix a prime  $p$ . Given a group  $G$  that can appear as  $E(\mathbb{Q}_{\infty,p})_{\text{tors}}$  for some  $E/\mathbb{Q}$ , do there exist infinitely many  $j$ -invariants such that there exists an  $E/\mathbb{Q}$  with such a  $j$ -invariant with  $E(\mathbb{Q}_{\infty,p})_{\text{tors}} \simeq G$  but  $E(\mathbb{Q})_{\text{tors}} \not\simeq G$ ? By Theorem 1.1 we need only consider the cases where  $p = 2, 3$ .

**Theorem 6.1.** *Let  $G$  be one of the following groups:*

$$\begin{aligned} \mathbb{Z}/N\mathbb{Z}, & \quad 3 \leq N \leq 10, \text{ or } N = 12, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N\mathbb{Z}, & \quad 1 \leq N \leq 4, \end{aligned}$$

*There exist infinitely many elliptic curves  $E/\mathbb{Q}$  with distinct  $j$ -invariants such that  $E(\mathbb{Q}_{\infty,2})_{\text{tors}} \simeq G$  and  $E(\mathbb{Q})_{\text{tors}} \not\simeq G$ .*

**Proof.** We break this down into cases depending on whether  $E(\mathbb{Q})[2]$  needs to be trivial or not.

Suppose that  $G = \mathbb{Z}/N\mathbb{Z}$  for some odd integer  $N$ . Then there exist infinitely many elliptic curves  $E/\mathbb{Q}$  with distinct  $j$ -invariants such that  $E(\mathbb{Q})_{\text{tors}} \simeq$

$\mathbb{Z}/N\mathbb{Z}$  and  $E$  has no additional isogenies over  $\mathbb{Q}$ . This is true because for each  $N$  in the statement the elliptic curves  $E/\mathbb{Q}$  with  $E(\mathbb{Q})_{\text{tors}} \simeq \mathbb{Z}/N\mathbb{Z}$  come in a non-isotrivial 1-parameter family that generically doesn't have any additional isogenies. So by Hilbert irreducibility, outside of a "thin" set every curve in the family also has no additional isogenies. For more details about Hilbert irreducibility and thin sets, see [23, Chapter 9]. Thus, for each of these  $E$  the quadratic twist  $E^2$  of  $E$  by 2 will have trivial torsion over  $\mathbb{Q}$ , because for odd  $N$  we have that  $E(\mathbb{Q}(\sqrt{2}))[N] \simeq E(\mathbb{Q})[N] \oplus E^2(\mathbb{Q})[N]$  and since  $\mathbb{Q}(\sqrt{2})$  does not contain any  $m^{\text{th}}$ -roots of unity for any  $2 < m \mid N$  the existence of the Weil-pairing gives that  $E^2(\mathbb{Q})[N] = \{\mathcal{O}\}$ . Further, since  $E$  and  $E^2$  become isomorphic over  $\mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}_{\infty,2}$ , it follows that  $E^2(\mathbb{Q}_{\infty,2})_{\text{tors}} \simeq \mathbb{Z}/N\mathbb{Z}$ . Notice that the torsion can't grow any further since  $E$  and hence  $E^2$  don't have any additional isogenies.

Next, suppose that  $G = \mathbb{Z}/2n\mathbb{Z}$  with  $n \geq 2$ . Again there are infinitely many elliptic curves  $E/\mathbb{Q}$  such that  $E(\mathbb{Q})_{\text{tors}} \simeq \mathbb{Z}/2n\mathbb{Z}$  and  $E$  has no additional isogenies over  $\mathbb{Q}$ . Now we have  $E^2(\mathbb{Q})_{\text{tors}} \simeq \mathbb{Z}/2\mathbb{Z}$  and  $E^2(\mathbb{Q}_{2,\infty})_{\text{tors}} \simeq \mathbb{Z}/2n\mathbb{Z}$  or  $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2n\mathbb{Z}$ . Again, the torsion can't grow any further since  $E$  and hence  $E^2$  don't have any additional isogenies. If  $n > 4$ , we can't have that  $E^2(\mathbb{Q}_{2,\infty})_{\text{tors}} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2n\mathbb{Z}$  by Theorem 1.2, while when  $n = 2, 3, 4$  both cases are possible depending on the class of the discriminant of  $E$  modulo squares. Checking the generic elliptic curves with a rational torsion subgroup isomorphic to  $\mathbb{Z}/2n\mathbb{Z}$  for  $n = 2, 3, 4$  we see that there are infinitely many curves whose discriminant are congruent to 2 mod squares and infinitely many curves whose discriminants are congruent to  $-1$  mod squares. In the first case we have that  $E^2(\mathbb{Q}_{2,\infty})_{\text{tors}} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2n\mathbb{Z}$ , while in the second case  $E^2(\mathbb{Q}_{2,\infty})_{\text{tors}} \simeq \mathbb{Z}/2n\mathbb{Z}$ . So all that remains is to check the case when  $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ .

To finish the last case we give a non-isotrivial family  $E_t$  with  $E_t(\mathbb{Q})_{\text{tors}} \simeq \mathbb{Z}/2\mathbb{Z}$  and  $\mathbb{Q}(E_t[2]) = \mathbb{Q}(\sqrt{2})$ . This family is

$$E_t : y^2 = x^3 - \frac{2}{t^2 - 1/2}x^2 - \frac{2}{t^2 - 1/2}x$$

and generically these curves have no other isogenies and so for infinitely many of them  $E_t(\mathbb{Q}_{2,\infty})_{\text{tors}} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ .  $\square$

We list in Table 1 examples of elliptic curves with minimal discriminant achieving growth to each possible torsion group over  $\mathbb{Q}_{\infty,2}$ .

*Remark.* Clearly it is impossible for an elliptic curve to have its torsion "grow" and become trivial so in Theorem 6.1  $G$  cannot be the trivial group and since an elliptic curve can only go from having trivial 2-torsion to having a point of order 2 in an extension degree divisible by 3, it cannot be that torsion grows to  $\mathbb{Z}/2\mathbb{Z}$  over  $\mathbb{Q}_{\infty,2}$ .

Now we consider the cases of torsion growth over  $\mathbb{Q}_{\infty,3}$ . By the results in Section 5 we need to consider the cases of torsion growth listed on Table 2 (we

Cremona Reference	$E(\mathbb{Q})_{\text{tors}}$	$E(\mathbb{Q}_{\infty,2})_{\text{tors}}$
704d1	$\{\mathcal{O}\}$	$\mathbb{Z}/3\mathbb{Z}$
24a6	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$
704a1	$\{\mathcal{O}\}$	$\mathbb{Z}/5\mathbb{Z}$
320c1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/6\mathbb{Z}$
832f	$\{\mathcal{O}\}$	$\mathbb{Z}/7\mathbb{Z}$
24a3	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/8\mathbb{Z}$
1728j3	$\{\mathcal{O}\}$	$\mathbb{Z}/9\mathbb{Z}$
768b1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/10\mathbb{Z}$
30a5	$\mathbb{Z}/6\mathbb{Z}$	$\mathbb{Z}/12\mathbb{Z}$
14a5	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
24a2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$
14a2	$\mathbb{Z}/6\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$
32a4	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$

TABLE 1. Elliptic curves of minimal conductor with torsion growth over  $\mathbb{Q}_{\infty,2}$ .

Cremona Reference	$E(\mathbb{Q})_{\text{tors}}$	$E(\mathbb{Q}_{\infty,3})_{\text{tors}}$
162b2	$\{\mathcal{O}\}$	$\mathbb{Z}/7\mathbb{Z}$
324a2	$\{\mathcal{O}\}$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
27a3	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/9\mathbb{Z}$
162b1	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/21\mathbb{Z}$
27a4	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/27\mathbb{Z}$
324a1	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$

TABLE 2. Elliptic curves of minimal conductor with torsion growth over  $\mathbb{Q}_{\infty,3}$ .

offer an example of minimal conductor for each type of growth in question). We prove that besides  $\mathbb{Z}/3\mathbb{Z}$  to  $\mathbb{Z}/21\mathbb{Z}$  and  $\mathbb{Z}/3\mathbb{Z}$  to  $\mathbb{Z}/27\mathbb{Z}$  (which are easily explained by  $X_0(21)$  and  $X_0(27)$  having finitely many rational points), all of these cases occur for infinitely many  $j$ -invariants.

First we have a theorem that gives the conductor of a cyclic cubic number field in terms of its defining polynomial.

**Theorem 6.2** ([11]). *Let  $K$  be a number field with  $[K : \mathbb{Q}] = 3$  and  $\text{Gal}(K/\mathbb{Q}) \simeq \mathbb{Z}/3\mathbb{Z}$ . Then  $K = \mathbb{Q}(\theta)$  for a  $\theta$  satisfying  $\theta^3 + A\theta + B = 0$  with  $A, B \in \mathbb{Z}$  and, for any  $R \in \mathbb{Z}$ , if  $R^2 \mid A$  and  $R^3 \mid B$ , then  $|R| = 1$ . Further, the conductor  $\mathfrak{f}(K)$  is given by*

$$\mathfrak{f}(K) = 3^\alpha \prod_{\substack{p(\text{prime}) \equiv 1 \pmod{3} \\ p \mid (A,B)}} p$$

where, letting  $C$  be the square root of the discriminant of  $K$ ,

$$\alpha = \begin{cases} 0 & \text{if } 3 \nmid A \text{ or } 3 \parallel A, \ 3 \nmid B, \ 3^3 \mid C \\ 2 & \text{if } 3^2 \parallel A, \ 3^2 \parallel B \text{ or } 3 \parallel A, \ 3 \nmid B, \ 3^2 \parallel C. \end{cases}$$

The following lemma gives a way to construct an elliptic curve with torsion growth  $\{\mathcal{O}\}$  over  $\mathbb{Q}$  to  $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$  over  $\mathbb{Q}_{\infty,3}$  and also an elliptic curve with torsion growth  $\mathbb{Z}/3\mathbb{Z}$  over  $\mathbb{Q}$  to  $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$  over  $\mathbb{Q}_{\infty,3}$ .

**Lemma 6.3.** *Let*

$$j_3(h) = \frac{(h+27)(h+3)^3}{h}.$$

*Suppose that we have  $u, v \in \mathbb{Z}$  with  $(u, v) = 1$  and  $u^2 + 27v^2 = 4 \cdot 3^k \cdot p^3$  for some  $k = 2, 3$  and some  $p \equiv 1 \pmod{3}$ . Then there is an elliptic curve  $E/\mathbb{Q}$  with  $j$ -invariant  $j_3(\frac{u^2}{v^2})$  such that*

$$E(\mathbb{Q})_{\text{tors}} = \{\mathcal{O}\} \quad \text{and} \quad E(\mathbb{Q}_{\infty,3})_{\text{tors}} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

*Moreover there is a quadratic twist  $E'$  of  $E$  such that*

$$E'(\mathbb{Q})_{\text{tors}} \simeq \mathbb{Z}/3\mathbb{Z} \quad \text{and} \quad E'(\mathbb{Q}_{\infty,3})_{\text{tors}} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}.$$

**Proof.** Let  $E/\mathbb{Q}$  be an elliptic curve with  $E(\mathbb{Q})_{\text{tors}} \simeq \mathbb{Z}/3\mathbb{Z}$  and  $E(\mathbb{Q}_{\infty,3})_{\text{tors}} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$ . Then  $E$  has a rational 3-isogeny and square discriminant, so  $E$  corresponds to a rational point on  $X_0(3)$ , and so

$$j(E) = \frac{(h+27)(h+3)^3}{h},$$

for some  $h \in \mathbb{Q}$ . A model for an elliptic curve with such a  $j$ -invariant is given by

$$E_h : y^2 = f(x) = x^3 + \frac{-27(h+3)^3(h+27)}{(h^2+18h-27)^2}x + \frac{54(h+3)^3(h+27)}{(h^2+18h-27)^2},$$

and by computing the discriminant of this model we can see that  $E_h$  has square discriminant if and only if  $h \in (\mathbb{Q}^*)^2$ . If we choose an  $h$  such that the discriminant of  $E_h$  is a square, then  $\text{Gal}(\mathbb{Q}(E_h[2])/\mathbb{Q}) \simeq \mathbb{Z}/3\mathbb{Z}$  and  $\mathbb{Q}(E_h[2]) = \mathbb{Q}(f)$ .

Now we examine  $\mathbb{Q}(f)$  to determine when  $\mathbb{Q}(f) = \mathbb{Q}_{1,3}$ . This will occur precisely when the conductor of  $\mathbb{Q}(f)$  is divisible only by 3. Through a change of variables we see that  $\mathbb{Q}(f) = \mathbb{Q}(x^3 + A(h)x + B(h))$  where

$$A(h) = -27(h+3)(h+27) \quad \text{and} \quad B(h) = 54(h+27)(h^2+18h-27).$$

We homogenize the equations by letting  $h = \frac{u^2}{v^2}$  be written in lowers terms, so that all parameters are integers to obtain

$$A(u, v) = -27(u^2 + 3v^2)(u^2 + 27v^2)$$

and

$$B(u, v) = 54(u^2 + 27v^2)(u^4 + 18u^2v^2 - 27v^4).$$

By Theorem 6.2 the conductor will be a power of 3 when the gcd of  $A(u, v)$  and  $B(u, v)$  is divisible only by 3, primes  $p \equiv 2 \pmod{3}$ , and by cubes, since by a change of variables we can remove cubes from the gcd of  $A(u, v)$  and  $B(u, v)$ .

We can see that  $(A(u, v), B(u, v)) = 2^a \cdot 3^b \cdot (u^2 + 27v^2)$  for some  $a, b \in \mathbb{Z}^{\geq 0}$ , since if a prime divides both  $u^2 + 3v^2$  and  $u^4 + 18u^2v^2 - 27v^4$  then it must be 2 or 3. Thus, if we choose  $u$  and  $v$  as in the statement of the lemma, we see that  $\mathbb{Q}(E_h[2])$  will have conductor a power of 3, and thus  $\mathbb{Q}(E_h[2]) = \mathbb{Q}_{1,3}$ .

Finally, by construction,  $E_h$  is defined over  $\mathbb{Q}$  and has a rational 3-isogeny. Thus, there is a quadratic twist of  $E_h$  that has a 3-torsion point over  $\mathbb{Q}$ . Note that taking a quadratic twist does not change the field of definition of the 2-torsion points, so this twist indeed has the growth we are looking for over  $\mathbb{Q}_{1,3}$ .  $\square$

Now, we have a lemma to ensure there are infinitely many non-isomorphic  $E/\mathbb{Q}$  with the above torsion growth.

**Lemma 6.4.** *For any prime  $p \equiv 1 \pmod{3}$  and  $k = 2, 3$  there exist  $u, v \in \mathbb{Z}$  with  $(u, v) = 1$  such that  $u^2 + 27v^2 = 4 \cdot 3^k \cdot p^3$ .*

**Proof.** Let  $K = \mathbb{Q}(\sqrt{-3})$ . Then

$$u^2 + 27v^2 = \text{Nm}_{K/\mathbb{Q}}(u + 3v\sqrt{-3})$$

and so we wish to prove that there are elements of the form  $u + 3v\sqrt{-3}$  with  $(u, v) = 1$  of norm  $4 \cdot 3^k \cdot p^3$  for any  $p \equiv 1 \pmod{3}$  and for  $k = 2, 3$ . Since norms are multiplicative, and we see that

$$4 \cdot 3^2 = \text{Nm}_{K/\mathbb{Q}}(3 + 3\sqrt{-3})$$

and

$$4 \cdot 3^3 = \text{Nm}_{K/\mathbb{Q}}(9 + 3\sqrt{-3})$$

it remains to show that there is an element of norm  $p^3$  in  $K$ .

Let  $\alpha$  be a root of  $x^2 + x + 1$ , so that the ring of integers of  $K$  is equal to  $\mathbb{Z}[\alpha]$ . Since  $p \equiv 1 \pmod{3}$ , this prime splits in  $K$  and so

$$p = \mathfrak{p}\bar{\mathfrak{p}} = (x + y\alpha)(x + y\alpha^2)$$

for some  $x, y \in \mathbb{Z}$ . We have  $\text{Nm}_{K/\mathbb{Q}}(\mathfrak{p}) = p$  and so we claim  $\mathfrak{p}^3$  is the element we want to take. We can find relatively prime  $a, b \in \mathbb{Z}$  such that

$$\mathfrak{p}^3 = a + 3b\alpha.$$

Indeed, simply writing  $\mathfrak{p}^3 = (x + y\alpha)^3$  for some  $x, y \in \mathbb{Z}$  and expanding shows that the coefficient of  $\alpha$  is divisible by 3. Further, if  $d = (a, b) > 1$ , then

$$\mathfrak{p}^3 = (d) \left( \frac{a}{d} + 3\frac{b}{d}\alpha \right).$$

However, taking norms on both sides shows that  $d = p$ , so then  $\bar{\mathfrak{p}} \mid \mathfrak{p}^3$ , which is impossible.

Thus, we have found elements in  $K$  of norm  $4 \cdot 3^2$ ,  $4 \cdot 3^3$ , and  $p^3$  for any prime  $p \equiv 1 \pmod{3}$ . The lemma follows from expanding the product of  $3 + 3\sqrt{-3}$  and  $9 + 3\sqrt{-3}$  with  $\mathfrak{p}^3$  to show that the product is indeed of the form  $u + 3v\sqrt{-3}$  with  $(u, v) = 1$ .  $\square$

An immediate corollary of this lemma is:

**Corollary 6.5.** *There are infinitely many  $j \in \mathbb{Q}$  such that there exists an elliptic curve  $E/\mathbb{Q}$  with  $j(E) = j$  that satisfies  $E(\mathbb{Q})_{\text{tors}} = \{\mathcal{O}\}$  and  $E(\mathbb{Q}_{\infty,3})_{\text{tors}} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ .*

*There are infinitely many  $j \in \mathbb{Q}$  such that there exists an elliptic curve  $E/\mathbb{Q}$  with  $j(E) = j$  that satisfies  $E(\mathbb{Q})_{\text{tors}} \simeq \mathbb{Z}/3\mathbb{Z}$  and  $E(\mathbb{Q}_{\infty,3})_{\text{tors}} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$ .*

Now we illustrate a parallel idea for torsion growth from  $\{\mathcal{O}\}$  to  $\mathbb{Z}/7\mathbb{Z}$  and from  $\mathbb{Z}/3\mathbb{Z}$  to  $\mathbb{Z}/9\mathbb{Z}$  over  $\mathbb{Q}_{\infty,3}$ , respectively.

**Lemma 6.6.** *Let*

$$j_7(h) = \frac{(h^2 + 13h + 49)(h^2 + 5h + 1)^3}{h}.$$

*Suppose that we have  $u, v \in \mathbb{Z}$  with  $(u, v) = 1$  and  $u^2 + 13uv + 49v^2 = 3^k \cdot p^3$  for some  $k = 2, 3$  and some  $p \equiv 1 \pmod{3}$ . Then there is an elliptic curve  $E/\mathbb{Q}$  with  $j$ -invariant  $j_7(\frac{u}{v})$  such that  $E(\mathbb{Q})_{\text{tors}} = \{\mathcal{O}\}$  and  $E(\mathbb{Q}_{\infty,3})_{\text{tors}} \simeq \mathbb{Z}/7\mathbb{Z}$ .*

**Proof.** Let  $E/\mathbb{Q}$  be an elliptic curve with  $E(\mathbb{Q}_{\infty,3})_{\text{tors}} \simeq \mathbb{Z}/7\mathbb{Z}$ . Then  $E/\mathbb{Q}$  has a 7-isogeny over  $\mathbb{Q}$ , so it corresponds to a rational point on  $X_0(7)$ , and so

$$j(E) = \frac{(h^2 + 13h + 49)(h^2 + 5h + 1)^3}{h}$$

for some  $h \in \mathbb{Q}$ . A model for an elliptic curve with such a  $j$ -invariant is given by

$$E_h : y^2 = x^3 - \frac{27(h^2 + 5h + 1)^3(h^2 + 13h + 49)}{(h^4 + 14h^3 + 63h^2 + 70h - 7)^2}x + \frac{54(h^2 + 5h + 1)^3(h^2 + 13h + 49)}{(h^4 + 14h^3 + 63h^2 + 70h - 7)^2}.$$

We can compute the 7<sup>th</sup> division polynomial of  $E_h$ , and obtain that it has one irreducible factor of degree 3, which we denote by  $f_7$ , and one irreducible factor of degree 21. We wish to determine for which values of  $h$  does this degree 3 factor define the extension  $\mathbb{Q}_{1,3}$ .

By a change of coordinates we see that  $\mathbb{Q}(f_3(h)) = \mathbb{Q}(x^3 + A(h)x + B(h))$  where

$$A(h) = -3(h^2 + 13h + 49) \text{ and } B(h) = -(2h + 13)(h^2 + 13h + 49).$$

We homogenize the equations by letting  $h = \frac{u}{v}$  with  $(u, v) = 1$  so that all parameters are integers to obtain

$$A(u, v) = -3(u^2 + 13uv + 49v^2)$$

and

$$B(u, v) = -(2u + 13v)(u^2 + 13uv + 49v^2).$$

By Theorem 6.2 the conductor will be a power of 3 when the gcd of  $A(u, v)$  and  $B(u, v)$  is divisible only by 3, primes  $p \equiv 2 \pmod{3}$ , and by cubes, since by a change of variables, we can remove cubes from the gcd of  $A(u, v)$  and  $B(u, v)$ .

We can see that  $(A(u, v), B(u, v)) = 2^a \cdot 3^b \cdot (u^2 + 13uv + 49v^2)$  for some  $a, b \in \mathbb{Z}^{\geq 0}$ . Thus, if we choose  $u$  and  $v$  as in the statement of the lemma, we see that  $\mathbb{Q}(f_7)$  will have conductor a power of 3, and thus  $\mathbb{Q}(f_7) = \mathbb{Q}_{1,3}$ . Now taking an appropriate quadratic twist, we can make  $\mathbb{Q}(f_7) = \mathbb{Q}(P)$  for a point  $P \in E[7]$  of order 7. □

**Lemma 6.7.** *Let  $h = \frac{u}{v}$  for  $u, v \in \mathbb{Z}$  with  $(u, v) = 1$  satisfying*

$$u^2 + 3uv + 9v^2 = 3^3 \cdot p^3$$

*for some prime  $p \equiv 1 \pmod{3}$  and let  $E_h/\mathbb{Q}$  be the elliptic curve given by*

$$E_h : y^2 = x^3 - 27h^5(h^3 - 24)^5x + 54h^6(h^3 - 24)^6(h^6 - 36h^3 + 216).$$

*Then  $E_h(\mathbb{Q})_{tors} \simeq \mathbb{Z}/3\mathbb{Z}$  and  $E_h(\mathbb{Q}_{\infty,3})_{tors} \simeq \mathbb{Z}/9\mathbb{Z}$ .*

**Proof.** For such torsion growth to occur, an elliptic curve must have a 3-torsion point over  $\mathbb{Q}$  as well as a 9-isogeny over  $\mathbb{Q}$  whose kernel contains this point of order 3. A model for elliptic curves over  $\mathbb{Q}$  with this level structure is given in [3] Table 6, and they are elliptic curves precisely of the form

$$E_h : y^2 = x^3 - 27h^5(h^3 - 24)^5x + 54h^6(h^3 - 24)^6(h^6 - 36h^3 + 216)$$

for some  $h \in \mathbb{Q}$ . We can compute the 9<sup>th</sup> division polynomial of  $E_h$  and divide it by the 3<sup>rd</sup> division polynomial and obtain one factor of degree 3, which we denote by  $f_9$ . We wish to determine for which values of  $h$  does this degree 3 factor define the extension  $\mathbb{Q}_{1,3}$ .

By a change of coordinates we see that  $\mathbb{Q}(f_9(h)) = \mathbb{Q}(x^3 + A(h)x + B(h))$  where

$$A(h) = -432(h^2 + 3h + 9) \text{ and } B(h) = -1728(2h + 3)(h^2 + 3h + 9).$$

We homogenize the equations by letting  $h = \frac{u}{v}$  with  $(u, v) = 1$  so that all parameters are integers to obtain

$$A(u, v) = -432(u^2 + 3uv + 9v^2) \text{ and } B(u, v) = -1728(2u + 3v)(u^2 + 3uv + 9v^2).$$

By Theorem 6.2 the conductor will be a power of 3 when the gcd of  $A(u, v)$  and  $B(u, v)$  is divisible only by 3, primes  $p \equiv 2 \pmod{3}$ , and by cubes, since by a change of variables, we can remove cubes from the gcd of  $A(u, v)$  and  $B(u, v)$ .

We can see that  $(A(u, v), B(u, v)) = 2^a \cdot 3^b \cdot (u^2 + 3uv + 9v^2)$  for some  $a, b \in \mathbb{Z}^{\geq 0}$ . Thus, if we choose  $u$  and  $v$  as in the statement of the lemma, we see that  $\mathbb{Q}(f_9)$  will have conductor a power of 3, and thus  $\mathbb{Q}(f_9) = \mathbb{Q}_{1,3}$ . Now, if  $P$  is a generator of the kernel of the isogeny, then  $G_{\mathbb{Q}}$  acts on  $\langle P \rangle$  by multiplication by 1, 4, or 7, since  $3P \in E(\mathbb{Q})$ . Hence,  $P$  is fixed by an index 3 subgroup of  $G_{\mathbb{Q}}$ , and thus defined over a cubic field, in particular, the cubic field where its  $x$ -coordinate is defined, i.e.  $\mathbb{Q}(f_9) = \mathbb{Q}_{1,3}$ .  $\square$

We remark that the criteria given in Lemma 6.6 and Lemma 6.7 are asking when  $3^3 p^3$  is primitively represented by some binary quadratic form. In both Lemmas, the binary quadratic forms have discriminant  $-27$ . We now prove that both the above criteria are satisfied for infinitely many primes  $p$  by proving a statement about integers represented by binary quadratic forms of discriminant  $-27$ .

**Lemma 6.8.** *Let  $f(x, y)$  be any binary quadratic form of discriminant  $-27$ . Then there exist primitive solutions to  $f(x, y) = 3^3 \cdot p^3$  for all  $p \equiv 1 \pmod{3}$ .*

**Proof.** Since  $f(x, y)$  has discriminant  $-27$ , there is an  $\mathrm{SL}_2(\mathbb{Z})$  transformation of variables so that

$$f(x, y) \sim u^2 + uv + 7v^2,$$

i.e.  $\begin{bmatrix} u \\ v \end{bmatrix} = M \begin{bmatrix} x \\ y \end{bmatrix}$  for some  $M \in \mathrm{SL}_2(\mathbb{Z})$ . Moreover,  $\mathrm{SL}_2(\mathbb{Z})$  transformations preserve the gcd of the coordinates, so we need only find primitive solutions to

$$u^2 + uv + 7v^2 = 3^3 \cdot p^3$$

for all primes  $p \equiv 1 \pmod{3}$ . We let  $K = \mathbb{Q}(\sqrt{-3})$  with ring of integers by  $\mathbb{Z}[\alpha]$  where  $\alpha = \frac{-1+\sqrt{-3}}{2}$  is a primitive  $3^{\mathrm{rd}}$  root of unity. Notice that

$$u^2 + uv + 7v^2 = \mathrm{Nm}_{K/\mathbb{Q}}(u + 3v\alpha),$$

so we want to find elements in  $K$  of the form  $u + 3v\alpha$  with  $(u, v) = 1$  and norm  $3^3 \cdot p^3$ .

Let  $p$  be a prime such that  $p \equiv 1 \pmod{3}$ . Then  $p$  is split in  $K$ , so

$$p = \mathfrak{p}\bar{\mathfrak{p}} = (x + y\alpha)(x + y\alpha^2)$$

for some  $x, y \in \mathbb{Z}$ . Recalling the argument given in the proof of Lemma 6.4,

$$\mathfrak{p}^3 = a + 3b\alpha$$

for some relatively prime integers  $a, b \in \mathbb{Z}$ . Further, we have that 3 is ramified in  $K$ ,

$$(3) = \mathfrak{p}_3^2 = (1 + 2\alpha)^2$$

and finally that  $\mathfrak{p}_3^3 = (3 + 2(3\alpha))$ . Thus, an element of norm  $3^3p^3$  is

$$\mathfrak{p}_3^3 p^3$$

and the lemma follows from expanding the product of the two elements to show that the product is indeed of the form  $u + 3v\alpha$  with  $(u, v) = 1$ .  $\square$

Thus, an immediate corollary is

**Corollary 6.9.** *There are infinitely many  $j \in \mathbb{Q}$  such that there exists an elliptic curve  $E/\mathbb{Q}$  with  $j(E) = j$  that satisfies  $E(\mathbb{Q})_{tors} = \{\mathcal{O}\}$  and  $E(\mathbb{Q}_{\infty,3})_{tors} \simeq \mathbb{Z}/7\mathbb{Z}$ .*

*There are infinitely many  $j \in \mathbb{Q}$  such that there exists an elliptic curve  $E/\mathbb{Q}$  with  $j(E) = j$  that satisfies  $E(\mathbb{Q})_{tors} \simeq \mathbb{Z}/3\mathbb{Z}$  and  $E(\mathbb{Q}_{\infty,3})_{tors} \simeq \mathbb{Z}/9\mathbb{Z}$ .*

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